Math. Ann. 293, 579–594 (1992)

Mathematische Annalen


# Bounds for stable bundles and degrees of Weierstrass schemes

**F. Catanese**[1] **and M. Schneider**[2]

[1] Dipartimento di Matematica, Universitá di Pisa, Via Buonarroti 2, 56127 Pisa, Italy
[2] Mathematisches Institut der Universität Bayreuth, Postfach 101251, W-8580 Bayreuth, Federal Republic of Germany

Received December 2, 1991

## 0 Introduction

Mumford and Takemoto [Mu1, Ta] introduced the notion of a stable vector bundle $E$ on a projective (or Kähler) variety $X$, in terms of the slope relative to a chosen polarization.

When $X$ is a curve, and $E$ is semistable, the condition of stability immediately implies that the degree of the zero scheme of a section $\sigma$ of $E$ is bounded by the slope of $E$ (which in this case is the quotient $d/r$, $d$ being the degree of $E$, $r$ being the rank of $E$).

The main purpose of this paper is to show that a similar result holds also in higher dimensions.

To be more precise, recall that if $n = \dim X$ and $r = \operatorname{rank} E$, and $\sigma$ is a section whose associated zero scheme $Z_\sigma$ has codimension $r$, then the $r$-th Chern class $c_r(E)$ gives precisely the cohomology class of $Z_\sigma$, hence, if we have chosen an ample divisor $H$ on $X$, the degree of $Z_\sigma$ equals $H^{n-r} \cdot c_r(E)$.

In general we show

**Theorem 2.2.** *Let $E$ be a semistable vector bundle on a Cohen-Macaulay projective variety $X$ of dimension $n$ which is non-singular in codimension 2 and is embedded by a very ample divisor $H$. There is a polynomial type function $P_{n,h}(E)$ in the class $H$, in the first $h$ Chern classes of $E$ and in the first two Chern classes of $X$, such that for every nonzero section $\sigma \in H^0(X, E)$, whose scheme of zeroes $Z$ has codimension $h$, the degree of $Z$ is bounded from above by $P_{n,h}(E)$.*

Here, by a polynomial type function $P$ in some cohomology classes, we mean a polynomial $P$ in the absolute values of the numbers obtained by taking top dimensional cup products of these cohomology classes.

The main motivation to establish the above estimate was to obtain upper bounds for the degrees of Weierstrass loci on varieties of general type, for which the cotangent bundle $\Omega^1_X$ is known to be semistable by a result of Tsuji [Tsu].

We obtain the desired upper bounds, namely

**Theorem 3.14.** *Let $X$ be a smooth projective variety of dimension $n$ with ample canonical bundle $K$.*

*Then there exist constants $C$ and $N$ such that the degrees of the Weierstrass schemes $W_k^m(K)$ (with respect to the ample divisor $K$), can be bounded by $C$ times the $N$-th power of $K^n$.*

We postpone to subsequent work the attempt to use these types of results in order to give polynomial upper bounds for the order of the group of automorphisms $\mathrm{Aut}(X)$ of varieties of general type (cf. the work of Corti in [Cor], where this program was carried out with success in the case of surfaces).

The above mentioned results are based on a sequel of technical results which are contained in the first section and that we would like to summarize in the following Theorem A, which incorporates results from Theorem 1.9, Proposition 1.11 and Corollary 1.15.

**Theorem A.** *Let $X$ be a Cohen-Macaulay projective variety which is non-singular in codimension 2, has dimension $n$ and is embedded in $\mathbb{P}^N$ with degree $d$ by a very ample divisor $H$.*

*Let $E$ be a semistable rank $r$ vector bundle with $H$-slope $\mu = \mu(E)$. There exist universal polynomial type functions $Q_n(E)$ and $P_n(E)$ in the class $H$ and in the first two Chern classes of $E$ and $X$, such that*

(i) $H^q(X, E(kH)) = 0$ *for each* $q \geqq 1$, *and* $k \geqq Q_n(E)$,

(ii) $E(kH)$ *is globally generated for* $k \geqq Q_n(E) + n$,

(iii) $\dim H^q(X, E) \leqq P_n(E)$ *for* $q \geqq 1$,

(iv) *there exists a universal polynomial $F_n(\mu, d)$ with positive coefficients (which depend only upon $r$ and $n$) in the variables $\mu$, $d$, and $d^{-1}$, and with degree equal to $n$ in $\mu$, such that*

$$\dim H^0(X, E) \leqq F_n(\mu, d) \quad \text{if } \mu \geqq 0 .$$

Bounds of this type for projective space have been given by Elencwajg and Forster in [E-F].

Two words concerning the techniques of proof of Theorem A: we use an effective version due to Flenner of the Mehta-Ramanathan semi-stable restriction theorem, and the Castelnuovo-Mumford theory of $m$-regularity of a sheaf.

Other consequences of our results here are

**Theorem 3.9.** *The dimension of the moduli space $M = M(c_1, \ldots, c_{\min(r,n)})$ of stable vector bundles of rank $r$ with given Chern classes can be bounded from above by a polynomial type function in the class of $H$, in the first two Chern classes of $X$ and in $c_1, c_2$.*

and

**Theorem 3.11.** *Let $X$ be a smooth projective variety of dimension $n \geqq 3$ and let $E$ be a semistable vector bundle on $X$.*

*For each multiindex I there exists a polynomial type function $P_I$ depending only on H, and the first two Chern classes of E and X, such that*

$$c_I(E) \cdot H^{n-|I|} \leqq P_I .$$

The second paragraph contains also an attempt to define a canonical sequence of residual subschemes for the zero scheme of a section of bundle, and to bound their degrees.

The last paragraph contains the above mentioned and some further applications.

## 1 Bounds for global generation and cohomology vanishing

Let $E$ be a rank $r$ vector bundle on a Cohen-Macaulay projective variety $X$ of dimension $n$ which we assume to be embedded by a very ample divisor $H$ and to be non-singular in codimension 2 (in particular $X$ is normal: varieties of this type are e.g. varieties with terminal singularities).

We denote by $d$ the degree of $X$, thus $d = \deg(X) = H^n$.

The ($H$-) slope $\mu(E)$ is defined to be $(1/r)\, c_1(E) \cdot H^{n-1}$, and $E$ is said to be ($H$-) semistable if $\mu(\mathscr{F}) \leqq \mu(E)$ for every nontrivial subsheaf $\mathscr{F}$ of $E$.

Notice that, if $Y$ is a hypersurface in the linear system $|mH|$, then the slope $\mu(E_{|Y}) = m\mu(E)$.

Also, the first two Chern classes of $Y$ are expressed in terms of the first two Chern classes of $X$ and $H$ as follows:

$$c_1(Y) = (c_1(X) - mH)_{|Y} ,$$

$$c_2(Y) = (c_2(X) - mHc_1(X) + m^2H^2)_{|Y} .$$

We remark moreover that if $X$ is Cohen-Macaulay and non-singular in codimension 2, the same is true for $Y$ if $Y$ is sufficiently general.

Before we explain what we are going to prove, we need to introduce the concept of a polynomial type function.

(1.1) A function $G: \mathbb{R}^n \to \mathbb{R}$ is said to be of polynomial type if there exists a polynomial $P(x) = P(x_1, \ldots, x_n)$ with real coefficients, such that $G(x) = P(|x_1|, \ldots, |x_n|)$.

In general, given a polynomial $P(x) = P(x_1, \ldots, x_n)$ with real coefficients, we denote by $P^+(x)$ the polynomial obtained by replacing the coefficients of $P$ by their absolute values.

One has then obviously

(1.2) $P(x) \leqq P^+(|x_1|, \ldots, |x_n|)$, and if $G_1(x), G_2(x), \ldots G_r(x)$ are of polynomial type, then there exists a function $G(x)$ of polynomial type such that $G(x) \geqq \max(G_1(x), G_2(x), \ldots, G_r(x))$.

We shall apply (1.2) in the following way:

(1.3) Assume that we are given a finite number of subsets of $\mathbb{R}^n$, $S_1, S_2, \ldots, S_s$, whose union is the whole of $\mathbb{R}^n$, and corresponding polynomials $P_1, P_2, \ldots, P_s$, such that the following holds: given an integer $k$ and a point $x$ in $S_i$, a certain property $(\mathscr{P}_k(X))$ holds true if the integer $k$ is at least $\geqq P_i(x)$.

Then there exists a polynomial type function $G$ on $\mathbb{R}^n$ such that, for each $x$, $(\mathscr{P}_k(x))$ holds true if the integer $k$ is at least $\geqq G(x)$.

Moreover, by slight abuse of language, when we shall speak of a polynomial $P$ in certain cohomology classes we shall mean that $P$ is a polynomial function in the numbers which are obtained from these cohomology classes by evaluating on the fundamental class all the monomials of maximal degree in these cohomology classes (similarly for polynomial type functions).

We shall show, by induction on $n$, that there exist a polynomial type function $Q_n(E) = Q_n(c_1(E), c_2(E), c_1(X), c_2(X), H)$ in the first two Chern classes of $E$, $X$ and in the class $H$, and polynomial type functions (except that for $q = 0$ we have indeed a polynomial). $G_{q,n}(E, k) = G_{q,n}(c_1(E), c_2(E), c_1(X), c_2(X), H, k)$ in $k$, in the first two Chern classes of $E$, $X$ and in the class $H$, such that

(1.4) $H^q(X, E(kH)) = 0$ for each $q \geqq 1$, and $k \geqq Q_n(E)$,

(1.5) $\dim(H^q(X, E(kH)) \leqq G_{q,n}(E, k)$.

The inductive procedure is made possible by virtue of the following result (see [Fl]):

**(1.6) Theorem** (Flenner). *Given a semistable rank $r$ vector bundle $E$ on a normal projective variety $X$ embedded in $\mathbb{P}^N$ by a very ample divisor $H$, assume that $m$ is an integer satisfying $1/m(C(m, n+m) - m - 1) > \deg(X) \max((r^2 - 1)/4, 1)$, $C(a, b)$ being the binomial coefficient, and assume also that $Y$ is a sufficiently general element of the linear system $|mH|$: then the restriction $E_{|Y}$ is semistable.*

*(1.7) Remark.* It is easy to see that, for fixed $r$ and $n$, Flenner's inequality holds if $1/m\,(\text{const.}\, m^n - m - 1) > \text{const.}\deg(X)$, i.e. for $m^{n-1} > \text{const.}\deg(X)$, a fortiori if $m > \text{const.}\deg(X)$. In the case of surfaces ($n = 2$), $m = 2d$ suffices for $r = 2$, otherwise for higher $r$, $m \geqq d(r^2 - 1)/2$.

We also recall the basic results of the Castelnuovo-Mumford theory of regularity (cf. [Mu2, lecture 14, p. 99 and foll.])

**(1.8) Theorem** (Castelnuovo-Mumford). *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^N$ which is $m$-regular, i.e. such that $H^q(\mathbb{P}^N, \mathscr{F}(m-q)) = 0$ for all $q \geqq 1$: then*

*(i) $H^0(\mathbb{P}^N, \mathscr{F}(k-1)) \otimes H^0(\mathbb{P}^N, \mathscr{O}(1)) \to H^0(\mathbb{P}^N, \mathscr{F}(k))$ is surjective for $k > m$,*

*(ii) $H^q(\mathbb{P}^N, \mathscr{F}(k)) = 0$ for each $q \geqq 1$, whenever $k + q \geqq m$, and thus in particular,*

*(iii) $\mathscr{F}(k)$ is generated by global sections for $k \geqq m$.*

**(1.9) Theorem.** *Let $X$ be a Cohen-Macaulay projective variety which is non-singular in codimension 2, has dimension $n$ and is embedded in $\mathbb{P}^N$ with degree $d$ by a very ample divisor $H$.*

*Let $E$ be a semistable rank $r$ vector bundle with $H$-slope $\mu(E)$. There exist universal polynomial type functions $Q_n(E) = Q_n(c_1(E), c_2(E), c_1(X), c_2(X), H)$ in the class $H$, in the first two Chern classes* of *$E$ and $X$, and $G_{q,n}(E, k) = G_{q,n}(c_1(E), c_2(E),$*

*$c_1(X), c_2(X), H, k)$ in $k$, in the class $H$, in the first two Chern classes of $E$ and $X$, such that the following* (1.4) *and* (1.5) *hold:*

(1.4) $H^q(X, E(kH)) = 0$ *for each* $q \geqq 1$, and $k \geqq Q_n(E)$ ,

(1.5) $\dim(H^q(X, E(kH))) \leqq G_{q,n}(E, k)$ .

*Moreover, the assertion* $\dim(H^0(X, E(kH))) \leqq G_{0,n}(E, k)$ *can be sharpened to the following one: there exists a polynomial* $G_n(E, k)$ *in the variables* $v = \mu + kd$ ($v$ *is the slope of* $E(kH)$), $d$, $d^{-1}$, *and of degree* $n$ *in* $v$, *such that* $\dim(H^0(X, E(kH))) \leqq G_n(E, k)$ *if* $v \geqq 0$,

*whereas obviously* $H^0(X, E(kH)) = 0$ *if* $v < 0$.

(There is by Serre duality an analogous assertion for $q = n$.)

*Proof.* The proof of Theorem 1.9 shall be given through a sequel of propositions and several intermediate steps.

*Step* $n = 1$ *($X$ is a curve).*
Let us consider the case $n = 1$. In this case we recall a fact we already observed in the introduction, namely that

(1.10) Any nonzero section of a semi-stable bundle $E$ of rank $r$ vanishes at most at $\mu(E)$ points. Whence

$$h^0(X, E) \leqq \deg(E) + r \text{ if } \deg(E) \geqq 0 ,$$

whereas,

$$\text{if } \deg(E) < 0, \quad \text{then } h^0(X, E) = 0 .$$

By Riemann-Roch we get (if $\deg(E) \geqq 0$), $h^1(X, E) \leqq rg$, where $g$ is the genus of $X$.
On the other hand, Serre duality yields (even when $\deg(E) \leqq 0$)

$$h^1(X, E) \leqq r(2g-1) - \deg(E), \quad \text{when } \deg(E) \leqq r(2g-2) ,$$

whereas

$$h^1(X, E) = 0 \quad \text{for } \deg(E) > r(2g-2) .$$

Therefore we can take $Q_1(E) = 2g - 1 + |\mu(E)|$,

$$G_{0,1}(E, k) = |\deg(E)| + r + r\deg(X)|k| ,$$

$$G_{1,1}(E, k) = r(2g-1) + |\deg(E)| + r\deg(X)|k| .$$

Whereas, $h^0(X, E(kH)) \leqq G_1(E, k) = r(v + 1)$ if $v \geqq 0$. Q.E.D.

**(1.11) Proposition.** *Given a semistable rank $r$ vector bundle $E$ on a normal and Cohen-Macaulay projective variety $X$ embedded in $\mathbb{P}^N$ by a very ample divisor $H$, let $\mu$ be the slope of $E$, and $d$ the degree of $X$.*

*Then there exists a universal polynomial $F_n(\mu, d)$ with positive coefficients (which depend only upon $r$ and $n$) in the variables $\mu$, $d$, and $d^{-1}$, with degree equal to $n$ in $\mu$, and such that*

$$h^0(X, E) \leqq F_n(\mu, d) \textit{ if } \mu \geqq 0 \textit{ (else, if } \mu < 0,\ h^0(X, E) = 0) .$$

*Proof.* We already dealt with the case $n = 1$, $G_1(\mu, d) = r(\mu + 1)$.

Assume now inductively that we have found the required polynomials for dimension $n-1$, and let $Y \in |mH|$ be as in Flenner's restriction Theorem 1.6. Note that, if $Y$ is general, $Y$ is again Cohen-Macaulay and non-singular in codimension 1, whence $Y$ is also normal and the inductive assumptions are fulfilled.

We consider the long cohmology sequence associated to the exact sequence of sheaves obtained by restricting $E(-\text{im}\,H)$ to $Y$:

$$0 \to E(-(i+1)mH) \to E(-imH) \to E(-imH)_{|Y} \to 0 \,. \tag{1.12}$$

We set $\mu = h(md) + x$, with $0 \leqq x < md$.

Notice once again that if $\mu$ is the slope of $E$ on $X$, then the slope of $E_{|Y}$ equals $m\mu$. In particular, $E(-imH)_{|Y}$ has no sections for $i \geqq h+1$.

By (1.12) and the induction hypothesis we obtain

$$h^0(X,E) \leqq \sum_{i=0,\dots,h} h^0(X, E(-imH)_{|Y}) \leqq$$
$$\leqq \sum_{i=0,\dots,h} F_{n-1}(m(\mu - imd), md) = \sum_{i=0,\dots,h} F_{n-1}(m(x+imd), md) \,.$$

Write $F_{n-1}(t, md) = \sum_{j=0,\dots,(n-1)} a_j(md, m^{-1}d^{-1})t^j$.

Then

$$h^0(X,E) \leqq \sum_{j=0,\dots,(n-1)} a_j(md, m^{-1}d^{-1}) \sum_{i=0,\dots,h} (m(x+imd))^j =$$
$$= \sum_{j=0,\dots,(n-1)} a_j \cdot (m^2 d)^j \sum_{i=0,\dots,h} (i + x/md)^j \,.$$

Since the polynomials $a_j$ have positive coefficients and $x/md < 1$, we can bound

$$h^0(X,E) \leqq \sum_{j=0,\dots,(n-1)} a_j \cdot (m^2 d)^j \left( \sum_{i=0,\dots,h+1} i^j \right).$$

Elementary calculus yields $\left( \sum_{i=0,\dots,h+1} i^j \right) \leqq (1/j+1)(h+2)^{j+1}$. Whence finally

$$h^0(X,E) \leqq \sum_{j=0,\dots,(n-1)} a_j \cdot (m^2 d)^j (1/j+1)(h+2)^{j+1} \leqq$$
$$\leqq \sum_{j=0,\dots,(n-1)} (1/j+1)\, a_j \cdot (m^2 d)^j (2 + \mu/md)^{j+1} \leqq$$
$$\leqq \sum_{j=0,\dots,(n-1)} (1/j+1)\, a_j(md, m^{-1}d^{-1})\, m^{j-1} d^{-1} \quad (\mu + 2md)^{j+1} \,.$$

By Remark 1.7 we may take $m = cd$, where $c$ is a constant $\geqq 1$ depending only on $r$ and $n$, thus

$$h^0(X,E) \leqq F_n(\mu, d) = (\text{definition}) =$$
$$= \sum_{j=0,\dots,(n-1)} (1/j+1) a_j(cd^2, c^{-1}d^{-2})\, c^{j-1} d^{j-2} \quad (\mu + 2cd^2)^{j+1} \,.$$

Q.E.D. for Proposition 1.11.

*(1.13) Remarks.* (i) We would like to spell out in detail the polynomial $F_2(\mu, d)$: since for $F_1(\mu, d)$ we have $a_0 = a_1 = r$, then

$$F_2(\mu, d) = r\{(\mu + 2cd^2)c^{-1}d^{-2} + 1/2d^{-1}(\mu + 2cd^2)^2\} = \\ = r\{(2 + 2c^2d^3) + (c^{-1}d^{-2} + 2cd)\mu + 1/2d^{-1}\mu^2\} .$$

(ii) it is clear that $G_{0,n}(E, k)$ is obtained from $F_n(\mu, d)$ by replacing $\mu$ by $|\mu| + |k|d$, and $d^{-1}$ by 1.

(iii) $G_{n,n}(E, k)$, in view of Serre duality, is obtained from $G_{0,n}(E, k)$ by replacing $|\mu|$ by $|\mu| + |\deg K|$, where $K$ is the canonical sheaf of $X$. $K$ is torsion free since by the assumption that $X$ is normal and Cohen-Macaulay $K$ coincides with the sheaf of Zariski differentials, which is a Weil divisorial sheaf (cf. [Re]). We can then apply the restriction estimates to $E^* \otimes K$.

As to the vanishing of $H^n(X, E(kH))$, this is equivalent to the vanishing of $H^0(X, E^* \otimes K(-kH))$, which holds true when the slope $-\mu + \deg K - kd$ is negative, i.e., when $k > d^{-1}(|\mu| + |\deg K|)$, e.g. for $k > (|\mu| + |\deg K|)$.

*Step $n = 2$ ($X$ is a surface).*
By 1.12 and 1.13 we have to worry only about $H^1(X, E(kH))$, and by our assumptions $X$ is a smooth surface.

*Substep 2.I: bounding $h^1(X, E(kH))$.*
We have by definition

(1.14) $h^1(X, E(kH)) = h^0(X, E(kH)) + h^2(X, E(kH)) - \chi(X, E(kH))$ .

By the Riemann-Roch theorem

$$\chi(X, E) = (r/12)(c_1(X)^2 + c_2(X)) + 1/2c_1(E)(c_1(E) + c_1(X)) - c_2(E) ,$$

whence also

$$\chi(X, E(kH)) = (r/12)(c_1(X)^2 + c_2(X)) + 1/2(c_1(E) + rkH)(c_1(E) + rkH + \\ + c_1(X)) - c_2(E) - (r-1)c_1(E)H - 1/2(r-1)rk^2H^2 .$$

We have now polynomial functions of $k$, $\mu = c_1(E) \cdot H/r, d = H^2$, $e = c_2(X)$, $\delta = \deg(K) = -c_1(X) \cdot H, \ldots$, and we have four basic regions.

(A) $\delta \geqq \mu + dk \geqq 0$

(B) $\mu + dk \geqq 0, \mu + dk > \delta$: here $h^2(X, E(kH)) = 0$

(C) $0 > \mu + dk > \delta$: here $h^0(X, E(kH)) = h^2(X, E(kH)) = 0$

(D) $0 > \mu + dk, \delta \geqq \mu + dk$: here $h^0(X, E(kH)) = 0$ .

In each region, by 1.12, 1.13 and 1.14, $h^1(X, E(kH))$ is bounded by a polynomial expression.

By 1.3 we conclude that there exists a polynomial type upper bound for $h^1(X, (kH))$.

We actually want to prove here more, namely that the degree in $k$ is at most 1. Then we observe firstly that in the regions (A) and (C) $k$ is bounded, whence the upper bound is given by an expression which is of degree zero in $k$.

Secondly, since regions (B) and (D) correspond to each other via Serre duality, it suffices to check that in case (B) we have an upper bound which is of degree at most 1 in $k$.

In fact, by 1.13(i)

$h^0(X, E(kH)) \leqq r/2\, dk^2 + \ldots$ (terms of lower degree in $k$), while

$$\chi(X, E(kH)) = 1/2(dr^2k^2 - (r-1)rdk^2) + \ldots \text{ (lower terms)}.$$

The assertion is thus proven.

*Substep 2.II: effective vanishing of $h^1(X, E(kH))$ for $|k| \gg 0$.*

As in Proposition 1.11 we let $Y \in |mH|$ be as in Flenner's restriction Theorem 1.6.

We firstly assume $k \geqq k_0 = (2g-1) + |m\mu(E)|$, where $g$ is the genus of $Y$.

We have the following long exact cohomology sequence

$$0 \to H^0(X, E((k-m)H)) \to H^0(X, E(kH)) \to H^0(X, E(kH)_{|Y}) \to$$
$$\to H^1(X, E((k-m)H)) \to H^1(X, E(kH)) \to 0 .$$

By step 2.I and the above exact sequence the dimension of $H^1(X, E((k-m)H))$ is at most $k_1 = G_{1,2}(E, k_0)$, thus there exists a $k'$ with $k_0 \leqq k' \leqq k_0 + k_1$ and such that $H^1(X, E((k'-m)H)) \to H^1(X, E(k'H))$ is an isomorphism, or equivalently $H^0(X, E(k'H)) \to H^0(X, E(k'H)_{|Y})$ is surjective.

We have the following diagram

$$\begin{array}{ccc} H^0(X, E(k'H)) \otimes H^0(\mathbb{P}^N, \mathcal{O}(1)) & \longrightarrow & H^0(X, E((k+1)H)) \\ \downarrow & & \downarrow \\ H^0(X, E(k'H)_{|Y}) \otimes H^0(\mathbb{P}^N, \mathcal{O}(1)) & \longrightarrow & H^0(X, E((k'+1)H)_{|Y}) . \end{array}$$

In the above, if $k'$ is $\geqq k_0 + 3$, then by Castelnuovo-Mumford's Theorem 1.8, the lower horizontal arrow is surjective.

The left vertical arrow is surjective by the choice of $k'$, therefore the right vertical arrow is also surjective, which means that

$$H^1(X, E(k'+1-m)H)) \to H^1(X, E(k'+1)H))$$

is also an isomorphism.

By iterating this procedure we get that the homomorphism $H^1(X, E((k-m)H)) \to H^1(X, E(kH))$ is an ismorphism for all $k \geqq k'$, and Serre's theorem B implies the vanishing of $H^1(X, E(kH))$ for all $k \geqq k_0 + k_1 + 3$.

It suffices then for $k$ positive to take $Q_2''(E, k) = \max((k_0 + k_1 + 3), (|\mu| + |\deg K|))$.

On the other hand, for $k$ negative we may consider $Q_2^*(E, k) = Q_2''(E^* \otimes K, |k|)$, and finally we set

$$Q_2(E, k) = \max(Q_2''(E, |k|), Q_2^*(E, k)) .$$

*Step III: vanishing of $h^q(X, E(kH))$ for $q \geqq 1$, $|k| \gg 0$, $n \geqq 3$.*

By induction (the case $n = 2$ being already settled) and by virtue of the exact sequence

$$\ldots \to H^{q-1}(X, E((k+m)H)_{|Y}) \to$$
$$H^q(X, E(kH)) \to H^q(X, E((k+m)H)) \to H^q(X, E((k+m)H)_{|Y}) \to \ldots$$

Serre's Theorem B implies immediately that

$$H^q(X, E(kH)) = 0 \quad \text{for each } q \geqq 2, \text{ and } k \geqq Q_{n-1}(E_{|Y}) .$$

The case of negative $k$ is treated similarly.

We only need at this point to remark that, since

$$c_1(Y) = (c_1(X) - mH)_{|Y}, c_2(Y) = (c_2(X) - mHc_1(X) + m^2H^2)_{|Y},$$

$\mu(E_{|Y}) = m\mu(E), \ldots Q_{n-1}(E_{|Y})$ is a polynomial type function $Q_n(c_1(E), c_2(E), c_1(X), c_2(X), H)$ as desired.

Finally, the case $q = 1$ is reduced by means of Serre duality to the case $q = n - 1 \geqq 2$.

*Step IV: bounding $h^q(X, E(kH))$ for all $q$ with $n - 1 \geqq q \geqq 1$.*
By Serre duality, it suffices to consider only the case where $k$ is positive.

We consider again the exact sequence

$$\ldots \to H^{q-1}(X, E((k+m)H)_{|Y}) \to$$
$$H^q(X, E(kH)) \to H^q(X, E((k+m)H)) \to H^q(X, E((k+m)H)_{|Y}) \to \ldots$$

We know that $H^q(X, E((k+im)H)) = 0$ for $(k+im) \geqq Q_n(E)$, therefore

$$h^q(X, E(kH)) \leqq \sum_{i=0,\ldots,(1/m)(Q_n(E)-k)} h^{q-1}(X, E((k+im)H)_{|Y}).$$

At this moment we proceed similarly as in Proposition 1.11, and since $h^{q-1}(X, E((k+md)H)_{|Y})$ is bounded by a polynomial type function $G_{q,n-1}(E, k+md)$ we finally obtain a polynomial type function $G_{q,n}(E, k)$ with $h^q(X, E(kH)) \leqq G_{q,n}(E, k)$.

Q.E.D. for Theorem 1.9.

**(1.15) Corollary.** *Given a semistable rank $r$ vector bundle $E$ on a Cohen–Macaulay projective variety $X$ of dimension $n$ which is non-singular in codimension 2 and is embedded in $\mathbb{P}^N$ by a very ample divisor $H$, then $E(kH)$ is globally generated for $k \geqq Q_n(E) + n$.*

*Moreover, there exists a polynomial type function $P_n(c_1(E), c_2(E), c_1(X), c_2(X), H)$ in the class $H$, in the first two Chern classes of $E$ and $X$, such that, for all $q$ with $1 \leqq q \leqq n - 1$, and for all natural numbers $k$ holds:* $\dim(H^q(X, E(kH))) \leqq P_n(E)$.

*Proof.* The proof of the first assertion follows immediately from Castelnuovo-Mumford's Theorem 1.8. and (1.4).

The second assertion follows easily from Theorem 1.9. Q.E.D.

## 2 Bounds for the degree of sections of semistable bundles

Throughout this section let $E$ be semistable rank $r$ vector bundle on a Cohen-Macaulay projective variety $X$ of dimension $n$ which is non-singular in codimension 2 and is embedded by a very ample divisor $H$.

Let $\sigma \in H^0(X, E)$ be a nonzero section, and let $Z = Z_\sigma$ be the scheme of zeroes of $\sigma$. We make no assumption whatsoever on the codimension $h$ of $Z$.

We can define the "top dimensional subscheme" $Z'$ of $Z$ as follows: given a local primary decomposition for the ideal sheaf $\mathcal{I}$ of $Z$, we consider the intersection $\mathcal{I}'$ of the primary ideals corresponding to the components of codimension $h$ (they are of minimal codimension, whence the corresponding primary ideals are uniquely determined).

**(2.1) Definition.** Given a nonzero section $\sigma \in H^0(X, E)$, with scheme of zeroes $Z$, we define its degree $d$ with respect to $H$ as the intersection product $Z' \cdot H^h$.

Our main result in this section gives an upper bound for $d = \deg(Z)$ which is of polynomial type in the Chern classes of $E$.

**(2.2) Theorem.** *Let $E$ be a semistable vector bundle on a Cohen-Macaulay projective variety $X$ of dimension $n$ which is non-singular in codimension 2 and is embedded by a very ample divisor $H$. There is a polynomial type function $P_{n,h}(E)$ in the class $H$, in the first $h$ Chern classes of $E$ and in the first two Chern classes of $X$, such that for every nonzero section $\sigma \in H^0(X, E)$, whose scheme of zeroes $Z$ has codimension $h$, the degree of $Z$ is bounded from above by $P_{n,h}(E)$.*

*Proof.* Notice firstly that, if $r$ is the rank of $E$, then clearly $h \leqq r$.

*Step I. Reduction to the case where $h = n$, i.e., $Z = Z'$ has dimension O.*
In fact the degree $d$ of $Z$ is given by the intersection product $Z' \cdot H^h$, and we can choose general hypersurfaces $Y_1, \ldots, Y_h$ such that the schematic intersections $Z \cap Y_1 \cap \ldots \cap Y_h$, respectively $Z' \cap Y_1 \cap \ldots \cap Y_h$ coincide and have dimension zero.

Setting $X^* = Y_1 \cap \ldots \cap Y_h$ and $Z^* = Z \cap Y_1 \cap \ldots \cap Y_h$ we have that $Z^*$ is the scheme of zeroes of the restriction of the section $\sigma$ to $X^*$.

It suffices to apply Flenner's Theorem 1.6 so that the restriction of $E$ to $X^*$ is semistable, and then notice that if $Y_i \in |mH|$, then $\deg Z^* = m^h \deg Z$; then we repeatedly apply the standard argument that polynomials in the class $H$, in $m$ (chosen according to Flenner's theorem), and in the Chern classes of $X^*$ and $E$ are polynomials in $H$ and in the Chern classes of $E$ and $X$.

*Step II: the case where* $\dim Z = 0$. By Corollary 1.15 there exists a positive integer $k$ such that $E(kH)$ is generated by global sections, with $k$ as usually depending polynomially on the Chern classes of $E$, $X$, and $H$.

We have thus an epimorphism $\mathcal{O}(-kH)^N \to E$, and it is a standard fact that there exists an exact sequence of vector bundles

(2.3) $$0 \to \mathcal{O}(-kH)^{r-n} \to E \to F \to 0\,.$$

Clearly every nonzero section $\sigma \in H^0(X, E)$ induces a section $\tau \in H^0(X, F)$.

Let $W$ be the zero scheme of $\tau$: clearly $Z = Z_\sigma$ is a subscheme of $W$, and we claim that $\dim W = 0 = \dim Z$.

Let in fact $\Gamma$ be a positive dimensional subvariety contained in $W$. It suffices then to show, by contradiction, that $\Gamma$ is contained also in $Z$. This follows since, by 2.3, $Z \subset W$ is defined by $r - n$ sections of $H^0(W, \mathcal{O}(-kH))$: $H$ being ample, these sections vanish identically on $\Gamma$, as we wanted to show.

The conclusion is that $\deg(Z) = \text{length}(\mathcal{O}_Z) \leqq \text{length}(\mathcal{O}_W) = c_n(F)$. Finally, by 2.3, $c_n(F)$ is clearly a polynomial in the Chern classes of $E$ and in $kH$, and alltogether we can bound $\deg(Z)$ by a polynomial type function in the Chern classes of $E$, $X$, and $H$. Q.E.D.

The following result is central for applications

**(2.4) Theorem.** *Let $X$ be a Cohen-Macaulay projective variety of dimension $n$ which is non-singular in codimension 2, and assume we are given a vector bundle $E$ equipped with an increasing filtration $F_i$ by subbundles with semistable quotients $E_i = F_{i+1}/F_i$. There is a polynomial type function $P_{n,h}(E)$ in the class $H$, in the first $h$ Chern classes of the $E_i$' s and in the first two Chern classes of $X$, such that for every nonzero section*

*$\sigma \in H^0(X, E)$, whose scheme of zeroes $Z$ has codimension $h$, the degree of $\sigma$ is bounded from above by $P_{n,h}(E)$.*

*Proof.* The proof proceeds in an entirely analogous fashion to the one given in 2.2, once we show that there exists a positive integer $k$ such that $E(kH)$ is generated by global sections, with $k$ depending polynomially on the Chern classes of the $E_i$'s and $X$, and on the class $H$. This follows inductively (on the length of the filtration) since if we have a short exact sequence of vector bundles $0 \to A \to B \to C \to 0$, and $k$ is an integer such that $A(k)$, $C(k)$ are generated by global sections, and moreover $H^1(X, A(k)) = 0$, then also $B(k)$ is generated by global sections.

It suffices then to apply Corollary 1.15 and Theorem 1.9. Q.E.D.

Next, we consider the rather particular case where $E$ is a rank 2 vector bundle over a surface $X$, and $\sigma$ is a nonzero section vanishing also in codimension 1.

**(2.5) Theorem.** *Let $E$ be a rank 2 semistable vector bundle on a smooth projective surface $X$, and let $\sigma$ be a nontrivial section of $E$ giving rise to an exact sequence*

$$0 \to \mathcal{O}(D) \to E \to \mathcal{I}_W(L - D) \to 0$$

*where $L = \det(E)$, $D$ is an effective divisor, $W$ is a 0-dimensional subscheme. Then, if we set $\delta = \mathrm{length}(W)$, we have:*

(i) *if $L$ is a nef divisor $\delta \leqq c_2(E) + (L \cdot H)^2/4H^2$,*

*otherwise*

(ii) *$\delta \leqq$ a polynomial type function in $H$, and in the Chern classes of $E$, $X$.*

*Proof.* By the above exact sequence we get the following equality for Chern classes:

$$\delta = c_2(E) - D(L - D) = c_2(E) + D^2 - D \cdot L \leqq c_2(E) + D^2$$

since $L$ is nef, and $D$ is effective.

By the Index theorem, we have $D^2 H^2 \leqq (D \cdot H)^2$. Now use the semistability of $E$ to infer $D \cdot H \leqq (1/2) L \cdot H$, and the first assertion is proven.

Otherwise, i.e. if $D \cdot L$ is $< 0$, we use 1.15 to find an integer $k$, given as usual by a polynomial in $H$, and in the Chern classes of $E$, $X$, and a homomorphism $\varphi$

$$\varphi \colon \mathcal{O}(-kH)^2 \to E$$

such that $\mathrm{coker}(\varphi)$ is supported on an effective divisor $C$ which has no common components with $D$.

Then we can write $C = 2kH + L$, whence $-D \cdot L = 2kH \cdot D - C \cdot D \leqq 2kH \cdot D \leqq$ (by semistability) $\leqq kL \cdot H$. Q.E.D.

*(2.6) Remark.* In the case of higher rank on a surface, $\sigma$ gives rise to a pair of exact sequences

$$0 \to \mathcal{O}(D) \to E \to \mathcal{F} \to 0$$

$$0 \to \mathcal{F} \to V \to \Delta \to 0$$

where $V$ is a vector bundle (of rank $r - 1$), $D$ is an effective divisor, and $\Delta$ is supported on a finite set.

We notice that if we set $\delta = \mathrm{length}(\Delta)$, then our definition coincides with the one given in 2.5, and a similar calculation with Chern classes yields:

$$\delta = c_2(E) - D(L - D) - c_2(V),$$

and the only problem is to bound $c_2(V)$ from below.

In the case where $\dim X > 2$, one can still, in the case when the scheme of zeroes $Z$ of a nonzero section $\sigma \in H^0(X, E)$ is not pure dimensional ($Z'$ different from $Z$ according to our previous notation), try to define a residual lower dimensional subscheme $Z''$ of $Z'$ in $Z$, and try to estimate the degree of $Z''$.

One possibility is, denoting as above by $\mathscr{I}'$ the ideal of the "top dimensional subscheme" $Z'$, and observing that $\mathscr{I}'$ contains the ideal sheaf $\mathscr{I}$ of $Z$, to consider the "top dimensional part" $\mathscr{I}''$ of $\mathrm{Ann}(\mathscr{I}'/\mathscr{I})$.

**(2.7) Lemma.** *Let $\mathscr{I}$ be a sheaf of $\mathcal{O}$ ideals, and consider, given a local primary decomposition of $\mathscr{I}$, the intersection $\mathscr{I}'$ of the primary ideals corresponding to the components of minimal codimension $= h$. Let $\mathscr{J}$ be $\mathrm{Ann}(\mathscr{I}'/\mathscr{I})$, and define $\mathscr{J}'$ analogously to $\mathscr{I}'$. Then the associated primes of $(\mathscr{I}'/\mathscr{I})$ coincide with the associated primes of $\mathcal{O}/\mathscr{I}$ of codimension $> h$, in particular the associated primes of $\mathscr{J}'$ coincide with the associated primes of $\mathcal{O}/\mathscr{I}$ of subminimal codimension $h'' > h$.*

*Proof.* Recall that $\mathscr{P}$ is an associated prime of a module $M$ if and only if there is an embedding of $\mathcal{O}/\mathscr{P}$ in $M$, or equivalently there is an element $x$ in $M$ such that $\mathscr{P} = \mathrm{Ann}(x)$; observe moreover that, $\mathcal{O}/\mathscr{P}$ being an integral domain, $\mathscr{P} = \mathrm{Ann}(y)$ for each $y$ in the image of $\mathcal{O}/\mathscr{P}$, whence if this image intersects nontrivially a submodule $N$ then $\mathscr{P}$ is also an associated prime of $N$.

Since $\mathscr{I}'/\mathscr{I}$ is contained in $\mathcal{O}/\mathscr{I}$, $\mathrm{Ass}(\mathscr{I}'/\mathscr{I}) \subset \mathrm{Ass}(\mathcal{O}/\mathscr{I})$, moreover if $\mathscr{P}$ is an associated prime of $\mathcal{O}/\mathscr{I}$, but not of $\mathscr{I}'/\mathscr{I}$, then $\mathscr{P}$ is an associated prime of $\mathcal{O}/\mathscr{I}'$, whence it is an associated prime of codimension $= h$. Conversely, if $\mathscr{P}$ is an associated prime of $\mathscr{I}'/\mathscr{I}$, and has codimension $= h$, then by localizing at $\mathscr{P}$ we obtain $\mathscr{I}_{\mathscr{P}} = \mathscr{I}'_{\mathscr{P}}$, whence $(\mathscr{I}'/\mathscr{I})_{\mathscr{P}} = 0$, contradicting the existence of an embedding of $\mathcal{O}/\mathscr{P}$ into $\mathscr{I}'/\mathscr{I}$.

The final assertion follows since, given a module $M$, the minimal primes of $\mathrm{Ass}(M)$ and of $\mathrm{Ann}(M)$ are the same (cf. [Mat, Theorem 9, p. 50]). Q.E.D.

*(2.8) Remark.* It is clear that the procedure can be continued to inductively define a sequence of ideal sheaves $\mathscr{I} = \mathscr{I}^{(1)}, \mathscr{I}'' = \mathscr{I}^{(2)}, \mathscr{I}^{(3)}, \ldots$ so that one can define residual schemes to the top dimensional part $Z'$ of the subscheme of zeroes $Z$ of a section $\sigma$ of $E$.

One can easily verify that in the situation of Theorem 2.5, if $\sigma$ vanishes in codimension 1, then the 0-dimensional scheme $W$ coincides with the residual subscheme $Z''$ we have just defined.

We notice moreover that if $Y$ is a sufficiently general hypersurface section of $X$, and we consider the restriction $\tau$ of the section $\sigma \in H^0(X, E)$ to $Y$, then the subschemes associated to $\tau$ coincide with the intersection with $Y$ of the subschemes associated to $\sigma$.

From this it follows that Theorem 2.5 has the following corollary

**(2.9) Corollary.** *Let $E$ be a rank 2 semistable vector bundle on a smooth projective variety $X$, and let $\sigma$ be a nontrivial section of $E$ vanishing in codimension 1. Then the degree of the residual subscheme $Z''$ can be bounded by a polynomial type function in $H$, and in the first two Chern classes of $E$, $X$.*

*(2.10) Question:* can one in general give upper bounds for the degrees of all the residual subschemes of the subscheme of zeroes $Z$ of a section of a semistable vector bundle?

## 3 Applications

In this section we shall give applications of the above techniques in several directions.

First we shall give bounds for the degrees of Weierstrass schemes, later on we shall show that the dimension of moduli spaces of semistable vector bundles with given Chern classes can be bounded by a polynomial expression in the Chern classes of the bundle, finally, after proving a restriction theorem, we shall show that the Chern numbers and the degrees of the higher Chern classes of a semistable vector bundle are bounded by polynomial type functions in the first two Chern classes: this applies in particular to varieties of general type with ample canonical bundle.

### *3.a. Weierstrass schemes*

Let $X$ be a smooth projective variety of dimension $n$, and let $E$ be a vector bundle over $X$.

We consider the bundle $J_k(E)$ of $k$-jets of $E$.

We have a natural bundle homomorphism

$$v_k(E)\colon H^0(X, E) \otimes \mathcal{O} \to J_k(E) \,. \tag{3.1}$$

For any integer $m$ we get the associated exterior $m$-power map:

$$\Lambda^m(v_k(E))\colon \Lambda^m(H^0(X, E)) \otimes \mathcal{O} \to \Lambda^m(J_k(E)), \quad \text{yielding a section} \tag{3.2}$$

$$w_k^m(E)\colon \mathcal{O} \to \Lambda^m(H^0(X, E))^* \otimes \Lambda^m(J_k(E)), \tag{3.3}$$

* denoting the dual vector space .

**3.4. Definition** (cf. [Oga]) The Weierstrass subscheme $W_k^m(E)$ is the scheme of zeroes of the section $w_k^m(E)$ given in 3.3.

**(3.5) Theorem.** *Let $E$ be a semistable rank $r$ vector bundle on a smooth projective variety $X$ of dimension $n$ with ample canonical divisor $K$.*

*Then the degrees of the Weierstrass schemes $W_k^m(E)$, defined as in* 2.1 *with respect to the ample divisor $K$, can be bounded by polynomial type functions in the first two Chern classes of $E$ and $X$.*

*Proof.* We recall that there is a natural exact sequence

$$0 \to S^k(\Omega_X^1) \otimes E \to J_k(E) \to J_{k-1}(E) \to 0 \,. \tag{3.6}$$

On $\Lambda^m(J_k(E))$ there is the natural $\Lambda$-filtration associated to the exact sequence 3.6 whose successive quotients are isomorphic to $\Lambda^{m-i}(S^k(\Omega_X^1) \otimes E) \otimes \Lambda^i(J_{k-1}(E))$ so that by induction we get a filtration on $\Lambda^m(J_k(E))$ with quotients isomorphic to tensor products of vector bundles of the form $\Lambda^j(S^h(\Omega_X^1) \otimes E)$ with $j, h$ non-negative integers.

Since by Tsuji's theorem [Tsu] $\Omega_X^1$ is $K$-semistable, we can apply Maruyama's theorem [Ma] asserting that the tensor product of semistable bundles is semistable and that bundles associated to a semistable bundle via a representation of the linear group are also semistable, and we obtain that these quotient bundles are $K$-semistable.

Now, all our explicit estimates were based on the datum of a very ample divisor $H$. But this is provided by a recent result of Demailly [De], showing that if $K$ is ample, then $12n^n K$ is indeed very ample.

We apply now Theorem 2.4. Q.E.D.

*(3.7) Remark.* The canonical filtrations on the vector bundles $\Lambda^m(J_k(E))$ give rise to a sequence of subschemes containing the Weierstrass subscheme $W_k^m(E)$.

*(3.8) Remark.* Theorem 3.5 has a natural application in the case where the bundle $E$ is invariant under the group $\mathrm{Aut}(X)$ of automorphisms of $X$. Obvious choices for such bundles $E$ are the canonical bundle $K$, or more generally the bundles $\Lambda^j(S^h(\Omega_X^1))$. In this way in fact one can produce invariant subschemes for the action of $\mathrm{Aut}(X)$.

### *3.b. Moduli spaces*

Fix a smooth projective variety $X$ of dimension $n$, and consider the moduli space $M = M(c_1, \ldots, c_{\min(r,n)})$ of stable vector bundles of rank $r$ with given Chern cohomology classes.

By deformation theory the Zariski tangent space at $E \in M$ is isomorphic to the vector space $H^1(X, \mathrm{End}(E))$.

Again by Maruyama's theorem $\mathrm{End}(E)$ is semistable, hence we can apply Theorem 1.9 to obtain the following result which answers a question posed by Balancheff.

**(3.9) Theorem.** *The dimension of the moduli space $M = M(c_1, \ldots, c_{\min(r,n)})$ of stable vector bundles of rank r with given Chern classes can be bounded from above by a polynomial type function in the class of H, in the first two Chern classes of X and in $c_1, c_2$.*

### *3.c. Restriction theorems for semistable vector bundles*

**(3.10) Theorem.** *Let X be a smooth projective variety and let E and F be semistable vector bundles on X.*

*Then there exists an integer $k_0$ depending polynomially in the class H and in the first two Chern classes of E, F and X such that if their restrictions $E_{|Y}, F_{|Y}$, to a general element $Y \in |kH|$, $k \geqq k_0$ are isomorphic, then E and F are isomorphic.*

*Proof.* The bundle $G = \mathrm{Hom}(E, F)$ is semistable by Maruyama's theorem, hence by Theorem 1.9. we find a number $k_0$ with the required polynomial dependence such that $H^1(X, G(-kH)) = 0$ and the restrictions $E_{|Y}, F_{|Y}$, to a general element $Y \in |kH|$ are semistable for all $k \geqq k_0$.

Whence $H^0(X, G)$ surjects onto $H^0(X, G_{|Y})$.

Lift the isomorphism which exists by our assumption to obtain a homomorphism $\alpha\colon E \to F$ whose determinant $\det(\alpha)$ is a section of $\det(G)$ which has no zeros on $Y$.

Since $Y$ is ample $\det(\alpha)$ never vanishes and $E$, $F$ are isomorphic. Q.E.D.

### 3.d. *Bounding the higher Chern classes in terms of $c_1, c_2$*

Let $X$ be a smooth projective variety of dimension $n \geqq 3$, and let $E$ be a semistable vector bundle on $X$.

Let us set up some notation: assume that, for each $j$ such that $1 \leqq j \leqq s = \min(r, n)$ we are given a non-negative integer $i_j$, then we let $I$ be the multiindex $(i_1, \ldots, i_s)$ and we denote by $|I|$ its weight $\left(|I| \text{ equals the weighted sum } \sum_{j=1,\ldots,s} (i_j)j\right)$, and by $c_I(E)$ the product $c_I(E) = \prod_{j=1,\ldots,s} (c_j(E))^{i_j}$, yielding a cohomology class in $H^{2|I|}(X, \mathbb{Z})$.

**(3.11) Theorem.** *Let $X$ be a smooth projective variety of dimension $n \geqq 3$, and let $E$ be a semistable vector bundle on $X$.*

*For each multiindex $I$ there exists a polynomial type function $P_I$ depending only on $H$, and the first two Chern classes of $E$ and $X$, such that*

$$c_I(E) \cdot H^{n-|I|} \leqq P_I .$$

*Proof.* We first recall a result of [D-P-S, Corollary 2.6], stating that for a nef bundle $F$ one has

$$c_I(F) \cdot H^{n-|I|} \leqq (c_1(F))^{|I|} \cdot H^{n-|I|} . \tag{3.12}$$

We know by Corollary 1.15 that the polynomial type function $Q_n(E)$ has the property that for $k = Q_n(E) + n$ the bundle $F = E(kH)$ is generated by global sections. Therefore in particular $F$ is nef and (3.12) holds.

The theorem is now readily proven by induction on $|I|$. In fact, for $|I| = 1$ there is nothing to prove. For the inductive step we use (3.12) and the fact that, with our choice of $F$, $c_I(F) = c_I(E(k)) = c_I(E) + \sum_{|J|<|I|} a_J(c_J(E) \cdot (kH)^{|I|-|J|})$, where the coefficients $a_J$ are universal constants and $k$ is a polynomial type function of the desired kind. Q.E.D.

**(3.13) Proposition.** *Let $X$ be a smooth projective variety of dimension $n$ with ample canonical bundle $K$. Then for each multiindex $I$ of weight $n$ there exists a constant $D_I$ such that*

$$c_I(\Omega_X^1) \leqq D_I K^n .$$

*Proof.* By the quoted result of Demailly, we have that $mK$ is very ample if $m \geqq 12n^n$. It follows immediately that $\Omega_X^1(2mK)$ is generated by global sections, whence (3.12) applies to the case: $F = \Omega_X^1(2mK)$, $H = K$, $I$ such that $i_1 = 0$. We can then apply the same induction on $|I|$ as in the proof of 3.11 Q.E.D.

**(3.14) Theorem.** *Let $X$ be a smooth projective variety of dimension $n$ with ample canonical bundle $K$.*

*Then there exist constants C and N such that the degrees of the Weierstrass schemes $W_K^m(K)$ (defined as in 2.1 with respect to the ample divisor K), can be bounded by C times the N-th power of $K^n$.*

*Proof.* Apply Theorem 3.5 and Proposition 3.13. Q.E.D.

We would like to make a concluding remark to the effect that most of the inequalities we wrote could be improved with a more careful analysis, especially if we would give up the desire to express everything by means of a single polynomial type function. On the other hand, one has to admit that the overall conceptual treatment is thus simplified for the reader.

*Acknowledgements.* The scientific cooperation of the two authors was made possible by the D.F.G. Schwerpunkt Programm "Komplexe Mannigfaltigkeiten", by the M.U.R.S.T. 40% Program "Geometria Algebrica" and by the SCIENCE Program (Contract SCI-0398-C(A)).

## References

[Bog] Bogomolov, F.A.: Holomorphic tensors and vector bundles on projective varieties. Math. USSR, Izv. **13**, 499–555 (1979)
[Cor] Corti, A.: Polynomial bounds for the number of automorphisms of a surface of general type. Ann. Ec. Norm. Supér. **24**, 113–137 (1991)
[De] Demailly, J.P.: A numerical criterion for very ample line bundles. J. Differ. Geom. (to appear)
[D-P-S] Demailly, J.P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. (Preprint 1991)
[E-F] Elencwajg, G., Forster, O.: Bounding cohomology groups of vector bundles on $\mathbb{P}^n$. Math. Ann. **246**, 251–270 (1980)
[Fl] Flenner, H.: Restrictions of semistable bundles on projective varieties. Comment. Math. Helv. **59**, 635–650 (1984)
[F-L] Fulton, W., Lazarsfeld, R.: Positive polynomials for ample vector bundles. Ann. Math. **118**, 35–60 (1983)
[Ko] Kobayashi, S.: Differential geometry of complex vector bundles, (Publ. Math. Soc. Japan, vol. 15) Princeton: Iwanami Shoten Publishers and Princeton University Press 1987
[Ma] Maruyama, M.: The theorem of Grauert-Mülich-Spindler. Math. Ann. **255**, 317–333 (1981)
[Mat] Matsumura, H.: Commutative Algebra, II. Edit. Reading, Mass.: Benjamin 1980
[M-R] Mehta, V.B., Ramanathan, A.: Semistable sheaves on projective varieties and their restrictions to curves. Math. Ann. **258**, 213–224 (1982)
[Mu1] Mumford, D.: Projective invariants of projective structures and applications. In: Proc. Int. Congr. Math., Stockholm, 1962, pp. 526–530. Djursholm: Institut Mittag-Leffler 1963
[Mu2] Mumford, D.: Lectures on curves on an algebraic surface. (Ann. Math. Stud., vol. 59), Princeton, N.J.: Princeton University Press 1966
[Oga] Ogawa, R.: On the points of Weierstrass in dimension greater than one. Trans. Am. Math. Soc. **184**, 401–417 (1973)
[Re] Reid, M.: Canonical 3-folds. In: Beauville, A. (ed.) Géométrie Algébrique Angers 1979, pp. 273–310. Alphen aan den Rijn: Sijthoff & Noordhoff 1980
[Ta] Takemoto, F.: Stable vector bundles on algebraic surfaces. Nagoya Math. J. **47**, 29–48 (1972)
[Tsu] Tsuji, H.: Stability of tangent bundles of minimial algebraic varieties. Topology **27**, 429–442 (1988)

**Note added in proof.** It was brought up to our attention by H. Flenner that the Cohen-Macaulay assumption in our theorems, e.g. in Theorem 2.2, can be relaxed to the assumption of normality (this is a consequence of the results of H. Flenner: Die Sätze von Bertini für lokale Ringe. Math. Ann. **229**, 97–111 (1977)).