VECTOR BUNDLES, LINEAR SYSTEMS AND EXTENSIONS OF π_1

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§0 Introduction

I. Reider ([Rei]) introduced a new method to prove that certain linear systems on algebraic surfaces are free from base points (respectively very ample).

He uses a construction due to Schwarzenberger, ([Sch1,2], [GH]) producing, if Z is a 0-cycle not imposing independent conditions on the linear system $|K+L|$, a certain rank 2 bundle occurring as an extension

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_Z L \to 0$$

and then derives a contradiction if the vector bundle $\mathcal{E}$ is numerically unstable according to Bogomolov ([Bog]).

As pointed out by D. Kotschick ([Kot]), if the numerical inequality $c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) \geq 0$ becomes an equality, i.e. $L^2 - 4 \deg Z = 0$, and the Chern class L is divisible by 2, then $\mathcal{E}$ is the twist (by a line bundle) of a vector bundle with trivial Chern classes; hence, by a deep theorem of Donaldson ([Do]), if $\mathcal{E}$ is Mumford-stable with respect to an ample divisor H, then this vector bundle arises from an irreducible SU(2)-representation of the fundamental group $\pi_1(S)$ of the surface.

In fact, cf. [Ko], when one has equality, and one can prove that the bundle $\mathcal{E}$ is stable for some ample divisor H, then the associated projective bundle $P(\mathcal{E})$ arises from a PU(2)-representation of the fundamental group $\pi_1(S)$ of the surface.
We thus get a central extension Γ of $\pi_1(S)$ by a cyclic group of order 2, whose extension class measures the obstruction to lifting the PU(2) to a SU(2)-representation.

We apply this method to the study of bicanonical systems on surfaces S with $K_S^2 = 4$, where $L = K_S$: in this case one cannot have a SU(2)-irreducible representation since the numerical class of $L = K_S$ is not divisible by 2 . Whence, we get in §1 the result that $|2K_S|$ is free from base points if $H^2(\pi_1(S), \mathbb{Z}/2\mathbb{Z})=0$.

This partial result is of some interest in view of the open problem (cf. §1) whether $|2K_S|$ has base points only when $K_S^2 = 1$, and $p_g = 0$ (the only cases which are left open being the ones where $p_g = 0$, $K_S^2 = 2,3,4$).

After the results of §2, we obtain a sharper theorem (Thm. 3.2) implying in particular that $|2K_S|$ is base point free if the pull back of K_S to the universal covering is not 2-divisible, or it gives rise to a trivial extension.

The second section is devoted to the geometrical analysis of the possible central extension of the fundamental group. This problem is treated in a greater generality, by considering the standard m^{th} root extraction covering trick (cf. [Mi2]), under which the pull-back of a divisor L becomes m-divisible; we first show (cf. Lemma 2.1) that in this situation the fundamental group changes up to a central extension by a cyclic group of order dividing m (this argument is essential for the main result of [Mi2]). Later on, we give a complete description of the extension which occurs, in terms of the divisibility properties of the pull-back of L to the universal covering of S.

Our first example where the first homology group would not change, but the fundamental group would, was the case of a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover of an Abelian surface: here L gives a polarization of type (1,4) and the fundamental group of the cover is the classical Heisenberg central extension of the fundamental group of the Abelian surface associated to the mod 2 reduction of the alternating form given by the Chern class of L.

We appealed again to Donaldson's theorem in order to calculate the fundamental group of the cover, just by providing the existence of some stable bundle with trivial Chern classes (the ideas here were influenced by the article [BLvS], whose results by the way can also be reproved using the above ideas (cf. work in progress by the second author)).

Later on, guided by the conjecture raised by this nice example, we worked out completely the general case, where we essentially investigate the spectral sequence describing the cohomology of the quotient S in terms of group cohomology.

The main result of the paper is the following

Main Theorem: Let $Y \to X$ be the $(\mathbb{Z}/m\mathbb{Z})^2$-Galois covering given by the m^{th} root extraction of the divisor D.

Then we have a central extension

$$0 \to \mathbb{Z}/r\mathbb{Z} \to \pi_1(Y) \to \pi_1(X) \to 0$$

where, if $\pi: \tilde{X} \to X$ is the universal covering of X, $\tilde{D} = \pi^*(D)$, d the divisibility index of $\tilde{D}$, then r = G.C.D.(m,d) and the extension class in $H^2(G,\mathbb{Z}/r\mathbb{Z})$, $(G = \pi_1(X))$, is given by the Chern class of (–D) modulo r, via the exact sequence

$$0 \to H^2(G,\mathbb{Z}/r\mathbb{Z}) \to H^2(X,\mathbb{Z}/r\mathbb{Z}) \to H^2(\tilde{X},\mathbb{Z}/r\mathbb{Z})^G \quad .$$

Acknowledgements : both authors would like to acknowledge support from M.U.R.S.T. ; the first author would like to acknowledge the warm hospitality of the Max Planck Institute in October '90, where the final part of the paper was prepared.

We want to thank Igor Reider and Ingrid Bauer Kosarew for some useful conversation.

Mathias Kreck showed us kindly how one could avoid the use of Lefschetz duality in 2.18.

Added in proof: we would like to call the reader's attention to related partial results, concerning base points of the bicanonical system, by Weng Lin ([We]).

§1 Bicanonical systems on surfaces of general type.

Let S be a smooth (complete) minimal surface of general type, and consider the bicanonical linear system $|2K_S|$, where K_S is a canonical divisor on S .

Through work of several people (Moishezon [Moi], Kodaira [Kod], Bombieri [Bo1,2], Francia [Fr], Reider [Rei] and others, e.g. [Ca-Ci], [Mi1])) it is known that the above linear system has no base points if p_g is ≥ 1, and also in the case $p_g = 0$, provided $K^2 \geq 5$, and particular cases when $K^2 = 2$ (cf. [Pet2], [Xi] Thm 5.5 page 77).

Since $|2K_S|$ is a pencil exactly when $K_S^2 = 1$, one may ask about the remaining cases $p_g = 0$, $K_S^2 = 3,4$.

There are no known examples of surfaces with invariants $p_g = 0$, $K^2 = 3,4$ such that the bicanonical linear system has base points.

In this paragraph we shall give some sufficient conditions, concerning the fundamental group $\pi_1(S)$, which ensure that $|2K_S|$ be (base point) free .

On the other hand, by looking at some examples of surfaces with the above invariants constructed by Burniat and Keum ([Bu], [Pet], [Ke]), we shall see however that those conditions are not necessary (but, using the results of the next paragraph one gets some weaker sufficient conditions).

As we remarked in the introduction, it remains an interesting question to know whether the case $p_g = 0$ and $K_S^2 = 1$ is indeed the only exception to $|2K_S|$ being (base point) free.

Theorem 1.1 Let S be a minimal surface with $p_g = 0$, $K_S^2 = 4$, and such that no nontrivial central extension Γ

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma \to \pi_1(S) \to 0$$

of $\pi_1(S)$ has an irreducible SU(2)-representation.
Then the bicanonical linear system $|2K_S|$ is (base point) free.

Corollary 1.2 In particular, the theorem holds if $H^2(\pi_1(S),\mathbb{Z}/2\mathbb{Z}) = 0$ (e.g. if $\pi_1(S)$ is cyclic, or it has odd order).

Proof of Theorem 1.1 The proof of the theorem will be divided in two different cases and will involve some assertions that will be justified at a later time.

Assume that the bicanonical linear system $|2K_S|$ has a base point x. Then (cf. [GH], [Rei]) there exists a vector bundle $\mathcal{E}$ of rank 2 on S occurring as an extension

(1.3) $$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{J}_x(K_S) \to 0$$

where $\mathcal{J}_x$ is the ideal sheaf of the given point x. We will show that (with no assumption on $\pi_1(S)$) the vector bundle $\mathcal{E}$ is K-stable (prop. 1.4) and, furthermore, it is stable with respect to a suitable ample line bundle H on S (see prop. 1.9).

Then, (cf. e.g. [Ko], thm. 10.19, page 236, and thm. 4.7, page 114) $\mathcal{E}$ admits a Hermite-Einstein structure, and, since $c_1^2(\mathcal{E}) - 4\,c_2(\mathcal{E}) = 0$, it is projectively flat, i.e., it comes from an irreducible PU(2)-representation of $\pi_1(S)$. We can lift thus the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to SU(2) \to PU(2) \to 0$$

to obtain

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma \to \pi_1(S) \to 0\,.$$

If this last extension were split, then the bundle $\mathcal{E}$ would arise from an irreducible SU(2)-representation; hence its Stiefel-Whitney class w_2 would vanish. Then, since w_2 is the mod 2 reduction of the first Chern class of $\mathcal{E}$, K_S would be 2-divisible.

But we claim that in fact K_S is not 2-divisible even in $\mathrm{Num}(S) = H^2(S,\mathbb{Z})/\mathrm{torsion}$. In fact, if K_S is numerically equivalent to 2L, then $L^2 = 1$ and $K_S \cdot L = 2$: by the genus formula we have then a contradiction.

Q.E.D. for theorem 1.1

Proposition 1.4 The vector bundle $\mathcal{E}$ is stable with respect to the canonical bundle K_S.

Proof Otherwise there exists an invertible and saturated subsheaf N of $\mathcal{E}$ which satisfies the inequality $N \cdot K_S \geq (1/2)\, K_S^2$ and gives a diagram of exact arrows of the following form

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & N & & & \\
 & & & \downarrow & & & \\
(1.5) \quad 0 \to & \mathcal{O}_S & \to & \mathcal{E} & \to & \mathcal{I}_x(K_S) & \to 0 \\
 & & & \downarrow & & & \\
 & & & \mathcal{I}_Z(M) = \mathcal{I}_Z(K_S - N) & & & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}
$$

where Z is a 0 dimensional subscheme of S and $M = K_S - N$ is then a line bundle on S with the property

(1.6) $\quad M \cdot K_S \leq 2$.

We can compare the two expressions of the Euler characteristic of the bundle $\mathcal{E}$ obtained from the two above exact sequences:

$$\chi(\mathcal{E}) = \chi(\mathcal{O}_S) + \chi(\mathcal{I}_x(K_S)) = 2\chi(\mathcal{O}_S) - 1$$
$$= \chi(N) + \chi(\mathcal{I}_Z(M)) = 2\chi(M) - \deg Z .$$

Hence we have the equality $1 + \deg Z = 2\chi(M)$ and therefore, applying the Riemann-Roch theorem to M and the fact that $M \cdot (K_S - M)$ is even, we infer that M satisfies the further inequality:

(1.7) $\quad M^2 \geq M \cdot K_S$.

By the Index theorem, we have

(1.8) $\quad M^2 K_S{}^2 - (M \cdot K_S)^2 \leq 0$

and so, by 1.6 and 1.7, $M^2 = M \cdot K_S = 0$. But, again by the Index theorem, this can only happen if $M = 0$.

Diagram 1.5 then gives a contradiction, since $N = K_S$. In fact there is no non zero morphism $K_S \to \mathcal{O}_S$ nor $K_S \to \mathcal{I}_x(K_S)$. Q.E.D.

<u>Proposition 1.9</u> There exists an ample line bundle H on S such that the vector bundle $\mathcal{E}$ is H-stable.

<u>Proof</u> If K_S is ample, it is enough to take $H = K_S$. Otherwise, let $E_1, \dots , E_\mu$ be the finitely many curves ($\cong \mathbb{P}^1$) on S such that $K_S \cdot E_i = 0$ for each $i = 1, \dots , \mu$. We recall that the intersection matrix $(E_i \cdot E_j)$ is negative definite.

One can easily construct an effective divisor W on S of the form $W = \Sigma_i n_i E_i$ $(n_i \in \mathbb{N})$ such that $W \cdot E_i < 0$ for each i: if D is a divisor on S such that dim (supp D $\cap$ supp E_i) = 0 and $D \cdot E_i > 0$ for each i, and π: $S \to X$ is the blow down of the E_i's, the divisor W can be defined by the condition $\pi^*(\pi_* D) = W + D$.

By the Nakai-Moishezon criterion the divisor $H_t = K_S - tW$ is then ample for $0 < t << 1$: in fact we can assume that D is ample and this implies

(1.10) $\quad H_t = K_S - tW = K_S - t\,\pi^*(\pi_* D) + tD =$
$\quad = \pi^*(K_X - t\,\pi_* D) + t\,D$;

moreover, $K_X - t\,\pi_* D$ is ample for $t << 1$ because K_X is ample on X.

For each effective divisor M on S we decompose M as the sum $M = M' + M''$ of two effective divisors, where $M'' \in \langle E_1, \dots , E_\mu \rangle$ and dim $(M' \cap E_i) = 0$ for each i .

If, for each t, there exists a line bundle N_t destabilizing $\mathcal{E}$ with respect to H_t, then for each t we have a diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & N_t & & & & \\
 & & & & \downarrow & & & & \\
(1.11) \quad 0 & \to & \mathcal{O}_S & \to & \mathcal{E} & \to & \mathcal{I}_x(K_S) & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{I}_{Z_t}(M_t) = \mathcal{I}_{Z_t}(K_S - N_t) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & & .
\end{array}
$$

If the "oblique arrow" $\mathcal{O}_S \to \mathcal{I}_{Z_t}(M_t)$ were zero, then $\mathcal{O}_S \subset N_t$, and, since both are saturated, $\mathcal{O}_S \cong N_t$ contradicting $N_t \cdot H_t > 0$.

It follows that M_t is an effective divisor for each t and, moreover, that

(1.12) $\quad M_t \cdot H_t \leq (1/2)\, K_S \cdot H_t = (1/2)\, K_S^2 - t\, W \cdot K_S =$
$= (1/2)\, K_S^2 = 2$.

But $M_t \cdot H_t = M_t \cdot K_S - t\, M_t'' \cdot W - t\, M_t' \cdot W$: the inequality $M_t \cdot K_S \leq 2$ would contradict the K_S-stability of $\mathcal{E}$ and so we have $M_t \cdot K_S$ $(=M_t' \cdot K_S) \geq 3$ and $M_t \cdot H_t \leq 2$ for each t. It follows that

(1.13) $\quad t\, (M_t \cdot W) \geq 1$

and then $0 < M_t \cdot W = M_t' \cdot W + M_t'' \cdot W$, where $M_t'' \cdot W < 0$ by the choice of W.

Let us fix now an index t_o such that H_{t_o} is ample. Then

$$M_t \cdot H_{t_o} = M_t \cdot (K_S - t_o W) = M_t \cdot (K_S - t\, W - (t_o - t)\, W) =$$
$$= M_t \cdot H_t - (t_o - t)\, M_t \cdot W < 2 \quad \text{for } 0 < t < t_o .$$

This implies that $\{M_t\}$ is a bounded family and so the set $\{W \cdot M_t\}$ is bounded too: but this is absurd, because then the inequality $W \cdot M_t \geq t^{-1}$ is impossible. Q.E.D.

We will now consider two examples of surfaces S of general type with $p_g = 0$, $K_S^2 = 4$, which have a base point free bicanonical system but fail to satisfy the hypotheses of Theorem 1.1.

<u>Example 1.14</u> The Burniat surface B(2) has fundamental group $\pi_1(B(2))$ isomorphic to $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2$, where $\mathbb{H}$ is the quaternion group of order 8 (cf. table 13 of [BPV], [Bu], [Pet]): so $\pi_1(B(2))$ has a non trivial central extension given by $\mathbb{H}^2$, which clearly admits an irreducible SU(2)-representation, since the group $\mathbb{H}$ admits the following irreducible SU(2)-representation, given by

$$i \to \begin{pmatrix} -\sqrt{(-1)} & 0 \\ 0 & -\sqrt{(-1)} \end{pmatrix}$$
$$j \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$k \to \begin{pmatrix} 0 & -\sqrt{(-1)} \\ -\sqrt{(-1)} & 0 \end{pmatrix}.$$

<u>Proposition 1.15</u> The bicanonical system is free from base points for the Burniat surface with K^2=4.

<u>Proof</u> By lemma 3.3. of ([Pet], (ii) page 118) it follows that, S being a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover

of the blow up Y of the plane in 5 points of which 3 lie on a line, all the sections of $H^0(S,2K_S)$ are Galois invariant and are pull-backs of rational tensor 2-forms with simple poles on the branch divisor. Hence the linear system $|2K_S|$ factors through the $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover and indeed through the anticanonical mapping of Y which is a birational morphism onto a quartic weak del Pezzo surface with a node corresponding to the line containing the 3 collinear points. Q.E.D.

Example 1.16 In [Ke], J.H. Keum gives an example of a surface S of general type with $p_g = 0$, $K_S^2 = 4$ and with fundamental group $\pi_1(S) \cong \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. Also here there exists a non split central extension $\tilde{\Gamma}$ of Γ by $\mathbb{Z}/2\mathbb{Z}$ which admits an irreducible SU(2)-representation, since there is a surjective map ψ obtained as a composition

$$\psi \colon \Gamma \to \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^3 \to (\mathbb{Z}/2\mathbb{Z})^2$$

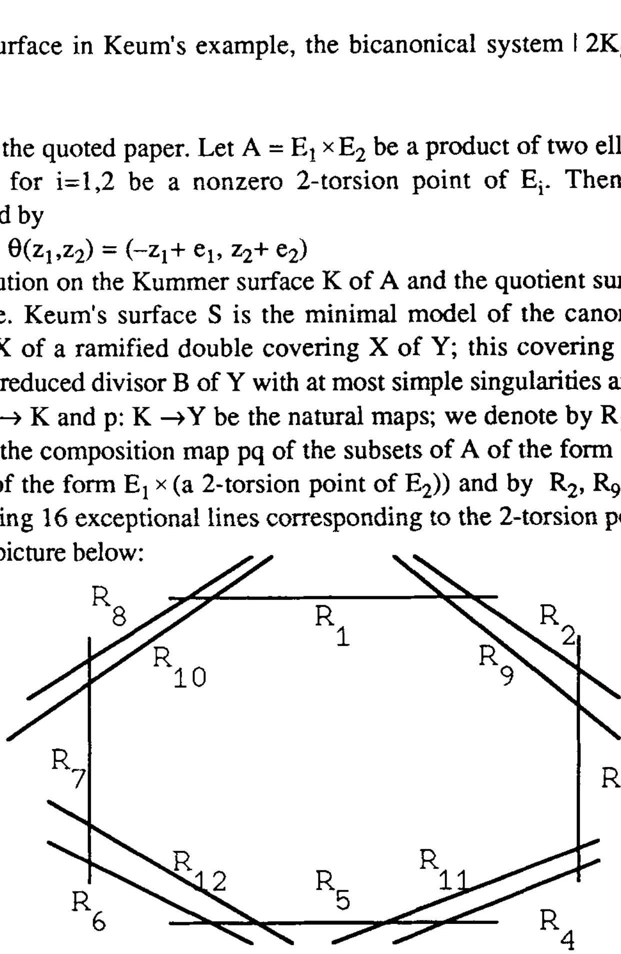
and as above one can pull-back by ψ the extension $1 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{H} \to (\mathbb{Z}/2\mathbb{Z})^2 \to 0$.

Proposition 1.17 If S is the surface in Keum's example, the bicanonical system $|2K_S|$ is base point free.

Proof We recall the notation of the quoted paper. Let $A = E_1 \times E_2$ be a product of two elliptic curves $E_i = \mathbb{C}/\mathbb{Z} + \tau_i\mathbb{Z}$ and e_i for i=1,2 be a nonzero 2-torsion point of E_i. Then the endomorphism θ: A → A defined by

$$\theta(z_1,z_2) = (-z_1 + e_1,\ z_2 + e_2)$$

induces a fixed-point-free involution on the Kummer surface K of A and the quotient surface $Y = K/\theta$ is an Enriques surface. Keum's surface S is the minimal model of the canonical resolution of singularities $\overline{X} \to X$ of a ramified double covering X of Y; this covering X is determined by a square root of a reduced divisor B of Y with at most simple singularities and B is defined as follows. Let q : A → K and p: K →Y be the natural maps; we denote by R_1, R_5 (resp. R_3, R_7) the images under the composition map pq of the subsets of A of the form (a 2-torsion point of E_1)$\times E_2$ (resp. of the form $E_1 \times$(a 2-torsion point of E_2)) and by R_2, R_9, R_4, R_{11}, R_6, R_{12}, R_8, R_{10} the remaining 16 exceptional lines corresponding to the 2-torsion points of A; the lines intersect as in the picture below:



The branch divisor B is defined by

$$B = R_2+R_4+R_6+R_8+R_9+R_{10}+R_{11}+R_{12}+F+G$$

where F, G are smooth elliptic curves belonging to the elliptic pencils $|2R_3+R_2+R_4+R_9+R_{11}|$ and $|2R_1+R_2+R_8+R_9+R_{10}|$ respectively. In particular, it holds

$$(pq)^*(F)= a\times E_2+(-a)\times E_2+(a+e_1)\times E_2+(-a+e_1)\times E_2$$
$$(pq)^*(G) = E_1\times b+E_1\times(-b)+E_1\times(b+e_2)+E_1\times(-b+e_2)$$

for some $a\in E_1$, $b\in E_2$.

To prove the proposition, it is enough to check that the bicanonical system $|2K_{\overline{X}}|$ of the surface $\overline{X}$ has base locus consisting entirely of the exceptional curves of the first kind counted with multiplicity one. Moreover, we can reduce the problem to the inspection of the linear system $|F+G|$ on Y, via the isomorphisms $H^0(2K_{\overline{X}}) \cong H^0(\mathcal{O}_Y(B)) \cong H^0(\mathcal{O}_Y(F+G))$. This follows since both $|F|$ and $|G|$ are base point free pencils. Q.E.D.

§2 m^{th} root extraction trick and change of fundamental group

In this paragraph we are going to sharpen the result of theorem 1.1, showing that the extension appearing there can be realized by the fundamental group of a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover Y of S . By pulling back $\mathcal{E}$ to Y, and showing that the pull back still remains stable (and not only semistable) we shall be able to apply Donaldson's theorem to a stable vector bundle on Y with trivial Chern classes ([Do], [Ko]).

In order to do so , we recall the m^{th} root extraction trick, which will produce the desired Y in the case m=2.

In the rest of the paragraph we shall explicitly describe how the fundamental group of Y can be computed, later on we shall apply this recipe in some concrete examples.

Lemma 2.1 (m^{th} root extraction trick) Let S be a smooth algebraic surface and $f : S \to \mathbb{P}^2$ be a holomorphic map associated to a base point free subsystem of a linear system $|D|$. Let Y be obtained as the fibre product $Y = S \times_{\mathbb{P}^2} \mathbb{P}^2$ of the previous $f : S \to \mathbb{P}^2$, and, for a general choice of coordinates, of the m^{th} power map $g : \mathbb{P}^2 \to \mathbb{P}^2$ (i.e., $g(y_0,y_1,y_2) = (y_0^m, y_1^m, y_2^m)$).

Then, if F is the natural morphism $F: Y \to S$, $F_*: \pi_1(Y) \to \pi_1(S)$ is surjective, and its kernel K is cyclic of order dividing m; moreover K is contained in the centre of $\pi_1(Y)$.

Proof We denote for simplicity by H_i (i=0,1,2) the coordinate lines on $\mathbb{P}^2$, by D_i the inverse image of H_i under f, by C_i the inverse image of D_i under F, by H' the union of the H_i's, and similarly we define D' and C'. By the genericity of the H_i's, all the above divisors have global normal crossings, Y is smooth, F is a Galois $(\mathbb{Z}/m\mathbb{Z})^2$-cover branched on D' and totally ramified at the singular points of D'. In particular $\pi_1(Y-C')\to\pi_1(S-D')$ is a normal subgroup with quotient group $(\mathbb{Z}/m\mathbb{Z})^2$.

We have the following diagram of sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
(\mathbb{Z}/m\mathbb{Z})^3 & \to & \mathcal{K}/\mathcal{H} & \to & (\mathbb{Z}/m\mathbb{Z})^2 & \to & 0 & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathcal{K} & \to & \pi_1(S-D') & \to & \pi_1(S) & \to & 0 \\
 & & \uparrow & & \uparrow F_* & & \uparrow F_* & & \\
0 & \to & \mathcal{H} & \to & \pi_1(Y-C') & \to & \pi_1(Y) & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2.2}$$

which are exact, with the exception of the first row. Here exactness of the middle column was already mentioned, whereas exactness of the second and third row is standard, and the inclusion $\mathcal{H} \subset \mathcal{K}$ is obvious.

Claim 2.3 $\mathcal{K}, \mathcal{H}$ are central and with respective generators $\gamma_0, \gamma_1, \gamma_2$ for $\mathcal{K}$, $\gamma_0^m, \gamma_1^m, \gamma_2^m$ for $\mathcal{H}$.

Proof of the Claim 2.3 : let γ_i, for i=0,1,2, be a loop consisting of the conjugate (under a path in S–D') of a small circle in the normal space to a smooth point of D_i . Argueing as in theorem 1.6 of [Ca1], one shows that $\mathcal{K}$ is generated by conjugates of the γ_i's, and that the γ_i's lie in the centre of $\pi_1(S-D')$.
An entirely similar argument applies to $\mathcal{H}$, since γ_i^m is obtained in Y–C' by the same procedure by which γ_i is gotten. Q.E.D. for the claim

It is worthwhile to notice that, since γ_i , γ_j can be chosen to be local generators of $\pi_1(U-D')$ for a suitable neighbourhood of $x \in D_i \cap D_j$, by looking at the local monodromy of F, we obtain that they map to 2 generators of $(\mathbb{Z}/m\mathbb{Z})^2$.

By diagram chasing, the surjectivity of F_* follows from the surjectivity of the map $\mathcal{K}/\mathcal{H} \to (\mathbb{Z}/m\mathbb{Z})^2$.

Clearly $K = \ker (F_*: \pi_1(Y) \to \pi_1(S)) = (\mathcal{K} \cap \pi_1(Y-C'))/\mathcal{H}$, is contained in $\mathcal{K}/\mathcal{H}$. But, as we mentioned above, we have two surjective maps $(\mathbb{Z}/m\mathbb{Z})^3 \to \mathcal{K}/\mathcal{H} \to (\mathbb{Z}/m\mathbb{Z})^2$ such that any 2 of the 3 standard generators of $(\mathbb{Z}/m\mathbb{Z})^3$ map to 2 generators of $(\mathbb{Z}/m\mathbb{Z})^2$. Hence K is cyclic of order dividing m.

We can be more precise since in fact K is isomorphic to the kernel of $\mathcal{K}/\mathcal{H} \to (\mathbb{Z}/m\mathbb{Z})^2$, therefore K is the image of the cyclic group $(\mathbb{Z}/m\mathbb{Z}) = \ker\big((\mathbb{Z}/m\mathbb{Z})^3 \to (\mathbb{Z}/m\mathbb{Z})^2\big)$ inside $\pi_1(Y)$. Q.E.D.

Corollary 2.4 If S is simply connected, $\pi_1(Y)$ is cyclic of order r where r = G.C.D. (m,d), and d is the divisibility index of the divisor D (i.e., d is the order of the cyclic group $(\mathbb{Q}D \cap H^2(S,\mathbb{Z}))/\mathbb{Z}D)$). More generally, the kernel K always admits a surjective homomorphism onto a cyclic group of order r.

Proof Hypotheses of proposition 1.8 of [Ca1] are satisfied. Applying this proposition, one can see that $\pi_1(Y)$ is the quotient of $\ker((\mathbb{Z}/m\mathbb{Z})^3 \to (\mathbb{Z}/m\mathbb{Z})^2)$ by the image of the map obtained as the composition of $(H^2(S,\mathbb{Z}) \to H^2(D,\mathbb{Z}) \cong \mathbb{Z}^3 \to (\mathbb{Z}/m\mathbb{Z})^3)$. By Poincare' duality, this image consists of the elements divisible by d and our assertion is proven.
In the general case, the above quotient represents exactly the natural image of K in $H_1(S-D',\mathbb{Z})$ (cf. the proof of cor. 1.7, prop. 1.8 ibidem). Q.E.D.

We can indeed prove a much more precise statement (see Th. 2.16 below), which is our main result. If m is a fixed positive integer, the assertion in lemma 2.1 shows that (it suffices, in general, to add to the divisor D m times a suitably very ample divisor H, such that $|D+mH|$ yields a finite morphism to $\mathbb{P}^2$) to each divisor D on a smooth surface S we can associate a central extension of the fundamental group of the surface by a cyclic group $\mathbb{Z}/r'\mathbb{Z}$, where r' is divisible by r = G.C.D. (m,d) and d is the divisibility index of D: this is the extension describing the fundamental group of a $(\mathbb{Z}/m\mathbb{Z})^2$-Galois covering of S under which the divisor D becomes m-divisible.

Looking now at S as a quotient of its universal covering space $\tilde{S}$, we will see that, by general properties of the cohomology of good quotient spaces, to each divisor D and to each integer m is uniquely associated another central extension of $\pi_1(S)$ by a cyclic group: we shall call the latter the "algebraic construction" of the extension.

The key fact is that this algebraic construction yields the same extension that we obtain by the geometrical construction of extracting the m^{th} root of D, as it will be shown in Th. 2.16.

We firstly recall some basic facts and notation concerning the spectral sequence for the cohomology of a quotient $X = \tilde{X}/G$ by a properly discontinuous group G (see [Mu], appendix to section 2, and [G], ch. 5).

If $\mathfrak{F}$ is a G-linearized sheaf on $\tilde{X}$ and π: $\tilde{X} \to X$ is the quotient map, one can describe the functor $\Gamma(\tilde{X},\mathfrak{F})^G$ as a composition in two different ways:

(2.5) $\quad \mathfrak{F} \to \Gamma(\tilde{X},\mathfrak{F}) \to \Gamma(\tilde{X},\mathfrak{F})^G$

(2.6) $\quad \mathfrak{F} \to \pi_*(\mathfrak{F})^G \to \Gamma(X,\pi_*(\mathfrak{F})^G)$.

The derived cohomology functors are then given with two filtrations whose associated gradings are the limits of two spectral sequences

(2.7) $\quad H^p(G,H^q(\tilde{X},\mathfrak{F})) = E_2{}^{p,q}$

(2.8) $\quad H^i(X,\mathcal{R}_G{}^j\,\pi_*(\mathfrak{F})) = E'_2{}^{i,j}$

where $H^p(G, -)$ is the group cohomology and $\mathcal{R}_G{}^j\,\pi_*(-)$ denote the derived cohomology functors of $\pi_*(-)^G$. If the action of G is free and gives indeed a covering space, then the functor $\mathfrak{F} \to (\pi_*\mathfrak{F})^G$ is exact and consequently $\mathcal{R}_G{}^j\pi_*(\mathfrak{F}) = 0$ for each $j \geq 0$. So the spectral sequence degenerates at E'_2 and the first spectral sequence converges to $H^i(X, (\pi_*\mathfrak{F})^G)$:

(2.9) $\quad H^p(G,H^q(\tilde{X},\mathfrak{F})) \Rightarrow H^*(X,(\pi_*\mathfrak{F})^G)$.

Assume now that $\mathfrak{F} = \mathbb{Z}_{\tilde{X}}$. Then $(\pi_*\mathbb{Z}_{\tilde{X}})^G = \mathbb{Z}_X$, thus

(2.10) $\quad H^p(G,H^q(\tilde{X},\mathbb{Z}_{\tilde{X}})) \Rightarrow H^*(X,\mathbb{Z}_X)$.

If we assume moreover that $H^1(\tilde{X},\mathbb{Z}_{\tilde{X}}) = 0$, e.g. if $\tilde{X}$ is the universal cover of X, the $E_2{}^{p,1}$-term in the spectral sequence 2.7 vanishes and so the differential $d_2 : E_2{}^{p,q} \to E_2{}^{p+2,q-1}$ is zero for each $q \leq 2$. So we can say that $E_2{}^{p,q} = E_3{}^{p,q}$ for each $q \leq 2$ and, finally, that

(2.11) $\quad d_3 : H^p(G, H^2(\tilde{X},\mathbb{Z}_{\tilde{X}})) \to H^{p+3}(G, H^0(\tilde{X},\mathbb{Z}_{\tilde{X}}))$

is the only non zero map for $q \leq 2$. The edge-morphisms give then an isomorphism

(2.12) $\quad H^1(X,\mathbb{Z}_X) \cong H^1(G,\mathbb{Z})$

and an exact sequence

(2.13) $\quad 0 \to H^2(G,\mathbb{Z}) \to H^2(X,\mathbb{Z}_X) \xrightarrow{\pi^*} H^2(\tilde{X},\mathbb{Z}_{\tilde{X}})^G \to H^3(G,\mathbb{Z})$

where the arrow $H^2(X,\mathbb{Z}_X) \to H^2(\tilde{X},\mathbb{Z}_{\tilde{X}})^G$ is given by the inverse image π^*. We similarly have an analogous sequence for any system $\mathbb{F}$ of coefficients ($\mathbb{F} = \mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q}$, ...) and in particular, associated to the exact sequence $0 \to m\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \to H^2(G,\mathbb{Z}) & \to & H^2(X,\mathbb{Z}_X) & \xrightarrow{\pi^*} & H^2(\tilde{X},\mathbb{Z}_{\tilde{X}})^G & \to & H^3(G,\mathbb{Z}) \\
m\downarrow & & m\downarrow & & m\downarrow & & m\downarrow \\
0 \to H^2(G,\mathbb{Z}) & \to & H^2(X,\mathbb{Z}_X) & \xrightarrow{\pi^*} & H^2(\tilde{X},\mathbb{Z}_{\tilde{X}})^G & \to & H^3(G,\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to H^2(G,\mathbb{Z}/m\mathbb{Z}) & \to & H^2(X,\mathbb{Z}_X/m\mathbb{Z}_X) & \to & H^2(\tilde{X},\mathbb{Z}_{\tilde{X}}/m\mathbb{Z}_{\tilde{X}})^G & \to & H^3(G,\mathbb{Z}/m\mathbb{Z})
\end{array} \tag{2.14}$$

which is exact in the rows and in the columns.

Algebraic construction Now let $\pi: \tilde{X} \to X$ be the universal covering, let G be the foundamental group of X and m be a fixed positive integer. By the universal coefficients formula, the group $H^2(\tilde{X}, \mathbb{Z}_{\tilde{X}})$ is torsion free, hence, if d is the divisibility index of a class $\tilde{D}$ in $H^2(\tilde{X}, \mathbb{Z}_{\tilde{X}})^G$, then the image $\tilde{\delta} \in H^2(\tilde{X}, \mathbb{Z}/m\mathbb{Z}_{\tilde{X}})^G$ under the map of diagram 2.14 has period exactly m/r, where r = G.C.D.(m,d). In particular, the image $\hat{\delta}$ in $H^2(\tilde{X}, \mathbb{Z}/r\mathbb{Z}_{\tilde{X}})^G$ is 0. Hence if $\tilde{D} \in H^2(\tilde{X}, \mathbb{Z}_{\tilde{X}})^G$ equals $\pi^*(D)$ for $D \in H^2(X, \mathbb{Z}_X)$, the class D maps to a cohomology class $\delta \in H^2(X, \mathbb{Z}/r\mathbb{Z}_X)$ coming from a cohomology class $\delta \in H^2(G, \mathbb{Z}/r\mathbb{Z})$, since we must have $\hat{\delta} = \pi^*(\delta) = 0$ in $H^2(\tilde{X}, \mathbb{Z}/r\mathbb{Z}_{\tilde{X}})^G$. So we get:

Construction To each class $D \in H^2(X, \mathbb{Z}_X)$ we can associate $\tilde{D}$, r, δ as above, hence a unique (up to isomorphism) central extension

(2.15) $$0 \to \mathbb{Z}/r\mathbb{Z} \to G(\delta) \to G \to 0 \quad .$$

Theorem 2.16 If Y is obtained from X via the m^{th} root extraction trick associated to $| D |$, then we have that $\pi_1(Y) \cong G(-\delta)$, i.e., as an extension of $G \cong \pi_1(X)$, $\pi_1(Y)$ is the extension associated to $-D$ by the "algebraic construction" described before.

Proof Let us take the fibre product Y' of the universal covering $\pi: \tilde{X} \to X$ and the m^{th} root extraction $(\mathbb{Z}/m\mathbb{Z})^2$-Galois covering $F: Y \to X$; we have the following diagram

(2.17)
$$\begin{array}{ccccc} \tilde{Y} & \to & Y' & \xrightarrow{q} & \tilde{X} \\ & & p \downarrow & F & \downarrow \pi \\ & & Y & \to & X \\ & & \psi \downarrow & (y_i^m) & \downarrow f \\ & & \mathbb{P}^2 & \to & \mathbb{P}^2 \end{array}$$

where

i) $0 \to \mathbb{Z}/r'\mathbb{Z} \to \pi_1(Y) \to \pi_1(X) = G \to 0$ is a central extension and r' divides m, as we know from lemma 2.1;

ii) p is the covering associated to $\mathbb{Z}/r'\mathbb{Z}$, so $\pi_1(Y') \cong \mathbb{Z}/r'\mathbb{Z}$;

iii) $f^* \mathcal{O}(1) \equiv D$;

iv) $(\mathbb{Z}/m\mathbb{Z})^2 \times \Gamma$ operates on Y'. The covering $\tilde{Y} \to Y'$ is an etale $(\mathbb{Z}/r'\mathbb{Z})$-Galois cover induced by the universal covering $\tilde{Y} \to Y$, since Γ operates freely.

Let $D' = D_1 \cup D_2 \cup D_3$ and $\tilde{D}' = \pi^{-1}(D')$, where the D_i's are the inverse image under the map f of the coordinate lines on $\mathbb{P}^2$.

Then $\pi_1(X - \tilde{D}') = \ker(\pi_1(X - D') \to \pi_1(X)) = \mathcal{K}$ is abelian and is generated by $\gamma_0, \gamma_1, \gamma_2$.

Step I We show that r = r'.

By Lefschetz's theorem $\pi_1(D_i)$ surjects onto $\pi_1(X)$, hence $\pi^{-1}(D_i) = \tilde{D}_i$ is connected and smooth.

The map $Y' \to \tilde{X}$ is an abelian $(\mathbb{Z}/m\mathbb{Z})^2$-cover which is unramified on $\tilde{X} - \tilde{D}'$. Since $\pi_1(Y')$ is cyclic, it suffices to calculate the first homology group $H_1(Y', \mathbb{Z})$.

We proceed as in [Ca1], sequel to cor. 1.7., and prop. 1.8. Here we have to apply Lefschetz's duality as in [Dol] prop. 7.14 page 297, by which it follows that, for a manifold M with boundary $L = \partial M$, and of dimension n, $H_{n-i}(X) \cong H_{n-i}(X - \partial X) \cong H_c^i(X, \partial X)$ the last group denoting cohomology with compact supports.

Hence, argueing as in loc. cit., $H_1(\tilde{X} - \tilde{D}') = H_c^3(\tilde{X}, \tilde{D}')$, which, by the exact sequence

(2.18) $\quad H_c^2(\tilde{X}) \xrightarrow{\rho} H_c^2(\tilde{D}') \to H_c^3(\tilde{X},\tilde{D}') \to 0$
is isomorphic to coker (ρ), $\tilde{X}$ being simply connected.
Now, by Poincare' duality $H_c^2(\tilde{X}) \cong H_2(\tilde{X})$ is a free $\mathbb{Z}$-module, whereas by Mayer-Vietoris and again Poincare' duality $H_c^2(\tilde{D}') \cong \oplus_i H_c^2(\tilde{D}'_i) \cong \oplus_i H_0(\tilde{D}'_i) \cong \mathbb{Z}^3$, the $\tilde{D}'_i$'s being as we mentioned smooth and connected. The map ρ is given by geometrical intersection and its dual sends $\mathbb{Z}^3 \to H^2(\tilde{X},\mathbb{Z})$ by mapping each generator e_1, e_2, e_3 to the class of $\tilde{D}_i$.

Argueing as in loc. cit. and as in claim 2.3, we obtain that, if d is the divisibility index of $\tilde{D}$, then firstly coker (ρ) is isomorphic to $\mathbb{Z}^3/\mathbb{Z}(de_1+de_2+de_3)$, hence secondly $\pi_1(Y') \cong (\mathbb{Z}/m\mathbb{Z})/(d) = \mathbb{Z}/r\mathbb{Z}$ if r = G.C.D.(m,d) as in our notation.

<u>Step II</u> We can reduce the proof of the theorem to the case where r = m.

In fact, we can factor F: Y $\to$ X as $Y \xrightarrow{F''} Z \xrightarrow{F'} X$ where F' is obtained by extracting the r^{th} root of D, hence $F'^*(D) \equiv rL$. We shall show in Step III that the divisibility of the pull back $\tilde{L}$ of L to the universal cover $\tilde{Z}$ of Z is precisely d/r.

Since F" is obtained by extracting the $(m/r)^{th}$ root of L, it follows by step I that $\pi_1(Y) \cong \pi_1(Z)$, thereby reducing the proof of the theorem to the case r = m.

<u>Proof of step II</u> (Proof of the theorem in the special case r=m)

We first introduce some notation to describe explicitly the cocycles on X and $\tilde{X}$.
Let $\{U_\alpha\}$ be a sufficiently fine cover of X, such that, for each U_α, $\pi^{-1}(U_\alpha) = \cup_{g\in G} g(V_\alpha)$ where the union is disjoint and we have made a non canonical choice of V_α, a connected component of $\pi^{-1}(U_\alpha)$. We shall also write, for further use,

(2.19) $\quad g(V_\alpha) = g \cdot V_\alpha = V_{(\alpha,g)} \qquad$ (so $V_{(\alpha,1)} = V_\alpha$)

and we let G act on the left.

One can observe the following facts:

a) For each (α,β) such that $U_\alpha \cap U_\beta \neq \emptyset$, there exists a unique element $h(\alpha,\beta)$ of G such that

(2.20) $\quad V_{(\alpha,1)} \cap V_{(\alpha,h(\alpha,\beta))} \neq \emptyset$.

b) If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, since π is a local homeomorphism

(2.21) $\quad \emptyset \neq V_{(\gamma,h(\alpha,\gamma))} \cap V_{(\beta,h(\alpha,\beta))} \qquad (= h(\alpha,\beta) \cdot (V_{(\gamma,h(\beta,\gamma))} \cap V_{(\beta,1)}))$

Hence, if G acts on the left, we have the relation

(2.22) $\quad h(\alpha,\gamma) = h(\alpha,\beta)\, h(\beta,\gamma) \qquad$ for each $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

In particular $\qquad h(\beta,\alpha) = h(\alpha,\beta)^{-1}$.

c) If $U_\alpha \cap U_\beta \neq \emptyset$, $V_{(\alpha,g)}$ intersects exactly $V_{(\beta, g\, h(\alpha,\beta))}$.

Therefore, if $(f_{\alpha\beta})$ is a cocycle for $L = \mathcal{O}_X(D)$ relative to the covering $\{U_\alpha\}$ on X, there exists, for the line bundle $\tilde{L}$ on $\tilde{X}$ such that $(\tilde{L}^{\otimes r}) \cong \pi^*(\mathcal{O}_X(D))$ (whose existence is guaranteed by our assumption), a cocycle $(\tilde{f}_{(\alpha,g)(\beta,\, g\, h(\alpha,\beta))})$ such that $(\tilde{f}_{(\alpha,g)(\beta,\, g\, h(\alpha,\beta))})^r = f_{\alpha\beta}$.
For short, we write $(\tilde{f}_{(\alpha,g)(\beta,g')})$ but we recall that $g' = g\, h(\alpha,\beta)$.

We write $z_{(\alpha,g)}$ for a local generator of on $V_{(\alpha,g)}$, so that

(2.23) $\quad z_{(\alpha,g)} = \tilde{f}_{(\alpha,g)(\beta,g')}\, z_{(\beta,g')}$.

Using the isomorphism $\tilde{L}^{\otimes r} \cong \pi^*(L)$ we can assume that G acts on $(\tilde{L}^{\otimes r})$ by sending the local generators $(z_{(\alpha,g)})^r$ of $(\tilde{L}^{\otimes r})$ one to another, i.e. for each $\bar{g} \in G$

(2.24) $\quad \bar{g}: \ (z_{(\alpha,g)})^r \to (z_{(\alpha,\bar{g} g)})^r$.

Since $\widetilde{L} \in H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})^G$, for each $\bar{g} \in G$ there is an isomorphism $\widetilde{L} \to \bar{g}_*(\widetilde{L})$, which we still denote by $\bar{g}$ and which induces the above action on $(\widetilde{L}^{\otimes r})$ (by which $(L^{\otimes r}) \cong \pi^*(L)$). Hence $\bar{g}$ acts on $\widetilde{L}$ by

$$(2.25) \qquad \bar{g}:\ z_{(\alpha,g)} \to c^{\bar{g}}_{(\alpha,g)}\ z_{(\alpha,\bar{g} g)}$$

where $c^{\bar{g}}_{(\alpha,g)}$ satisfies the identity $(c^{\bar{g}}_{(\alpha,g)})^r = 1$. The constants $c^{\bar{g}}_{(\alpha,g)}$ must satisfy some compatibility condition, since

$$(2.26) \qquad \begin{array}{ccc} z_{(\alpha,g)} = \widetilde{f}_{(\alpha,g)(\beta,g')}\ z_{(\beta,g')} & \xrightarrow{\bar{g}} & \widetilde{f}_{(\alpha,g)(\beta,g')}\ c^{\bar{g}}_{(\beta,g')}\ z_{(\beta,\bar{g} g')} \\ \downarrow \bar{g} & & \\ c^{\bar{g}}_{(\alpha,g)}\ z_{(\alpha,\bar{g} g)} = c^{\bar{g}}_{(\alpha,g)}\ \widetilde{f}_{(\alpha,\bar{g} g)(\beta,\bar{g} g')}\ z_{(\beta,\bar{g} g')} & & \end{array}$$

where $g' = g\, h(\alpha,\beta)$ as before. Hence:

$$(2.27) \qquad c^{\bar{g}}_{(\alpha,g)}\ \widetilde{f}_{(\alpha,\bar{g}g)(\beta,\bar{g}\, gh(\alpha,\beta))} = \widetilde{f}_{(\alpha,g)(\beta,gh(\alpha,\beta))}\ c^{\bar{g}}_{(\beta,gh(\alpha,\beta))}\ .$$

The above formula shows that $c^{\bar{g}}_{(\beta,gh(\alpha,\beta))}$ is completely determined by $c^{\bar{g}}_{(\alpha,g)} \in \mu_r$, where μ_r is the group of the r[th] roots of the unity.

Since $\widetilde{X}$ is connected, the $c^{\bar{g}}_{(\alpha,g)}$ are completely determined by one of them. Moreover, once fixed a local generator $z_{(\alpha,1)}$ for the bundle L such that $(z_{(\alpha,g)})^r$ is G-invariant as before, for each $\bar{g} \in G$ one can also choose the root $z_{(\alpha,\bar{g})}$ such that the action is given by

$$\bar{g}:\ z_{(\alpha,1)} \to z_{(\alpha,\bar{g})}\ .$$

In other words, we may assume:

$$(2.28) \qquad c^{g}_{(\alpha,1)} = 1 \qquad \text{for each } g \in G\,.$$

We can now check how the composite action of $(g_1\, g_2)^{-1} \cdot g_1 \cdot g_2$ fails to act as the identity:

$$(2.29) \qquad \begin{array}{l} z_{(\alpha,g)} \xrightarrow{g_2} c^{g_2}_{(\alpha,g)}\ z_{(\alpha,g_2 g)} \xrightarrow{g_1} c^{g_2}_{(\alpha,g)}\ c^{g_1}_{(\alpha,g_2 g)}\ z_{(\alpha,g_1 g_2 g)} \\ \qquad\qquad \downarrow (g_1 g_2)^{-1} \\ c^{g_2}_{(\alpha,g)}\ c^{g_1}_{(\alpha,g_2 g)}\ c^{(g_1 g_2)^{-1}}_{(\alpha,g_1 g_2 g)}\ z_{(\alpha,g)} = \\ = c^{g_2}_{(\alpha,g)}\, c^{g_1}_{(\alpha,g_2 g)} \left(c^{(g_1 g_2)}_{(\alpha,g)}\right)^{-1} z_{(\alpha,g)} \end{array}$$

where the last equality follows by observing that we can assume:

$$(2.30) \qquad c^{\bar{g}^{-1}}_{(\alpha,g)} = \left(c^{\bar{g}}_{(\alpha,(\bar{g})^{-1} g)} \right)^{-1}.$$

By 2.27 and the connectedness of $\widetilde{X}$ we get then:

$$(2.31) \qquad c^{g_2}_{(\alpha,g)}\, c^{g_1}_{(\alpha,g_2 g)} \left(c^{(g_1 g_2)}_{(\alpha,g)}\right)^{-1} \text{ is independent of } (\alpha,g).$$

This is in fact an element of $H^2(G,\mu_r)$ classifying the extension of groups ("theta group" extending G): by the assumption 2.28, we know that for each α:

(2.32) $c^{g_2}_{(\alpha,g)} c^{g_1}_{(\alpha,g_2g)} \left(c^{(g_1g_2)}_{(\alpha,g_1g_2g)}\right)^{-1} = c^{g_1}_{(\alpha,g_2)} = c(g_1, g_2)$

and we found that:

(2.33) $c^{g_1}_{(\alpha,g_2)} = c^{g_1}_{g_2} = c\,(g_1, g_2)$ is independent of α.

We must then explicitly write the cocycle $c_{\alpha\beta\gamma} \in H^2(X,\mu_r)$ which is associated to the cocycle $c(g_1, g_2)$ given in 2.32. We use the description proven in ([Mu], page 23) of the image of $c(g_1, g_2)$ which yields the following formula:

(2.34) $c_{\alpha\beta\gamma} = c^{h(\alpha,\beta)}_{h(\beta,\gamma)}$.

We want now to show that, taking Chern classes modulo r, the inverse $(c_{\alpha\beta\gamma})^{-1}$ gives exactly the Chern class of L.

We start by describing more explicitly this class, in terms of the chosen cocycles for L and $\tilde{L}$. We will use the exact sequence

(2.35) $0 \to H^2(G,\mu_r) \to H^2(X,\mu_r) \xrightarrow{\pi^*} H^2(\tilde{X},\mu_r)^G \to H^3(G,\mu_r)$

obtained as in 2.13.

We can apply the theory of spectral sequences for G-linearized sheaves discussed above to the case $\mathcal{F} = \mathcal{O}_{\tilde{X}}, \mathcal{O}^*_{\tilde{X}}$: specializing 2.9, we get

(2.36) $H^p(G,H^q(\tilde{X},\mathcal{O}_{\tilde{X}})) \Rightarrow H^*(X,\mathcal{O}_X)$

(2.37) $H^p(G, H^q(\tilde{X},\mathcal{O}^*_{\tilde{X}})) \Rightarrow H^*(X,\mathcal{O}^*_X)$ (since $\pi_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_X$ and $\pi_*(\mathcal{O}^*_{\tilde{X}}) = \mathcal{O}^*_X$) and an exact sequence

(2.38) $0 \to H^1(G,H^0(\mathcal{O}^*_{\tilde{X}})) \to H^1(X,\mathcal{O}^*_X) \xrightarrow{\pi^*} H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}})^G \xrightarrow{\partial} H^2(G,H^0(\mathcal{O}^*_{\tilde{X}}))$

In this sequence, to each G-linearized line bundle $\tilde{\mathcal{L}}$ on $\tilde{X}$ is associated the "theta group" $\mathcal{G}(\tilde{\mathcal{L}})$: this is a central extension

(2.39) $0 \to \mathrm{Aut}(\tilde{\mathcal{L}}) = H^0(\mathcal{O}^*_{\tilde{X}}) \to \mathcal{G}(\tilde{\mathcal{L}}) \to G \to 0$

classified by $\partial(\tilde{\mathcal{L}}) \in H^2(G, H^0(\mathcal{O}^*_{\tilde{X}}))$. In particular this applies when $\tilde{\mathcal{L}}$ is a pull back bundle from X.

We will firstly consider the $(2\pi i)$-twisted exponential sequence

(2.40) $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{e} \mathcal{O}^*_X \to 0$

on X (and the corresponding on $\tilde{X}$) that gives rise to the diagram

$$\begin{array}{ccccccc}
0 \to H^1(G,H^0(\mathcal{O}_{\tilde{X}})) & \to & H^1(X,\mathcal{O}_X) & \xrightarrow{\pi^*} & H^1(\tilde{X},\mathcal{O}_{\tilde{X}})^G & \to & H^2(G,H^0(\mathcal{O}_{\tilde{X}})) \\
\downarrow e & & \downarrow e & & \downarrow & & \downarrow \\
0 \to H^1(G,H^0(\mathcal{O}^*_{\tilde{X}})) & \to & H^1(X,\mathcal{O}^*_X) & \xrightarrow{\pi^*} & H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}})^G & \xrightarrow{\partial} & H^2(G,H^0(\mathcal{O}^*_{\tilde{X}})) \\
\downarrow & & \downarrow \delta & & \downarrow \delta_G & & \downarrow \\
0 \to H^2(G,\mathbb{Z}) & \to & H^2(X,\mathbb{Z}) & \xrightarrow{\pi^*} & H^2(\tilde{X},\mathbb{Z})^G & \to & H^3(G,\mathbb{Z})
\end{array} \qquad (2.41)$$

in which rows and columns are exact and the map δ is the first Chern class: so Im δ is the Neron Severi group NS(X) of X and Im δ_G is NS^G .

On the other hand, we can consider the Kummer sequence

(2.42) $0 \to \mu_r \xrightarrow{r} \mathcal{O}^*_X \xrightarrow{r} \mathcal{O}^*_X \to 0$,

where by $\mathcal{O}^*_X \xrightarrow{r} \mathcal{O}^*_X$ we denote the r^{th} power (and the corresponding sequence on $\tilde{X}$): the exact sequence of cohomology groups gives then

(2.43) $H^1(X,\mathcal{O}^*_X) \xrightarrow{r} H^1(X,\mathcal{O}^*_X) \to H^2(X,\mu_r)$

and, again, we have a diagram with exact rows and colums

$$
\begin{array}{lccccccc}
 & 0 \to H^1(G,H^0(\mathcal{O}^*_{\tilde X})) & \to & H^1(X,\mathcal{O}^*_X) & \xrightarrow{\pi^*} & H^1(\tilde X,\mathcal{O}_{\tilde X})^G & \to & H^2(G,H^0(\mathcal{O}^*_{\tilde X})) \\
 & r\downarrow & & r\downarrow & & \downarrow & & \downarrow \\
(2.44) & 0 \to H^1(G,H^0(\mathcal{O}^*_{\tilde X})) & \to & H^1(X,\mathcal{O}^*_X) & \xrightarrow{\pi^*} & H^1(\tilde X,\mathcal{O}^*_{\tilde X})^G & \xrightarrow{\partial} & H^2(G,H^0(\mathcal{O}^*_{\tilde X})) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 \to H^2(G,\mu_r) & \to & H^2(X,\mu_r) & \xrightarrow{\pi^*} & H^2(\tilde X,\mu_r)^G & \to & H^3(G,\mu_r)\ .
\end{array}
$$

By diagram chasing, using 2.44, we check that the Chern class $\hat c_1(L)$ (this is the first Chern class $c_1(L)$ modulo r) of L is given by

$$
(2.45)\quad \hat c_1(L) = f_{\alpha\beta}^{\ 1/r} f_{\beta\gamma}^{\ 1/r} f_{\alpha\gamma}^{\ -1/r} = \tilde f_{(\alpha,1)(\beta,h(\alpha,\beta))} \tilde f_{(\beta,1)(\gamma,h(\beta,\gamma))} (\tilde f_{(\alpha,1)(\gamma,h(\alpha,\gamma))})^{-1}.
$$

(we can take the cocycle $(1/2\pi i)[\log f_{\alpha\beta} + \log f_{\beta\gamma} + \log f_{\gamma\alpha}]$ as a representative for $c_1(L)$).

Since $f_{\alpha\beta} = (\tilde f_{(\alpha,g)(\beta,gh(\alpha,\beta))})^r$ the class is zero in $H^2(\tilde X,\mu_r)$ as follows from the equality

$$
(2.46)\quad \tilde f_{(\alpha,g)(\beta,gh(\alpha,\beta))} \tilde f_{(\beta,gh(\alpha,\beta))(\gamma,gh(\alpha,\gamma))} = \tilde f_{(\alpha,g)(\gamma,gh(\alpha,\gamma))}.
$$

Hence $\hat c_1(L)$ is cohomologous to a class coming from $H^2(G,\mu_r)$ in 2.35, and our claim is that this class is the inverse $(c_{\alpha\beta\gamma})^{-1}$ of the class $c_{\alpha\beta\gamma}$ described in 2.34.

So we have to show that

$$
(2.47)\quad \hat c = \hat c_1(L) = \tilde f_{(\alpha,1)(\beta,h(\alpha,\beta))} \tilde f_{(\beta,1)(\gamma,h(\beta,\gamma))} (\tilde f_{(\alpha,1)(\gamma,h(\alpha,\gamma))})^{-1}
$$

is cohomologous (in $H^2(X,\mu_r)$) to

$$
(2.48)\quad (c_{\alpha\beta\gamma})^{-1} = (c^{h(\alpha,\beta)}_{h(\beta,\gamma)})^{-1} = c(h(\alpha,\beta),h(\beta,\gamma))^{-1},
$$

where, for each g, $\bar g$, α,β it holds (notice that $c^{\bar g}_{(\alpha,g)} \in \mu_r$)

$$
(2.49)\quad c^{\bar g}_{g}\ \tilde f_{(\alpha,\bar g g)(\beta,\bar g gh(\alpha,\beta))} = c^{\bar g}_{gh(\alpha,\beta)}\ \tilde f_{(\alpha,g)(\beta,gh(\alpha,\beta))}.
$$

In particular

$$
(2.50)\quad \tilde f_{(\alpha,\bar g)(\beta,\bar g h(\alpha,\beta))} = c^{\bar g}_{h(\alpha,\beta)}\ \tilde f_{(\alpha,1)(\beta,h(\alpha,\beta))}.
$$

We shall use again the cocycle condition

$$
(2.51)\quad 1 = \tilde f_{(\alpha,1)(\beta,h(\alpha,\beta))} \tilde f_{(\beta,h(\alpha,\beta))(\gamma,h(\alpha,\gamma))} (\tilde f_{(\alpha,1)(\gamma,h(\alpha,\gamma))})^{-1}.
$$

$$
(2.52)\quad \hat c = \frac{\hat c}{1} = \frac{\tilde f_{(\beta,1)(\gamma,h(\beta,\gamma))}}{\tilde f_{(\beta,h(\alpha,\beta))(\gamma,h(\alpha,\gamma))}}
$$

which, using 2.50, gives the desired equality:

$$
(2.53)\quad \hat c = \frac{1}{c^{h(\alpha,\beta)}_{h(\beta,\gamma)}}\ .
$$

<u>Step III</u> We must now prove that, if F': Z → X is obtained by extracting the r^{th} root of D, and we thus have $(F')^*(D) \equiv rL$, then the pull back $\tilde L$ of L to the universal cover $\tilde Z$ of Z is exactly (d/r)-divisible, if the pull back $\tilde D$ of D to the universal cover $\tilde X$ is exactly d-divisible.

We have thus $\tilde M$ such that $\tilde D \equiv d\tilde M$, and we remark that in the previous steps we have proved the following:

Fact: the map f: $\tilde{Z} \to \tilde{X}$ is a $(\mathbb{Z}/r\mathbb{Z})^3$-Galois cover, obtained as the fibre product of three elementary cyclic covers $f_i : \tilde{Y}_i \to \tilde{X}$ (hence f is the composition of three elementary cyclic covers). Each f_i is gotten by taking the r^{th} root of a smooth and connected divisor $\tilde{D}_i$, linearly equivalent to $\tilde{D}$, inside the line bundle associated to the divisor $(d/r)\tilde{M}$.

Since $\tilde{L} = f^*((d/r)\tilde{M})$, the desired result shall follow by iterated application of the following:

Proposition 2.55 Let f: $Y \to X$ be an elementary cyclic covering of connected and simply connected complex manifolds, i.e. there is an effective smooth irreducible divisor D given by a section σ of $\mathcal{L}^{\otimes r}$, and Y is the submanifold of the total space of the line bundle associated to $\mathcal{L}$, obtained by extracting the r^{th} root of D. Then, NS(X) denoting the Neron Severi group of X, the map f^* induces an isomorphism between NS(X) and $NS(Y)^{\mathbb{Z}/r\mathbb{Z}}$, and in particular the divisibility index of the class of a divisor M equals the one of $f^*(M)$.

Proof: We shall argue as in lemma 4 of [Ca2], recalling that $NS(X) \subset H^2(X,\mathbb{Z}_X)$ is given by the Chern classes of invertible sheaves, for which, though, we shall use the notation as for divisors (by real abuse of notation).

First of all, f_*f^*: NS(X)→NS(X) is given by multiplication by r, hence clearly, $H^2(X,\mathbb{Z}_X)$ being free, f^* is injective.

Moreover, if $f^*(M) = kN$, then first of all $N \in NS(Y)^{\mathbb{Z}/r\mathbb{Z}}$. In fact, $N = (1/k) f^*(M)$, hence $N \in H^2(S,\mathbb{Z}) \cap \mathbb{Q}NS(Y)^{\mathbb{Z}/r\mathbb{Z}} = NS(Y)^{\mathbb{Z}/r\mathbb{Z}}$. In view of the exact sequence

(2.56) $\quad 0 \to H^1(Y,\mathcal{O}_Y) \to H^1(Y,\mathcal{O}_Y^*) \to NS(Y) \to 0$,

we can assume that, if $f^*(M) = kN$, and M is an invertible sheaf on X, we have an invertible sheaf N such that $f^*(M) \equiv N^{\otimes k}$. Since $H^1(\mathbb{Z}/r\mathbb{Z}, H^1(Y,\mathcal{O}_Y)) = 0$ (these are homomorphisms of $\mathbb{Z}/r\mathbb{Z}$ into a $\mathbb{C}$-vector space), we can achieve that $N \in H^1(Y,\mathcal{O}_Y^*)^{\mathbb{Z}/r\mathbb{Z}}$.

To N we associate the theta group of automorphisms of the line bundle associated to N which cover the action of $\mathbb{Z}/r\mathbb{Z}$ on Y, we have thus a (non central) extension

(2.57) $\quad 0 \to H^0(Y,\mathcal{O}_Y^*) \to \mathcal{G}(N) \to \mathbb{Z}/r\mathbb{Z} \to 0$.

Claim: The sequence 2.57 splits.

Assuming the claim, we obtain that there is an action of $(\mathbb{Z}/r\mathbb{Z})$ on the invertible sheaf N, and it suffices then to show that the invariant direct image sheaf $\mathcal{N} = f_*(N)^{\mathbb{Z}/r\mathbb{Z}}$ is invertible, since then $N \equiv f^*(\mathcal{N})$. The sheaf $\mathcal{N}$ is clearly invertible outside the branch divisor D, whereas, locally at D, $f_*\mathcal{O}_Y = \{\Sigma_{i=0}^{r-1} f_i(x,\sigma) z^i\}$, where $z^r = \sigma$, $\sigma = 0$ being the local equation of D, and (x,σ) is a local coordinate vector for X at a point of D.

Locally a generator g of $\mathbb{Z}/r\mathbb{Z}$ acts on Y by $z \to \varepsilon z$, $\varepsilon = \exp(2\pi i/r)$, and, if N is locally trivialized with w a fibre variable, by $(z, w) \to (\varepsilon z, \varphi(x,z)w)$: here $\Pi_{i=0}^{r-1}\varphi(x,\varepsilon^i z) = 1$, since $g^r = 1$.

Writing $\varphi(x,z) = \Sigma_{i=0}^{r-1} \varphi_i(x) z^i$, we obtain $\varphi_0(x)^r = 1$, hence there exists h such that $0 \le h \le r-1$ and

(2.58) $\quad \varphi(x,z) = \varepsilon^h (1+\Sigma_{i\ge1} \varphi_i(x)z^i) = \varepsilon^h \exp(\Sigma_{i\ge1} \psi_i(x) z^i)$.

Changing the local trivialization of N by

(2.59) $\quad w \to \exp(\Sigma_{j\ge1} a_j(x) z^j)$

we replace $\varphi(x,z)$ by $\varepsilon^h \exp(\Sigma_{j\geq 1} z^j(\psi_j(x) - a_j(x) + \varepsilon^j a_j(x)))$, hence we may assume $\psi_i(x) \equiv 0$ for i not divisible by r.

Finally, since $\varphi = \varphi(x,\sigma)$, the equality $\Pi_{i=0}^{r-1} \varphi(x,\varepsilon^i z) = 1$ implies that $\varphi = \varepsilon^h$. But then the locally invariant sections are given by functions $\zeta(x,z)$ such that $\zeta(x,\varepsilon z) = \varepsilon^h \zeta(x,z)$, hence we can write $\zeta(x,z) = z^h \xi(x,z)$, thereby proving that $f_*(N)^{\mathbb{Z}/r\mathbb{Z}}$ is invertible.

There remains to prove the claim.

<u>Proof of the claim</u>: We choose an acyclic cover $\{U_\alpha\}$ of X such that each inverse image $\varphi^{-1}(U_\alpha) = V_\alpha$ is also acyclic. Then V_α is defined by local equations in $U_\alpha \times \mathbb{C}$, $z_\alpha{}^r = \sigma_\alpha$, with $z_\alpha = g_{\alpha\beta} z_\beta$, $g_{\alpha\beta}$ being a cocycle in $H^1(X,\mathcal{O}_X^*)$ for $\mathcal{L}$.

Then $H^0(\mathcal{O}_Y) \cong H^0(\mathcal{O}_X) \oplus (\oplus_{i=1}^{r-1} H^0(L^{-i}))$, and we have just written the eigenspace decomposition for the action of $\mathbb{Z}/r\mathbb{Z}$ on Y.

Let $(n_{\alpha\beta})$ be a cocycle for N relative to the cover $\{V_\alpha\}$ of Y: saying that $N \in H^1(Y,\mathcal{O}_Y^*)^{\mathbb{Z}/r\mathbb{Z}}$ means that, if $n_{\alpha\beta} = \Sigma_{i=0}^{r-1} n_{i,\alpha\beta} z_\alpha{}^i$, $\varepsilon = \exp(2\pi i/r)$, then the cocycle $\hat{n}_{\alpha\beta} = \Sigma_{i=0}^{r-1} n_{i,\alpha\beta} \varepsilon^i z_\alpha{}^i$ is cohomologous to $n_{\alpha\beta}$. I.e.,

(2.60) $\quad n_{\alpha\beta} = \hat{n}_{\alpha\beta} (\Sigma_{i=0}^{r-1} \psi_{i,\alpha} z_\alpha{}^i)^{-1} (\Sigma_{i=0}^{r-1} \psi_{i,\beta} (g_{\alpha\beta} z_\alpha)^i)$.

The equation 2.60 is indeed equivalent to the assertion: if $w_\alpha = n_{\alpha\beta} w_\beta$ is a fibre coordinate for the line bundle associated to N, then $w_\alpha \to \psi_\alpha w_\alpha$ ($\psi_\alpha = \Sigma_{i=0}^{r-1} \psi_{i,\alpha} z_\alpha{}^i$) lifts the action of the generator g of $\mathbb{Z}/r\mathbb{Z}$ from Y to N.

One can lift this action in a different way, just by multiplying ψ_α by a global invertible function ν on Y, and what we have to show amounts to prove that we can choose ν in such a way that this action has period r. In other words, we want

(2.60) $\quad \Pi_{i=0}^{r-1} g^i(\psi_\alpha \nu) = 1$.

We notice that $\xi_\alpha = \Pi_{i=0}^{r-1} g^i(\psi_\alpha)$ is an invariant invertible function on V_α, hence $\xi_\alpha \in \mathcal{O}_X^*(U_\alpha)$; moreover, by the previous equation 2.60, $\xi_\alpha = \xi_\beta$ and we have $\xi \in H^0(X,\mathcal{O}_X^*)$. Since X is simply connected, we can choose ν to be the inverse of an rth root of ξ, whence $\Pi_{i=0}^{r-1} g^i(\psi_\alpha \nu) = \nu^r \cdot \xi = 1$. Q.E.D.

<u>§3 Back to stable bundles and linear systems</u>

In order to apply the previous theorem, let now $\pi: Y \to S$ be a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover of S as in Lemma 2.1, such that $\pi^*(K_S) \equiv 2L$ for a line bundle L on Y and $\pi_*(\mathcal{O}_Y) = \mathcal{O}_S \oplus (\mathcal{O}_S(-3K_S))^3$.

<u>Proposition 3.1</u> The pullback $\pi^*(\mathcal{E})$ is $\pi^*(H)$-stable if H is an ample line bundle on S such that $\mathcal{E}$ is H-stable.

<u>Proof</u> By pullback under π we have the exact sequence on Y

(*) $\quad 0 \to \mathcal{O}_Y \to \pi^*\mathcal{E} \to \mathcal{I}_{\pi^{-1}(x)}(\pi^* K_S) \to 0$.

Let $0 \to N' \to \pi^*\mathcal{E} \to \mathcal{I}_Z(M') \to 0$ be a $H'(=\pi^*H)$-semi-destabilizing sequence for $\pi^*\mathcal{E}$ on Y. The Galois group $G = (\mathbb{Z}/2\mathbb{Z})^2$ acts on Y and $\pi^*\mathcal{E}$ has a natural G-linearization. There are two different cases

i) $g^*N' = N'$ for each $g \in G$;
ii) $g^*N' \neq N'$ for some $g \in G$.

In case i) the line bundle inherits from $\pi^*\mathcal{E}$ a G-linearization and then there exists an invertible subsheaf N of $\mathcal{E}$ on S such that $N' = \pi^*N$. But then $H \cdot N = (1/4)\, H'\cdot N' \geq (1/8)\, \pi^*K_S \cdot H' = (1/2)\, K_S \cdot H$ and $\mathcal{E}$ is not H-stable, which is absurd.

In case ii), let us set $g^*N' = N''$. The line bundle N'' is still a subsheaf of $\mathcal{E}$ and satisfies also the equalities $(N')^2 = (N'')^2$, $H' \cdot N' = H' \cdot N''$.

By hypothesis N' and N'' are distinct, so the map obtained as a composition $\beta\colon N'' \to \pi^*\mathcal{E} \to \mathcal{I}_S(M')$ gives a non zero element of $H^0(\mathcal{I}_Z(M'-N''))$.

But $\mathrm{div}(\beta) \cdot H' = H'\cdot M' - H'\cdot N'' \leq 0$ implies $N' \cdot H' = M' \cdot H'$, $N'' \equiv M'$, $Z = \emptyset$. So the bundle $\pi^*\mathcal{E}$ splits as a direct sum $\pi^*\mathcal{E} \cong N' \oplus N''$ and there exists two respective global holomorphic sections of N' and N'' such that the sequence (*) has the following form

$$0 \to \mathcal{O}_Y \xrightarrow{{}^t(n',n'')} \pi^*\mathcal{E} \xrightarrow{(-n'',n')} \mathcal{I}_{(\pi^{-1}(x))}(\pi^*K_S) \to 0 .$$

In particular, $(n'=0) \cap (n''=0) = \pi^{-1}(x)$ and $N'\cdot N'' = 4$. But the long exact sequence of cohomology associated to the sequence (*) gives

$$0 \neq h^0(\mathcal{I}_{(\pi^{-1}(x))}(\pi^*K_S)) \leq h^0(\pi_*\pi^*K_S) = h^0(K_S \oplus \mathcal{O}_Y(-2K_S)^3) = 0.$$

Q.E.D.

Since, as we saw in theorem 1.1, the canonical bundle is not 2-divisible, we consider the $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover $\pi\colon Y \to S$ of S described in Lemma 2.1 and associated to the linear system $|D| = |3K_S|$. Then there exists a line bundle L on Y such that $\pi^*(K_S) \equiv 2L$ and $\pi_*(\mathcal{O}_Y) = \mathcal{O}_S \oplus (\mathcal{O}_S(-3K_S))^3$. By Prop. 3.1 the pullback $\pi^*(\mathcal{E})$ is $\pi^*(H)$-stable for any ample line bundle H on S such that $\mathcal{E}$ is H-stable. Moreover, $\pi^*(\mathcal{E})(-L)$ has trivial Chern classes, hence it gives rise to an irreducible SU(2)-representation of $\pi_1(Y)$ not induced by $\pi_1(Y) \to \pi_1(S)$.

We get thus the theorem

<u>Theorem 3.2</u> Let S be a minimal surface with $p_g = 0$, $K_S^2 = 4$. Then $|2K_S|$ is base point free if, $\pi\colon \tilde{S} \to S$ being the universal cover, either
i) $\pi^*(K_S)$ is not 2-divisible,
ii) $\pi^*(K_S)$ is 2-divisible and either its Chern class modulo 2 is trivial or it gives a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma \to \pi_1(S) \to 0$$

not associated to any irreducible PU(2)-representation of $\pi_1(S)$.

We give now an alternative proof (using Donaldson's theorem) of the existence of an example where $m = 2$, D is not 2-divisible, but $\pi_1(Y) \neq \pi_1(S)$.

Let A be a simple minimal abelian surface admitting a polarization L of type (1,4). Consider now a $G=(\mathbb{Z}/2\mathbb{Z})^2$-Galois ramified cover $\pi\colon Y \to A$ over A associated to $|L|$ as in Lemma 2.1. The pullback of L under π is 2-divisible and $\pi_*(\mathcal{O}_Y) = \mathcal{O}_S \oplus (\mathcal{O}_S(-L))^3$. The map associated to the linear system $|L|$ is a well defined birational but not injective morphism $\varphi = \varphi_{|L|}$ on A (cf. [BLvS]); let $Z = \{x_1, x_2\}$ be a subset of A such that $\varphi(x_1) = \varphi(x_2)$. There exists then a vector bundle $\mathcal{E}$ of rank 2 on S occurring as an extension

(3.3) $$0 \to \mathcal{O}_A \to \mathcal{E} \to \mathcal{I}_Z(L) \to 0 .$$

Proposition 3.4 The vector bundle $\mathcal{E}$ is L-stable (since A is simple).

Proof Otherwise there exists an invertible and saturated subsheaf N of $\mathcal{E}$ which satisfies the inequality $N \cdot L \geq (1/2) L^2 = 4$ and gives a diagram of exact arrows of the following form

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & N & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathcal{O}_A & \to & \mathcal{E} & \to & \mathcal{I}_Z(L) & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathcal{I}_W(M) = \mathcal{I}_W(L-N) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{3.5}$$

for a suitable 0-dimensional subset of A.

The composition of maps $\alpha: N \to \mathcal{E} \to \mathcal{I}_Z(L)$ is non-zero and so M is an effective non-zero divisor. We have that $N^2 \geq M^2$, because of the inequality

(3.6) $\quad 0 \leq (N - M) \cdot L = (N - M) \cdot (N + M) = N^2 - M^2$,

and by computing the first Chern class of $\mathcal{E}$, we get the equality $M \cdot N + \deg W = 2$ which implies that $M \cdot N \leq 2$.

Then we get $M^2 \leq 2$ from the inequality $M \cdot L \leq 4$.

By the hypothesis that A is simple, since M^2 is even, the only possibility is $M^2 = 2$. But then by the Index theorem $M^2 N^2 \leq 4 \Rightarrow N^2 \leq 2 \Rightarrow M^2 = N^2 = 2$, $M \cdot N = 2$, and again by the Index theorem M and N differ by a topologically trivial line bundle on A: but this is impossible, because the line bundle L is not 2-divisible on A. Q.E.D.

Theorem 3.7 The pullback bundle $\pi^*(\mathcal{E})$ on Y is $\pi^*(L)$-stable.

Corollary 3.8 The corresponding surjective map $\pi_1(Y) \to \pi_1(A)$ between fundamental groups has a kernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof of corollary 3.9 Otherwise, Lemma 2.1 would imply that $\pi_1(Y) \cong \pi_1(A) \cong \mathbb{Z}^4$ and, by Donaldson's Theorem, Y does not admit any stable rank two vector bundle with trivial Chern classes because every SU(2)-representation of $\pi_1(Y)$ is reducible. But $\pi^*(L) \equiv 2L''$ for some line bundle L" on Y and the vector bundle $\pi^*(\mathcal{E}) \otimes (L'')^{-1}$ cannot be stable, contradicting the theorem. Q.E.D. for corollary 3.8

Proof of theorem 3.8 As in Prop. 3.1, we can assume otherwise that the bundle $\pi^*(\mathcal{E})$ splits as a direct sum $\pi^*\mathcal{E} \cong N' \oplus N''$ of line bundles on Y such that $(N')^2 = (N'')^2 = N' \cdot N'' = 8$ and so by the Index theorem N' is homologous to N". In particular, by pullback under π, the sequence 3.3 gives rise to an exact sequence on Y of the form

$$0 \to \mathcal{O}_Y \xrightarrow{{}^t(n',n'')} \pi^*\mathcal{E} \xrightarrow{(-n'',n')} \mathcal{I}_{(\pi^{-1}(x))}(\pi^* K_S) \to 0. \tag{3.10}$$

where n' and n" are respective holomorphic sections of N' and N" whose divisors have no common components and meet (tranversally) in 8 points.

Since in any case $H^1(A,\mathbb{Z}) \cong H^1(Y,\mathbb{Z})$, then $Pic^0(Y) \cong Pic^0(A)$ and there exists a topologically trivial line bundle M such that $N'' \cong N' \otimes \pi^*(M)$.

It follows that $(\pi_*(N'\oplus N'')) \cong \pi_*(\pi^*\mathcal{E}) \cong \mathcal{E}\otimes\pi_*\mathcal{O}_Y \cong \mathcal{E}\oplus\mathcal{E}(-L)^3$. This is absurd: in fact, being also $(\pi_*(N'\oplus N'')) \cong \pi_*(N')\oplus(\pi_*(N')\otimes M))$, for each stable quotient $\mathcal{F}$ in the Harder-Narasimhan filtration of $(\pi_*(N'\oplus N''))$, a corresponding quotient $\mathcal{F}\otimes M$ must also appear (cf. [Ko], ch. 5), whereas $\mathcal{E}$ and $\mathcal{E}(-L)$ are stable.

Q.E.D. for Theorem 3.9

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