Open problems

(collected by E. Ballico, C. Ciliberto and F. Catanese)

First we spend a few lines on the fate of some of the problems collected in the problem list appeared at the end of a previous Proceedings volume ([BC]) edited by two of us.

(i) Problem (34) of [BC] was solved in [EH].

(ii) Problem (9) of [BC] had a negative answer by Ph. Ellia , A. Hirschowitz and E. Mezzetti (work in preparation).

(iii) Problem (43) of [BC] (on Clifford theory for vector bundles on curves) had many developments. Among related published papers, see [La], [Su], [Te1] [Te2]. At Liverpool, July 24-27, 1991, there was a workshop on these topics (organized by P. Newstead, University of Liverpool and supported by Europroj, a net/organization to whom many european algebraic geometers belong); you can obtain from Newstead a "Brill-Noether problem list" which contains the state of the art on this theory (with its ramifications), a careful discussion of involved problems and several new open problems.

(v) Related to problems (1), (5) and (6) of [BC] on linear series, see several papers by M. Coppens, G. Martens, S. Greco and coworkers (see e.g. [CK1], [GR], and their references; see also the references [CKM1], [CKM2] and [CK2] given for problems (3) and (4) of this volume (and in the next few years check future works by M. Coppens, G. Martens and coworkers)).

(vi) About problem (31) of [BC] on the Wahl map, see [CM1] and [CM2].

(vii) Related to problem (16) of [BC], one can see [CGT], [CG1] and [CG2].

Here is the new list of problems and questions. Problem (1) is on modular forms and theta-functions. Problem (2) is on K-theory. Problems (3), (4) and (5) are concerned with the behaviour of linear systems on complex projective curves (and were communicated to us by M. Coppens). Problems (i) with $6 \leq i \leq 16$ were a "private" list worked out by the participants of a workshop on "Hyperplane sections and bounds of the genus of curves in $\mathbf{P}^n$" (Nice, 8 - 12 april 1991) organized by Ph. Ellia and R. Strano as part of the activities of Europroj. We owe many thanks to the organizers of the workshop and to G. Bolondi. These problems seems to be of very different level of generality and expected difficulty, but are collected together here (with only very small editing, just to update the "state of the art " on them to September 1991) since it seems the best way to show to outsiders and potentially interested mathematicians who is interested in what).

(1) (Salvati Manni) Let τ be a point of the Siegel upper half space H_g and let $\Gamma_g(4,8)$ be the subgroup of $Sp(g,\mathbf{Z})$ defined by the congruences

$$\sigma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1_{2g} \bmod 4 \text{ and } \mathrm{diag}(b) \equiv \mathrm{diag}(c) \equiv 0 \bmod 8;$$

then a modular form of weight k (half - integer) relative to $\Gamma_g(4,8)$ is a holomorphic function f defined on H_g such that

$$f(\sigma\cdot\tau) = \det(c\tau+d)^k f(\tau)$$

and such that for g = 1 it satisfies certain conditions at the cusps. $A(\Gamma_g(4,8))$ denotes the graded ring generated by such functions; it is finitely generated over **C** and integrally closed. For any (column) vector $m \in ((1/2)\mathbf{Z}^{2g}/\mathbf{Z}^{2g})$ we denote by ϑ_m the thetanullwert with characteristic m. It is a well known fact that $\vartheta_m \in A(\Gamma_g(4,8))$. We put $\mathfrak{R}_g = \mathrm{Proj}(\mathbf{C}[\vartheta_m])$, $m \in ((1/2)\mathbf{Z}^{2g}/\mathbf{Z}^{2g})$, $S_g = \mathrm{Proj}(A(\Gamma_g(4,8)))$ and we shall denote by α_g the morphism induced by the inclusion. The following results are known: (i) α_g is bijective for every g; (ii) α_g is an isomorphism for g = 1, 2; (iii) α_g is not an isomorphism for $g \geq 6$ (see [Ig], [SM]). It is an open problem to know if α_g is an isomorphism when g = 3, 4, 5.

(2) (Barbieri Viale - Srinivas) Let X be an integral algebraic scheme over **C** (with arbitrary singularities). Let H^i_X be the sheaf on X, for the Zariski topology, associated to the presheaf $U \to H^i(U_{an},\mathbf{Z})$, where U_{an} is the analytic space associated to U, and the cohomology is singular cohomology. Is $H^2(X,H^1_X)$ a finitely generated abelian group? For non-singular projective X, $H^1(X,H^1_X)$ is the Neron - Severi group of X. Barbieri Viale and Srinivas have shown (see [Bv], ch. III, §4, or [BvS]) that $H^1(X,H^1_X)/\mathrm{Im}(\mathrm{Pic}(X))$ is a free abelian group, say of rank r, and that there is an exact sequence

$$0 \to \mathbf{Z}^{\wedge \oplus r} \to T(H^2(X,\mathcal{O}^*_X)) \to T(H^2(X,H^1_X)) \to 0$$

where T(A) is the Tate module of the abelian group A. A positive answer to the question implies $T(H^2(X,H^1_X)) = 0$, so that r can be computed purely in terms of $H^2(X,\mathcal{O}^*_X)$, which depends only on the structure of X as an algebraic variety. For a discussion on what was known at the end of 1990, see [Bv], ch. III, §4. In particular, the question has a positive answer if X has only isolated singularities. There is an example (see [Bv], ch. III, esempio 4.15.3)) of a normal surface with two elliptic singularities for which r>0.

(3) (Coppens) Let V be complete base point free special linear system g^r_d on a smooth curve C; V is called *primitive* if $|K_C - V|$ has no base points. The primitive length of a curve C is the number of elements in the set {c: c is the Clifford index of some primitive linear system on C}; l will denote the primitive length.

(a) What is the behaviour of l on special subloci of M_g (e.g. the locus $M_{g,k}$ of k-gonal curves) ?

(b) What is the value of l at general points of special subloci of M_g ? (Especially on $M_{g,k}$; this is known for $k \leq 5$).

(c) Is it possible to characterize double covering as curves with small values of the primitive length l ? A very explicit related question: if C is a double covering of a curve of genus 2, then does C have a base point free complete g^1_{g-3} ?

(4) (M. Coppens and, for part (c) T. Kato) Let C be a smooth plane curve of degree d. A point $P \in C$ is called an e-inflectional point of C if the tangent line to C at P intersects C with multiplicity e at P.

(a) Study the stratification of such pairs (C,P) according to the Weierstrass gap sequence of C at P. Coppens is able to describe this gap sequence if (C,P) is general (and d>>d-e) (for $e \in \{d, d-1, d-2\}$ there is no stratification, for $e \le d-3$ there is).

(b) Study similar questions on the normalization of nodal curves.

(c) (T. Kato) Is there any smooth plane curve of degree $d \ge 5$ having more then 3d d-inflectional points ? If no, is every smooth plane curve with 3d d-inflectional points birationally equivalent to the Fermat curve ?

(5) (Coppens) Let C be a general smooth curve of genus g and assume that the Brill - Noether number $\rho(d,g,r) \ge 0$ with $r \ge 3$. For a linear system g^r_d, A, on C and for $e \le d$, $n \le r$, define $V^n_e(A) = \{D \in C^{(e)}$: D imposes at most n conditions to A$\}$. These sets have a natural scheme structure.

(a) How do the schemes $V^n_e(A)$ behave if A is a general point of W^r_d ?

(b) Stratify W^r_d according to the exceptional behaviour of these schemes $V^n_e(A)$.

(6) (Main problem of lifting) For references, see [GP], [ES], [St1], [St2]; key word: Laudal's lemma). If $C \subset \mathbf{P}^3$ is a smooth connected curve, and X a general hyperplane section, set $\sigma := \min\{t\colon H^0(\mathcal{I}_X(t)) \neq 0\}$. Find a (sharp) function $f(\sigma,h)$ such that if $d > f(\sigma,h)$ and $h^0(\mathcal{I}_X(\sigma)) \neq 0$, then $h^0(\mathcal{I}_C(\sigma+h)) \neq 0$.

Remarks: (a) $f(\sigma,0) = \sigma^2+1$ (Laudal's lemma).

(b) (Bolondi) A lower bound for the function $f(\sigma,h)$ is given by the following family of curves (constructed using reflexive sheaves). Fix $h \ge 0$; let k be the smallest integer bigger or equal to $(2h-7 + (12h^2+12h+25)^{1/2})/2$. Then for every $\sigma \ge k+h+2$ there exists a smooth connected curve C such that $h^0(\mathcal{I}_X(\sigma)) \neq 0$, $h^0(\mathcal{I}_C(\sigma+h)) = 0$ and $\deg(C) = \sigma^2 - (k+1)\sigma + (k^2+5k+6)/2$. Note that $k \approx (1+\sqrt{3})h$, and that it gives $f(\sigma,1) \ge \sigma^2 - 2\sigma + 6$ and $f(\sigma,2) \ge \sigma^2 - 5\sigma + 21$. In [St2] it was proved that $f(\sigma,2) \ge \sigma^2 - 6\sigma + 9$.

Peskine suggested another way for having a lower bound for f, i.e. to look at suitable sections of $S^2(E)$, where E is the null-correlation bundle.

Subproblem: classify the extremal curves, i.e. for which $d = f(\sigma,h)$, $h^0(\mathcal{I}_X(\sigma)) \neq 0$ but $h^0(\mathcal{I}_C(\sigma+h)) = 0$. For $h = 0$ and $\sigma \ge 24$ these curves all belongs to the same liaison class, and this seems to be true also for $h = 1$. Find a reasonable conjecture for the Hartshorne - Rao module of an extremal curve (knowing that for $t < s(C)$ $H^0(\mathcal{I}_X(t))$ injects into $H^0(\mathcal{I}_X(t-1))$) and then use it for finding a conjecture for $f(\sigma,h)$.

(7) Improve the classical Laudal's lemma considered in problem (6) by putting the genus g into the picture (and/or $h^0(\mathcal{I}_X(\sigma))$). For example:

a) do union of lines have the expected σ ?

b) do rational curves have the expected σ ?

c) in particular, if C is a rational curve of degree 10, can it be $h^0(\mathcal{I}_X(3)) \neq 0$ but $h^0(\mathcal{I}_C(3)) = 0$?

(8) (Halphen problem) (see also problem (11))

For the general definitions, see [Ha] or [BE1], §1. For range A (sharpness of the bound, non existence of gaps for the genus below the upper bound and maximal rank) see work in preparation by Ch. Walter (Rutgers) which supersedes the asymptotic results announced in [BE1], th. 1.5, and [BE2], end of the introduction.

a) range B: prove Hartshorne - Hirschowitz conjecture (see e.g. [BE1]); classify the extremal curves (are all in the same irreducible component of the Hilbert scheme?). Is the conjecture true at least for maximal rank curves ?

b) work out a conjecture for gaps in the range B and C (it was suggested to look at curves with maximal and comaximal rank). How to use the fact that curves with the maximal conjectured genus must actually lie on a surface of degree s ? Try to prove this fact. In general, study the Hilbert scheme of curves with maximal genus or with maximal index of speciality.

(9) (Ellia) Classify curves (i.e. locally Cohen - Macaulay 1-dimensional schemes) with small σ.

(10) Stratify the Hilbert scheme of unions of disjoint lines according to σ and s (see problem (6) for the notations). Do the same for union of rational curves.

(11) Halphen's gaps: determine $s(d,g):= \min\{k$: every smooth space curve of degree d and genus g lies on a surface of degree $k\}$. The general principle is: $G(d,s)$ should be decreasing in s (it is so in the range C). If $C \subset \mathbf{P}^3$ is a curve of degree d and genus g with $G(d,s) \geq g > G(d,s+1)$ we expect $s = s(d,g)$ and we know that $s \geq s(d,g)$. We say that (d,g,s) is a gap if $s > s(d,g)$.

If we allow singularities for curves and g is the geometric genus instead of the arithmetic one, is there any gap (say, in the range C) ? It was suggested to see if the proofs in [GP] give obstructions for p_a at least for curves with only nodes as singularities.

(12) (A problem related to lifiting) (Laudal)

a) consider a flat family of plane curves of degree d in $\mathbf{P}^3$, of dimension 3 and whose planes dominates $\mathbf{P}^{3*}$. Choose a general line L of $\mathbf{P}^3$ and consider the surface F(L) described by the curves of the family in the planes through L. Determine the degree of L.

b) Consider the above question from an infinitesimal point of view, and give conditions in order that deg(F(L)) = d.

(13) (Special plane sections) General problem: in which codimension the plane sections fail to have the same postulation, for example the same σ, than the general one. More specifically: is there any curve such that for a 2-dimensional family of plane sections one has $\sigma = 2$ without the curve being on a quadric ? The expected answer is: NO.

8b) (Ragusa) Let X be a set of points with the strong uniform position on a quadric Q and with card(X) even; is there a curve $C \subset \mathbf{P}^3$ with $X = C \cap Q$ (as schemes) ?

(14) (Curves in $\mathbf{P}^r$ for $r>3$) For references and notations for this problem, see [HEi], ch. III, and [Ci2] (in this refined form this type of problems was first considered by G. Fano in [F]:

see [Ci1] for several historical references). C is a smooth non degenerate curve in $\mathbf{P}^r$ with deg(C) = d and genus g; X is a general hyperplane section of C.

a) Extend Fano's results by assuming $h_X(i) \geq i(r-1)$ for $i \geq 3$ and $i(r-1) < d$.

b) Improve Fano's "trisecant lemma": if $d > (k+2)(r-1)+2$ with $k := \pi_0 - g$, then C lies on a surface of degree r-1.

c) More generally: under which conditions we can lift curves (or subvarieties) through X to surfaces (or subvarieties with one dimension more) through C ? For a very different approach, see [Li].

d) Compute the analogous of Harris $\pi_\alpha(d,r)$ for $\alpha \geq t$. Prove that they are the analogous of G(d,s) for range C of curves in $\mathbf{P}^3$. Relate them with Harris' conjecture ([HEi], ch. III, and [Ci2]).

e) Study and classify curves for which $d = 2r-2+2\alpha$ and $g = \pi_\alpha$ ($\alpha = 1,\dots,r-1$).

f) Prove Harris' conjecture for $\alpha = 2$ improving [Ci2], th. 3.7.

h) Prove (or disprove) that the bounds $\pi_\alpha(d,r)$, $\alpha = 1,\dots,r-1$, are sharp.

i) Find an upper bound for the genus of a curve in $\mathbf{P}^r$ with degree d and not lying on a hypersurface of degree s.

j) More specifically: determine an upper bound for the genus of curves of degree d in $\mathbf{P}^r$ which, say, lie on a cubic hypersurface, but not on a quadric.

k) Try to work out Halphen - Harris framework working with subvarieties of higher dimensions (not only surfaces) through the curve.

l) Find an upper bound for the (geometric) genus of a surface of degree d in $\mathbf{P}^r$ lying on a hypersurface of degree s.

m) Classify curves in $\mathbf{P}^r$ with high genus (say $g > \pi_0(d-1,r)$). This is related to work of Comessatti (see the following question).

n) A related question: Given d and g, determine a sharp function R(d,g) such that for every smooth, irreducible, non degenerate curve C of degree d and genus g in $\mathbf{P}^r$, we have $r \leq R(d,g)$. And: Given r, and g determine a sharp function D(r,g) such that for C as above we have $d \geq D(r,g)$. Note: Comessatti determines R(d,g) and D(d,g) but but allowing C singular and taking as g the geometric genus. He proves that in such case r = R(d,g) if and only if $g > \pi_0(d,r+1)$, while d = D(r,g) if and only if $g > \pi_0(d-1,r)$.

o) Consider the question of existence of smooth curves of given (d,g) in $\mathbf{P}^r$ for $r \geq 7$. Improve [Ci3]. In particular examine the case $g > \pi_{r/3}$.

p) A related question: determine a function D(r) such that all surfaces of degree $d \leq D(r)$ in $\mathbf{P}^r$ with hyperplane section of positive genus are cones (this is related to results on Gaussian maps (see [BM])).

(15) Lifting in higher dimension (Mezzetti) Consider $Y \subset \mathbf{P}^{r+2}$ with dim(Y) = r. Let $C = X \cap H$ be the general hyperplane section and let σ the first integer t with $H^0(\mathcal{I}_X(t)) \neq 0$. Give conditions implying $H^0(\mathcal{I}_C(\sigma)) \neq 0$.

(16) (Ellia) Find a sharp function $f(\sigma)$ such that every curve is contained in a surface of degree $f(\sigma)$ (Conjecture: $f(\sigma) = 2\sigma-2$; it is easy to prove that $f(\sigma)\leq 3\sigma$).

References

[BC] E. Ballico, C. Ciliberto (eds): *Open problems,* in Algebraic Geometry and Projective Curves, Proceedings Trento 1988, Lect. Notes in Math. **1389**, Springer-Verlag, 1989.

[BE1] E. Ballico, Ph. Ellia: *A program for space curves,* Rend. Sem. Mat.Torino **44** (1986), 25-42.

[BE2] E. Ballico, Ph. Ellia: *On the existence of curves with maximal rank in* $\boldsymbol{P}^n$, J. reine angew. Math. **397** (1989), 1-22.

[Bv] L. Barbieri Viale: *Teorie coomologiche e cicli algebrici sulle varietà singolari,* Tesi di dottorato, 1991.

[BvS] L. Barbieri Viale, V. Srinivas: in preparation.

[BM] A. Beauville, J. Merindol: *Sections hyperplanes des surfaces K3,* Duke Math. J. **55** (1987), 873-878.

[Ci1] C. Ciliberto: *Alcune applicazioni di un classico procedimento di Castelnuovo,* in: Seminari di Geometria, Università di Bologna, 1982-1983.

[Ci2] C. Ciliberto: *Hilbert functions of finite sets of points and the genus of a curve in a projective space,* in: Space Curves, Proc. Rocca di Papa 1985, pp. 24-73, Lect. Notes in Math. **1266**, Springer-Verlag, 1987.

[Ci3] C. Ciliberto: *On the degree and genus of smooth curves in a projective space,* Advances in Math. **81** (1990), 198-248.

[CM1] C. Ciliberto, R. Miranda: *On the Gaussian map for canonical curves of low genus,* Duke Math. J. **61**(1990), 417-443.

[CM2] C. Ciliberto, R. Miranda: *Gaussian maps for certain families of canonical curves,* Proc. Bergen Conference (to appear).

[CG1] C. Ciliberto, G. van der Geer: *Subvarieties of the moduli space of curves parametrizing jacobians with non-trivial endomorphisms,* preprint.

[CG2] C. Ciliberto, G. van der Geer: *On the Jacobian of a hyperplane section of a surface,* this volume.

[CGT] C. Ciliberto, G. van der Geer, M. Teixidor i Bigas: *On the number of parameters of curves whose Jacobians possess non-trivial endomorphism,* preprint.

[CK1] M. Coppens, T. Kato: *The gonality of smooth curves with plane models,* Manuscripta Math. **70** (1990), 5-25, and *Corrections,* Manuscripta Math. **71** (1991), 337-338.

[CK2] M. Coppens, T. Kato: *Weierstrass points on plane curves,* in preparation.

[CKM1] M. Coppens, C. Keem, G. Martens: *Primitive linear series on curves,* preprint.

[CKM2] M. Coppens, C. Keem, G. Martens: *Primitive linear systems on the general k-gonal curve,* in preparation.

[EH] Ph. Ellia, A. Hirschowitz: *Une remarque sur la lissification des courbes gauches,* C. R. Acad. Sci. Paris **312** (1991), 979-981.

[ES] Ph. Ellia, R. Strano: *Sectiones planes et majoration du genre des courbes gauches,* preprint.

[F] G. Fano: *Sopra le curve di dato ordine e dei massimi generi in uno spazio qualunque,* Mem. Acad. Sci. Torino **44** (1894), 335-382.

[GR] S. Greco, G. Raciti: *Gap orders of rational functions on plane curves with few singular points,* Manuscripta Math. **70** (1991), 441-447.

[GP] L. Gruson, Ch. Peskine: *Genre des courbes de l'espace projectif,* in: Algebraic Geometry, Tromsø 1977, pp. 31-60, Lect. Notes in Math. **687**, Springer-Verlag, 1978.

[HEi] J. Harris (chapter III is in collaboration with D. Eisenbud): *Curves in projective space,* Les Presses de l'Université de Montreal, 1982.

[Ha] R. Hartshorne: *On the classification of space curves,* in: Vector bundles and differential equations (Nice 1979), pp. 82-112, Progress in Math. **7**, Birkhäuser, Boston, 1980.

[Ig] J. Igusa: *On the variety associated with the ring of Thetanullwerte,* Amer. J. Math **103** (1981), 377-398.

[La] G. Laumon: *Fibrés vectoriels spéciaux,* Bull. Soc. Math. France **119** (1991), 97-119.

[Li] J. Little: *Translation manifolds and the converse of Abel's theorem,* Compos. Math. **49** (1983), 147-171.

[SM] R. Salvati Manni: *On modular forms of weight $\leq 5/2$ relative to $\Gamma_g(4,8)$,* preprint.

[St1] R. Strano: *A characterization of complete intersection curves in P^3,* Proc. Amer. Math. Soc. **104** (1988), 711-715.

[St2] R. Strano: *Plane sections of curves of P^3 and a conjecture of Hartshorne and Hirschowitz,* preprint.

[Su] N. Sundararam: *Special divisors and vector bundles,* Tohoku Math. J. **39** (1987), 175-213.

[Te1] M. Teixidor i Bigas: *Brill - Noether theory for stable vector bundles,* Duke Math. J. **62** (1991), 385-400.

[Te2] M. Teixidor i Bigas: *Moduli spaces of (semi)stable vector bundles on tree-like curves,* Math. Ann. **290** (1991), 341-348.