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FABRIZIO CATANESE
MICHAEL SCHNEIDER

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Polynomial bounds for Abelian groups of automorphisms

Dedicated to Frans Oort on the occasion of his 60th birthday

FABRIZIO CATANESE[1] and MICHAEL SCHNEIDER[2]
[1] *Dipartimento di Matematica, Università di Pisa, via Buonarroti 2, I-56127 Pisa, Italy catanese@vm.cnuce.cnr.it, catanese@dm.unipi.it*
[2] *Department of Mathematics, University of Bayreuth, Universitätsstrasse 30, D-95440 Bayreuth, Germany. michael.schneider@uni-bayreuth.de*

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Introduction

Let $X = X^n_d$ be a projective variety of dimension n and degree d embedded in $\mathbb{P}^N$: then the group $\mathrm{Lin}(X)$ of projectivities sending X to itself is a linear algebraic group defined by equations of degree at most d.

In fact, one can find polynomials $P_1, \ldots, P_r$ of degree at most d whose common zero set is X and one can observe then that $\mathrm{Lin}(X) \subset \mathrm{PGL}(N+1)$ is defined by the infinite set of equations $P_i(gx) = 0$ for each $i = 1, \ldots, r$ and for each x in X, where $g \in \mathrm{PGL}(N+1)$. But these equations $P_i(gx) = 0$ have the same degree in g and x.

There are two possibilities now, viz., either

(i) $\mathrm{Lin}(X)$ is a finite group or
(ii) $\mathrm{Lin}(X)$ has dimension bigger than zero.

In case (i), as was essentially observed by Andreotti in [An50], a variant of the Bezout theorem shows then that $\mathrm{card}(\mathrm{Lin}(X)) \leqslant d^{(N+1)^2}$. In fact more generally if a variety $X \subset \mathbb{P}^N$ is the unique irreducible component of top dimension of the intersection of hypersurfaces of degree $\leqslant m$ then we have the bound

$$\mathrm{card}(\mathrm{Lin}(X)) \leqslant m^{(N+1)^2},$$

provided that $\mathrm{Lin}(X)$ is finite. Taking for m the degree d of X gives the above estimate.

In particular, by the same argument the intersection of $\mathrm{Lin}(X)$ with the linear subspace of diagonal matrices has card $\leqslant m^N$.

In case (ii), since any linear algebraic group of positive dimension contains a subgroup isomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$, by Rosenlicht's cross-section theorem ([Ro56]) it follows that (cf. [Ma63]) X is birationally ruled .

Our first result is a particular issue of this alternative.

THEOREM 0.1. *Let $X = X_d^n$ be a projective variety of dimension n and degree d in $\mathbb{P}^N$, and let G be an abelian subgroup of* $\mathrm{GL}(N+1)$ *acting faithfully on X. Then either*

- $\mathrm{card}(G) \leqslant d^{2^n}$ *or*
- *X admits a linear $\mathbb{C}^*$-action.*

At first glance it seems that the bound $\mathrm{card}(G) \leqslant d^{2^n}$ is often less effective than the Andreotti-type estimate $\mathrm{card}(G) \leqslant d^N$, but we should observe that our point of view is to consider n fixed (whereas N can go to infinity): thus only our bound gives a polynomial bound in the degree. In the later applications to varieties of general type N moreover depends rather badly on n and d.

We may remark that $\mathbb{C}^*$ contains finite subgroups of arbitrary order, but these subgroups are all cyclic: in the rest of the first section we elaborate on this remark as follows. If there is a linear action of a finite abelian group having subgroups with a small number of generators, we relate bounds for the index of such subgroups to the existence of linear $(\mathbb{C}^*)^k$-actions on X.

At this point the reader might wonder why we are interested about the linear actions of finite abelian groups? To answer this question we need a small historical digression. For a curve C the groups $\mathrm{Bir}(C)$ of birational automorphisms and the group $\mathrm{Aut}(C)$ of biregular automorphisms coincide. After Schwarz and Klein proved the finiteness of the group $\mathrm{Aut}(C)$ of automorphisms of a curve C of genus at least 2, Hurwitz [Hu93] showed that one has indeed a sharp estimate

$$\mathrm{card}(\mathrm{Aut}(C)) \leqslant 42(2g-2).$$

In higher dimension, after it was realized that even the group $\mathrm{Aut}(X)$ could be discontinuous and infinite, thus not an algebraic group (cf. [PS97]), emerged the idea that the right analogue of a curve of genus g at least 2 would be a variety X of general type. For these, $\mathrm{Bir}(X)$ acts linearly on a pluricanonical system which yields a birational image of X. Matsumura showed ([Ma63]) that for varieties of general type $\mathrm{Bir}(X)$ is finite.

Earlier, Andreotti ([An50]) had given an explicit bound for $\mathrm{card}(\mathrm{Bir}(X))$ when X is a surface of general type. Unfortunately his bound is exponential in the birational invariants of X, whereas Hurwitz's bound is linear.

The first progress concerning polynomial bounds in higher dimension was obtained by Howard and Sommese [HS82], who pointed out the important role of the study of the actions of *abelian* groups. They proved a quadratic bound for the cardinality of an abelian group acting on a surface of general type, and remarked that by a classical result of Jordan the stabilizer of a point contains an abelian subgroup whose index is bounded by a constant depending only on the dimension n of X.

Using this approach, and the existence of invariant loci of bounded degree for the action of the group $\mathrm{Bir}(X)$ (Weierstrass and discriminant loci) Corti [Co91] was able to give a polynomial upper bound in the case of surfaces, immediately followed by Huckleberry and Sauer [HS90] who used some deep results on the structure of finite groups. Finally Xiao [Xi90] first obtained a linear estimate in the abelian case, later ([Xi93] and [Xi]) extended Hurwitz's result to surfaces of general type proving a linear bound with coefficient $(42)^2$. Since it is not clear to us that the approach à la Hurwitz of considering the quotient $X/\mathrm{Bir}(X)$ has any chance of being carried out in dimension 3 or more, and since for surfaces all other approaches have as a initial step the polynomial bounds in the abelian case, we provide in Section 2 a solution to this step in higher dimension:

THEOREM 0.2. *Let X be a variety of general type which has at worst log-terminal Gorenstein singularities and with K_X nef. Let $G \subset \mathrm{Bir}(X)$ be an abelian group of birational automorphisms. Then we have the following bound*

$$\mathrm{card}(G) \leqslant C(n)(K_X^n)^{2^n},$$

where $C(n) = [2(n+1)(n+2)!(n+2)]^{n2^n}$.

Concerning the hypotheses of the above theorem, we should remark that an essential ingredient in the proofs by Andreotti and the others was the theorem about the existence of a fixed explicit m such that the mth pluricanonical map is birational (this number is 5 for surfaces, cf. [Bo73]). To our knowledge, a general result of this sort is still missing in higher dimension (even in dimension 3, in spite of Mori's result about the finite generation of the canonical ring). Therefore we have to restrict to the case (K_X nef) where the Riemann-Roch formula guarantees that the plurigenera grow fast enough, and we use results of Demailly and Kollár yielding an explicit but very large m depending only on the dimension. Since our bounds in Section 2 (probably also in Section 1) are far from being optimal, we devote Section 3 to the analysis of an inductive method, using the semipositivity of direct images of relative canonical sheaves, to obtain a lower degree bound.

Our result there is limited for simplicity to dimension 3:

THEOREM 0.3. *Let X be a 3-fold of general type and let G be an abelian subgroup of $\mathrm{Bir}(X)$. Let m be a positive integer such that $H^0(X, mK_X)$ contains an eigenspace for the action of G of dimension at least 2. Then we have*

$$\mathrm{card}(G) \leqslant \max(6P_2(X), P_{3m+2}(X)).$$

Here $P_k(X)$ denotes the kth plurigenus of X.

If one would be able to show the existence of such an integer m depending only on the dimension (cf. [Xi90]), then we would get the desired linear bound.

Combining the above theorem with the techniques introduced in the previous sections, we are able to prove a polynomial estimate of degree 4 in the 3-dimensional case.

In the final section we pose some questions whose answer would be important in the study of higher dimensional varieties and their groups of automorphisms.

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Added in Proof

In the mean time three more papers have been written [Ji95], [Sz95], [Xi95], the last two inspired by our paper, making substantial progress. In [Ji95] it is stated that abelian automorphism groups of minimal 3-dimensional smooth projective varieties X can be bounded linearly in K_X^3. Xiao Gang [Xi95], combining ideas from our paper and his technique from [Xi90], does the same in arbitrary dimension. Finally Szabó [Sz95] proves the following results:

- Let $X_n^d \subset \mathbb{P}^N$ be a projective variety of dimension n and degree d such that $\mathrm{Lin}(X)$ is finite. Then there is the bound

  $$\mathrm{card}(\mathrm{Lin}(X)) \leqslant d^{8n3^n}.$$

  If $\mathrm{Lin}(X)$ is in addition abelian, there is the optimal bound

  $$\mathrm{card}(\mathrm{Lin}(X)) \leqslant d^{n+1},$$

  as suggested in Remark 1.2 and conjectured independently by C. Peters.
- Let X be a minimal n-dimensional variety of general type and let r denote its index. Then

  $$\mathrm{card}(\mathrm{Bir}(X)) \leqslant f(n)(rK_X^n)^{g(n)},$$

  where f and g are functions depending only on n and not on X.

We like to thank the referee for suggesting to formulate Corollary 1.6.

1. Abelian groups acting linearly on projective varieties

In this section $X = X_d^n \subset \mathbb{P}^N$ will be an embedded projective variety of dimension n and degree d. A linear $\mathbb{C}^*$ - action on $\mathbb{P}^N$ is given as follows: there exist coordinates $x_0, \ldots, x_N$, integers $a_0, \ldots, a_N$ such that the ideal generated by their differences $(a_i - a_j)$ is the whole ring of integers and the action of $\mathbb{C}^*$ on $\mathbb{P}^N$

$$\mathbb{C}^* \times \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

is given by

$$(t,(x_0,\dots,x_N)) \to (t^{a_0}x_0,\dots,t^{a_N}x_N).$$

The above condition that the differences $(a_i - a_j)$ be coprime guarantees that we have an injective homomorphism into $\mathrm{PGL}(N+1)$.

We shall say that X admits a linear $\mathbb{C}^*$-action if there is a linear $\mathbb{C}^*$-action on $\mathbb{P}^N$ leaving X invariant.

We shall say that X admits an abelian linear action of order k if there is an abelian subgroup G of $\mathrm{GL}(N+1)$ of order k which embeds in $\mathrm{PGL}(N+1)$ and leaves X invariant.

Since $\mathbb{C}^*$ contains cyclic groups of any order, if X admits a linear $\mathbb{C}^*$-action then X admits an abelian linear action of arbitrary order.

We have the following striking converse:

THEOREM 1.1. *Let $X = X_d^n$ be a projective variety of dimension and degree d in $\mathbb{P}^N$, and let G be an abelian group acting linearly on X as above. Then either*

- $\mathrm{card}(G) \leqslant d^{2^n}$ *or*
- *X admits a linear $\mathbb{C}^*$-action.*

REMARK 1.2.

- Let X be the Fermat hypersurface in $\mathbb{P}^{n+1}$. Then for $d \geqslant 3$ we have the exact sequence
$$1 \to (\mu_d)^{n+1} \to \mathrm{Lin}(X) \to S_{n+2} \to 1,$$
where μ_d is the group of dth roots of unity and S_n is the symmetric group. In particular from this we see that the order of an abelian subgroup of $\mathrm{Lin}(X)$ cannot be bounded by a polynomial in d of degree smaller than $n+1$.
Moreover in this example all the monomials of degree strictly smaller than d correspond to distinct characters of the abelian group. Therefore (compare the first step of the proof of the above theorem) in general there is no invariant rational function of degree strictly smaller than d.
- If X is a smooth hypersurface, then $K_X^n = d(d-n-2)^n$. More generally for a smooth complete intersection $K_X^n \leqslant d^{n+1}$.
In analogy with Xiao's conjecture, one may ask whether there exists a constant $C(n)$, such that in the above theorem the sharper estimate $\mathrm{card}(G) \leqslant C(n)d^{n+1}$ holds.
At least for hypersurfaces this was proven by Howard and Sommese [HS82].

Proof of Theorem 1.1. Without loss of generality we may assume that X is nondegenerate. We shall prove the result essentially by induction on n.

Step (1) We shall show that either there exists a linear $(\mathbb{C}^*)^n$-action on $\mathbb{P}^N$ such that X is the closure of an orbit, or there exist polynomials F_1, F_0 of degree at most d such that the rational function $\phi = F_1/F_0$ is G-invariant and non constant on X.

Step (2) Assume ϕ is as in Step (1) , let $f\colon X \to C$ be its Stein factorization, and let Y be a general fibre of f. Let γ be the degree of $C \to \mathbb{P}^1$, δ the degree of Y in $\mathbb{P}^N$ and G' be the subgroup of G leaving Y invariant. Then

$$\operatorname{card}(G) \leqslant \operatorname{card}(G')\gamma$$

and moreover

$$\gamma\delta \leqslant d^2.$$

Proof of Step (1). Let $x_0, \dots, x_n, y_{n+1}, \dots, y_N$ be a coordinate system on $\mathbb{P}^N$ given by eigenvectors of the linear action of G and such that $x_0, \dots, x_n$ are algebraically independent on X. Therefore the projection

$$\pi_j\colon X \to \mathbb{P}^{n+1},$$

given by the coordinates $x_0, \dots, x_n, y_j$ has as image an irreducible hypersurface Σ_j of degree at most d.

Let $x = (x_0, \dots, x_n)$ and let $P_j = P_j(x, y_j)$ be the equation of Σ_j. Since X is nondegenerate, the degree of P_j is at least 2, whence P_j, being irreducible, is not a monomial. Let F_0 and F_1 be two distinct monomials appearing in P_j: since the projection π_j commutes with the action of G, P_j is an eigenvector for G, whence the monomials F_0 and F_1 have the same eigenvalue, and the rational function $\phi = F_1/F_0$ is G-invariant.

Clearly ϕ is constant on X if and only if there exists λ such that $F_1 - \lambda F_0 = 0$ on X. But then $F_1 - \lambda F_0$, being a function of x and y_j in the ideal of X is a multiple of P_j; since its degree equals the one of P_j, $P_j = F_1 - \lambda F_0$ is a binomial. Notice that y_j must appear in P_j but cannot divide it: therefore P_j will be of the form

$$P_j = x^{D(j)} - \lambda y_j^{a_j} x^{B(j)}.$$

We are therefore done unless for each $j = n+1, \dots, N$ our polynomial P_j is a binomial. Up to multiplying y_j by a constant we may assume that X is contained in the locus of zeroes Z of the polynomials $P_j = x^{D(j)} - y_j^{a_j} x^{B(j)}$. Let a be the least common multiple of the a_j's and write $a = a_j b_j$. Then Z is stable under the $(\mathbb{C}^*)^n$-action which makes $t = (t_0, \dots, t_n)$ operate on $\mathbb{P}^N$ by

$$x_i \to t_i^a x_i,$$

$$y_j \to t^{b_j(D(j)-B(j))} y_j.$$

Let π be the linear projection to $\mathbb{P}^n$ given by the coordinates x and let $U \simeq (\mathbb{C}^*)^n \subset \mathbb{P}^n$ be the open set where all coordinates are nonzero. Then

(i) $\pi^{-1}(U) \cap X$ is dense in X since the restriction of π to X is dominant
(ii) $\pi^{-1}(U) \cap X$ is contained in $\pi^{-1}(Z)$ which is a finite union of orbits of dimension n.

Therefore X is the closure of an orbit of dimension n and Step (1) is proven.

Proof of Step (2). G acts trivially on $\mathbb{P}^1$ by construction. Therefore the G-orbit of any point of the curve C has cardinality bounded by γ. We conclude the proof of the first assertion since the subgroup G' is the inverse image of the stabilizer of a general point of C. The inequality $\gamma\delta \leqslant d^2$ follows since $\gamma\delta$ is the degree of the hypersurface on X cut out by a polynomial $F_1 - \mu F_0$ which is of degree at most d.

End of the proof. By induction either we are as in Step (2) with

$$\operatorname{card}(G') \leqslant (\delta)^{2^{n-1}},$$

which implies the desired bound

$$\operatorname{card}(G) \leqslant \gamma \operatorname{card}(G') \leqslant \gamma(\delta)^{2^{n-1}} \leqslant (\gamma\delta)^{2^{n-1}} \leqslant d^{2^n},$$

or there exist invariant rational functions $\phi_1, \dots, \phi_k \colon X \to \mathbb{P}^1$ such that

(i) $\psi \colon X \to \mathbb{P}^k$ given by $\phi_1, \dots, \phi_k$ is dominant
(ii) the general fibre W of the Stein factorization of ψ dominates $\mathbb{P}^{n-k}$ under the projection p given by $(x_0, \dots, x_{n-k})$
(iii) for each general fibre W there exists a $(\mathbb{C}^*)^{n-k}$-action sending $t = (t_0, \dots, t_{n-k})$ to a diagonal matrix $\operatorname{diag}(t_0^a, \dots, t_{n-k}^a, t^{E(n-k+1)}, \dots, t^{E(N)})$ with $|a|, |E(j)| \leqslant d^{nN}$ and with W the closure of an orbit of dimension $n-k$.

The $(\mathbb{C}^*)^{n-k}$-actions as in (iii) are only finitely many, therefore there exists a single $(\mathbb{C}^*)^{n-k}$-action as above such that the general fibre W is an orbit closure. In particular this action leaves X invariant as required. Q.E.D.

We are going to show now that the above proof can be used to derive a more precise statement. For this purpose we need a well known lemma.

LEMMA 1.3. *Let $T = (\mathbb{C}^*)^m$ and assume that we are given a T-action on $\mathbb{P}^N$ and a nondegenerate orbit Tx. Diagonalize the action of T and assume that g is a diagonal projectivity which leaves Tx invariant. Then g belongs to the T-action.*

Proof. We view $\mathbb{P}^N$ as the projective space of diagonal matrices. Since Tx is nondegenerate, x is an invertible diagonal matrix. For general $t \in T$ there exists $t' \in T$ such that $gtx = t'x$, here we identify t and t' with their images in $\mathrm{PGL}(N+1)$. But then $g = t't^{-1}$, as we wanted to show. Q.E.D.

DEFINITION 1.4. Let us say that a projective variety X satisfies property P_h if for all abelian linear actions of a group G on X there exists a subgroup $G' \subset G$ of index $[G : G'] \leqslant d^{2^{n-h}}$ and admitting a set of h generators.

COROLLARY 1.5. *Let G and X be as in Theorem 1.1. Then the maximal integer m such that X admits a linear effective $(\mathbb{C}^*)^m$-action (i.e., with a general orbit on*

X of dimension m) equals the minimal integer h such that property P_h holds for X.

Proof. If there exists an effective $(\mathbb{C}^*)^m$-action, since $(\mathbb{C}^*)^m \supset (\mathbb{Z}/p\mathbb{Z})^m$ for each positive integer p, you see immediately that properties $P_0, \dots, P_{m-1}$ do not hold. Conversely, if moreover there is no $(\mathbb{C}^*)^{m+1}$-action, then in the proof of Theorem 1.1 we construct a rational map $\psi \colon X \to \mathbb{P}^k$ with k maximal such that the general fibre W of ψ is the closure of an orbit of an effective $(\mathbb{C}^*)^{m'}$-action with $m' = n - k$. If we let G' be the stabilizer of W we see that the property $P_{m'}$ holds by Lemma 1.3. So by the above argument we have $m' \geqslant m$. On the other hand we have found an effective $(\mathbb{C}^*)^{m'}$-action, hence $m \geqslant m'$. This completes the proof of the Corollary.

In fact the proof of Theorem 1.1 shows the following result.

COROLLARY 1.6. *Let $X_n^d \subset \mathbb{P}^N$ be a projective variety of degree d and dimension n. Assume that* $\mathrm{Lin}(X) \supset G$, *a finite Abelian subgroup of $GL(n+1)$. Then there exists a G-invariant pencil of hypersurfaces of degree at most d.*

2. Abelian groups of automorphisms of varieties of general type

We assume throughout this section that X is a variety of general type having at worst log-terminal Gorenstein singularities and with K_X nef. By the results of Demailly [De93] and Kollár [Ko93] mK_X is globally generated for $m \geqslant 2(n+2)!(n+2)$, where $n = \dim X$. But then consider the finite morphism $\Phi_{|mK_X|} : X \to \Delta$, and take a point $z \in \Delta_{\mathrm{reg}}$ such that on its inverse image Φ is etale, finite, and take then a general complete intersection curve $C \supset \Phi^{-1}(z)$. C is smooth, and $|nmK_X| \to |nmK_X|_C|$ is onto. Since $|nmK_X|_C|$ is very ample we have the birationality of $|nmK_X|$ (cf. [Wi81,87]). Since the sum of a birational and free linear system with a free linear system is a birational and free linear system, $|mK_X|$ gives a birational morphism onto its image as soon as $m \geqslant 2(n+1)(n+2)!(n+2)$. If thus m satisfies the above inequality we have a morphism $X \to Y \subset \mathbb{P}^N$ such that the degree d of Y satisfies $d \leqslant m^n K_X^n$. We want to give polynomial bounds for the order of an *abelian* subgroup G of $\mathrm{Bir}(X)$.

Clearly G has a faithful linear action on $\mathbb{P}^N$ which leaves Y invariant, and therefore we can apply the results of section 1 to obtain the following theorem.

THEOREM 2.1. *Let X be a variety of general type which has at worst log-terminal Gorenstein singularities and with K_X nef and let $G \subset$* $\mathrm{Bir}(X)$ *be an abelian group of birational automorphisms. Then we have the following bound*

$$\mathrm{card}(G) \leqslant C(n)(K_X^n)^{2^n},$$

where $C(n) = [2(n+1)(n+2)!(n+2)]^{n2^n}$.

COROLLARY 2.2. *Assume that X is an n-dimensional variety of general type admitting a birational model X' with at worst log-terminal Gorenstein singularities and $K_{X'}$ nef. Let $G \subset \mathrm{Bir}(X)$ be an abelian group of birational automorphisms. Then, if we let $P_m(X)$ be the mth plurigenus of any desingularization of X, and we define y as*

$$y = \lim_{m\to\infty} \frac{P_m(X')n!}{m^n},$$

we have

$$\mathrm{card}(G) \leqslant C(n)y^{2^n},$$

where $C(n) = [2(n+1)(n+2)!(n+2)]^{n2^n}$.

Proof. In fact $K_{X'}^n = y$ since the plurigenera are birational invariants, thus we simply apply Theorem 2.1 to X'. Q.E.D.

3. Sharper bounds in the 3-dimensional case

In this section we want to improve our bounds in the 3-dimensional case, keeping essentially the same hypotheses as in Section 2.

In fact in Section 2 we have seen that if an n-dimensional variety of general type has a birational model X with at worst Gorenstein log-terminal singularities and K_X nef, then there exists an integer

$$m \leqslant c(n)K_X^n = [2(n+1)(n+2)!(n+2)]^n K_X^n$$

such that the following property holds.

For each abelian group $G \subset \mathrm{Bir}(X)$ let $V = H^0(X, \mathcal{O}_X(mK_X))$, so that G acts linearly on V, and we can diagonalize the action of G to get the eigenspace decomposition

$$V = \bigoplus_{\chi\in G^*} V_\chi.$$

Then we have:

- $\dim V_\chi \geqslant 2$ for some character $\chi \in G^*$.

We assume from now on that X is 3-dimensional and we consider a G-invariant pencil

$$\lambda\colon X \dashrightarrow \mathbb{P}^1.$$

given by a 2-dimensional subspace of V_χ.

Let $X' \subset X \times \mathbb{P}^1$ be the graph of the rational map λ. G acts on X' through $g\cdot(x,t) = (g\cdot x, t)$. Taking a resolution of singularities $\widetilde{X}$ we obtain a morphism

$$\widetilde{X} \xrightarrow{f} \mathbb{P}^1,$$

whose Stein factorization will be

$$\widetilde{X} \xrightarrow{\hat{f}} C \xrightarrow{g} \mathbb{P}^1.$$

The morphism $\widetilde{X} \to X$ will be denoted by π. Let $u \in \mathbb{P}^1$ be a general point. Then $g^{-1}(u) = \{t_1, \ldots, t_k\}$ and $f^{-1}(u) = Y_{t_1} \cup \cdots \cup Y_{t_k}$. The smooth surfaces Y_i are of general type by the "easy" addition formula

$$\kappa(\widetilde{X}) \leq \kappa(Y_i) + \dim(C).$$

Clearly the group G acts birationally on $\widetilde{X}$ and biregularly on C compatibly with the above factorization diagram. We have a subgroup G' of G of index $\leqslant k$ which acts as the identity on C. Therefore we can reduce to the case where $G \subset \mathrm{Bir}(Y)$, Y a general fibre of $\widetilde{X} \to C$. Xiao [Xi90] proved for minimal surfaces of general type S a linear bound

$$\mathrm{card}(G) \leqslant 52\, K_S^2 + 32,$$

provided $K_S^2 \geqslant 140$. Taking for S the minimal model of Y we observe that

$$\mathrm{Bir}(Y) = \mathrm{Bir}(S) = \mathrm{Aut}(S)$$

and that $K_S^2 = P_2(S) - \chi(\mathcal{O}_S) = P_2(Y) - \chi(\mathcal{O}_S)$. Since $\chi(\mathcal{O}_S) \geqslant 1$ it is therefore enough to give an upper bound for $P_2(Y)$. To do this consider the map

$$\hat{f} \colon \widetilde{X} \to C.$$

By Fujita [Fu78], Kawamata [Ka82] and Viehweg [Vi82] the locally free sheaf

$$E := \hat{f}_* \left(\omega_{\widetilde{X}/C}^{\otimes 2} \right)$$

is strictly semipositive on C.

LEMMA 3.1. *The bundle $E \otimes \omega_C \otimes \mathcal{O}(D)$ is globally generated for any divisor D on C of degree at least one.*

Proof. Take any point $q \in C$ and consider the exact sequence

$$0 \to E \otimes \omega_C(D-q) \to E \otimes \omega_C(D) \to (E \otimes \omega_C(D))_q \to 0.$$

It is enough to show

$$H^1(C, E \otimes \omega_C(D-q)) = 0.$$

By Serre-duality this means

$$H^0(C, E^* \otimes \mathcal{O}_C(-D+q)) = 0.$$

A nonvanishing section of $E^* \otimes \mathcal{O}_C(-D+q)$ gives an injection

$$\mathcal{O}_C(D-q) \hookrightarrow E^*.$$

Since E^* is strictly seminegative this implies that $\deg \mathcal{O}_C(D-q) < 0$ what is absurd. Q.E.D.

REMARK 3.2. The same proof shows more generally that

$$f_*\left(\omega_{\widetilde{X}/C}^{\otimes k}\right) \otimes \omega_C \otimes L$$

is globally generated for $k \geqslant 1$ and all line bundles L on C with $\deg L \geqslant 2$, and $\deg L \geqslant 1$ for $k \geqslant 2$.

The lemma implies that

$$P_2(Y) = \operatorname{rank} E \leqslant \dim H^0(C, E \otimes \omega_C(D))$$

for each effective divisor of degree at least one. But

$$h^0(C, E \otimes \omega_C(D)) = h^0\left(\widetilde{X}, \omega_{\widetilde{X}/C}^{\otimes 2} \otimes \hat{f}^*\omega_C(D)\right),$$

and since

$$\omega_{\widetilde{X}/C}^{\otimes 2} = \omega_{\widetilde{X}}^{\otimes 2} \otimes \hat{f}^*\omega_C^{-2},$$

we get

$$h^0(C, E \otimes \omega_C(D)) = h^0\left(\widetilde{X}, \omega_{\widetilde{X}}^{\otimes 2} \otimes \hat{f}^*(D - K_C)\right).$$

If g, the genus of C, is at least 2 we observe that the index of G' in G is bounded by $4g+4$, cf. [Na87]. We can choose D of degree 1 and D' effective and linearly equivalent to $K_C - D$. Since for every effective divisor $D'' \leqslant D'$ we have $H^1(C, E \otimes \omega_C(D + D'')) = 0$, we have

$$h^0(C, E \otimes \omega_C(D)) \leqslant h^0\left(\widetilde{X}, \omega_{\widetilde{X}}^{\otimes 2}\right) - P_2(Y)(2g-3),$$

whence

$$P_2(Y) \leqslant \frac{P_2(X)}{2g-2}.$$

Therefore in this case

$$\operatorname{card}(G) \leqslant \left(\frac{4g+4}{2g-2}\right) P_2(X) \leqslant 6P_2(X).$$

Assume now $g \leqslant 1$. We take $D = \mathcal{O}(p)$ for some point $p \in C$ and then $D - K_C$ is an effective divisor of degree at most 3. Since

$$\pi^*(mK_X) = Y_{t_1} \cup \cdots \cup Y_{t_k} \cup \Xi,$$

we get the inequality

$$h^0(C, E \otimes \omega_C(D)) \leqslant h^0(\tilde{X}, 2\widetilde{K} + 3m\pi^*(K_X)) - (3k-3)P_2(Y).$$

Since

$$\widetilde{K} = \pi^*(K_X) + R$$

we get

$$(3k-2)P_2(Y) \leqslant h^0(\tilde{X}, (2+3m)\widetilde{K}) = h^0(X, (3m+2)K_X).$$

Therefore

$$\mathrm{card}(G) \leqslant \left(\frac{k}{3k-2}\right) P_{3m+2}(X) \leqslant P_{3m+2}(X).$$

Hence we have proved the following result.

THEOREM 3.3. *Let X be a 3-fold of general type and let G be an abelian subgroup of* $\mathrm{Bir}(X)$. *Let m be a positive integer such that $H^0(X, mK_X)$ contains an eigenspace for the action of G of dimension at least* 2. *Then we have*

$$\mathrm{card}(G) \leqslant \max(6P_2(X), P_{3m+2}(X)).$$

COROLLARY 3.4. *Let X be a 3-fold of general type with at worst log-terminal Gorenstein singularities and nef canonical bundle K_X. Let G be an abelian subgroup of* $\mathrm{Bir}(X)$. *Then we have*

$$\mathrm{card}(G) \leqslant c\,(K_X^3)^4,$$

where the constant c can be effectively computed and is in any case strictly smaller than 10^{47}.

Proof. By what we have seen in Sections 1 and 2, for

$$m = (2(n+1)(n+2)!(n+2))^n K_X^n$$

the assumptions of the previous theorem are satisfied and it suffices to bound the plurigenera $P_2(X), P_{3m+2}(X)$. Let us carry out the estimate for $P_{3m+2}(X)$, the estimate for $6P_2(X)$ yielding a smaller number. Let W be a general surface in $(3m+2)K_X$ and let C be a general curve section of W by a hypersurface in $(3m+2)K_X$. By restriction and Kawamata–Viehweg vanishing we get that

$$P_{3m+2}(X) \leqslant h^0(C, (3m+2)K_X) \leqslant (3m+3)^2(3m+2)K_X^3,$$

where the last inequality just comes from bounding by the degree.

Since m is large, the last term is bounded by $28m^3$. This proves the Corollary. Q.E.D.

REMARK 3.5. Observe that for each fixed r there are constants a and b such that for every smooth 3-fold X with K_X nef

$$bK_X^3 \geqslant P_r(X) \geqslant aK_X^3.$$

The first inequality is true in all dimensions.

Proof. The estimate $bK_X^n \geqslant P_r(X)$ follows by restriction as in the above corollary. The second estimate is obtained as follows: By the Kawamata–Viehweg vanishing theorem we have for $r \geqslant 2$

$$P_r(X) = \chi(X, rK_X).$$

Since X is smooth, the Riemann-Roch formula yields

$$\chi(X, rK_X) = \frac{r(2r-1)(r-1)}{12}K_X^3 + \frac{2r-1}{24}K_X c_2(X).$$

Using Miyaoka's result $c_2(X)K_X \geqslant 0$ we get for $r \geqslant 3$ the estimate

$$P_r(X) \geqslant \frac{r(r-1)(2r-1)}{12}K_X^3.$$

REMARK 3.6. By the above we see that morally any polynomial bound in a fixed plurigenus $P_r(X)$ is equivalent to a polynomial bound of the same degree in K_X^3. If the number m in Theorem 3.3 could be chosen to be a constant, then we would have a linear bound.

4. Final comments

Replacing the invariant pencil of Section 3 by an equivariant pencil (cf. [HS82]), certainly has the advantage that we only need $P_r \geqslant 2$, but creates the difficulty of bounding the index of G' when the genus g of the base curve C of the pencil is 0 (the proof of Theorem 3.3 works for an equivariant pencil when $g \geqslant 2$ verbatim and for $g = 1$ up to multiplication by 6, which is an upper bound for the group of automorphisms of an elliptic curve which have a fix point). This approach would be feasible if the following questions would have a positive answer.

QUESTION 4.1. Given a smooth variety $\widetilde{X}$ of general type and a fibration $f: \widetilde{X} \to \mathbb{P}^1$ (i.e. f has connected fibres), are there at least 3 singular fibres?

QUESTION 4.2. Be given a smooth variety $\widetilde{X}$ of general type and a fibration $f: \widetilde{X} \to \mathbb{P}^1$. Can one give an upper bound for the number of singular fibres which produces a linear bound in terms of birational invariants in the case where f is obtained from a pluricanonical mapping ϕ_{mK}?

Both questions have a positive answer in the surface case (cf. [HS82] and in fact for Question 4.2 one simply applies the classical Zeuthen–Segre formula). In dimension 3 L. Migliorini [Mi] proved the existence of at least one singular fibre, cf. also [Kov] in higher dimension.

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