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Rational Surfaces in $\mathbb{P}^4$ Containing a Plane Curve (*).

F. CATANESE - K. HULEK

Abstract. – *The families of smooth rational surfaces in $\mathbb{P}^4$ have been classified in degree ≤ 10. All known rational surfaces in $\mathbb{P}^4$ can be represented as blow-ups of the plane $\mathbb{P}^2$. The fine classification of these surfaces consists of giving explicit open and closed conditions which determine the configurations of points corresponding to all surfaces in a given family. Using a restriction argument originally due independently to Alexander and Bauer we achieve the fine classification in two cases, namely non-special rational surfaces of degree 9 and special rational surfaces of degree 8. The first case completes the fine classification of all non-special rational surfaces. In the second case we obtain a description of the moduli space as the quotient of a rational variety by the symmetric group S_5. We also discuss in how far this method can be used to study other rational surfaces in $\mathbb{P}^4$.*

I. – Introduction.

The families of smooth rational surfaces in $\mathbb{P}^4$ have been classified in degree ≤ 10 ([A1], [I1], [I2], [O1], [O2], [R1], [R2], [PR]). In his thesis POPESCU [P] constructed further examples of rational surfaces in degree 11. The existence of these surfaces has been proved in various ways, using linear systems, vector bundles and sheaves or liaison arguments. All known rational surfaces can be represented as a blowing-up of $\mathbb{P}^2$. Although it would seem the most natural approach to prove directly that a given linear system is very ample, this turns out to be a very subtle problem in some cases, in particular when the surface S in $\mathbb{P}^4$ is special (i.e. $h^1(\mathcal{O}_S(H)) \neq 0$). On the other hand, being able to handle the linear system often means that one knows the geometry of the surface very well.

The starting point of our paper is the observation that every known rational surface in $\mathbb{P}^4$ contains a plane curve C. Using the hyperplanes through C one can construct a residual linear system $|D|$. I.e., we can write $H \equiv C + D$ with $\dim|D| \geq 1$.

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Indirizzo degli AA.: F. CATANESE: Dipartimento di Matematica, Università di Pisa, via Buonarroti 2, I-56127 Pisa, Italy; e-mail: catanese@dm.unipi.it; K. HULEK: Institut für Mathemathik, Universität Hannover, Postfach 6009, D-30060 Hannover, Germany; e-mail: hulek@math.uni-hannover-de

This situation was studied in particular by ALEXANDER [A1], [A2] and BAUER [B]: if $|H|$ restricts to complete linear systems on C and D' where D' varies in a 1-dimensional linear subsystem of $|D|$, then H is very ample on S if and only if it is very ample on C and the curves D' (cf. Theorem II.1). In this way one can reduce the question of very ampleness of H to the study of linear systems on curves. In [CFHR] the following curve embedding theorem was proved which we shall state here only for the (special) case of curves contained in a smooth surface.

THEOREM I.1. – *A divisor H is very ample on C if for every subcurve Y of C of arithmetic genus $p(Y)$*

(i) $H.Y \geqslant 2p(Y) + 1$ *or*

(ii) $H.Y \geqslant 2p(Y)$ *and there is no 2-cycle ξ of Y such that* $I_\xi \mathcal{O}_Y \cong \omega_Y(-H)$.
More generally

(iii) *If ξ is an r-cycle of C, then $H^0(C, \mathcal{O}_C(H))$ surjects onto $H^0(\mathcal{O}_C(H) \otimes \mathcal{O}_\xi)$ unless there is a subcurve Y of C and a morphism φ: $I_\xi \mathcal{O}_Y \to \omega_Y(-H)$ which is «good» (i.e. φ is injective with a cokernel of finite length) and which is not induced by a section of $H^0(Y, \omega_Y(-H))$.*

The method described above was used in [CF] to characterize exactly all configurations of points in $\mathbb{P}^2$ which define non-special rational surfaces of degree $\leqslant 8$. In these cases $H.D \geqslant 2p(D) + 1$. This left the case open of one non-special surface, namely the unique non-special surface of degree 9. In this case one has a decomposition $H \equiv C + D$ where C is a plane cubic, and $|D|$ is a pencil of curves of genus $p(D) = 3$ and $H.D = 6$. Section II is devoted to this surface. In Theorem II.2 we classify all configurations of points in the plane which lead to non-special surfaces of degree 9 in $\mathbb{P}^4$. This completes the fine classification of non-special surfaces.

In Section III we show that this method can also be applied to study special surfaces. We treat the (unique) special surface of degree 8. In this case there exists a decomposition $H \equiv C + D$ where C is a conic and $|D|$ is a pencil of curves of genus 4 with $H.D = 6$. It turns out that for the general element D' of $|D|$ (but not necessarily for all elements) H is the canonical divisor on D'. In Theorem III.14 we give a characterization of these configurations of points which define smooth special surfaces of degree 8 in $\mathbb{P}^4$. We then use this result to give an existence proof (in fact we construct the general element in the family) of these surfaces using only the linear system $|H|$ (Theorem III.17), and in particular to describe the moduli space of the above surfaces modulo projective equivalence (Theorem III.20).

Finally in Section IV we discuss some possibilities how this method can be used to study other rational surfaces in $\mathbb{P}^4$, suggesting some explicit decompositions $H \equiv C + D$ of the hyperplane class as the sum of divisors.

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II. - The non-special rational surface of degree 9.

In this section we want to give an application of Theorem I.1 to non-special rational surfaces. These surfaces have been classified by ALEXANDER [A1]. CATANESE and FRANCIOSI treated all non-special rational surfaces of degree ≤ 8 by studying suitable decompositions $H = C + D$ of the embedding linear systems. The crucial observation here is the following result, originally due to J. ALEXANDER and I. BAUER [B].

THEOREM II.1 (Alexander-Bauer). - *Let X be a smooth projective variety and let C, D be effective divisors with* $\dim |D| \geq 1$. *Let H be the divisor $H \equiv C + D$. If $|H|_{|C}$ is very ample and for all D' in a 1-dimensional subsystem of $|D|$, $|H|_{|D'}$ is very ample, then $|H|$ is very ample on X.*

By Alexander's list there is only one non-special rational surface of degree bigger than 8. This surface is a $\mathbb{P}^2$ blown up in 10 points $x_1, \ldots, x_{10}$ embedded by the linear system $|H| = \left| 13L - 4 \sum_{i=1}^{10} x_i \right|$. Alexander showed that for general position of the points x_i the linear system $|H|$ embeds $S = \widetilde{\mathbb{P}}^2(x_1, \ldots, x_{10})$ into $\mathbb{P}^4$. Clearly the degree of S is 9. Here we show that using Theorem I.1 one can also apply the decomposition method to this surface. In fact we obtain necessary and sufficient conditions for the position of the points x_i for $|H|$ to be very ample. Our result is the following

THEOREM II.2. - *The linear system $|H| = |13L - 4\sum x_i|$ embeds the surface $S = \widetilde{\mathbb{P}}^2(x_1, \ldots, x_{10})$ into $\mathbb{P}^4$ if and only if*

(0) *no x_i is infinitely near*,

(1) $|L - \sum_{i \in \Delta} x_i| = \emptyset$ *for* $|\Delta| \geq 4$,

(2) $|2L - \sum_{i \in \Delta'} x_i| = \emptyset$ *for* $|\Delta'| \geq 7$,

(3) $|3L - \sum_i x_i| = \emptyset$,

$(3)'_{ij}$ $|3L - \sum_{k \neq i,j} x_k - 2x_i| = \emptyset$ *for all pairs* (i, j),

$(4)_{ijk}$ $|4L - 2x_i - 2x_j - 2x_k - \sum_{l \neq i,j,k} x_l| = \emptyset$ *for all triples* (i, j, k),

$(6)_i$ $$|6L - x_i - 2\sum_{j\neq i} x_j| = \emptyset\,,$$

$(10)_1$ *If* $D = 10L - 4x_1 - 3\sum_{i\geq 2} x_i$, *then* $\dim|D| = 1$.

REMARKS. – (i) Clearly conditions (0) to (6) are open conditions. The expected dimension of $|D|$ is 1, hence condition $(10)_1$ is also open.

(ii) The last condition is asymmetrical. If $|H|$ is very ample condition $(10)_i$ is necessarily fulfilled for all i. On the other hand, our theorem shows that in order to prove very ampleness for $|H|$ it suffices to check only one of the conditions $(10)_i$.

PROOF. – We shall first show that the conditions stated are necessary. Clearly (0) follows since $H.(x_i - x_j) = 0$. Similarly the ampleness of H immediately implies conditions (1) to (4). Assume the linear system $\left|6L - x_i - 2\sum_{j\neq i} x_j\right|$ contains some element A. Then $H.A = 2$, and $p(A) = 1$ which contradicts very ampleness of H. For (10) we consider $C \equiv H - D \equiv 3L - \sum_{i\geq 2} x_i$. Clearly $|C|$ is non empty. For $C' \in |C|$ we consider the exact sequence

(11) $$0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(H) \to \mathcal{O}_{C'}(H) \to 0\,.$$

If $h^0(\mathcal{O}_S(D)) \geq 3$, then either $h^0(\mathcal{O}_S(H)) \geq 6$ and $|H|$ does not embed S into $\mathbb{P}^4$ or $|H|$ maps C' to a line. But since $p(C) = 1$ this means that $|H|$ cannot be very ample.

Now assume that conditions (0) to $(10)_1$ hold. We shall first show

(I) $$h^1(\mathcal{O}_S(D)) = 0\,,$$

(II) $$h^1(\mathcal{O}_S(C)) = 0\,,$$

(III) $$h^0(\mathcal{O}_S(H)) = 5\,.$$

Ad(I): By condition $(10)_1$ we have $h^0(\mathcal{O}_S(D)) = 2$. Clearly $h^2(\mathcal{O}_S(D)) = h^0(\mathcal{O}_S(K - D)) = 0$. Hence the claim follows from Riemann-Roch, since $\chi(\mathcal{O}_S(D)) = 2$.

Ad(II): We consider $-K \equiv 3L - \sum_i x_i \equiv C - x_1$. By condition (3) $h^0(\mathcal{O}_S(-K)) = 0$. Clearly also $h^2(\mathcal{O}_S(-K)) = h^0(\mathcal{O}_S(2K)) = 0$. Hence by Riemann-Roch $h^1(\mathcal{O}_S(-K)) = -\chi(\mathcal{O}_S(-K)) = 0$. Now consider the exact sequence

(12) $$0 \to \mathcal{O}_S(-K) \to \mathcal{O}_S(C) \to \mathcal{O}_S(C)|_{x_1} = \mathcal{O}_{x_1} \to 0\,.$$

This shows $h^1(\mathcal{O}_S(C)) = 0$. Note that this also implies (by Riemann-Roch) that $h^0(\mathcal{O}_S(C)) = 1$, i.e. the curve C' is uniquely determined.

Ad(III): In view of (I) and sequence (11) it suffices to show that $h^0(\mathcal{O}_{C'}(H)) = 3$. By Riemann-Roch on C' this is equivalent to $h^1(\mathcal{O}_{C'}(H)) = 0$. Since (by (3)) $K_{C'}$ is trivial this in turn is equivalent to $h^0(\mathcal{O}_{C'}(-H)) = 0$. By conditions (3), (3)′, the curve C' contains no exceptional divisor. As a plane curve C' can be irreducible or it can decompose into a conic and a line or three lines. In view of conditions (1) and (2), however, C' cannot have multiple components and moreover H has positive degree on every component. This proves $h^0(\mathcal{O}_{C'}(-H)) = 0$ and hence the claim.

This shows that $|H|$ maps S to $\mathbb{P}^4$ and that, moreover, $|H|$ restricts to complete linear systems on C' and all curves $D' \in |D|$. We shall now show

(IV) For every subcurve $A \leqslant C'$ we have $H.A \geqslant 2p(A) + 1$.

(V(i)) For every proper subcurve $B' \subset D'$ of an element $D' \in |D|$ we have $H.B' \geqslant 2p(B') + 1$.

(V(ii)) H does not restrict to a «$(2 + K)$»-divisor on D', i.e. $\mathcal{O}_{D'}(H - K_{D'})$ does not have a good section defining a degree 2-cycle.

It then follows from (IV) and [CF, Theorem 3.1] that $|H|$ is very ample on C'. Because of (V(i)) and (V(ii)) it follows from Theorem I.1 that $|H|$ is very ample on every element D' of $|D|$. It then follows from Theorem II.1 that $|H|$ is very ample.

Ad (V(ii)): Let $H_{D'}$ be the restriction of H to D', and denote the canonical bundle of D' by $K_{D'}$. It suffices to show that $h^0(\mathcal{O}_{D'}(H_{D'} - K_{D'})) = 0$. Now

$$H_{D'} - K_{D'} = (H - K - D)|_{D'} = (C - K)|_{D'} = (2C - x_1)|_{D'}.$$

There is an exact sequence

$$(13) \qquad 0 \to \mathcal{O}_S(2C - x_1 - D) \to \mathcal{O}_S(2C - x_1) \to \mathcal{O}_{D'}(H_{D'} - K_{D'}) \to 0.$$

Since

$$2C - x_1 \equiv 6L - x_1 - 2\sum_{i=2}^{10} x_i$$

it follows from condition $(6)_1$ that $h^0(\mathcal{O}_S(2C - x_1)) = 0$. Clearly $h^0(\mathcal{O}_S(2C - x_1 - D)) = 0$. Now

$$2C - x_1 - D \equiv -4L + 3x_1 + \sum_{i=2}^{10} x_i$$

resp.

$$K - (2C - x_1 - D) \equiv L - 2x_1.$$

Hence $h^2(\mathcal{O}_S(2C - x_1 - D)) = h^0(\mathcal{O}_S(K - (2C - x_1 - D))) = 0$. Since moreover $\chi(\mathcal{O}_S(2C - x_1 - D)) = 0$ it follows that $h^1(\mathcal{O}_S(2C - x_1 - D)) = 0$. The assertion follows now from sequence (13).

Ad (IV) and (V(i)): We have to show that for all curves A with $A \leqslant C'$, resp. $A < D'$, $D' \in |D|$ the following holds

$$(14) \qquad H.A \geqslant 2p(A) + 1.$$

We first notice that it is enough to prove (14) for divisors A with $p(A) \geqslant 0$. Assume in fact we know this and that $p(A) < 0$. Then A is necessarily reducible. For every irreducible component A' of A we have $p(A') \geqslant 0$ and hence $H.A' > 0$. This shows $H.A > 0$ and hence (14). Clearly (14) also holds for the lines x_i. Hence we can assume that A is of the form

(15) $$A \equiv aL - \sum_i b_i x_i \quad \text{with} \quad 1 \leqslant a \leqslant 10\,.$$

Note that

(16) $$2p(A) = a(a-3) - \sum_i b_i(b_i - 1) + 2$$

(17) $$H.A = 13a - 4\sum_i b_i\,.$$

We proceed in several steps

CLAIM 1. – Let A be as in (15) with $1 \leqslant a \leqslant 3$. Assume that $p(A) \geqslant 0$. Then (14) is fulfilled.

PROOF OF CLAIM 1. – After possibly relabelling the x_i we can assume that $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{10}$. If $a = 1$ or 2 then $b_1 \leqslant 1$ and $b_{10} \geqslant 0$ by (16) since we assume $p(A) \geqslant 0$. Moreover $p(A) = 0$. If $H.A \leqslant 2p(A)$ we get immediately a contradiction to conditions (1) or (2). If $a = 3$ then we have two cases. Either $b_1 \leqslant 1$, $b_{10} \geqslant 0$ as above and $p(A) = 1$. Then $H.A \leqslant 2p(A)$ violates condition (3). Or $b_1 = 2$ or $b_{10} = -1$ and the other b_i are 0 or 1. Then $H.A \leqslant 2p(A)$ is only possible for $b_1 = 2$, but this would violate condition (3)$'$.

CLAIM 2. – H is ample on C and D, i.e. for every irreducible component A of C', resp. D', $D' \in |D|$ we have $H.A > 0$.

PROOF OF CLAIM 2. – Assume the claim is false. Let A be an irreducible component with $H.A \leqslant 0$. Since A is irreducible, $p(A) \geqslant 0$. By (16), (17) this leads to the two inequalities

(18) $$13a \leqslant 4\sum b_i\,,$$

(19) $$\sum b_i(b_i - 1) \leqslant a(a-3) + 2\,.$$

Multiplying (19) by 13^2 and using (18) we obtain

(20) $$169\left(\sum b_i^2 - \sum b_i\right) \leqslant 16\left(\sum b_i\right)^2 - 156\sum b_i + 338\,.$$

Now

(21) $$\left(\sum b_i\right)^2 = 10\sum b_i^2 - \sum_{i<j}(b_i - b_j)^2$$

and using this (20) becomes

(22) $$\sum_i (9b_i^2 - 13b_i) + 16 \sum_{i<j} (b_i - b_j)^2 \leqslant 338 .$$

The function $f(b) = 9b^2 - 13b$ for integers b is non positive only for $b = 0$ or 1. It is minimal for $b = 1$. Since $f(1) = -4$ we derive from (22)

(23) $$16 \sum_{i<j} (b_i - b_j)^2 \leqslant 378$$

resp.

(24) $$\sum_{i<j} |b_i - b_j|^2 \leqslant 23 .$$

At this point it is useful to introduce the following integer valued function

$$\delta = \delta(A) = \max_{i<j} |b_i - b_j| .$$

We have to distinguish several cases:

$\delta \geqslant 3$: Assume there is a pair (i, j) with $|b_i - b_j| \geqslant 3$. Then for all $k \neq i, j$:

$$|b_i - b_k|^2 + |b_j - b_k|^2 \geqslant 5 .$$

Hence

$$\sum_{i<j} |b_i - b_j|^2 \geqslant 9 + 5 . 8 = 49$$

contradicting (24).

$\delta = 2$: After possibly relabelling the x_i we can assume that $b_2 = b_1 + 2$ and $b_1 \leqslant$ $\leqslant b_k \leqslant b_2$ for $k \geqslant 3$. Then

$$|b_k - b_1|^2 + |b_k - b_2|^2 = \begin{cases} 2 & \text{if } b_k = b_1 + 1 \\ 4 & \text{if } b_k = b_1 \text{ or } b_k = b_2 . \end{cases}$$

Let t be the number of b_k which are either equal to b_1 or to b_2. Then

$$\sum_{i<j} |b_i - b_j|^2 \geqslant 4 + 4t + 2(8 - t) + t(8 - t) = 20 + t(10 - t) .$$

It follows from (24) that $t = 0$. But then (22) gives

$$\sum (9b_i^2 - 13b_i) \leqslant 18 .$$

Looking at the values of $f(b) = 9b^2 - 13b$ one sees immediately that this is only possible for $b_1 = -1$ or $b_1 = 0$. In the first case it follows from (18) that $a < 0$ which is absurd. In the second case we obtain $a \leqslant 3$ and hence we are done by Claim 1.

$\delta \leqslant 1$: Here we can assume

$$b_1 = \ldots = b_k = m , \qquad b_{k+1} = \ldots = b_{10} = m + 1 .$$

Since $f(b) \geqslant 42$ for $b \geqslant 3$ it follows immediately from (22) that $m \leqslant 2$. If $m \leqslant 0$ then (18) gives $a \leqslant 3$ and we are done by Claim 1. It remains to consider the subcases $m = 1$ or 2.

$m = 2$: Since $f(2) = 10$ and $f(3) = 42$ formula (22) implies

$$10k + 42(10 - k) + 16k(10 - k) \leqslant 338 .$$

One checks easily that this is only possible for $k = 9$ or 10. In this case (18) gives $a \leqslant 6$. If $k = 9$ then (18) gives $22 \leqslant a(a - 3)$, i.e. $a \geqslant 7$, a contradiction. If $k = 10$, then (18) implies $18 \leqslant a(a - 3)$. This is only possible for $a = 6$. But now the existence of A would contradict condition (6).

$m = 1$: Since $f(1) = -4$ and $f(2) = 10$ formula (22) reads

$$-4k + 10(10 - k) + 16k(10 - k) \leqslant 338$$

or equivalently

$$k(73 - 8k) \leqslant 119 .$$

It is straightforward to check that this implies $k \leqslant 2$ or $k \geqslant 7$. If $k \leqslant 2$ then $\sum b_i(b_1 - 1) \geqslant 16$ and (19) shows that $a \geqslant 6$. On the other hand $\sum b_i \leqslant 19$ and this contradicts (18). Now assume $k \geqslant 7$. Then $\sum b_i \leqslant 13$. It follows from (18) that either $a \leqslant 3$—and this case is dealt with by Claim 1—or $a = 4$ and $\sum b_i = 13$. Then $k = 7$ and the existence of A contradicts condition (4).

END OF PROOF. – It follows immediately from Claim 1 that (14) holds for subcurves $A \leqslant C'$. It remains to consider subcurves $A < D'$, $D' \in |D|$. Since H is ample on D we have $H.A > 0$, hence it suffices to consider curves with $p(A) \geqslant 1$. Also by ampleness of H on D it follows that

$$1 \leqslant H.A \leqslant 5 \tag{25}$$

since $H.D = 6$. Also note that, as an immediate consequence of (17):

$$a \equiv H.A \pmod 4 . \tag{26}$$

Finally we remark the following

OBSERVATION. – If $A < D$ is not one of the exceptional lines x_i, then $H.A \leqslant 4$ implies $b_i \geqslant 0$ for all i. Otherwise at most one $b_i = -1$ and all other $b_i \geqslant 0$.

This follows from the ampleness of H on x_i, since $H.x_i = 4$.

From now on we set

$$B := D - A . \tag{27}$$

By adjunction

$$p(A) + p(B) = p(D) + 1 - A.B = 4 - A.B . \tag{28}$$

We write

$$B \equiv bL - \sum c_i x_i .$$

We shall now proceed by discussing the possible values of the coefficient a of A in decreasing order.

$a = 10$: Then $B = \sum d_i x_i$, $d_i \geq 0$ and since $H.B \leq 5$ we must have $B = x_i$. Then $A.B = 4$ or 5 and $p(A) \leq 0$ by (28).

$a = 9$: By (25), (26) we have to consider two cases

$$(\alpha) \qquad H.A = 5, \quad H.B = 1 ,$$

$$(\beta) \qquad H.A = 1, \quad H.B = 5 .$$

Using our above observation for B in case (α) we find that

$$B \equiv L - x_i - x_j - x_k .$$

But now $A.B \geq 2$ and hence $p(A) \leq 1$. Hence $H.A = 5 \geq 2p(A) + 1$.

Using condition (1) we have to consider the following cases for (β):

$$B \equiv L - x_i - x_j ,$$
$$B \equiv L - x_i - x_j - x_k + x_l .$$

In the first case $A.B \geq 4$ and $p(B) = 0$, hence $p(A) \leq 0$. In the second case $A.B \geq 5$ and $p(B) = -1$, hence again $p(A) \leq 0$.

$a = 8$: Here by (26) the only possibility is

$$H.A = 4 , \qquad H.B = 2 .$$

Using our observation for B we find that

$$B \equiv 2L - x_{i_1} - \dots - x_{i_6} .$$

Either the x_{i_j} are all different or we have 1 double point (and B is a pairs of lines) or 3 double points (and B is a double line). Then $A.B \geq 3$ (resp. 4, resp. 8) and $p(B) = 0$ (resp. -1, resp. -3). In either case $p(A) \leq 1$ and hence $H.A \geq 2p(A) + 1$.

$a = 7$: In this case

$$H.A = H.B = 3 .$$

All coefficients satisfy $b_i \geq 0$. It is enough to consider divisors A with $p(A) \geq 2$. Together with $H.A = 3$ this leads to the following conditions on the b_i:

$$\sum b_i = 22 , \qquad \sum b_i (b_i - 1) \leq 26 .$$

Let $\beta_i = \max(0, b_i - 1)$. Then these conditions become

$$\sum \beta_i \geqslant 12, \qquad \sum(\beta_i + \beta_i^2) \leqslant 26$$

and it is easy to check that no solutions exist.

$a = 6$: We now have to consider

$$H.A = 2, \qquad H.B = 4.$$

We have to consider divisors A with $p(A) \geqslant 1$. Arguing as in the case $a = 7$ this leads to

$$\sum b_i = 19, \qquad \sum b_i(b_i - 1) \leqslant 18$$

resp.

$$\sum \beta_i \geqslant 9, \qquad \sum(\beta_i + \beta_i^2) \leqslant 18.$$

The only solution is $b_j = 1$ for one b_j and $b_i = 2$ for $j \neq i$. But then $A \in \left| 6L - x_j - 2\sum_{i \neq j} x_i \right|$ contradicting condition (6).

$a = 5$: Then we have two possible cases

$$(\alpha) \qquad H.A = 5, \qquad H.B = 1,$$

$$(\beta) \qquad H.A = 1, \qquad H.B = 5.$$

We shall treat (α) first. Then by the ampleness of H the curve B must be irreducible. Set

$$B = 5L - \sum c_i x_i, \qquad c_i \geqslant 0.$$

Then $H.B = 1$ and irreducibility of B gives:

$$\sum c_i = 16, \qquad \sum c_i(c_i - 1) \leqslant 12.$$

One easily checks that this is only possible if 6 of the c_i are 2, and the others are 1. Hence

$$B \in \left|5L - 2\sum_{i \in \Delta} x_i - \sum_{i \notin \Delta} x_i\right|, \qquad |\Delta| = 6.$$

Then $p(B) = 0$. Moreover $A.B \geqslant 3$, hence $p(A) \leqslant 1$ and hence $H.A \geqslant 2p(A) + 1$.

In case (β) we apply the above argument to A and find $p(A) = 0$, i.e. again $H.A \geqslant 2p(A) + 1$.

$a = 4$: Then $H.A = 4$ and $H.B = 2$. We are done if $p(A) \leqslant 1$, and otherwise $H.A \geqslant 52 - 44 = 8$, a contradiction.

$1 \leqslant a \leqslant 3$: This follows immediately from Claim 1.

By (15) this finishes the proof of the theorem. ∎

III. - The special rational surface of degree 8 in $\mathbb{P}^4$.

In this section we want to show how the decomposition method can be employed to obtain very precise geometric information also about special surfaces. We consider the rational surface in $\mathbb{P}^4$ of degree 8, sectional genus $\pi = 6$ and speciality $h = h^1(\mathcal{O}_S(1)) = 1$. This surface was first constructed by OKONEK [O2] using reflexive sheaves. In geometric terms it is $\mathbb{P}^2$ blown-up in 16 points embedded by a linear system of the form

$$|H| = \left| 6L - 2\sum_{i=1}^{4} x_i - \sum_{k=5}^{16} y_k \right| .$$

Our aim is to study the precise open and closed conditions which the points x_i, y_k must fulfill for $|H|$ to very ample. If $|H|$ is very ample, the exceptional lines x_i are mapped to conics. Their residual intersection with the hyperplanes gives a *pencil* $|D_i|$. Hence we immediately obtain the (closed) necessary condition

$$(D_i) \qquad |D_i| \equiv \left| 6L - 3x_i - 2\sum_{j \neq i} x_j - \sum_{k=5}^{16} y_k \right| \quad \text{is a pencil.}$$

By Riemann-Roch this is equivalent to $h^1(\mathcal{O}_S(D_i)) = 1$. We first want to study the linear system $|H|$ on the elements of the pencil $|D_i|$. Note that

$$p(D_i) = 4 , \qquad H.D_i = 6 .$$

If $D = A + B$ is a decomposition of some element $D \in |D_i|$, then

$$(29) \qquad p(A) + p(B) + A.B = 5 ,$$

$$(30) \qquad A.H + B.H = 6 .$$

The first equality can be proved by adjunction, the second is obvious.

LEMMA III.1. – *Assume $|H|$ is very ample. Then for every proper subcurve Y of an element $D \in |D_i|$, $h^1(\mathcal{O}_Y(H)) \leq 1$ and $p(Y) \leq 3$.*

PROOF. – Riemann-Roch on Y gives

$$(31) \qquad h^0(\mathcal{O}_Y(H)) = h^1(\mathcal{O}_Y(H)) + H.Y + 1 - p(Y).$$

Consider the sequence

$$(32) \qquad 0 \to \mathcal{O}_S(H - Y) \to \mathcal{O}_S(H) \xrightarrow{\alpha} \mathcal{O}_Y(H) \to 0 .$$

Since $h^2(\mathcal{O}_S(H - Y)) = h^0(\mathcal{O}_S(K - (H - Y))) = 0$ and $h^1(\mathcal{O}_S(H)) = 1$ we have $h^1(\mathcal{O}_Y(H)) \leq 1$. We now consider the rank of the restriction map $H^0(\alpha)$. Since Y is a curve contained in a hyperplane section $2 \leq \operatorname{rank}(\alpha) \leq 4$. If $\operatorname{rank} \alpha = 2$, then Y is a line, hence $p(Y) = 0$. Next assume $\operatorname{rank}(\alpha) = 3$. In this case Y is a plane curve of degree $d = Y.H$. Since Y is a proper subcurve of D which is not a line $2 \leq d \leq 5$. Then

$h^1(\mathcal{O}_Y(H)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d-4))$. Since $h^1(\mathcal{O}_Y(H)) \leqslant 1$ this shows in fact $d \leqslant 4$. But then $p(Y) \leqslant 3$. Finally assume that $\operatorname{rank}(\alpha) = 4$, i.e. Y is a space curve. By (31)

$$p(Y) = h^1(\mathcal{O}_Y(H)) - h^0(\mathcal{O}_Y(H)) + H.Y + 1 \leqslant 3$$

Since $H.Y \leqslant 5$. ■

REMARK III.2. – Note that the above proof also shows the following: If Y is a proper subcurve of D with $p(Y) = 3$, then Y is a plane quartic with $H_Y = K_Y$ or Y has degree 5.

Before proceeding we note the following result from [CF] which we shall use frequently in the sequel.

PROPOSITION III.3. – *Let Y be a curve contained in a smooth surface with $p(Y) \leqslant 2$. If H is very ample on S, then $H.Y \geqslant 2p(Y) + 1$.*

PROOF. – [CF, Prop. 5.2]. ■

PROPOSITION III.4. – *If $|H|$ is very ample, then every element $D \in |D_i|$ is 2-connected. Moreover, either*

(i) *D is 3-connected or*

(ii) *Every decomposition of D which contradicts 3-connectedness is either of the form $D = A + B$ with $H.B = 4$, $H_B = K_B$ or of the form $D = A + B$ with $H.B = 5$. In the latter case $B = B' + B''$ with $H.B' = 4$, $H_{B'} = K_{B'}$.*

PROOF. – Let $D = A + B$. We first consider the case $p(A), p(B) > 0$. Since $|H|$ is very ample, it follows that $H.A \geqslant 3$, $H.B \geqslant 3$. But then $H.A = H.B = 3$ and hence $p(A) = p(B) = 1$. By (29) this shows $A.B = 3$.

Now assume $p(A) \leqslant 0$. Since $p(B) \leqslant 3$ by Lemma III.1 it follows from (29) that $A.B \geqslant 2$. The only case where $A.B = 2$ is possible is $p(A) = 0$, $p(B) = 3$. In this case $H.B \geqslant 4$ since Riemann-Roch for B gives

$$h^0(\mathcal{O}_B(H)) = h^1(\mathcal{O}_B(H)) + H.B - 2$$

and we know that $h^0(\mathcal{O}_B(H)) \geqslant 3$. We first treat the case $H.B = 4$. Then $h^1(\mathcal{O}_B(H)) = 1$ and $h^0(\mathcal{O}_B(H)) = 3$. In this case B is a plane quartic and $H_B = K_B$. Now assume $H.B = 5$. If $h^1(\mathcal{O}_B(H)) = 0$ then B is a plane quintic. But in this case $p(B) = 6$, a contradiction. It remains to consider the case $h^1(\mathcal{O}_B(H)) = 1$. By duality $h^0(\mathcal{O}_B(K_B - H)) = 1$. Let σ be a non-zero section of $\mathcal{O}_B(K_B - H)$. As usual we can write $B = Y + Z$ where Z is the maximal subcurve where σ vanishes. Note that $Z \neq \emptyset$, since $K_B - H$ has negative degree. Then $Y.(K_Y - H) \geqslant 0$. By the very ampleness of H this implies $p(Y) \geqslant 3$ and hence $p(Y) = 3$. Then we must have $H.Y = 4$ and by the previous analysis Y is a plane quartic with $H_Y = K_Y$. ■

At this point it is useful to introduce the following concept.

DEFINITION. – We way that an element $D \in |D_i|$ fulfills condition (C) if for every decomposition $D = A + B$:

(i) $p(A),\ p(B) \leqslant 2$.

(ii) $H.A \geqslant 2p(A) + 1,\ H.B \geqslant 2p(B) + 1$.

REMARK III.5. – It follows immediately from (29) that an element $D \in |D_i|$ which fulfills condition (C) is 3-connected.

For future use we also note

LEMMA III.6. – *Let D be a curve of genus 4, and let H be divisor on D of degree 6 with $h^0(\mathcal{O}_D(H)) \geqslant 4$. Assume that for every proper subcurve Y of D we have $H.Y \geqslant$ $\geqslant 2p(Y) - 1$. Then H is the canonical divisor on D.*

PROOF. – By Riemann-Roch and duality $h^0(\mathcal{O}_D(K_D H)) \geqslant 1$. Let σ be a non-zero section of $\mathcal{O}_D(K_D - H)$. As usual this defines a decomposition $D = Y + Z$ where Z is the maximal subcurve where σ vanishes. If $Z = \emptyset$ the claim is obvious. Otherwise $(K_D - H).Y \geqslant Z.Y$ and by adjunction this gives $H.Y \leqslant 2p(Y) - 2$, a contradiction. ■

Our next aim is to analyze the condition $h^0(\mathcal{O}_S(H)) = 5$. For this we introduce the divisor

$$\Delta_i \equiv H - (L - x_i).$$

LEMMA III.7. – *The following conditions are equivalent:*

(i) $h^0(\mathcal{O}_S(H)) = 5$ *(resp.* $h^1(\mathcal{O}_S(H)) = 1$*).*

(ii) $h^0(\mathcal{O}_D(H)) = 4$ *(resp.* $h^1(\mathcal{O}_D(H)) = 4$*) for some (every) element* $D \in |D_i|$.

(iii) $h^1(\mathcal{O}_D(K_D - H)) = 1$ *for some (every) element* $D \in |D_i|$.

Moreover assume that $D \in |D_i|$ fulfills condition (C). *Then the following conditions are equivalent to* (i)-(iii):

(iv) $O_D(H) = K_D$.

(v) $\Delta_i|_D \equiv (2L - \sum x_j)|_D$.

PROOF. – Since $h^0(\mathcal{O}_S(D_i)) \geqslant 1$ we have an exact sequence

$$0 \to \mathcal{O}_S(x_i) \to \mathcal{O}_S(H) \to \mathcal{O}_D(H) \to 0\,.$$

Since $h^1(\mathcal{O}_S(x_i)) = 0$ the equivalence of (i) and (ii) follows. The equivalence of (ii) and (iii) is a consequence of Serre duality. It follows from Lemma III.6 that (iii) implies (iv) if (C) holds. Conversely if $\mathcal{O}_D(H) = K_D$ then $h^0(\mathcal{O}_S(K_D - H)) = h^0(\mathcal{O}_D) = 1$, since D is 3-connected. To show the equivalence of (iv) and (v) note that by adjunction

$$K_D \equiv (K_S + D)|_D \equiv \Big(3L - 2x_i - \sum_{j \neq i} x_j\Big)\Big|_D .$$

Hence $K_D \equiv H|_D \equiv (\Delta_i + (L - x_i))|_D$ if and only if $\Delta_i|_D \equiv (K_D - (L - x_i))|_D \equiv \equiv (2L - \sum x_j)|_D$. ■

We want to discuss necessary open conditions which must be fulfilled if $|H|$ is ample.

DEFINITION. – We say that $|H|$ fulfills condition (P) if for every divisor Y on S with $Y.L \leq 6$, $p(Y) \leq 2$, $H.Y \leq 2p(Y)$ the linear system $|Y|$ is empty.

REMARK III.8. – (i) By Proposition III.3 this condition is necessary for $|H|$ to be very ample.

(ii) Note that in order to check (P) one only need check *finitely many* open conditions.

(iii) For $Y.L = 0$ condition (P) implies that the only points which can have infinitely near points are the x_i. The only possibility is that at most one of the points y_k is infinitely near to some point x_i.

(iv) If $Y.L = 1$ then (P) implies

$$\Big|L - \sum_{i \in \Delta} x_i - \sum_{k \in \Delta'} y_k\Big| = \emptyset \quad \text{for } 2|\Delta| + |\Delta'| \geq 6 .$$

In particular no three of the points x_i can lie on a line.

(v) If $Y.L = 6$ then (P) gives

$$|D_i - x_j| = \emptyset \ (j \neq i), \qquad |D_i - y_k - y_l| = \emptyset \ (k \neq l).$$

There are, however, two more open conditions which are not as obvious to see.

PROPOSITION III.9. – *If $|H|$ embeds S into $\mathbb{P}^4$ then the following open conditions hold:*

(Q) $|D_i - 2x_i| = \emptyset, \quad |D_i - x_i - y_k| = \emptyset, \quad |D_i - 2y_k| = \emptyset$

(R) *For any effective curve C with $C \equiv L - x_i - x_j - y_k$, $C \equiv L - x_i - x_j$ or $C \equiv y_k$ one has* $\dim |D_i - C| \leq 0$. *Moreover* $\dim |H - (L - x_i - x_j)| \leq 1$.

PROOF. – We start with (R). We already know that $\dim |D_i| = 1$. Hence we have to see that such a curve C is not contained in the plane spanned by the conic x_i. But this would contradict very ampleness since $C.x_i = 1$ or 0. If $|H|$ is very ample then it embeds $\Lambda_{ij} = L - x_i - x_j$ as a plane conic (irreducible or reducible but reduced). The claim then follows from the exact sequence

$$0 \to \mathcal{O}_S(H - (L - x_i - x_j)) \to \mathcal{O}_S(H) \to \mathcal{O}_{\Lambda_{ij}}(H) \to 0 .$$

Next we consider the linear system $|D_i - 2x_i|$. Assume there is a curve $B \in |D_i - 2x_i|$. Then $p(B) = -3$. Since $H.B = 2$ we have the following possitibilities: B is a reduced conic (either smooth or reducible). Then $p(B) = 0$, a contradiction. If B is the union of 2 skew lines, then $p(B) = -1$ which is also not possible. Hence B must be a non-reduced line. But this is not possible, since the class of B in S is not divisible by 2.

The crucial step is to prove the

CLAIM. – Se $D = D_i$. If $|D|$ contains $y_k + B$, then B is of the form $B = B' + (L - x_i - x_j - y_k)$ with $H_{B'} = K_{B'}$.

It follows from Lemma III.7 that there exists a non-zero section $0 \neq \sigma \in H^0(\mathcal{O}_D(K_D - H))$. As usual this defines a decomposition $D = Y + Z$. Since $(K_D - H).y_k = -1$ the curve Z must contain the irreducible curve y_k. Moreover since $y_k.B = 2$ and $(K_D - H).B = 1$ it follows that Z contains some further curve Z' contained in B, i.e. $B = B' + Z'$. Now as in proof of Lemma III.6 $H.B' \leq 2p(B') - 2$ and very ampleness of $|H|$ together with III.1 implies $p(B') = 3$. As in the proof of Proposition III.4 one concludes that $H.B' = 4$, $H_{B'} = K_{B'}$. In particular Z' is a line. Since $p(D_i - 2y_k) = 1$ it follows that $Z' \neq y_k$. First assume that $Z'.y_k = 0$. Then $p(Z' + y_k) = -1$ and $B'.y_k = 2$. It follows from (29) that $B'.Z' = 1$. But now the decomposition $Z' + (B' + y_k)$ contradicts 2-connectedness. Hence Z' and y_k are two lines meeting in a point. This gives $p(y_k + Z') = 0$, $B'.(y_k + Z') = 2$. We can write

$$Z' = aL - \beta_i x_i - \sum_{j \neq i} \beta_j x_j - y_k - \sum_{l \neq k} \alpha_l y_l .$$

If $a = 0$ then $Z' = x_i - y_k$ or $Z' = x_j - y_k$, $j \neq i$. The first is impossible since $p(D_i - x_i) = 1$ the second contradicts $|D_i - x_j| = \emptyset$. Hence $1 \leq a \leq 6$. Since Z' is mapped to a line in $\mathbb{P}^4$ we find $Z'.y_l \leq 1$, $Z'.x_j \leq 2$, i.e.

$$(33) \qquad 0 \leq \alpha_l \leq 1 , \qquad 0 \leq \beta_i ,\ \beta_j \leq 2 .$$

It follows from (33) and from $p(Z') = 0$ that $a \leq 4$; moreover $p(Z') = 0$, $p(B') = 3$ and $p(B) = 3$ imply $Z'.B' = 1$. Using $0 \leq \alpha_l \leq 1$ this gives

$$(34) \qquad a(6 - a) - \beta_i(3 - \beta_i) - \sum_{j \neq i} \beta_j(2 - \beta_j) = 2 .$$

In view of (33) this shows $a(6 - a) \leq 7$ and since $a \leq 4$ it follows that $a = 1$. Then $\beta_i, \beta_j \leq 1$. If $\beta_i = 0$ then by (34) $\beta_j = 1$ for $j \neq i$, but no three of the points x_i can be collinear by (34). Hence $\beta_i = 1$ and exactly one β_j is 1. Together with $H . Z' = 1$ this gives $Z' = L - x_i - x_j - y_k$ as claimed.

We are now in a position to prove that $|D_i - x_i - y_k| = \emptyset$ and $|D_i - 2y_k| = \emptyset$. For this we have to show that B' cannot contain x_i or y_k. In the first case $B' = x_i + B''$. Then $H . x_i = 2$ and $K_{B'} . x_i = 1$ contradicting $H|_{B'} = K_{B'}$. Similarly in the second case $B' = y_k + B''$ with $H . y_k = 1$ and $K_{B'} . y_k = 0$ giving the same contradiction. ■

Observe for future use that in the following proposition the assumption that $|H|$ is very ample is not made.

PROPOSITION III.10. – *Assume that the open conditions* (P) *and* (Q) *hold. Then an effective decomposition $D = A + B$ either fulfills condition* (C) *and hence is not* 3-*disconnecting or* (*after possibly interchanging A and B*) $A = y_k$, $L - x_i - x_j$ or $L - x_i - x_j - y_k$.

PROOF. – Let $D = A + B$. Clearly we can assume $A . L \leqslant 3$. We shall first treat the case $A . L = 0$, i.e. A is exceptional with respect to the blowing down map $S \to \mathbb{P}^2$. Then $p(A) \leqslant 0$ and $A . H > 0$ by (P). By conditions (Q) and (P) (cf. Remark III.8 (iv)) if $A . H = 1$, then either $A = x_j - y_k$ or $A = x_i - y_k$ or $A = y_k$. In the first two cases $A . B \geqslant 3$ and $p(B) \leqslant 2$, the third is one of the exceptions stated. If $A . H \geqslant 2$ then $p(B) \leqslant 2$ and the claim follows from (P).

Hence we can now write

$$A \equiv aL - \sum \alpha_j x_j - \sum a_k y_k ,$$

$$B \equiv bL - \sum \beta_j x_j - \sum b_k y_k ,$$

with $a, b > 0$. Using the open conditions from Remark III.8 (v) (which are a consequence of (P)) and (Q) it follows that

$$\begin{aligned} &a_k, b_k \geqslant -1 , && a_k + b_k = 1 , \\ &\alpha_j, \beta_j \geqslant 0 , && \alpha_j + \beta_j = 2 , \ (j \neq i) , \\ &\alpha_i, \beta_i \geqslant -1 , && \alpha_i + \beta_i = 3 , \end{aligned}$$

and moreover that at most one of the integers a_k, b_k, α_i, β_i can be negative. If $\beta_i = -1$ then $\alpha_i = 4$. In this case A cannot be effective since we have assumed $a \leqslant 3$. If $\alpha_i = -1$ then $\beta_i = 4$ and hence $b \geqslant 4$. We have to consider the cases $a = 1$ or 2. In either case $p(A) \leqslant 0$ and $H . A \geqslant 2p(A) + 1$ follows from (P). On the other hand

$$H . B - (2p(B) + 1) =$$

$$= (9b - b^2 + 1) + \sum_{j \neq i} \beta_j(\beta_j - 3) + \sum_k b_k(b_k - 2) \geqslant (9b - b^2 + 1) - 6 - 12 \geqslant 3$$

since $b = 4, 5$. Hence we can now assume $\alpha_i, \beta_i \geqslant 0$.

$a = 1$. We first treat the case $a_k \geq 0$ for all k. Then

$$A \equiv L - \sum_{j \in \Delta} x_j - \sum_{k \in \Delta'} y_k .$$

Clearly $p(A) \leq 0$. Let $\delta_{i\Delta} = 0$ (resp. 1) if $i \notin \Delta$ (resp. $i \in \Delta$). Then

$$p(B) = |\Delta| + \delta_{i\Delta} .$$

We only have to treat the cases where $p(B) \geq 3$. The either $\delta_{i\Delta} = 0$, $|\Delta| \geq 3$ or $\delta_{i\Delta} = 1$, $|\Delta| \geq 2$. In the first case

$$H.A = 6 - 2|\Delta| - |\Delta'| \leq 0$$

contradicting (P) for A. In the second case the only possibility is $|\Delta| = 2$, $|\Delta'| \leq 1$. But then $L - x_j - x_j$ or $A = L - x_i - x_j - y_k$. Now assume that one a_k is negative. We can assume $a_{16} = -1$. Then

$$A \equiv L - \sum_{j \in \Delta} x_j - \sum_{k \in \Delta'} y_k + y_{16} .$$

In this case $p(A) = -1$ and

$$p(B) = |\Delta| + \delta_{i\Delta} - 1 .$$

Using the same arguments as before we find that $p(B) \leq 2$ in all cases.

$a = 2$. Again we first assume that all $a_k \geq 0$. Then

$$A \equiv 2L - \sum_{j \in \Delta} x_j - \sum_{k \in \Delta'} 2x_k - \sum_{l \in \Delta''} y_l - \sum_{m \in \Delta'''} 2y_m .$$

Clearly $p(A) \leq 0$. If $i \notin \Delta \cup \Delta'$ then $p(B) \leq 0$. If $i \in \Delta$ then $p(B) \leq 2$. Now assume that $i \in \Delta'$. In this case $p(B) \leq 2$ with one possible exception: $|\Delta| = 3$ and $|\Delta'''| = 0$. But then

$$A \equiv 2L - 2x_i - x_j - x_k - x_l - \sum_{l \in \Delta''} y_l .$$

In this case A splits into two lines meeting x_i. But then one of these lines must contain 3 of the points x_j contradicting condition (P). Finally let $a_{16} = -1$. The above arguments show that in this case $p(B) \leq 2$.

$a = 3$. Since in this case $p(A)$, $p(B) \leq 1$ condition (C) follows. ∎

Propositions III.4 and III.10 have provided us with a fairly good understanding of the behaviour of H on the pencil $|D_i|$.

COROLLARY III.11. – *Assume $|H|$ embeds S into $\mathbb{P}^4$. For every element $D \in |D_i|$ either:*

(i) *D is 3-connected and $H_D = K_D$ or*

(ii) *$D = B + (L - x_i - x_j)$ with $H|_B = K_B$.*

REMARK III.12. – The conic $L - x_i - x_j$ can be irreducible or reducible in which case it splits as $(L - x_i - x_j - y_k) + y_k$.

At this point we can also conclude our discussion about the linear system $|\Delta_i| = |H - (L - x_i)|$ (cf. III.7).

PROPOSITION III.13. – *If* $|H|$ *embeds* S *into* $\mathbb{P}^4$, *then* $\dim|\Delta_i| = 0$.

PROOF. – We first claim that the general element $D \in |D_i|$ is 3-connected. Indeed if D is not 3-connected, then $D = B + (L - x_i - x_j)$. The conic $L - x_i - x_j$ spans a plane E'. If E is the plane spanned by x_i then $E \neq E'$ since $(L - x_i - x_j).x_i = 1$. Hence D is cut out by the hyperplane spanned by E and E'. Varying the index j there are at most 3 such hyperplanes.

Clearly $L - x_i$ is effective. Consider the exact sequence

$$0 \to \mathcal{O}_S(\Delta_i) \to \mathcal{O}_S(H) \to \mathcal{O}_S(H)|_{L - x_i} \to 0 .$$

Since $H.(L - x_i) = 4$ and $p(L - x_i) = 0$ it follows that $|H|$ cannot map $L - x_i$ to a plane curve. This shows $h^0(\mathcal{O}_S(\Delta_i)) \leq 1$.

On the other hand choose an element $D \in |D_i|$ which is 3-connected. We have an exact sequence

$$0 \to \mathcal{O}_S(2x_i - L) \to \mathcal{O}_S(\Delta_i) \to \mathcal{O}_D(\Delta_i) \to 0 .$$

Now $h^0(\mathcal{O}_S(2x_i - L)) = h^2(\mathcal{O}_S(2x_i - L)) = 0$ and hence $h^1(\mathcal{O}_S(2x_i - L)) = 1$ by Riemann-Roch. Since $|H|$ is ample no 3 of the points x_i lie on a line. Hence $|2L - \sum x_i|$ is a base point free pencil. Since $|(2L - \sum x_i) - D| = \emptyset$ this shows that $|2L - \sum x_i|$ cuts out a base-point free pencil on D. Since D is 3-connected $(2L - \sum x_i)|_D \equiv \Delta_i|_D$ by Lemma III.7 and hence $h^0(\mathcal{O}_D(\Delta_i)) \geq 2$. By the above sequence this implies $h^0(\mathcal{O}_S(\Delta_i)) \geq 1$. ■

We are now ready to characterize very ample linear systems which embed S into $\mathbb{P}^4$.

THEOREM III.14. – *The linear system* $|H|$ *embeds into* $\mathbb{P}^4$ *if and only if*

(i) *The open conditions* (P), (Q) *and* (R) *hold.*

(ii) *The following closed conditions hold:*

(D_i) $\dim|D_i| = 1$,

(Δ_i) *For a 3-connected element* $D \in |D_i|$ *(whose existence follows from the above conditions)* $\Delta_i . D \equiv (2L - \sum x_i) . D$.

REMARK III.15. – As the proof will show, it is enough to check the closed conditions (D_i), (Δ_i) for one i.

PROOF. – We have already seen that these conditions are necessary. Next we shall show that a 3-connected element $D \in |D_i|$ exists if the open conditions and (D_i) are fulfilled. Assume that no element $D \in |D_i|$ is 3-connected. Then by Proposition III.10 every element D is of the form $D = B + C$ with $C = L - x_i - x_j, L - x_i - x_j - y_k$ or y_k. But by condition (R) there are only finitely many such elements in $|D_i|$.

We shall now proceed in several steps.

Step 1: $h^0(\mathcal{O}_S(H)) = 5$.

We have seen in the proof of Lemma III.7 that for a 3-connected element D the equality $\Delta_i . D \equiv (2L - \sum x_i) . D$ implies $K_D = H_D$ and hence $h^0(\mathcal{O}_D(K_D - H)) = 1$, resp. $h^1(\mathcal{O}_D(H)) = 1$. Now the claim follows from the equivalence of (i) and (ii) in Lemma III.7.

In order to prove very ampleness of $|H|$ we want to apply the Alexander-Bauer Lemma to the decomposition

$$H \equiv D_i + x_i .$$

We first have to show that $|H|$ cuts out complete linear systems on x_i and $D \in |D_i|$. Recall that x_i is either a $\mathbb{P}^1$ or consists of two $\mathbb{P}^1$'s meeting transversally (cf. Remark III.8 (iii)). Moreover $H . x_i = 2$ and if x_i is reducible then H has degree 1 on every component. Hence $h^0(\mathcal{O}_{x_i}(H)) = 3$. The claim for x_i then follows from the exact sequence

$$0 \to \mathcal{O}_S(D_i) \to \mathcal{O}_S(H) \to \mathcal{O}_{x_i}(H) \to 0 .$$

and condition (D_i), i.e. $h^0(\mathcal{O}_S(D_i)) = 2$. The corresponding claim for D follows from the sequence

$$0 \to \mathcal{O}_S(x_i) \to \mathcal{O}_S(H) \to \mathcal{O}_S(H)|_D \to 0 .$$

Our above discussion also shows that $|H|$ embeds x_i as a conic (which can be irreducible or consist of two different lines).

Step 2: If $D \in |D_i|$ is 3-connected then $H_D = K_D$ and $|H|$ is very ample on D.

We have already seen the first claim. We have to see that K_D is very ample. For this we consider the pencils $|\Sigma_1| = |L - x_i|$, resp. $|\Sigma_2| = |2L - \sum x_j|$. Clearly $|\Sigma_1|$ is base point free and the same is true for $|\Sigma_2|$ as no three of the points x_i lie on a line (by (P)). Hence

$$|\Sigma_1 + \Sigma_2| = \left| 3L - 2x_i - \sum_{j \neq i} x_j \right| = |D_i + K_S|$$

is base point free. By adjunction $(D_i + K_S)|_D \equiv K_D$ and the exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + D_i) \to \mathcal{O}_D(K_D) \to 0$$

shows that restriction defines an isomorphism $|\Sigma_1 + \Sigma_2| \cong |K_D|$. Let X be the blow-up of $\mathbb{P}^2$ in the points x_j and $\pi \colon S \to X$ the map blowing down the exceptional curves

y_k. The linear system $|\Sigma_1 + \Sigma_2|$ defines a morphism

$$f = \varphi_{|\Sigma_1 + \Sigma_2|} : X \to \mathbb{P}^3 .$$

It is easy to understand the map f: clearly f contracts the three (-1)-curves $\Lambda_{ij} = L - x_i - x_j, j \neq i$. Let $\pi' : X \to X'$ be the map which blows down the curves Λ_{ij} (this makes also sense if $\Lambda_{ij} = (L - x_i - x_j - y_k) + y_k$ where we first contract y_k and then $L - x_i - x_j - y_k$). Then X' is a smooth surface and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & f(X) \\ & {\scriptstyle \pi'}\searrow \quad \nearrow{\scriptstyle f'} & \\ & X' & \end{array}$$

where f' maps X' isomorphically onto a smooth quadric. This shows that $\varphi_{|K_D|} : D \to \mathbb{P}^3$ is the composition of the blowing down maps π: $S \to X$ and π': $X \to X' = \mathbb{P}^1 \times \mathbb{P}^1$ followed by an embedding of X'. Now $D . y_k = 1$, hence $\pi|_D$ can only fail to be an isomorphism if D contains y_k. But this is impossible if D is 3-connected. Similarly $D \cdot \Lambda_{ij} = 1$ and D cannot contain a component of Λ_{ij}. Hence we are done in this case.

It remains to treat the case when D is not 3-connected.

Step 3: If D is not 3-connected, then $D = B + (L - x_i - x_j)$, $H_B = K_B$ and $|H|$ restricts onto $|K_B|$.

We have already seen that $h^0(\mathcal{O}_S(H)) = 5$ and hence $h^0(\mathcal{O}_D(K_D - H)) = 1$. As usual a non-zero section σ defines a decomposition $D = Y + Z$. Our first claim is that Z is different from 0. In fact if $Z = 0$ then $K_D - H$ would be trivial on D. On the other hand D is not 3-connected, thus it splits as $D = A + B$ with A as in Proposition III.10, in particular $p(A) = 0$, $A . B = 2$. Then $K_D . A = 0$ contradicting $H . A > 0$ which follows from (P). Thus Z is different from 0 and since the section σ defines a good section σ' of $H^0(\mathcal{O}_Y(K_Y - H))$ it follows that $2p(B) - 2 \geqslant H . Y$, and hence $p(Y) \geqslant 3$, $Y . Z \leqslant 2$. Then Proposition III.10 applies and $Z = y_k$ or $L - x_i - x_j - y_k$ or $L - x_i - x_j$. If $Z = y_k$ or $L - x_i - x_j - y_k$ then $(K_Y - H) . Y = -1$, a contradiction. Hence $Z = L - x_i - x_j$ and $H|_Y = K_Y$. We next claim that B is 2-connected. Assume we have a decomposition $B = B_1 + B_2$ with $B_1 . B_2 \leqslant 1$. Then $(B_1 + B_2) . (L - x_i - x_j) = 2$, hence we can assume that $B_1 . (L - x_i - x_j) \leqslant 1$. But then $B_1 . (B_2 + L - x_i - x_j) \leqslant 2$ contradicting Proposition III.10. This shows that $h^1(\mathcal{O}_B(K_B)) = 1$ and $h^0(\mathcal{O}_B(K_B)) = 3$. The claim then follows from the exact sequence

$$0 \to \mathcal{O}_S(L - x_j) \to \mathcal{O}_S(H) \to \mathcal{O}_B(H) \to 0 .$$

Step 4: $|H|$ embeds D.

Our first claim is that $|H|$ embeds B as a plane quartic. Since $B - y_k$ is not effective by condition (P) and $B . y_k = 1$ it follows that the curve B is mapped isomorphically onto its image under the blowing down map π: $S \to X$. On X

$$B \equiv 5L - 2x_i - x_j - 2x_k - 2x_l, \qquad K_B \equiv (2L - x_i - x_k - x_l)|_B.$$

Thus $|K_B|$ is induced by a standard Cremona transformation centered at x_j, x_k and x_l. Again by (P) it follows that $B - \Lambda_{ik}$ for $k \neq i$ and $B - \Lambda_{kl}$ for $k, l \neq i$ are not effective. Since $B.\Lambda_{ik} = B.\Lambda_{kl} = 1$ it follows that B is mapped isomorphically onto a plane quartic.

It follows from condition (R) that $|H|$ embeds Λ_{ij} as a plane conic Q. The planes containing B and Q intersect in a line and span a $\mathbb{P}^3$. The line of intersection cannot be a component of Q since, by taking residual intersection with hyperplanes containing B, this would contradict $h^0(\mathcal{O}_S(x_i - y_k)) = 1$, resp. $h^0(\mathcal{O}_S(L - x_j - y_k)) = 1$. Hence the schematic intersection of the embedded quartic B and the conic Q has length at most 2. Let D' be the schematic image of D. Then $\mathcal{O}_{D'}$ is contained in the direct image of $\mathcal{O}_D$. But the former has colength ≤ 2 in $\mathcal{O}_Q \oplus \mathcal{O}_B$, the latter has colength 2, thus $D = D'$. ■

REMARK III.16. - We have already remarked that conditions (P) and (Q) lead to finitely many open conditions. Going through the proof of Proposition III.10 one sees that it is sufficient to check that no decomposition $A + B = D \in |D_i|$ exists where A (or B) contradicts one of the following conditions below: here Δ and Δ' are always disjoint subsets of $\{1, \ldots, 4\}$ whereas Δ'' is a subset of $\{5, \ldots, 16\}$. We set $\delta_{i\Delta} = 1$ (resp. 0) if $i \in \Delta$ (resp. $i \notin \Delta$). Similarly we define $\delta_{i\Delta'}$. Moreover $\delta_m = 1$ for at most one $m \in \{5, \ldots, 16\}$ and $\delta_m = 0$ otherwise. If $\delta_m = 1$ then $m \notin \Delta''$.

(0) $\quad |x_j - x_k| = \emptyset \ (j \neq k), \quad |y_k - y_l| = \emptyset \ (k \neq l), \quad |y_k - x_j| = \emptyset, \quad |x_j - y_k - y_l| = \emptyset.$

(1) $$\Big|L - \sum_{j \in \Delta} x_j - \sum_{k \in \Delta''} y_k\Big| = \emptyset \quad \text{for } 2|\Delta| + |\Delta'| \geq 6$$

(2) $$\Big|2L - \sum_{j \in \Delta} x_j - \sum_{k \in \Delta''} y_k\Big| = \emptyset \quad \text{for } 2|\Delta| + |\Delta'| \geq 12.$$

(3) $$\Big|3L - 2x_j - \sum_{k \in \Delta} x_k - \sum_{l \in \Delta''} y_l\Big| = \emptyset \quad \text{for } 2|\Delta| + |\Delta''| \geq 14,$$

$$\Big|3L - \sum_{j \in \Delta} x_j - \sum_{k \in \Delta''} y_k\Big| = \emptyset \quad \text{for } 2|\Delta| + |\Delta''| \geq 16.$$

(4) $$\Big|4L - (3 - \delta_{i\Delta} - 2\delta_{i\Delta'})x_i - \sum_{\substack{j \neq i \\ j \in \Delta}} x_j - 2 \sum_{\substack{k \neq i \\ k \notin (\Delta \cup \Delta')}} x_k - \sum_{l \notin \Delta''} y_l - \delta_m y_m\Big| = \emptyset$$

for $|\Delta| + |\Delta'| + \delta_{i\Delta} + 2\delta_{i\Delta'} - \delta_m \leq 5$, $2|\Delta'| + |\Delta''| - 2\delta_{i\Delta} - 4\delta_{i\Delta'} + \delta_m \leq 0$, $2|\Delta| + 4|\Delta'| + |\Delta''| \leq 11$.

(5) $$\left| 5L - (3 - \delta_{i\Delta})x_i - \sum_{\substack{j \neq i \\ j \in \Delta}} x_j - 2 \sum_{\substack{k \neq i \\ k \notin \Delta}} x_k - \sum_{l \notin \Delta''} y_l - \delta_m y_m \right| = \emptyset$$

for $|\Delta| + \delta_{i\Delta} - \delta_m \leq 2, \quad |\Delta''| - 2\delta_{i\Delta} + \delta_m \leq 0, \quad 2|\Delta| + |\Delta''| \leq 5.$

(6) $|D_i - x_j| = \emptyset \ (i \neq j), \quad |D_i - 2x_i| = \emptyset, \quad |D_i - x_i - y_k| = \emptyset, \quad |D_i - 2y_k| = \emptyset,$

$|D_i - y_k - y_l| = \emptyset \ (k \neq l).$

Now we want to show how Theorem III.14 can be used to prove the existence of the special surfaces of degree 8 by explicitly constructing a very ample linear system $|H|$. Let $x_1, \ldots, x_4$ be points in general position in $\mathbb{P}^2$, and blow them up. The linear system $\left| 5L - x_1 - 2 \sum_{j \geq 2} x_j \right|$ is 10-dimensional, its elements have arithmetic genus 3. Let Δ_1 be a general (and hence smooth) element of the 10-dimensional linear system $\left| 5L - x_1 - 2 \sum_{j \geq 2} x_j \right|$ on $\hat{\mathbb{P}}^2 = \mathbb{P}^2(x_1, \ldots, x_4)$. Note that the image of Δ_1 in $\mathbb{P}^2$ is the image of the canonical model of Δ_1 under a standard Cremona transformation. The linear system $\left| 2L - \sum_j x_j \right|$ cuts out a g_3^1 on Δ_1, since $H^1(\hat{\mathbb{P}}^2, \mathcal{O}_{\hat{\mathbb{P}}^2}(-3L + \sum_{j \geq 2} x_j)) = 0$.

The linear system

$$|L_0| := \left| \left(6L - 3x_1 - 2 \sum_{j \geq 2} x_j\right) \Big|_{\Delta_1} - g_3^1 \right| = \left| \left(4L - 2x_1 - \sum_{j \geq 2} x_j\right) \Big|_{\Delta_1} \right|$$

on Δ_1 has degree 12 and dimension 9. The linear system $\left| 4L - 2x_1 - \sum_{j \geq 2} x_j \right|$ on $\hat{\mathbb{P}}^2$ cuts out a subsystem of codimension 1 in $|L_0|$. We consider the variety

$$\mathcal{M} := \{(\Delta_1, \textstyle\sum y_k); \ \Delta_1 \text{ smooth}, \textstyle\sum y_k \in |L_0|\}.$$

$\mathcal{M}$ is rational of dimension 19.

THEOREM III.17. – *There is a non-empty open set $\mathcal{U}$ of the rational variety $\mathcal{M}$ for which the linear system $|H|$ embeds S into $\mathbb{P}^4$.*

PROOF. – We have to show that for a general choice of Δ_1 and $\sum y_k \in |L_0|$ the linear system $|H|$ embeds S into $\mathbb{P}^4$. We shall first treat the closed conditions. Since Δ_1 is smooth we can identify it with its strict transform on S. Consider the exact sequence

$$0 \to \mathcal{O}_S(L - 2x_1) \to \mathcal{O}_S(D_1) \to \mathcal{O}_{\Delta_1}(D_1) \to 0.$$

Since $\sum y_k \in |L_0|$ we have

(35) $$6L - 3x_1 - 2 \sum_{j \geq 2} x_j - \sum y_k \equiv g_3^1 \text{ on } \Delta_1$$

and hence $h^0(\mathcal{O}_S(D_1)) = h^0(\mathcal{O}_{\Delta_1}(D_1)) = 2$. This is condition ($\mathrm{D}_1$). Condition ($\Delta_1$) holds by construction.

In order to treat the open conditions we will first consider special points in $\mathfrak{M}$ which give us all open conditions but two. These we will then treat afterwards. The linear system $\left|4L - 2x_1 - \sum_{j \geq 2} x_j\right|$ is free on $\hat{\mathbb{P}}^2$. Hence a general element Γ is smooth and intersects Δ_1 transversally in 12 points y_k which neither lie on an exceptional line, nor on a line of the form $\Lambda_{kl} = L - x_k - x_l$. Moreover a general element Γ is irreducible. This follows since $\Gamma^2 = 9$ and $|\Gamma|$ is not composed of a pencil, since the class of Γ is not divisible by 3 on $\hat{\mathbb{P}}^2$. Let Γ' be the smooth transform of Γ on S. Since Γ is smooth, Γ' is isomorphic to Γ.

CLAIM. – $|D_1| = \Gamma' + \left|2L - \sum_j x_j\right|$.

This follows immediately since $D_1 \equiv \Gamma' + \left(2L - \sum_j x_j\right)$ and $\dim|D_1| = 1 = \dim(\Gamma' + |2L - \sum x_j|)$.

The only curves contained in an element of $|D_1|$ are Γ', conics $C \equiv 2L - \sum x_j$ and lines $\Lambda_{kl} = L - x_k - x_l$. The latter only happens for finitely many elements of $|D_1|$. This shows immediately that conditions (Q) and (R) are fulfilled with the possible exception that $\dim|H - \Lambda_{1j}| \geq 2$. To exclude this we consider w.l.o.g. the case $j = 2$. Note that $H - \Lambda_{12} \equiv \Delta_2 + x_1 \equiv \Gamma' + \Lambda_{34} + x_1$. Since Γ' is smooth of genus 2 and $\Gamma'.(\Delta_2 + x_1) = 1$ it follows that $h^0(\mathcal{O}_{\Gamma'}(\Delta_2 + x_1)) \leq 1$. The claim now follows from the exact sequence

$$0 \to \mathcal{O}_S(\Lambda_{34} + x_1) \to \mathcal{O}_S(\Delta_2 + x_1) \to \mathcal{O}_{\Gamma'}(\Delta_2 + x_1) \to 0$$

together with the fact that $h^0(\mathcal{O}_S(\Lambda_{34} + x_1)) = 1$. It remains to consider (P). The curve Γ' contradicts condition (P) since $p(\Gamma') = 2$, $H.\Gamma' = 4$. Similarly the decomposition $(\Gamma' + \Lambda_{ij}) + \Lambda_{kl}$ contradicts (P) if $k, l \neq 1$. On the other hand the above construction shows that for one (and hence the general) pair $(\Delta_k, \sum y_k)$ all open conditions given by (P) are fulfilled for a decomposition $D = A + B$ of an element in $|D_1|$ with the possible exception of $|\Gamma'| \neq \emptyset$ or $|D_1 - \Lambda_{kl}| \neq \emptyset$ for $k, l \neq 1$. The first case is easy, we can simply take an element $\sum y_k \in |L_0|$ which is not in the codimension 1 linear subsystem given by $\left|4L - 2x_1 - \sum_{j \geq 2} x_j\right|$ on $\hat{\mathbb{P}}^2$. Next we assume that there is an element $A \in |D_1 - \Lambda_{kl}|$ where $k, l \neq 1$. Then $A.\Delta_1 = 2$. Since Δ_1 cannot be a component of A this means that A intersects Δ_1 in two points Q_0, Q_1. If j is the remaining element of the set $\{1, \ldots, 4\}$ then $L - x_1 - x_j \equiv Q_0 + Q_1$ on Δ_1. The linear system $|L|$ cuts out a g_5^2 on Δ_1 and is hence complete. Hence $Q_0 + Q_1$ is the intersection of Λ_{1j} with Δ_1. In particular Λ_{1j} intersects A in at least 2 points, namely Q_0 and Q_1. Since $A.\Lambda_{1j} = 0$ this implies that Λ_{1j} is a component of A (we can assume that Λ_{1j} is irreducible). Hence $A = A' + \Lambda_{1j}$ with $A' \in |D_1 - \Lambda_{kl} - \Lambda_{1j}| = |\Gamma'|$ and we are reduced to the previous case. ■

REMARKS III.18. – (i) Originally OKONEK [O2] cosntructed surfaces of degree 8 and sectional genus 6 with the help of reflexive sheaves.

(ii) According to [DES] the rational surfaces of degree 8 with $\pi = 6$ arise as the locus where a general morphism $\varphi\colon \Omega^3(3) \to \mathcal{O}(1) \oplus 4\mathcal{O}$ drops rank by 1. The space of such maps has dimension 80. Taking the obvious group actions into account we find that the moduli space has dimension $43 = 19 + \dim \operatorname{Aut} \mathbb{P}^4$. Moreover this description shows that the moduli space is irreducible and unirational.

(iii) These surfaces are in $(3, 4)$-liaison with the Veronese surface [O2]. Counting parameters one finds again that they depend on 19 parameters (modulo $\operatorname{Aut}(\mathbb{P}^4)$).

(iv) It was pointed out to us by K. RANESTAD that ELLINGSRUD and PESKINE (unpublished) also suggested a construction of these surfaces via linear systems. They start with a smooth quartic $K_4 = \{f_4 = 0\}$ and a smooth quintic $K_5 = \{f_5 = 0\}$ touching in 4 points $x_1, \ldots, x_4$. Let $y_5, \ldots, y_{16}$ be the remaining points of intersection. Let

$$\mathfrak{J}' = \mathcal{O}_{\mathbb{P}^2}\left(-\sum x_i\right), \qquad \mathfrak{J} = \mathcal{O}_{\mathbb{P}^2}\left(-2\sum x_i - \sum y_k\right).$$

Then we have an exact sequence

$$0 \to \mathfrak{J}'(-4) \to \mathfrak{J} \to \mathcal{O}_{K_4}(-5) \to 0\,.$$

Twisting this by $\mathcal{O}(6)$ and taking global section gives

$$0 \to \Gamma(\mathfrak{J}'(2)) \to \Gamma(\mathcal{O}_S(H)) \to \Gamma(\mathcal{O}_{K_4}(1)) \to 0\,.$$

Since $h^0(\mathfrak{J}'(2)) = 2$ and $h^0(\mathcal{O}_{K_4}(1)) = 3$ this shows $h^0(\mathcal{O}_S(H)) = 5$. One can easily see that $|\Delta_i| \neq \emptyset$ and $\dim |D_i| \geqslant 1$ in this construction: counting parameters one shows that $\Delta_i = \{lf_4 + f_5 = 0\}$ for some suitable linear form and that there is at least a 1-dimensional family of curves in $|D_i|$ which are of the form $D = \{qf_4 + lf_5\}$ where q is of degree 2 and l is a linear form. This construction, too, depends on 19 parameters.

Finally we want to discuss the moduli space of smooth special surfaces of degree 8 in $\mathbb{P}^4$ (modulo $\operatorname{Aut}(\mathbb{P}^4)$). Recall the set $\mathfrak{M}$ consisting of pairs $(\Delta_1, \sum y_k)$ where $\Delta_1 \in |H - (L - x_1)|$ is smooth and $\sum y_k \in |L_0|$. We have proved in Theorem III.17 that for a general pair $(\Delta_1, \sum y_k)$ the linear system $|H|$ embeds S into $\mathbb{P}^4$. Indeed in this way we obtain the general smooth surface of degree 8 in $\mathbb{P}^4$. The surface $X = \widehat{\mathbb{P}}^2$, i.e. $\mathbb{P}^2$ blown up in $x_1, \ldots, x_4$ is the del Pezzo surface of degree 5. It is well known that $\operatorname{Aut} X \cong S_5$ the symmetric group in 5 letters ($\operatorname{Aut} X$ acts transitively on the 5 maximal sets of disjoint rational curves on X, see [M, Chapter IV]).

PROPOSITION III.19. – *For general S the only lines contained in S are the y_k's.*

PROOF. – Let l be a line on S. The statement is clear if l is π-exceptional as the x_i are mapped to conics and since we can assume that there are no infinitely near points. If l is not skew to the plane spanned by x_i then l is contained in a reducible member of $|D_i|$. But for general choice there is no decomposition $A + B$ with A (or B) a line.

Hence we can assume that $l.x_i = 0$ for $i = 1, \ldots, 4$ and $l.y_k \leq 1$ for all k. Thus $l \equiv aL - \sum_{k \in \Delta} y_k$ with $a \leq 2$. Since $H.l = 1$ we have either $a = 1$ and $|\Delta| = 5$ or $a = 2$ and $|\Delta| = 11$. In the first case 5 of the y_k are collinear. But then it follows with the monodromy argument of [ACGH, p. 111] that all the y_k's are collinear which is absurd. In the same way the case $a = 2$ would imply that all the y_k's are on a conic which also contradicts very ampleness of $|H|$.

THEOREM III.20. – *The moduli space of polarized rational surfaces* (S, H) *where* $|H|$ *embeds* S *into* $\mathbb{P}^4$ *as a surface of degree* 8, *speciality* 1 *and sectional genus* 6 *is birationally equivalent to* $\mathcal{M}/S_5$.

PROOF. – Let $\mathcal{V}$ be the open set of $\mathcal{M}$ where $|H|$ embeds S into $\mathbb{P}^4$ and where all the Δ_i's are smooth. Let $(\Delta_1, \sum y_k)$ and $(\Delta_1', \sum y_k')$ be two elements which give rise to surfaces $S, S' \subset \mathbb{P}^4$ for which a projective transformation $\bar{g}: S \to S'$ exists. Since obviously $\bar{g}$ carries lines to lines, it follows from Proposition III.19 that $\bar{g}$ is induced by an automorphism $g: X \to X$ carrying the set $\{y_k\}$ to $\{y_k'\}$. Conversely, the group $S_5 = \mathrm{Aut}(X)$ acts on $\mathcal{V}$ as follows. Let S correspond to $(\Delta_1, \sum y_k)$ and let $g \in \mathrm{Aut}(X)$: then, since $6L - 2\sum x_j = -2K_X$ which is invariant under the action of S_5, we set $\{y_k'\} = g\{y_k\}$, $H' = -2K_X - \sum y_k'$. Then H' embeds $S' = X(y_1', \ldots, y_{12}')$ and we set Δ_1' to be the unique curve in $|H' - L + y_1|$. ∎

IV. - Further outlook.

In this section we want to discuss how this method can possibly be applied to other surfaces. For smooth surfaces of degree ≤ 8 it is rather straightforward to give a decomposition $H \equiv C + D$ which allows to apply the Alexander-Bauer lemma. This was done in [B], [CF] and Section III of this article. In degree 9 there is one non-special surface, which was treated in Section II of this article, and a special surface with sectional genus $\pi = 7$ which was found by ALEXANDER [A2]. Here S is $\mathbb{P}^2$ blown up in 15 points $x_1, \ldots, x_{15}$ and $H \equiv 9L - 3\sum_{i=1}^{6} x_i - 2\sum_{j=7}^{9} x_j - \sum_{k=10}^{15} x_k$. As pointed out by Alexander one can take the decomposition $H \equiv C + D$ where $C \equiv 3L - \sum_{i=1}^{9} x_i$ and $D \equiv H - C$. Then C is a plane cubic and $|D|$ is a pencil of canonical curves of genus 4.

Rational surfaces of degree 10 were treated by RANESTAD [R1], [R2], POPESCU and RANESTAD [PR] and ALEXANDER [A2]. There is one surface with $\pi = 8$. In this case S is $\mathbb{P}^2$ blown up in 13 points and $H \equiv 14L - 6x_1 - 4\sum_{i=2}^{10} x_i - 2x_{11} - x_{12} - x_{13}$. Following ALEXANDER [A2] the curve $C \equiv 7L - 3x_1 - 2\sum_{i=2}^{10} x_i - \sum_{j=11}^{13} x_j$ is a plane quartic and the residual pencil $|D|$ has $p(D) = 3$ and degree 6. For sectional genus $\pi = 9$ there are two possibilities. The first is $\mathbb{P}^2$ blown in 18 points with $H \equiv 8L - 2\sum_{i=1}^{12} x_i - \sum_{j=13}^{18} x_j$. One the can take $C \equiv 4L - \sum_{i=1}^{16} x_i$ which becomes a plane quartic. For the residual intersection $|D|$ one finds $p(D) = 3$, $H.D = 6$. (For more details of this geometrically interesting situation see [PR, Proposition 2.2]). The second surface with $\pi = 9$ is more difficult. Again we have $\mathbb{P}^2$ blown up in 18 points, but this time $H \equiv 9L - 3\sum_{i=1}^{4} x_i - 2\sum_{j=5}^{11} x_j - \sum_{k=12}^{18} x_k$. Clearly S contains plane curves, e.g. the conics x_j. But then for the residual pencil $|D|$ one has $p(D) = 7$, $H.D = 9$ and this case seems difficult to handle. Numerically it would be possible to have a decomposition $H \equiv C + D$ with $C \equiv 3L - \sum_{i=1}^{3} x_i - \sum_{j=5}^{11} x_j - x_{12}$ which would be a plane cubic. In this case $p(D) = 4$, $H.D = 6$. It might be interesting to check whether one can actually construct surfaces with such a decomposition.

Of course, one can try and attempt to approach the problem of finding suitable decompositions $H \equiv C + D$ more systematically. Let us assume S is a rational surface and $H \equiv C + D$ a decomposition to which the Alexander-Bauer lemma can be applied. Let $h = h^1(\mathcal{O}_S(H))$ be the speciality of S. Since C is mapped to a plane curve the exact sequence

$$0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H) \to 0$$

is exact on global sections, and hence

$$h = h^1(D) + \delta(C)$$

where $h^1(D) = h^1(\mathcal{O}_S(D))$ and $\delta(C) = h^1(\mathcal{O}_C(H))$. The analogous sequence for D and the assumption that $|H|$ restricts to a complete system on the curves $D' \in |D|$ gives

$$h = h^1(C) + \delta(D)$$

where $h^1(C)$ and $\delta(D)$ are defined similarly. In general if C is a curve of genus $(d-1)(d-2)/2$ and $\mathcal{O}_C(H)$ is a line bundle of degree d it is difficult to show that $(C, \mathcal{O}_C(H))$ is a plane curve. Hence it is natural to assume $H.C \leq 4$. In order to be able to control the linear system $|H|$ on the curves $D' \in |D|$ one is normally forced to assume that $H.D \geq 2p(D) - 2$ and $H|_D = K_D$ in case of equality. Hence $\delta(D) = 0$ if $H.D > 2p(D) - 2$ and $\delta(D) = 1$ otherwise. Since $|H|$ is complete on D we have $h^0(\mathcal{O}_D(H)) \leq 4$. Now using our assumption that $H.D \geq 2p(D) - 2$ and Riemann-Roch on D we find

$$2p(D) - 2 \leq H.D \leq p(D) + 3 + \delta(D)$$

and from this

$$p(D) \leq 5 + \delta(D).$$

If $\delta(D) = 0$ then $p(D) \leq 5$. If $\delta(D) = 1$ then $H|_D = K_D$ and $h^0(\mathcal{O}_D(H)) = p(D)$, i.e. $p(D) \leq 4$ in this case. But now

$$d = H.C + H.D \leq p(D) + 7 + \delta(D).$$

This shows that one can find such a decomposition only if the degree $d \leqslant 12$. The case $d = 12$ can only occur for $H.C = 4$.

Finally we want to discuss the case $d = 11$. In his thesis POPESCU [P] gave three examples of rational surfaces of degree 11. In each case it is $\mathbb{P}^2$ blown up in 20 points. The linear systems are as follows:

$$H \equiv 10L - 4x_1 - 3\sum_{i=2}^{4} x_i - 2\sum_{j=5}^{14} x_j - \sum_{k=15}^{20} x_k, \tag{35}$$

$$H \equiv 11L - 5x_1 - 3\sum_{i=2}^{7} x_i - 2\sum_{j=8}^{13} x_j - \sum_{k=14}^{20} x_k, \tag{36}$$

$$H \equiv 13L - 5x_1 - 4\sum_{i=2}^{8} x_i - 2\sum_{j=9}^{11} x_j - \sum_{k=12}^{20} x_k. \tag{37}$$

In each of these cases S contains a plane quintic. The residual intersection gives a pencil of rational (cases (35) and (36)), resp. elliptic (case (37)) sextics. Since the linear system $|H|$ is not complete on the curves of this linear system, one cannot immediately apply the Alexander-Bauer lemma to this decomposition. One can ask whether there are decompositions fulfilling the conditions given above. A candidate in case (35) is given by $C \equiv 4L - x_1 - \sum_{i=2}^{4} x_i - \sum_{j=5}^{14} x_j - \sum_{k=15}^{17} x_k$ and $D \equiv H - C$. We do not know whether surfaces with such a decomposition actually occur. In the other cases one can show that no such decompositions exist.

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