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EVEN SETS OF NODES ARE BUNDLE SYMMETRIC

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0. Introduction

Let k be an algebraically closed field of characteristic $p \neq 2$, and let $F := \{f = 0\} \subseteq \mathbb{P}^3_k$ be a normal surface of degree d. Let $\pi\colon \widetilde{F} \to F$ be a minimal resolution of singularities. We denote by $H \subseteq F$ a general plane section of F defined by a general linear form h. Assume, for simplicity, that F is a nodal surface (i.e., its singularities are only ordinary quadratic, nodes for short).

Let Δ be a subset of the set of nodes of F, and let $\widetilde{\Delta} := \pi^{-1}(\Delta)$. Δ is said to be a $\delta/2$*–even set of nodes*, $\delta = 0, 1$, if the class of $\widetilde{\Delta} + \delta\pi^*H$ in $\mathrm{Pic}(\widetilde{F})$ is 2–divisible (when $\delta = 0$ we shall simply say that Δ is even).

The condition that Δ is $\delta/2$–even is equivalent to the existence of a double cover $\widetilde{p}\colon \widetilde{S} \to \widetilde{F}$ branched exactly along $\widetilde{\Delta} + \delta\pi^*H$ and (cf. [6, 2.11, 2.13]) it is possible to blow down $\widetilde{p}^{-1}(\widetilde{\Delta})$ getting a commutative diagram

$$(0.1) \qquad \begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{\pi}} & S \\ \downarrow{\scriptstyle \widetilde{p}} & & \downarrow{\scriptstyle p} \\ \widetilde{F} & \xrightarrow{\pi} & F \end{array}$$

where S is a nodal surface and p is finite of degree 2 branched exactly on Δ when $\delta = 0$ (respectively on Δ and H when $\delta = 1$; in this case d has to be even). The surface S is then endowed with a natural involution i such that $F \cong S/i$ and p is the quotient map. Thus we have an $\mathcal{O}_F$–linear map $i^{\#}\colon p_*\mathcal{O}_S \to p_*\mathcal{O}_S$ giving rise to a splitting of $\mathcal{O}_F$–modules $p_*\mathcal{O}_S \cong \mathcal{O}_F \oplus \mathcal{F}$ where $\mathcal{O}_F$ and $\mathcal{F}$ are the $+1$ and -1 eigenspaces of $i^{\#}$.

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The sheaf $\mathcal{F}$ is not locally free because of the nodes, but we shall see later in Section 3 that it is reflexive and Cohen–Macaulay. Moreover the multiplication map $\mathcal{O}_S \times \mathcal{O}_S \to \mathcal{O}_S$ induces a non-degenerate pairing $\mathcal{F} \times \mathcal{F} \to \mathcal{O}_S(-\delta)$. Therefore $\mathcal{F}$ is a $\delta/2$-quadratic sheaf in the sense of the following definition.

Definition 0.2. Let X be a locally Cohen–Macaulay projective scheme. We say that a reflexive, coherent, locally Cohen–Macaulay $\mathcal{O}_X$-sheaf $\mathcal{F}$ is a $\delta/2$-quadratic sheaf on X, $\delta \in \mathbb{Z}$, if there exists a symmetric isomorphism $\sigma\colon \mathcal{F}(\delta) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

The aim of Sections 1 and 2 is to prove in dimension 3 the following characterization of quadratic sheaves on hypersurfaces in projective space.

Theorem 0.3. *Let $F \subseteq \mathbb{P}^3_k$ be a surface of degree d, and let $\mathcal{F}$ be a $\delta/2$-quadratic sheaf on F. Then $\mathcal{F}$ fits into an exact sequence of the form*

$$0 \to \check{\mathcal{E}}(-d-\delta) \xrightarrow{\varphi} \mathcal{E} \to \mathcal{F} \to 0, \tag{0.3.1}$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathbb{P}^3_k}$-sheaf and φ is a symmetric map.

An entirely analogous proof with more complicated notation gives the same result in all dimensions.

Theorem 0.3 and the above discussion yield the following.

Corollary 0.4. *Let $F \subseteq \mathbb{P}^3_k$ be a nodal surface of degree d. Then every $\delta/2$-even set of nodes Δ on F, $\delta = 0, 1$, is the degeneracy locus of a symmetric map of locally free $\mathcal{O}_{\mathbb{P}^3_k}$-sheaves $\check{\mathcal{E}}(-d-\delta) \xrightarrow{\varphi} \mathcal{E}$ (i.e., F is the locus where $\operatorname{rk}(\varphi) \leq \operatorname{rk}\mathcal{E} - 1$, Δ is the locus where $\operatorname{rk}(\varphi) = \operatorname{rk}\mathcal{E} - 2$).*

In the above setting we say that Δ is a *bundle-symmetric* set of nodes. If it is possible to find such an $\mathcal{E}$ which is the direct sum of invertible $\mathcal{O}_{\mathbb{P}^3_k}$-sheaves, then we say that Δ is a *symmetric* set of nodes (see [6]).

Corollary 0.4 was conjectured in 1979 independently by W. Barth and the second author. Barth proved in [1] that bundle-symmetric sets are even while in [6] the converse result was proved under a cohomological assumption which gives a complete characterization of symmetric sets of nodes.

As soon as Walter's beautiful solution of Okonek's conjecture came out, it was immediately clear that his method would also work in our case.

Finally, in Section 3 we apply Corollary 0.4 to the study of nodal surfaces of low degree d, namely $d = 4, 5, 6$. This study ties up to an interesting history for which we defer the reader e.g. to [8], [6], [4], [7], [15], [2], [12]. In particular we get the following result.

Theorem 0.5. *Let $F \subseteq \mathbb{P}^3_k$ be a nodal surface of degree* 6. *Then every even set of nodes Δ on F has cardinality either* 24 *or* 32 *or* 40.

Using the above result, J. Wahl (see [17]) was able to give a simple proof of the result of D. Jaffe and D. Ruberman stating that a nodal surface of degree 6 can have at most 65 nodes.

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1. A resolution of $\mathcal{F}$

In this section we deal with the construction of a resolution $\mathcal{R}^*$ of any $\delta/2$–quadratic sheaf $\mathcal{F}$ on a surface $F \subseteq \mathbb{P}^3_k$ of degree d. We shall make use of notation and results proved in [18] about Horrocks' correspondence.

Lemma 1.1. *Let $F \subseteq \mathbb{P}^3_k$ be a surface and let $\mathcal{F}$ be a $\delta/2$–quadratic sheaf on F. Then $\mathrm{pd}_{\mathcal{O}_{\mathbb{P}^3_k,x}} \mathcal{F}_x = 1$ for each $x \in F$ and $\mathcal{E}xt^1_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F) =$ 0.*

Proof. By definition $\operatorname{depth} \mathcal{F}_x = \dim \mathcal{F}_x = 2$. Then the equality $\mathrm{pd}_{\mathcal{O}_{\mathbb{P}^3_k,x}} \mathcal{F}_x = 1$ follows from Auslander–Buchsbaum formula taking account that the depths of $\mathcal{F}_x$ as $\mathcal{O}_{F,x}$–module and as $\mathcal{O}_{\mathbb{P}^3_k,x}$–module coincide. $\mathcal{E}xt^1_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F) = 0$ follows from [10, Theorem 6.1]. q.e.d.

Let $d + \delta$ be even. From the spectral sequence of the Ext's, Lemma 1.1 and Serre's duality, follows the existence of isomorphisms for

$i = 1, 2$:

$$
\begin{aligned}
H^i(F, \mathcal{F}(m)) &\cong \operatorname{Ext}^{2-i}_{\mathcal{O}_F}(\mathcal{F}(m), \omega_{F|k})\check{} \\
&\cong H^{2-i}(F, \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(d-4-m)))\check{} \\
&\cong H^{2-i}(F, \mathcal{F}(d-4-m+\delta))\check{}.
\end{aligned}
\tag{1.2}
$$

In particular, if $m := (d-4+\delta)/2$ is an integer, there exists a non–degenerate alternating form

$$\Phi \colon H^1(F, \mathcal{F}((d-4+\delta)/2)) \times H^1(F, \mathcal{F}((d-4+\delta)/2)) \to H^2(F, \omega_{F|k}),$$

and we denote by U a fixed maximal isotropic subspace with respect to Φ.

Define

$$W := \begin{cases} \bigoplus_{m<(d-4+\delta)/2} H^1(F, \mathcal{F}(m)) & \text{if } d+\delta \text{ is odd,} \\ \bigoplus_{m<(d-4+\delta)/2} H^1(F, \mathcal{F}(m)) \oplus U & \text{if } d+\delta \text{ is even.} \end{cases}$$

As usual H^i_* is the Serre functor associating to a quasi–coherent sheaf $\mathcal{G}$ the graded module $H^i_*(\mathbb{P}^3_k, \mathcal{G}) := \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^3_k, \mathcal{G}(n))$. Let $\Gamma_* := H^0_*$, and let $\mathbf{R}\Gamma_*$ be its right derived functor in the derived category. Then $H^i_*(\mathbb{P}^3_k, \mathcal{G})$ is the i–th cohomology module of the complex $\mathbf{R}\Gamma_*(\mathcal{G})$.

As in [18, Section 2] one considers the truncation. Let D^* be a complex with differentials $\delta^i \colon D^i \to D^{i+1}$, let $r, s \in \mathbb{Z}$, $r < s$ and let $W \subseteq H^s(D^*)$ be a subspace. Then W may be pulled back to $\overline{W}$ satisfying $\operatorname{im}(\delta^{s-1}) \subseteq \overline{W} \subseteq \ker(\delta^s)$. We denote by $\tau_{>r}\tau_{\le s,W}(D^*)$ (if $W = H^s(D^*)$ we will omit it in the subscripts) the complex C^* defined as follows:

$$C^i := \begin{cases} 0 & \text{if } i \le r-1 \text{ or } i \ge s+1, \\ D^i & \text{if } r+1 \le i \le s-1, \\ D^r/\ker(\delta^r) & \text{if } i = r, \\ \overline{W} & \text{if } i = s. \end{cases}$$

Let us now consider the truncated complex $C^* := \tau_{>0}\tau_{\le 1,W}\mathbf{R}\Gamma_*(\mathcal{F})$. By definition we have

$$H^i(C^*) = \begin{cases} W & \text{if } i = 1, \\ 0 & \text{elsewhere,} \end{cases}$$

and there is a natural map

$$\beta\colon C^* \to \tau_{>0}\tau_{\leq 2}\mathbf{R}\Gamma_*(\mathcal{F}).$$

Since $H^1_*(F,\mathcal{F})$, hence W, has finite length we can apply to the complex C^* the syzygy bundle functor ([18, Theorem 0.4]. Cf. also the construction given after Corollary 2.8). We obtain a locally free sheaf $\mathcal{S}yz(C^*)$ and a morphism of quasi-coherent sheaves

$$\widetilde{\beta}\colon \mathcal{S}yz(C^*) \to \mathcal{F}$$

such that $\beta = \tau_{>0}\tau_{\leq 2}\mathbf{R}\Gamma_*(\widetilde{\beta})$ ([18, Proposition 2.10]).

Let $Q := \operatorname{coker}\left(H^0_*(\widetilde{\beta})\right)$. This means that we have an exact sequence of the form

$$H^0_*(\mathbb{P}^3_k, \mathcal{S}yz(C^*)) \xrightarrow{H^0_*(\widetilde{\beta})} H^0_*(F,\mathcal{F}) \to Q \to 0.$$

Let $d_1,\ldots,d_r$ be the degrees of a minimal set of generators of Q. These generators lift to $H^0_*(F,\mathcal{F})$, allowing us to define an epimorphism

$$\gamma\colon \mathcal{E} := \mathcal{S}yz(C^*) \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3_k}(-d_i) \twoheadrightarrow \mathcal{F}$$

which is surjective on global sections. By construction $\mathcal{E}$ is locally free.

If $\mathcal{K} := \ker(\gamma)$, then we have an exact sequence

$$\text{(1.3)} \qquad \mathcal{R}^*: \quad 0 \to \mathcal{K} \xrightarrow{d} \mathcal{E} \xrightarrow{\gamma} \mathcal{F} \to 0.$$

Proposition 1.4. *In the above sequence (1.3) $\mathcal{K}$ is locally free and* $\operatorname{rk}\mathcal{K} = \operatorname{rk}\mathcal{E}$.

Proof. We know from Lemma 1.1 that for each $x \in \mathbb{P}^3_k$ one has an exact sequence of the form

$$0 \to \mathcal{K}'_x \to \mathcal{E}'_x \to \mathcal{F}_x \to 0,$$

where $\mathcal{K}'_x$ and $\mathcal{E}'_x$ are free (and depend upon $x \in \mathbb{P}^3_k$). Moreover, since $\mathcal{F}$ is supported on F,

$$\operatorname{ann}_{\mathcal{O}_{\mathbb{P}^3_k,x}} \mathcal{F}_x \neq 0.$$

Therefore $\operatorname{rk}\mathcal{K}'_x = \operatorname{rk}\mathcal{E}'_x$ (see [13, Theorem 195]). The statement now follows from [13, Theorem A, Chapter 4] (Schanuel's lemma). q.e.d.

Proposition 2.10 of [18] implies for $i = 1, 2$ that

$$H^i(C^*) = H^i_*(\mathbb{P}^3_k, \mathcal{S}yz(C^*)) = H^i_*(\mathbb{P}^3_k, \mathcal{E}),$$

and that $H^i_*(\gamma)$ coincides with $H^i_*(\widetilde{\beta})$. Thus the above construction yields:

i) $H^0_*(\gamma)\colon H^0_*(\mathbb{P}^3_k, \mathcal{E}) \to H^0_*(F, \mathcal{F})$ is surjective by construction;

ii) $H^1_*(\gamma)\colon H^1_*(\mathbb{P}^3_k, \mathcal{E}) \to H^1_*(F, \mathcal{F})$ is injective since

$$H^1_*(\mathbb{P}^3_k, \mathcal{E}) = H^1(C^*) = W$$

(in particular $H^1(\mathbb{P}^3_k, \mathcal{E}((d-4+\delta)/2)) = U$);

iii) $H^2_*(\gamma)\colon H^2_*(\mathbb{P}^3_k, \mathcal{E}) \to H^2_*(F, \mathcal{F})$ is zero, in fact $H^2_*(\mathbb{P}^3_k, \mathcal{E}) = 0$.

From the above remarks taking the cohomology of the sequence (1.3) we then get:

iv) $H^1_*(\mathbb{P}^3_k, \mathcal{K}) = 0$;

v) $H^2_*(\mathbb{P}^3_k, \mathcal{K}) \cong H^1_*(F, \mathcal{F})/W$.

Recall that $\mathcal{F}(\delta) \cong \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F)$. On the other hand one has an exact sequence

$$\text{(1.5)} \qquad C^*: \quad 0 \to \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta) \xrightarrow{\lambda(f)} \mathcal{O}_{\mathbb{P}^3_k}(-\delta) \to \mathcal{O}_F(-\delta) \to 0.$$

Applying $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \cdot)$ to sequence (1.5) gives

$$0 \to \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(-\delta)) \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)) \xrightarrow{f} \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3_k}(-\delta)),$$

and since the multiplication by f is zero on $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta))$ we obtain an isomorphism

$$s_0\colon \mathcal{F} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(-\delta)) \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)).$$

Proposition 1.6. *Let s_0 be as above. If there is a commutative diagram*

$$\begin{array}{ccccccc} 0\to & \mathcal{K} & \xrightarrow{d} & \mathcal{E} & \xrightarrow{\gamma} & \mathcal{F} & \to 0 \\ & \downarrow{s_2} & & \downarrow{s_1} & & \downarrow{s_0} & \\ 0\to & \check{\mathcal{E}}(-d-\delta) & \xrightarrow{\check{d}} & \check{\mathcal{K}}(-d-\delta) & \xrightarrow{\overline{\gamma}} & \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)) & \to 0, \end{array} \tag{1.6.1}$$

then the maps s_i are isomorphisms.

Proof. By construction s_0 is an isomorphism, thus it suffices to prove the bijectivity of s_1. This will follow from Lemma 2.12 of [18] if we show that conditions (i), (ii) and (iii) of that lemma are satisfied for the composition

$$\mathcal{E} \xrightarrow{s_1} \check{\mathcal{K}}(-d-\delta) \xrightarrow{\overline{\gamma}} \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)).$$

Condition (iii) that $\mathcal{E}$ and $\check{\mathcal{K}}(-d-\delta)$ have the same rank was already shown in Proposition 1.4.

Since $\overline{\gamma} \circ s_1$ coincides with the map $\gamma\colon \mathcal{E} \twoheadrightarrow \mathcal{F}$, condition (ii) holds by the very definition of $\mathcal{E}$.

We have to verify (i) namely that

$$H^i_*(s_1)\colon H^i_*(\mathbb{P}^3_k, \mathcal{E}) \to H^i_*(\mathbb{P}^3_k, \check{\mathcal{K}}(-d-\delta))$$

are isomorphisms for $i = 1, 2$. Diagram (1.6.1) yields the equality

$$H^i_*(\overline{\gamma}) \circ H^i_*(s_1) = H^i_*(s_0) \circ H^i_*(\gamma).$$

Note that $H^i_*(s_0)$ are isomorphisms and the maps $H^i_*(\gamma)$ are injective, hence the same is true for the maps $H^i_*(s_1)$.

Since $\mathcal{K}$ and $\mathcal{E}$ are locally free, both $H^i_*(\mathbb{P}^3_k, \check{\mathcal{K}}(-d-\delta))$ and $H^i_*(\mathbb{P}^3_k, \mathcal{E})$ have finite length, thus we have only to prove that their lengths coincide.

We begin with $i = 2$. Here both modules are 0:

$$h^2(\mathbb{P}^3_k, \check{\mathcal{K}}(t)) = h^1(\mathbb{P}^3_k, \mathcal{K}(-4-t)) = 0$$

by iv) while $h^2(\mathbb{P}^3_k, \mathcal{E}(t)) = 0$ (cf. iii)).

Let now $i = 1$. One has

$$\begin{aligned} h^1(\mathbb{P}^3_k, \check{\mathcal{K}}(t)) &= \dim\big(H^2(\mathbb{P}^3_k, \mathcal{K}(-4-t))\check{\ }\big) \\ &= \dim\big(H^1(F, \mathcal{F}(-4-t))/W_{-4-t}\big) \\ &\cong \begin{cases} 0 & \text{if } -4-t < (d-4+\delta)/2, \\ \dim(U) & \text{if } -4-t = (d-4+\delta)/2, \\ h^1(F, \mathcal{F}(-4-t)) & \text{if } -4-t > (d-4+\delta)/2. \end{cases} \end{aligned}$$

By (1.2), $h^1(\mathbb{P}^3_k, \mathcal{F}(-4-t)) = h^1(\mathbb{P}^3_k, \mathcal{F}(t+d+\delta))$ and so the desired conclusion follows. q.e.d.

2. Proof of Theorem 0.3

This section is devoted to the proof of the following.

Claim 2.1. It is possible to construct diagram (1.6.1) in such a way that s_2 is the transpose of s_1.

Assuming 2.1 we have the

Proof of Theorem 0.3. Just set $\varphi := s_1^{-1} \circ \check{d}$ which is obviously symmetric. q.e.d.

Proof of Claim 2.1. Our first step is to extend the natural map $\eta\colon \mathcal{S}^2\mathcal{F} \to \mathcal{O}_F(-\delta)$, induced by the symmetric map σ, to a chain map $\phi\colon \mathcal{S}^2\mathcal{R}^* \to \mathcal{C}^*$ (see sequences (1.3) and (1.5)):

$$\begin{array}{ccccccccc} 0 \to \Lambda^2\mathcal{K} & \xrightarrow{\delta_1} & \mathcal{K}\otimes\mathcal{E} & \xrightarrow{\delta_0} & \mathcal{S}^2\mathcal{E} & \to & \mathcal{S}^2\mathcal{F} & \to 0 \\ \downarrow{\scriptstyle\phi_2} & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_0} & & \downarrow & \\ 0 & \to & \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta) & \xrightarrow{\lambda(f)} & \mathcal{O}_{\mathbb{P}^3_k}(-\delta) & \to & \mathcal{O}_F(-\delta) & \to 0. \end{array}$$

Assume that ϕ does exist. Then we get a map $s_1\colon \mathcal{E} \to \check{\mathcal{K}}(-d-\delta)$. It is obtained from ϕ_1 through the natural isomorphism

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{K}\otimes\mathcal{E}, \mathcal{O}_{\mathbb{P}^3_k}(m)) \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3_k}}(\mathcal{E}, \check{\mathcal{K}}(m)).$$

Let s_2 be the transpose of s_1.

We claim that $s_1 \circ d = \check{d} \circ s_2$, i.e., that the above diagram (1.6.1)actually commutes. It suffices to verify this equality at every point $x \in \mathbb{P}^3_k$. Choose $\alpha, \beta \in \mathcal{K}_x$. Since $\langle s_1 \circ d(\alpha), \beta\rangle = \phi_1(d(\alpha)\otimes\beta)$ and $\langle \check{d}\circ s_2(\alpha), \beta\rangle = \langle \alpha, s_1 \circ d(\beta)\rangle = \phi_1(\alpha \otimes d(\beta))$, our claim follows from

$$0 = \phi_2(\alpha\wedge\beta) = \phi_1 \circ \delta_1(\alpha\wedge\beta) = \phi_1(\alpha\otimes d(\beta) - \beta\otimes d(\alpha)).$$

There remains only to prove the following proposition.

Proposition 2.2. *ϕ exists.*

Proof. In order to have ϕ it suffices to define ϕ_0. Indeed the image of $\phi_0 \circ \delta_0$ is contained in the kernel of $\mathcal{O}_{\mathbb{P}^3_k}(-\delta) \to \mathcal{O}_F(-\delta)$ which coincides with the submodule $f\mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)$.

Then we simply set $\phi_1 := \frac{1}{f}\phi_0 \circ \delta_0$.

We want to lift the composition

$$\psi := \eta \circ \mathcal{S}^2\gamma\colon\ \mathcal{S}^2\mathcal{E} \to \mathcal{S}^2\mathcal{F} \to \mathcal{O}_F(-\delta)$$

to a map $\phi_0\colon\ \mathcal{S}^2\mathcal{E} \to \mathcal{O}_{\mathbb{P}^3_k}(-\delta)$.

From the exact sequence (1.5) we obtain the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^3_k}}\big(\mathcal{S}^2\mathcal{E}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)\big) \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^3_k}}\big(\mathcal{S}^2\mathcal{E}, \mathcal{O}_{\mathbb{P}^3_k}(-\delta)\big) \\ &\to \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^3_k}}\big(\mathcal{S}^2\mathcal{E}, \mathcal{O}_F(-\delta)\big) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3_k}}\big(\mathcal{S}^2\mathcal{E}, \mathcal{O}_{\mathbb{P}^3_k}(-d-\delta)\big) \\ &\cong H^1\big(\mathbb{P}^3_k, \mathcal{S}^2\check{\mathcal{E}}(-d-\delta)\big) \cong H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E}(d-4+\delta)\big)\check{}. \end{aligned}$$

We conclude that ψ is liftable if and only if

$$\partial(\psi) \in H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E}(d-4+\delta)\big)\check{}$$

is the zero map. First of all notice that, interchanging the roles of Λ^2 and $\mathcal{S}^2$ in Section 4 of [18] and imitating word by word the proofs of Lemmas 4.1, 4.2 and Corollary 4.3 of [18] we easily obtain

$$H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E}(d-4+\delta)\big) \cong \begin{cases} 0 & \text{if } d+\delta \text{ is odd,} \\ \Lambda^2 H^1\big(\mathbb{P}^3_k, \mathcal{E}((d-4+\delta)/2)\big) & \text{if } d+\delta \text{ is even.} \end{cases}$$

$\partial(\psi) \in H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E}(d-4+\delta)\big)\check{}$ is identified with the map

$$\lrcorner(\cdot \smile \partial(\psi))\colon\ H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E}(d-4+\delta)\big) \to H^3\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4)\big),$$

where

$$\begin{aligned} \smile\colon\ H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E}(d-4+\delta)\big) \times H^1\big(\mathbb{P}^3_k, \mathcal{S}^2\check{\mathcal{E}}(-d-\delta)\big) \\ \to H^3\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E} \otimes \mathcal{S}^2\check{\mathcal{E}}(-4)\big) \end{aligned}$$

is the cup–product and

$$\lrcorner\colon\ H^3\big(\mathbb{P}^3_k, \mathcal{S}^2\mathcal{E} \otimes \mathcal{S}^2\check{\mathcal{E}}(-4)\big) \to H^3\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4)\big)$$

is the natural contraction.

Thus $\partial(\psi) = 0$ if and only if

$$\lrcorner(\alpha \smile \beta \smile \partial(\psi)) = 0 \tag{2.2.1}$$

in $H^3\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4)\big)$ for each $\alpha, \beta \in H^1\big(\mathbb{P}^3_k, \mathcal{E}((d-4+\delta)/2)\big)$.

We have a commutative diagram:

$$\begin{array}{ccc}
H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\,\mathcal{E}(d-4+\delta)\big) & = & \Lambda^2 H^1\big(\mathbb{P}^3_k, \mathcal{E}((d-4+\delta)/2)\big) \\
\big\downarrow{\scriptstyle H^2(\psi)} & & \big\downarrow{\scriptstyle \Lambda^2 H^1(\gamma)} \\
H^2\big(F, \mathcal{O}_F(d-4)\big) & \xleftarrow{\Phi} & \Lambda^2 H^1\big(F, \mathcal{F}((d-4+\delta)/2)\big) \\
\big\downarrow{\scriptstyle \partial'} & & \\
H^3\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4)\big) & &
\end{array}$$

We claim that

Lemma 2.3. $\lrcorner(\cdot \smile \partial(\psi)) = -\partial' \circ H^2(\psi)$.

The above assertion implies $\partial(\psi) = 0$ since, by formula 2.2.1, $\partial' \circ H^2(\psi)(\alpha \smile \beta) = 0$ because $\alpha, \beta \in H^1\big(\mathbb{P}^3_k, \mathcal{E}((d-4+\delta)/2)\big) = U$ which was chosen to be isotropic with respect to Φ.

Proof of Lemma 2.3. Let $\mathcal{U} := \{U_i\}_{i=0,\dots,3}$ be the standard open covering of $\mathbb{P}^3_k$. For each i we fix a lifting $\widehat{\psi}_i\colon\ \mathcal{S}^2\,\mathcal{E}_{|U_i} \to \mathcal{O}_{U_i}(-\delta)$ of $\psi_{|U_i}$. Notice that $\widehat{\psi}_i - \widehat{\psi}_j$ maps to $f\mathcal{O}_{\mathbb{P}^3_k}(-d-\delta) \subseteq \mathcal{O}_{\mathbb{P}^3_k}(-\delta)$. On the other hand,

$$\partial(\psi) \in H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\,\mathcal{E}(d-4+\delta)\big)^\vee \cong H^1\big(\mathbb{P}^3_k, \mathcal{S}^2\,\check{\mathcal{E}}(-d-\delta)\big)$$

represents the obstruction to lifting ψ to ϕ_0, whence

$$\partial(\psi) = \frac{1}{f}(\widehat{\psi}_i - \widehat{\psi}_j) \in H^1\big(\mathcal{U}, \mathcal{S}^2\,\check{\mathcal{E}}(-d-\delta)\big).$$

We now compute explicitly $\partial' \circ H^2(\psi)$ and $\lrcorner(\cdot \smile \partial(\psi))$ using the fact that each element inside $H^2\big(\mathbb{P}^3_k, \mathcal{S}^2\,\mathcal{E}(d-4+\delta)\big)$ can be written as a sum of terms $\alpha \smile \beta$ where $\alpha, \beta \in H^1\big(\mathcal{U}, \mathcal{E}((d-4+\delta)/2)\big)$. $\alpha \smile \beta \smile \partial(\psi)$ is represented by the cocycle

$$\big(\alpha \smile \beta \smile \partial(\psi)\big)_{i_0,i_1,i_2,i_3} = \frac{1}{f}(\alpha_{i_0,i_1}\beta_{i_1,i_2}) \otimes (\widehat{\psi}_{i_2} - \widehat{\psi}_{i_3}),$$

hence

$$\big(\lrcorner(\alpha \smile \beta \smile \partial(\psi))\big)_{i_0,i_1,i_2,i_3} = \frac{1}{f}\big(\widehat{\psi}_{i_2}(\alpha_{i_0,i_1}\beta_{i_1,i_2}) - \widehat{\psi}_{i_3}(\alpha_{i_0,i_1}\beta_{i_1,i_2})\big).$$

On the other hand, $H^2(\psi)(\alpha \smile \beta)_{i,j,h} = \psi(\alpha_{i,j}\beta_{j,h})$, whence

$$\begin{aligned}\partial' \circ H^2(\psi)(\alpha \smile \beta)_{i_0,i_1,i_2,i_3} =& \frac{1}{f}\big(\widehat{\psi}_{j_0}(\alpha_{i_1,i_2}\beta_{i_2,i_3}) - \widehat{\psi}_{j_1}(\alpha_{i_0,i_2}\beta_{i_2,i_3}) \\ &+ \widehat{\psi}_{j_2}(\alpha_{i_0,i_1}\beta_{i_1,i_3}) - \widehat{\psi}_{j_3}(\alpha_{i_0,i_1}\beta_{i_1,i_2})\big),\end{aligned}$$

where $j_h \in \{i_0, i_1, i_2, i_3\} \setminus \{i_h\}$. In particular choosing $j_0 = j_1 = j_2 = i_3$ and $j_3 = i_2$ and using that $\alpha_{i_1,i_2} - \alpha_{i_0,i_2} + \alpha_{i_0,i_1} = 0$, $\beta_{i_1,i_2} - \beta_{i_1,i_3} + \beta_{i_2,i_3} = 0$ we get

$$\begin{aligned}&\partial' \circ H^2(\psi)(\alpha \smile \beta)_{i_0,i_1,i_2,i_3} \\ &\quad = \frac{1}{f}\big(\widehat{\psi}_{i_3}(-\alpha_{i_0,i_1}(\beta_{i_2,i_3} - \beta_{i_1,i_3})) - \widehat{\psi}_{i_2}(\alpha_{i_0,i_1}\beta_{i_1,i_2})\big) \\ &\quad = \frac{1}{f}\big(\widehat{\psi}_{i_3}(\alpha_{i_0,i_1}\beta_{i_1,i_2}) - \widehat{\psi}_{i_2}(\alpha_{i_0,i_1}\beta_{i_1,i_2})\big).\end{aligned}$$

Then the proof is complete. q.e.d.

Remark 2.2. Theorem 0.3 holds without the assumption $F \subseteq \mathbb{P}^3_k$. It suffices to consider any hypersurface $F \subseteq \mathbb{P}^n_k$ endowed with a $\delta/2$–quadratic sheaf $\mathcal{F}$.

3. Examples and applications

In this section we shall consider the case of a nodal surface of small degree $d = 4, 5, 6$ and we shall see how our main theorem can be used to classify $\delta/2$–even sets of nodes, going beyond [6], Section 3 and [4].

Moreover from now on the ground field k will be equal to the field $\mathbb{C}$ of complex numbers.

Following the notation used in the introduction we begin by proving the following.

Proposition 3.1. *The sheaf $\mathcal{F}$ defined in the introduction is $\delta/2$–quadratic.*

Proof. Notice that $\mathcal{F}_{|F\setminus\Delta}$ is invertible, hence reflexive.

The multiplication map $\mathcal{O}_S \times \mathcal{O}_S \to \mathcal{O}_S$ induces a symmetric non–degenerate bilinear form $\mathcal{F} \times \mathcal{F} \to \mathcal{O}_F$. If $\delta = 1$, such a map factors through the multiplication by h, $\lambda(h)\colon \mathcal{O}_F(-1) \to \mathcal{O}_F$. Thus, in both cases, we get a symmetric bilinear map $\mathcal{F} \times \mathcal{F} \to \mathcal{O}_F(-\delta)$ inducing a monomorphism $\sigma\colon \mathcal{F}(\delta) \rightarrowtail \check{\mathcal{F}} := \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F)$ which is obviously an isomorphism outside Δ, whence a global isomorphism since S is normal.

Hence $\mathcal{F}$ is isomorphic to $\mathcal{F}^{\check{}\check{}}$ and, indeed, $\mathcal{F}$ is reflexive which means that the natural map $\mathcal{F} \xrightarrow{\mu} \mathcal{F}^{\check{}\check{}}$ is an isomorphism (in fact μ is injective since $\mathcal{F}$ is torsion-free, and $\operatorname{coker}(\mu) = 0$ since $\mathcal{F}$ and $\mathcal{F}^{\check{}\check{}}$ have the same Hilbert polynomial).

If $x \in \Delta$, the completion $\widehat{\mathcal{F}}_x$ fits into the exact sequence

$$0 \to \widehat{\mathcal{O}}^{\oplus 2}_{\mathbb{P}^3_k,x} \xrightarrow{\mu} \widehat{\mathcal{O}}^{\oplus 2}_{\mathbb{P}^3_k,x} \to \widehat{\mathcal{F}}_x \to 0,$$

where μ is induced by a matrix of the form

$$\begin{pmatrix} w & z \\ z & y \end{pmatrix},$$

w, z, y being local parameters in $\widehat{\mathcal{O}}_{\mathbb{P}^3_k,x}$. Therefore we see that $\mathcal{F}_x$ is Cohen–Macaulay as a module over $\mathcal{O}_{\mathbb{P}^3_k,x}$. q.e.d.

Imitating the construction of $\mathcal{F}$, there exists a decomposition $p_*\mathcal{O}_{\widetilde{S}} \cong \mathcal{O}_{\widetilde{F}} \oplus \mathcal{L}$ where $\mathcal{L}$ is now invertible since $\widetilde{F}$ is smooth.

Moreover $\pi_*\mathcal{L} \cong \mathcal{F}$, $\widetilde{p}_*\mathcal{O}_S \cong \mathcal{O}_{\widetilde{F}} \oplus \mathcal{L}$. Since

$$R^1\widetilde{\pi}_*\mathcal{O}_{\widetilde{S}} = R^1\pi_*\mathcal{O}_{\widetilde{F}} = 0,$$

diagram (0.1) and the spectral sequences of the composite functors $R^p\pi_*R^q\widetilde{p}_*$ and $R^pp_*R^q\widetilde{\pi}_*$ yield $R^1\pi_*\mathcal{L} = 0$ thus

$$h^i\big(\widetilde{F}, \mathcal{L} \otimes \pi^*\mathcal{O}_F(m)\big) = h^i\big(F, \mathcal{F}(m)\big).$$

Note that

$$(3.2) \qquad H^1\big(F, \mathcal{F}(-m)\big) \cong H^1\big(F, \mathcal{F}(m + d - 4 + \delta)\big)\check{} = 0, \quad m > 0$$

(Theorem 1 of [14] applied to $\widetilde{p}^*\pi^*\mathcal{O}_{\widetilde{S}}(1)$ and formula (1.2)). Moreover by [6], Theorem 2.19, we get that Δ is symmetric if and only if $h^1\big(F, \mathcal{F}(m)\big) = 0$ for $0 \le m \le (d-4)/2$.

We have the long exact sequence

$$(3.3) \qquad \begin{aligned} 0 &\to H^0\big(F, \mathcal{F}\big) \to H^0\big(F, \mathcal{F}(1)\big) \to H^0\big(H, \mathcal{F}(1)_{|H}\big) \\ &\to H^1\big(F, \mathcal{F}\big) \xrightarrow{\lambda(h)} H^1\big(F, \mathcal{F}(1)\big) \to H^1\big(H, \mathcal{F}(1)_{|H}\big) \\ &\to H^2\big(F, \mathcal{F}\big) \to H^2\big(F, \mathcal{F}(1)\big) \to 0. \end{aligned}$$

associated to every $h \in H^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)\big)$ (defining a plane section $H \subseteq \mathbb{P}^3_k$). Notice that in any case $h^0\big(F, \mathcal{F}\big) = 0$ since S is connected.

Moreover, by (3.2) we have $H^1(F, \mathcal{F}(-1)) = 0$, and thus $H^1(H, \mathcal{F}_{|H}) = 0$ for each H.

Since h does not divide $f := \det(\varphi)$, the restriction to H of the exact sequence 0.3.1 remains exact. Hence $\mathcal{F}_{|H}$ is again a $\delta/2$–quadratic sheaf. Mutatis mutandis, Theorem 2.16 and Proposition 2.28 of [6] apply; therefore $\mathcal{F}_{|H}$ has a free resolution

$$(3.4)\qquad 0 \to \bigoplus_{j=1}^{p} \mathcal{O}_{\overline{H}}(-\ell_j) \xrightarrow{\alpha} \bigoplus_{i=1}^{p} \mathcal{O}_{\overline{H}}(-r_i) \to \mathcal{F}_{|H} \to 0,$$

where $\overline{H} \subseteq \mathbb{P}^3_k$ is the plane defined by h. In (3.4) $\alpha := (\alpha_{i,j})_{i,j=1,\dots,p}$ is a symmetric matrix of homogeneous polynomials $\alpha_{i,j}$ of degrees $(d_i+d_j)/2$, where the d_i's are in not–decreasing order, $d_i \equiv d_j$, $d \equiv \delta + d_i \pmod 2$, $\ell_j = (d+\delta+d_j)/2$ and $r_i = (d+\delta-d_i)/2$.

As in [6] we see the following:

i) $d_i + d_{p+1-i} > 0$ since $\det(\alpha) \neq 0$;

ii) $d_i + d_{p-i} > 0$ if $\det(\alpha)$ is square free;

iii) $r_i > 0$ since $h^0(H, \mathcal{F}_{|H}) = 0$, i.e., $d_i \leq d + \delta - 2$;

iv) $d_i + d_{p-1-i} > 0$ if $H = \{\det(\alpha) = 0\}$ is smooth.

Notice finally that

$$d = \sum_{i=1}^{p} d_i = \sum_{i \leq p/2} (d_i + d_{p+1-i}) + d_{(p+1)/2}.$$

Here $d_\lambda = 0$ if λ is not an integer. We then get the following cases for the p–tuple $(d_1, \dots, d_p)$:

$d = 4$, $\delta = 0$ $(2,2)$, $(0,2,2)$, $(0,0,2,2)$;

$d = 4$, $\delta = 1$ $(1,3)$, $(1,1,1,1)$, $(-1,1,1,3)$, $(-1,-1,3,3)$;

$d = 5$, $\delta = 0$ $(1,1,3)$, $(-1,3,3)$, $(-1,1,5)$, $(1,1,1,1,1)$, $(-1,1,1,1,3)$, $(-1,-1,1,3,3)$;

$d = 6$, $\delta = 0$ $(2,4)$, $(2,2,2)$, $(-2,4,4)$, $(0,2,2,2)$, $(-2,2,2,4)$, $(0,0,2,2,2)$.

On the other hand, if we assume that H is smooth, then most of the above possibilities disappear and we are only left with:

$d = 4$, $\delta = 0$ $(2,2)$;

$d = 4$, $\delta = 1$ $(1,3)$, $(1,1,1,1)$;

$d = 5$, $\delta = 0$ $(1,1,3)$, $(1,1,1,1,1)$.
$d = 6$, $\delta = 0$ $(2,4)$, $(2,2,2)$, $(0,2,2,2)$.

3.5. The case $d = 4$, $\delta = 0$

For all H we have $h^1\big(H, \mathcal{F}(1)_{|H}\big) = h^0\big(H, \mathcal{F}_{|H}\big) = 0$ whence $h^0\big(H, \mathcal{F}(1)_{|H}\big) = 2$ (Riemann–Roch), hence formula (3.2) yields

$$h^1\big(F, \mathcal{F}(1)\big) = h^1\big(F, \mathcal{F}(-1)\big) = 0.$$

Moreover $h^1\big(F, \mathcal{F}\big)$ is dual to itself, whence it has even dimension. Recall that

$$\chi(\mathcal{F}) = (8-t)/4,$$

where t is the number of nodes of F (see [6, Proposition 2.11]).

If $h^1\big(F, \mathcal{F}\big) = 0$, then Δ is symmetric of type $(2,2)$, $h^0\big(F, \mathcal{F}(1)\big) = 2$ and $t = 8$.

If $h^1\big(F, \mathcal{F}\big) = 2$, then $h^0\big(F, \mathcal{F}(1)\big) = 0$, $t = 16$. It follows from condition ii) of Section 1 that $h^1\big(\mathbb{P}^3_k, \mathcal{E}\big) = 1$ and $h^1\big(\mathbb{P}^3_k, \mathcal{E}(m)\big) = 0$ for $m \neq 0$. Moreover recall that $h^2_*\big(\mathbb{P}^3_k, \mathcal{E}\big) = 0$ (condition iii) of Section 1).

The Horrocks correspondence shows that $\mathcal{E}$ is stably equivalent to $\Omega^1_{\mathbb{P}^3_k}$ (stably equivalent means that adding respective direct sums of invertible sheaves we get isomorphic sheaves). Therefore, by the construction of $\mathcal{E}$, since $h^0\big(\mathbb{P}^3_k, \Omega^1_{\mathbb{P}^3_k}\big) = h^0\big(\mathbb{P}^3_k, \Omega^1_{\mathbb{P}^3_k}(1)\big) = 0$, we may assume that $h^0\big(\mathbb{P}^3_k, \mathcal{E}(1)\big) = 0$. Since we have $h^i\big(F, \mathcal{F}(1)\big) = 0$ for $i = 0, 1$,

$$h^0\big(F, \mathcal{F}(2)\big) = h^0\big(H, \mathcal{F}(2)_{|H}\big) = h^0\big(\mathbb{P}^3_k, \Omega^1_{\mathbb{P}^3_k}(2)\big) = 6.$$

Recall (Section 1 ii)) that there is an epimorphism

$$H^0\big(\mathbb{P}^3_k, \mathcal{E}(2)\big) \twoheadrightarrow H^0\big(\mathbb{P}^3_k, \mathcal{F}(2)\big).$$

Let $6 - r$ be the rank of the induced map

$$H^0\big(\mathbb{P}^3_k, \Omega^1_{\mathbb{P}^3_k}(2)\big) \to H^0\big(\mathbb{P}^3_k, \mathcal{F}(2)\big).$$

Then

$$\mathcal{E} \cong \Omega^1_{\mathbb{P}^3_k} \oplus \mathcal{O}_{\mathbb{P}^3_k}(-2)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^3_k}(-3)^{\oplus r_3} \oplus \dots.$$

We claim that $r_3 = r_4 = \dots = 0$.

Indeed, by Beilinson's theorem (see [5]) applied to $\mathcal{E}(1)$, since

$$H^3\big(\mathbb{P}^3_k, \mathcal{E}(-1)\big)^{\check{}} = H^0\big(\mathbb{P}^3_k, \check{\mathcal{E}}(-3)\big),$$

which, by Theorem 0.3, injects into $H^0\big(\mathbb{P}^3_k, \mathcal{E}(1)\big) = 0$, we get

$$\mathcal{E}(1) \cong \Big(H^1\big(\mathbb{P}^3_k, \mathcal{E}\big) \otimes \Omega^1_{\mathbb{P}^3_k}(1)\Big) \oplus \Big(H^3\big(\mathbb{P}^3_k, \mathcal{E}(-2)\big) \otimes \mathcal{O}_{\mathbb{P}^3_k}(-1)\Big).$$

Thus $\mathcal{E} \cong \Omega^1_{\mathbb{P}^3_k} \oplus \mathcal{O}_{\mathbb{P}^3_k}(-2)^{\oplus r}$ and $\check{\mathcal{E}}(-4) \cong \Omega^2_{\mathbb{P}^3_k} \oplus \mathcal{O}_{\mathbb{P}^3_k}(-2)^{\oplus r}$. It follows that

$$\varphi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \check{\varphi_{1,2}} & \varphi_{2,2} \end{pmatrix},$$

where $\varphi_{2,2}$ is a constant $r \times r$ symmetric matrix which must be zero by the minimality of $\mathcal{E}$.

If $r \geq 4$, then $\det(\varphi) = 0$ which is absurd.

If $r = 3$, then $\operatorname{rk}(\varphi_{1,2}) \leq 2$ on a surface Y of degree $c_1\big(\Omega^1_{\mathbb{P}^3_k}(2)\big) = 2$, and $F = 2Y$, again a contradiction.

If $r = 2$, then there is a curve Γ of degree $c_2\big(\Omega^2_{\mathbb{P}^3_k}(2)\big) = 2$ where $\operatorname{rk}(\varphi_{1,2}) \leq 1$, whence Γ is a double curve for F, which is again absurd.

If $r = 1$, then $\varphi_{1,2}$ is a section of $H^0\big(\mathbb{P}^3_k, \Omega^1_{\mathbb{P}^3_k}(2)\big) \cong \Lambda^2 H^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)\big)$.

If the rank of this alternating map is 2 then $\varphi_{1,2}$ should vanish on a line Γ (since $\varphi_{1,2} = x_0 dx_1 - x_1 dx_0$ for suitable coordinates), which is a double line for F, absurd.

We conclude that $\varphi_{1,2}$ corresponds to a non-degenerate alternating form. With a proper choice of the coordinates we can assume that $\varphi_{1,2}$ corresponds to $x_0 dx_1 - x_1 dx_0 + x_2 dx_3 - x_3 dx_2$. Then $\mathcal{O}_{\mathbb{P}^3_k}(-2)$ is identified to a subbundle of $\Omega^1_{\mathbb{P}^3_k}$, and dually $\Omega^2_{\mathbb{P}^3_k} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-2)$ is surjective. As in [1] we define the null-correlation bundle $\mathcal{V}_0$ as $\mathcal{V}_0 := \Omega^1_{\mathbb{P}^3_k}(2)/\operatorname{im}(\varphi_{1,2}(2))$ and we obtain a self-dual resolution

$$0 \to \check{\mathcal{V}}_0 \to \mathcal{V}_0 \to \mathcal{F}(2) \to 0.$$

Finally if $r = 0$ we get

$$0 \to \big(\Omega^1_{\mathbb{P}^3_k}(2)\big)\check{} \to \Omega^1_{\mathbb{P}^3_k}(2) \to \mathcal{F}(2) \to 0.$$

We have therefore shown that for quartic surfaces all even sets of nodes are either symmetric or $t = 16$ and we have exactly the two cases described in [1].

3.6. The case $d = 4$, $\delta = 1$

As in the case $\delta = 0$ we have again $h^1\big(F, \mathcal{F}(1)\big) = 0$. We get by (1.2) that $h^1\big(F, \mathcal{F}\big) = h^1\big(F, \mathcal{F}(1)\big) = 0$. It then follows that Δ is symmetric of type either $(1,1,1,1)$ or $(1,3)$.

3.7. The case $d = 5$

In this case $\delta = 0$ as we already noticed. Sequence (3.3) is self-dual and

$$\lambda\colon H^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)\big) \to \operatorname{Hom}_k\big(H^1\big(F,\mathcal{F}(1)\big), H^1\big(F,\mathcal{F}(1)\big)\check{}\,\big)$$

maps to the subspace of alternating bilinear forms.

If H is smooth, $\mathcal{F}_{|H}$ is of type either $(1,1,1,1,1)$ or $(1,1,3)$, and thus $h^0\big(H,\mathcal{F}(1)_{|H}\big) = h^1\big(H,\mathcal{F}(1)_{|H}\big) \le 1$ by (3.4).

The map $\lambda(h)$ is an isomorphism if either $h^0\big(H,\mathcal{F}(1)_{|H}\big) = 0$ or $h^0\big(F,\mathcal{F}(1)\big) = 1$. In fact in both cases, $h^0\big(F,\mathcal{F}(1)\big) = h^0\big(H,\mathcal{F}(1)_{|H}\big)$ and the assertion follows easily from the self–duality of sequence (3.3).

If for a general H the map $\lambda(h)$ is an isomorphism, then $h^1\big(F,\mathcal{F}\big)$ is even, and if it is not zero the pfaffian of $\lambda(h)$ defines a surface $B \subseteq \check{\mathbb{P}}^3_k$ of degree $h^1\big(F,\mathcal{F}\big)/2$ which is contained in the dual surface $\check{F}$ of F. Since F is nodal, then it is of general type, and by biduality also $\check{F}$ is of general type; therefore one has $\deg(\check{F}) \ge 5$, whence $h^1\big(F,\mathcal{F}\big) \ge 10$. We conclude that

$$t/4 - 5 = -\chi(\mathcal{F}) \ge 10,$$

hence $t \ge 60$ which implies $\deg(\check{F}) \le d(d-1)^2 - 2t \le -40$, an absurd. Thus the only possibility is $h^1\big(F,\mathcal{F}\big) = 0$ and we get that Δ is symmetric.

Finally if, for H smooth, $h^0\big(H,\mathcal{F}(1)_{|H}\big) = 1$ and $h^0\big(F,\mathcal{F}(1)\big) = 0$ then $\dim(\ker(\lambda(h))) = 1$, thus $h^1\big(F,\mathcal{F}\big) \ne 0$ is odd ($\lambda(h)$ is alternating).

In any case $h^1\big(F,\mathcal{F}(2)\big) = 0$ (again (3.2)), hence we have the exact sequence

$$0 \to H^0\big(F,\mathcal{F}(2)\big) \to H^0\big(H,\mathcal{F}(2)_{|H}\big) \to H^1\big(F,\mathcal{F}(1)\big) \to 0.$$

In particular $h^0\big(H,\mathcal{F}(2)_{|H}\big)$ does not depend on H. If H is general, then H is either of type $(1,1,1,1,1)$ or $(1,1,3)$, hence sequence (3.4) implies $h^0\big(H,\mathcal{F}(2)_{|H}\big) = 5$. On the other hand if $h^1\big(F,\mathcal{F}\big) \ge 3$, there exists H such that $\dim(\ker(\lambda(h))) \ge 3$, whence H is of type $(-1,1,5)$. Thus looking at sequence (3.4) we get $h^0\big(H,\mathcal{F}(2)_{|H}\big) = 6$, a contradiction.

We have therefore restricted ourselves to the case $h^1\big(F,\mathcal{F}\big) = 1$.

In this case $h^1\big(F,\mathcal{F}(-1)\big) = 0$ (formula (3.2)) and $h^0\big(F,\mathcal{F}(-1)\big) = 0$; thus $h^0\big(F,\mathcal{F}(2)\big) = h^2\big(F,\mathcal{F}(-1)\big) = 4$. Beilinson's theorem then yields a sequence of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(-1)^{\oplus 4} \oplus \Omega^2_{\mathbb{P}^3_k}(2) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^3_k}^{\oplus 4} \oplus \Omega^1_{\mathbb{P}^3_k}(1) \to \mathcal{F}(2) \to 0,$$

where

$$\varphi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{pmatrix}.$$

Moreover Beilinson's spectral sequence and the fact that $\lambda(h) = 0$ for each h imply that $\varphi_{2,2}\colon \Omega^2_{\mathbb{P}^3_k}(2) \to \Omega^1_{\mathbb{P}^3_k}(1)$ is zero. In particular we see that $\mathcal{O}^{\oplus 4}_{\mathbb{P}^3_k} \oplus \Omega^1_{\mathbb{P}^3_k}(1)$ has the required properties for $\mathcal{E}$; thus we may assume that

$$\mathcal{E}(2) \cong \mathcal{O}^{\oplus 4}_{\mathbb{P}^3_k} \oplus \Omega^1_{\mathbb{P}^3_k}(1)$$

and that φ is symmetric.

However, the determinantal quintic F should be singular along the set $D \subseteq \mathbb{P}^3_k$ of points where $\mathrm{rk}(\varphi_{1,2}) \le 2$ which has dimension at least 1 (generically D is a pair of skew lines). In fact $\det(\varphi)$ belongs to the square of the sheaf of ideals of D. We have reached the conclusion that, for $d = 5$, Δ is always symmetric (as shown in [4]).

3.8. The case $d = 6$, $\delta = 0$

For each maximal isotropic subspace $U \subseteq H^1(F, \mathcal{F}(1))$ we can define the locally free $\mathcal{O}_{\mathbb{P}^3_k}$-sheaf $\mathcal{E}$ as in Section 1 satisfying conditions i), ii) and iii) of that section. Since the square

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ \downarrow{\scriptstyle\lambda(h)} & & \downarrow{\scriptstyle\lambda(h)} \\ \mathcal{E}(1) & \longrightarrow & \mathcal{F}(1) \end{array}$$

is commutative for each $h \in H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1))$, we obtain that the image of $H^1(\mathbb{P}^3_k, \lambda(h))$ is contained in U. Taking into account the arbitrariness of U, we finally obtain the following proposition.

Proposition 3.8.1. *The multiplication map*

$$H^1(F, \mathcal{F}) \times H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)) \to H^1(F, \mathcal{F}(1))$$

is zero.

We must make some remarks on the possible dimensions of certain cohomology groups. Notice that the theorem of Riemann–Roch applied to $\widetilde{F}$ yields

$$\chi(\mathcal{F}(1)) = 8 - \frac{t}{4}.$$

If H is smooth, then $\mathcal{F}_{|H}$ is of type either $(2,4)$ or $(2,2,2)$ or $(0,2,2,2)$, so that $h^0(H, \mathcal{F}(1)_{|H}) \leq 1$; whence, by (3.3),

$$h^0(F, \mathcal{F}(1)) \leq h^0(H, \mathcal{F}(1)_{|H}) \leq 1.$$

If $h^0(F, \mathcal{F}(1)) = 1$ then

$$h^0(S, \pi^*\mathcal{O}_F(1) \otimes \mathcal{L}^{-1}) = h^0(S, \pi^*\mathcal{O}_F(1) \otimes \mathcal{L}) = 1;$$

thus there exists a curve $C \subseteq F$ of degree 6 passing through the set of nodes Δ. Since Sing(F) is finite, for a general choice of the coordinates $x_1, \dots, x_3$ in $\mathbb{P}^3_k$, the partial derivative $\partial f/\partial x_1$ does not vanish on any component of C. Denoting the corresponding surface by G, we obtain $t := \mathrm{card}(\Delta) \leq C \cdot G = 30$. Since t is divisible by 8, we conclude that $t \leq 24$.

It follows that

$$0 \leq h^1(F, \mathcal{F}(1)) = 2 - \chi(\mathcal{F}(1)) = \frac{t}{4} - 6 \leq 0,$$

and sequence (3.3) then becomes

$$0 \to H^0(F, \mathcal{F}(1)) \to H^0(H, \mathcal{F}(1)_{|H}) \to H^1(F, \mathcal{F}) \to 0.$$

If H is smooth, the only possible case is that H is of type $(2,4)$ and $h^0(F, \mathcal{F}(1)) = h^0(H, \mathcal{F}(1)_{|H}) = 1$, i.e., $h^1(F, \mathcal{F}) = 0$. Hence Δ is symmetric of type $(2,4)$.

From now on we shall therefore assume $h^0(F, \mathcal{F}(1)) = h^2(F, \mathcal{F}(1)) = 0$, so that

$$\chi(\mathcal{F}(1)) = -h^1(F, \mathcal{F}(1)) = 8 - \frac{t}{4} \leq 0.$$

It follows that $t \geq 32$ and the equality holds if Δ is symmetric of type either $(2,2,2)$ or $(0,2,2,2)$. Set $2\tau := h^1(F, \mathcal{F}(1)) = -8 + t/4$. Sequence (3.3) becomes

$$\begin{aligned} 0 \to H^0(H, \mathcal{F}(1)_{|H}) \to H^1(F, \mathcal{F}) \xrightarrow{\lambda(h)} H^1(F, \mathcal{F}(1)) \to \\ \to H^1(H, \mathcal{F}(1)_{|H}) \to H^2(F, \mathcal{F}) \to H^2(F, \mathcal{F}(1)) \cong H^0(F, \mathcal{F}(1))^{\check{}} \to 0. \end{aligned}$$

Since $\lambda(h) = 0$ by Proposition 3.8.1, we obtain $a := h^1(F, \mathcal{F}) = h^0(H, \mathcal{F}(1)_{|H}) \leq 1$, $2\tau \leq 3 + a = h^1(H, \mathcal{F}(1)_{|H})$, $b := h^2(F, \mathcal{F}) = 3 + a - 2\tau$. Finally, notice that $h^i(F, \mathcal{F}(-1)) = 0$, for $i = 0, 1$.

The Beilinson's table for $\mathcal{F}(2)$ is

$$\begin{matrix} c & b & 0 & 0 \\ 0 & a & 2\tau & a \\ 0 & 0 & 0 & b \end{matrix}$$

Since $b \geq 0$, $\tau \leq 2$. If $\tau = 2$, then $b = 0$ and $a = 1$. If $b = 0$, since also $\lambda(h) = 0$, then the differential d_1 is zero. Thus $E_\infty^{-2,1} = \Omega^2_{\mathbb{P}^3_k}(2)$, contradicting $E_\infty^{p,q} = 0$ if $p \neq -q$.

We can summarize the above results in the following statement.

Theorem 3.8.2. *Let $F \subseteq \mathbb{P}^3_k$ be a nodal surface of degree 6. Then for each even set of nodes Δ on F, $t := \mathrm{card}(\Delta) = 24, 32, 40$. Δ is not symmetric if $t = 40$.*

Let us briefly examine the case $t = 40$. Then we can choose e.g. $\mathcal{E} := \Omega^1_{\mathbb{P}^3_k}(-1) \oplus \mathcal{O}_{\mathbb{P}^3_k}(-2)$. We thus get the sequence

$$0 \to \Omega^3(2) \oplus \mathcal{O}_{\mathbb{P}^3_k}(-1) \to \Omega^1_{\mathbb{P}^3_k}(2) \oplus \mathcal{O}_{\mathbb{P}^3_k}(1) \to \mathcal{F}(3) \to 0.$$

In particular we have a sextic surface everywhere tangent to a Kummer quartic. This example was already described in [6, Proposition 2.24].

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