Fibred surfaces, varieties isogenous to a product and related moduli spaces

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FIBRED SURFACES, VARIETIES ISOGENOUS TO A PRODUCT AND RELATED MODULI SPACES

By Fabrizio Catanese

This article is dedicated to the memory of Fernando Serrano

Abstract. A fibration of an algebraic surface S over a curve B, with fibres of genus at least 2, has constant moduli iff it is birational to the quotient of a product of curves by the action of a finite group G. A variety isogenous to a (higher) product is the quotient of a product of curves of genus at least 2 by the free action of a finite group. Theorem B gives a characterization of surfaces isogenous to a higher product in terms of the fundamental group and of the Euler number. Theorem C classifies the groups thus occurring and shows that, after fixing the group and the Euler number, one obtains an irreducible moduli space. The result of Theorem B is extended to higher dimension in Theorem G, thus generalizing (cf. also Theorem H) results of Jost-Yau and Mok concerning varieties whose universal cover is a polydisk. Theorem A shows that fibrations where the fibre genus and the genus of the base B are at least 2 are invariants of the oriented differentiable structure. The main Theorems D and E characterize surfaces carrying constant moduli fibrations as surfaces having a Zariski open set satisfying certain topological conditions (e.g., having the right Euler number, the right fundamental group and the right fundamental group at infinity).

0. Introduction. The study of fibrations of a smooth algebraic surface S over a smooth algebraic curve B lies at the heart of the classification theory and of the geometry of algebraic surfaces. Our main results in this paper (Theorems D, E) concern a topological characterization of surfaces admitting a constant moduli fibration. These results are intimately related to a series of variations on the same theme (also in higher dimension), and to a circle of allied techniques which we present in the course of the paper. The first new result we have in this direction, and which is presented in Section 2 after we recall several known results (some classical, some quite recent), is the following theorem on the differentiable invariance of the genus of fibres (cf. Theorem 2.9 for a more precise statement).

Theorem A. *Let $f\colon S \to B$ be a fibration of the surface S over a curve B of genus $b \geq 2$, and with fibre F of genus $g > 1$. Assume that S' is another smooth compact Kähler surface which is orientedly diffeomorphic to S. Then the subspace of $H^1(S', \mathbb{C})$ corresponding to $f^*(H^1(B, \mathbb{C}))$ determines a fibration $f'\colon S' \to B'$ with*

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the same invariants as f (namely, the base curve B' has genus b, and the fibre genus is equal to g).

The result above, whose proof uses the results of Seiberg Witten theory, was conjectured in [Cat5], of which the present paper is in some sense a continuation. In fact, as in that paper one of the main themes was the topological characterization of the existence of fibrations (in dimension 2 and greater), here, as already remarked, the leitmotiv is the topological characterization of constant moduli fibrations. Recall that $f\colon\ S \to B$ is said to be a constant moduli fibration if all the smooth fibres are isomorphic, and that in this case S is bimeromorphic to a quotient $C_1 \times C_2/G$ of a product of curves by the action of a finite group G. The first case which deserves attention is the one where the group G acts freely, and we say then that S is strongly isogenous to a product. More generally, we shall say that a complex manifold X is isogenous to a higher product if it admits a finite unramified covering which is isomorphic to a product of curves of respective genera at least 2. Indeed, as we see in Proposition 3.11, this weaker property is equivalent to the stronger property of having a Galois such cover (thus the notions of a variety isogenous, resp. strongly isogenous to a product are equivalent). Our second result in the case of surfaces is (see Theorems 3.4, 4.13, 4.14.)

THEOREM B. *A surface S is isogenous to a higher product if and only if $\pi_1(S)$ admits a finite index subgroup Γ isomorphic to a direct product $\Pi_{g_1} \times \Pi_{g_2}$ of fundamental groups of curves of genera $g_1, g_2 > 1$, and moreover, d denoting the index of Γ, either*

(e) $e(S) = \frac{4(g_1-1)(g_2-1)}{d}$ *or*

(k) *S is minimal and $K_S^2 = \frac{8(g_1-1)(g_2-1)}{d}$ or*

(c) *S is minimal and $\chi(\mathcal{O}_S) = \frac{(g_1-1)(g_2-1)}{d}$.*

Setting $\Pi = \Pi_{g_1} \times \Pi_{g_2}$, one can indeed relax the hypothesis $\Gamma \cong \Pi$ to the weaker assumption that $\Gamma/\Gamma_3 \cong \Pi/\Pi_3$, where, for a group Γ, Γ_3 is the third group in the nilpotent series (i.e., Γ_2 is the commutator subgroup $[\Gamma,\Gamma]$ and $\Gamma_3 = [\Gamma,\Gamma_2]$).

Moreover, let Π be the fundamental group of a surface S isogenous to a product as above. Then the moduli space $\mathfrak{M}(\Pi, e(S))$ of the surfaces with fundamental group isomorphic to Π, and e.g. Euler number = $e(S) = 4(g_1-1)(g_2-1)/d$ is irreducible.

In fact the result can be made stronger, as to classify the fundamental groups Π which occur. For instance we have (see Theorem 4.13):

THEOREM C. *Let* Π *be a group such that:*

(i) *there is an exact sequence*

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \Pi \to G \to 1,$$

where G is a finite group of order d;

(b) *the individual factors* Π_{g_1}, Π_{g_2} *are normal in* Π*;*

(c) *the induced quotients*

$$1 \to \Pi_{g_i} \to \Pi(i) \to G \to 1$$

are such that

(c1) *there is no element of* Π *mapping in each* $\Pi(i)$ *to an element of finite order,*

(c′) *by the above exact sequences G embeds in* $\mathrm{Out}(\Pi_{g_i})$.

Then the moduli space of the surfaces S with $\pi_1(S) \cong \Pi$*, and with Euler number* $e(S) = \frac{4(g_1-1)(g_2-1)}{d}$ *is nonempty and irreducible.*

Theorems B and C are a sharpening in two directions of two earlier theorems of Jost and Yau [J-Y 1,2] asserting that if S' is homotopically equivalent to a surface S whose universal cover is the product $\mathcal{H}^2$ of two discs, then S' is also a quotient $\mathcal{H}^2/\Pi$ ($\Pi = \pi_1(S)$). Secondly, Jost and Yau show that if Π acts irreducibly (i.e., S is not isogenous to a product, in our terminology), then S' is either biholomorphic or antibiholomorphic to S (strong rigidity).

The assumption of Jost and Yau on S' is clearly that $\pi_1(S') = \pi_1(S)$, and moreover that $\pi_j(S') = 0$ for $j \geq 2$. The advantage of Theorem B is thus to involve the Euler number, which is easy to compute, unlike the higher homotopy groups. Also, we complement the strong rigidity result by showing weak rigidity (irreducibility of the moduli space) and an existence result in the reducible case: here we rely on the properties of Teichmüller spaces, and more precisely on a variant of the solution of the Nielsen realization problem (cf. Section 4) given in [Tro]. Moreover, we show indeed (cf. Theorem 7.7 and Remark 7.8) that we can get the first result of Jost and Yau also in the irreducible case with the weaker assumptions $\pi_1(S') = \pi_1(S)$, and $e(S) = e(S')$ (or, S' is minimal and $K_{S'}^2 = K_S^2$). Of course, one could ask whether the result of Theorem B could be further improved, and we show in Section 1 that this is not the case. Namely, we show that only an assumption on the fundamental group cannot be sufficient. Because for any surface X (Corollary 1.3 considers for simplicity the case where X is minimal and nonruled) there exists a finite (ramified) covering $S \to X$, such that S is of general type with K_S very ample, $\pi_1(S) \cong \pi_1(X)$, but S is not birationally equivalent to X. We can obtain such a surface S with negative index, but it is not clear whether it is possible to achieve positive index.

The above remark (based on a simple construction used by Bogomolov) has some implications on the moduli spaces of surfaces of general type. For those, the main numerical invariants are the holomorphic Euler Poincaré characteristic $\chi(\mathcal{O}_S)$ and the self intersection $K^2 = K_S^2$ of the canonical divisor K_S of the minimal model S (Noether's formula gives, for the Euler number, $e(S) = 12\chi(\mathcal{O}_S) - K_S^2$). Given a pair of invariants $x, y \geq 1$ the surfaces of general type S with those invariants (i.e. $\chi(\mathcal{O}_S) = x$, $K_S^2 = y$) are parametrized by a moduli space $\mathfrak{M}(x, y)$ which is a quasiprojective variety.

The following (cf. [Miy], [Yau1,2])

Noether's inequality: $K_S^2 \geq 2p_g - 4$, whence $K_S^2 \geq 2\chi - 6$ and
Bogomolov-Miyaoka-Yau inequality: $K_S^2 \leq 9\chi$

impose essentially the only restrictions on x, y. It is known (cf. [Cat6], [Ch]) that the number $i(x, y)$ of irreducible components of $\mathfrak{M}(x, y)$ has a limsup which grows exponentially ($y^{\frac{1}{4}y} \leq \limsup i(x, y) \leq C y^{cy^2}$), but most of the examples are based on the case of surfaces which are simply connected.

There are of course two parts into which the world of surfaces of general type can be divided: based (instead of upon political faith) on the property of the fundamental group being either finite or infinite. Proposition 6.11 shows that if we take two families $\mathcal{X}, \mathcal{Y}$ of surfaces, parametrized by smooth respective bases $\mathcal{B}, \mathcal{B}'$, and with surjective Kodaira Spencer map, then by the Bogomolov construction we get also a family $\mathcal{S}$ parametrized by a smooth base T, and with surjective Kodaira Spencer map.

Whence, from each irreducible component of the moduli space of 1-connected surfaces Y, we produce an irreducible component of the moduli space of surfaces S with $\pi_1(S) \cong \pi_1(X)$ (here, we can take e.g. X isogenous to a product). Therefore, the "geographical" meaning of Theorem B is that $\pi_1(S)$ determines the geometry only if we restrict ourselves to move on certain lines, x = constant, y = constant,or $y - 12x$ = constant. It would be interesting to show that being on the watershed line ($y = 8x$) and having fundamental group of a surface isogenous to a product does not imply that S is isogenous to a product (in a similar vein Moishezon and Teicher in [M-T] have shown the existence of simply connected surfaces on the watershed line, contrary to a previous expectation).

We find that the Bogomolov construction deserves to be better analyzed. For instance, assume that we "marry" two 1- connected (orientedly) homeomorphic surfaces Y, Y' with such an X (having infinite fundamental group): can we have an example where Y, Y' are not diffeomorphic but the resulting surfaces S, S' are diffeomorphic? Moreover (cf. [Cat1,2], [Man1,2]) assume that we consider (1-connected) surfaces Y, Y' for which the moduli space has several connected components, and let us take X isogenous to a product. Then, as observed, we produce new irreducible components of the moduli space for S, S': are these also connected components?

Leaving aside speculations and going to the technique, our method is founded on the classical Hodge Theory rather than on Jost and Yau's method of harmonic maps. In this way the algebraic geometric methods come into play (offering easier proofs and more precise results), and the advantage of doing so is particularly evident in the noncompact case, where we can make full use of the assumption of quasiprojectivity.

One of the basic tools in the characterization of the more general case of surfaces carrying a constant moduli fibration are two theorems of I. Bauer [Ba] and one of D. Arapura [Ara]. The first gives a bijection between the maps of a quasiprojective manifold $U = X - D$ to a quasiprojective curve C (with first Betti number $b_1 \geq 3$) and the real maximal isotropic subspaces V of $H^1(U, \mathbb{R})$, the second one (obtained independently by Arapura but under the weaker assumption $b_1 \geq 2$) generalizes results of several authors in the compact case and asserts the existence of such maps under the assumption on $\pi_1(U)$ of admitting a surjection onto a nonabelian free group.

With this in hand, we get our two main results on the "topological" characterization of constant moduli fibrations (cf. Theorems 5.4, 5.7 for more complete statements). We write "topological" with quotation marks because, as we show in Remark 2.5, the existence of these fibrations is not a topological property of a surface S (actually the property is not even invariant by deformation). But indeed this property is a combined topological property, based on the interplay of the Hausdorff topology and the analytic Zariski topology.

THEOREM D. *Let $U' = S' - D$ be a quasiprojective surface, and assume that U' is proper homotopically equivalent to the good locus Y^0 of a constant moduli fibration $f\colon S \to B$ having fibre genus $g \geq 2$ and base genus $b \geq 2$. Then S' carries a constant moduli fibration with the same invariants as S. Moreover, the birational classes of all such surfaces S' form an irreducible subvariety of the moduli space.*

The previous result is clearly in the spirit of the one by Jost-Yau, whereas the next Theorem (E) is a generalization of Theorem B. Except that (in the proof only, as pointed out by L. Kaup) we have to replace the Euler number by the Euler number in Borel Moore homology (this is the Euler number of the Alexandroff compactification, diminished by 1), and moreover we have also to consider the fundamental group at infinity (this is a disjoint union of groups, one for each end of U).

THEOREM E. *Let $U' = S' - D$ be a quasiprojective surface. Assume that $\pi_1(U') \cong \pi_1(Y^0)$, where Y^0 is the good locus of a constant moduli fibration $f\colon S \to B$ having fibre genus $g \geq 2$ and base genus $b \geq 2$, and that $\pi_1^\infty(U') \cong \pi_1^\infty(Y^0)$, compatibly with the natural homomorphism $\pi_1^\infty \to \pi_1$. Assume further that $e_{BM}(U') = e_{BM}(Y^0)$ (or, equivalently,$e(U') = e(Y^0)$). Then S' also carries a constant moduli fibration which, in case S' is minimal, is a deformation of the one of S.*

In Section 3 we proceed to a detailed classification of the surfaces isogenous to a product, which allows a precise description of their fundamental group and of their moduli spaces. Surfaces isogenous to a product are divided into 3 major types:

(M) Surfaces of mixed type, quotient of a product $C \times C$ by a group of automorphisms exchanging the two factors.

(D) Surfaces of double type for which the image of the Albanese map is either a product of curves, or a curve, or a point (in this last class, which we label by D^0, are contained some surfaces constructed by Beauville, and which turn out, by Theorem C, to be strongly rigid: cf. 3.22).

(GH) Generalized Hyperelliptic surfaces, quotients $C_1 \times C_2/G$ such that $C_2/G \cong \mathbb{P}^1$, while G acts freely on C_1.

This last case, generalizing the case of the classical hyperelliptic surfaces (where C_1, C_2 have genus 1) which play such a central role in the Enriques classification of algebraic surfaces, admits another very easy characterization (cf. Theorem 3.18).

THEOREM F. *Let S be a surface with index* 0, $e(S) > 0$, *irregularity* $q \geq 2$ *and Albanese map a pencil. S is a Generalized Hyperelliptic surface if and only if the genus g of the Albanese fibres satisfies* $g = 1 + \frac{\chi(S)}{(q-1)}$.

Now, an entirely similar classification for varieties isogenous to a product can be given also in higher dimension, but we prefer to stick to the 2-dimensional case just to limit ourselves to giving the flavor of the arguments used and to avoid a more complicated algebraic apparatus in the mixed case.

On the other hand, one can extend many of the above results from the surface case to higher dimension (here, to avoid touching upon the theory of minimal models, we restrict ourselves to imposing the natural condition that K should be ample). This was first done by Mok [Mok1,2] who extended the results of Jost and Yau to quotients of polydiscs, again under an assumption of homotopy equivalence. In Section 7 we show that one can relax these assumptions also in higher dimension.

THEOREM G. *Let X be a compact complex manifold of dimension n with ample canonical bundle and assume that:*

(i) $\Gamma = \pi_1(X)$ *admits a finite index subgroup* Γ' *isomorphic to* $\Pi_{g_1} \times \Pi_{g_2} \times \cdots \times \Pi_{g_n}$, *with* $g_1, \dots, g_n > 1$*;*

(ii) *the image of* $H^{2n}(\Gamma', \mathbb{Z})$ *in* $H^{2n}(X', \mathbb{Z})$ *is nonzero;*

(k) $K_X^n = \frac{n! 2^n (g_1-1)(g_2-1)\dots(g_n-1)}{d}$.

Then X is isogenous to a higher product, and the moduli space of such varieties is irreducible (weak rigidity).

THEOREM H. *Let X be a compact manifold of dimension n with ample canonical bundle, and assume that:*

(i′) $\pi_1(X) \cong \Gamma$, *where* Γ *is a cocompact torsion free subgroup of* $\mathrm{Aut}(\mathcal{H}^n)$ *(i.e.,* $\Gamma \cong \pi_1(W')$*, where* $\mathcal{H}^n/\Gamma = W'$ *is a compact manifold);*

(ii′) $H^{2n}(\Gamma, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})$ *is an isomorphism; or*

(ii″) $H^{2n}(\Gamma, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})$ *is nonzero and* $K_X^n = K_{W'}^n$.

Then X is also biholomorphic to a quotient $W = \mathcal{H}^n/\Gamma$.

It would be interesting to consider the higher dimensional topological characterization of varieties bimeromorphic to a quotient of a product of curves (cf., however, Remark 5.5). For the time being we felt that treating here all the possible generalizations would have been too demanding for the reader (and for the writer).

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Some notation:

Π_g = the fundamental group of a compact complex curve C of genus g.

$e(S)$ = the topological Euler number.

$e_{BM}(Y)$ = the Euler number in Borel Moore homology.

$\mathcal{H}$ = the unit disc in $\mathbb{C}$, $\mathcal{H} = \{z \mid |z| < 1\}$.

In a complex vector space V with a real structure, $\bar{U}$ denotes the complex conjugate of the subspace U.

1. Fundamental groups cannot determine. We start this section by reminding the reader of a consequence of the Lefschetz theorem on hyperplane sections.

LEMMA 1.1 *Let X, Y be smooth complete algebraic surfaces: then there exists a smooth complete surface $S \subset X \times Y$, mapping finitely to both X and Y with degree > 0, and such that $\pi_1(S) \cong \pi_1(X) \times \pi_1(Y)$.*

Proof. Let A be (the pullback of) a very ample divisor on X, and B a very ample divisor on Y: then $A + B$ is very ample on $X \times Y$, and any surface S which is a general complete intersection of two divisors H, H' in $|A + B|$ is smooth and has $\pi_1(S) \cong \pi_1(X) \times \pi_1(Y)$, by the theorems of Bertini and Lefschetz. The degree of $S \to X$ equals the intersection product B^2, thus we simply have to exclude the case where Y is $\mathbb{P}^2$, B is the hyperplane divisor, and similarly for X. To show e.g. the finiteness of $S \to X$ we argue as follows:

Claim 1. For H general, $H \to X$ has all fibres of dimension 1.

Proof. $H = \{(x, y) \mid \sum_{j=0}^{n} \alpha_j(x)\beta_j(y) = 0\}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are sections of $H^0(\mathcal{O}_X(A))$, and $\beta_0, \beta_1, \ldots, \beta_n$ form a basis of $H^0(\mathcal{O}_Y(B))$. It suffices to show that for each p in X, $\{H \mid H \supset \{p\} \times Y\}$ has codimension ≥ 3. But $H \supset \{p\} \times Y$ iff $\alpha_j(p) = 0$ for each j, thus we get a linear subspace of codimension $n + 1 \geq 3$.

Claim 2. For H' general, H' does not contain any component D of a fibre $H \cap (\{p\} \times Y)$ of $H \to X$.

Proof. If $H' = \{(x, y) \mid \sum_{j=0}^{n} \alpha'_j(x)\beta_j(y) = 0\}$, then $H' \supset D$ iff $\sum_{j=o}^{n} \alpha'_j(p)\beta_j(y) \equiv 0$ on D. Thus $\{H' \supset D\}$ is a linear subspace of codimension 3 unless D is embedded as a line. Therefore we can find a H' as desired, unless (H being irreducible) all the fibres of $H \to X$ are lines, which was excluded by setting $B^2 > 1$. □

Remark 1.2. Let X, Y, S be as in the construction of Lemma 1.1. Assume X, Y are minimal: then S is also minimal, and indeed S has a nef canonical divisor K_S, which is indeed very ample if X and Y are not ruled. Moreover, if we set $\chi = \chi(\mathcal{O}_X)$, $\chi' = \chi(\mathcal{O}_Y)$, $\chi(S) = \chi(\mathcal{O}_S)$, we have

$$\chi(S) = \chi B^2 + \chi' A^2 + c(A, B),$$

$$K_S^2 = K_X^2 B^2 + K_Y^2 A^2 + 8c(A, B) - 4A^2 B^2,$$

where

$$c(A, B) = \frac{7}{2}A^2B^2 + \frac{3}{2}\{(AK_X)B^2 + A^2(BK_Y)\} + \frac{1}{2}(AK_X)(BK_Y).$$

Proof. Let $Z = X \times Y$, use the Koszul sequence

$$(*) \qquad 0 \to \mathcal{O}_Z(-2A-2B) \to \mathcal{O}_Z(-A-B)^2 \to \mathcal{O}_Z \to \mathcal{O}_S \to 0$$

to calculate $\chi(\mathcal{O}_S)$, and then keep in mind that

$$K_S = (K_X + 2A + K_Y + 2B)|_S.$$

For instance, by a more general result of Fujita ([Fu], Thm. 11.2, p. 93) $K_X + 2A$ is nef on X, since the pair (X, A) is not the pair $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, and similarly for (Y, B). Clearly if X, Y are not ruled then K_X and K_Y are nef, thus K_S is then very ample. □

COROLLARY 1.3. *Let X be a nonruled minimal surface. Then there exists a finite (ramified) covering $S \to X$ of degree > 1, such that S is of general type with K_S very ample, $\pi_1(S) \cong \pi_1(X)$, but S is not birationally equivalent to X. We can moreover obtain that S has negative index, i.e., $K_S^2 - 8\chi(S) < 0$.*

Proof. Take X, Y as in Remark 1.2 but with Y simply connected, and A sufficiently very ample. Then, in particular, $K_S^2 - 8\chi(S) = B^2\{K_X^2 - 8\chi(X)\} + A^2\{K_Y^2 - 8\chi(Y)\} - 4A^2B^2$. Clearly, if the index of Y is nonpositive (there is no problem finding such a surface), for $A \gg 0$, $K_S^2 - 8\chi(S)$ is negative. □

Remarks 1.4.

(i) Indeed (cf. [Miy2], Prop. 3) the above construction can yield a surface of general type S with negative index and Ω_S^1 ample.

(ii) If X has positive (resp. zero index) we can ask whether we can also obtain the case that S has positive index (resp. zero index): it would suffice to find a 1-connected Y with a very ample divisor B such that $\{K_Y^2 - 8\chi(Y)\} \geq 4B^2$. (Can one find this? Cf. [Chen1,2].)

(iii) If X is a Yau surface, i.e. $K_X^2 = 9\chi(X)$ (by [Ya1,2] and [Miy2] then X is a quotient $X = \mathcal{B}/\Gamma$ of the unit ball in $\mathbb{C}^2$), then certainly S does not satisfy $K_S^2 = 9\chi(S)$. In fact, then S would also be homeomorphic to X (indeed isomorphic, by Mostow's rigidity [Mos]) whereas $K_S^2 > K_X^2$.

2. Basic results on fibred surfaces, and differentiable invariance of the fibre. For the benefit of the reader, we collect together a series of results which will be used in the sequel: throughout, by a fibration, we shall mean a surjective morphism with connected fibres between complete smooth varieties. Later on, when our varieties will no longer be complete, but only quasiprojective, a fibration will denote a surjective morphism with irreducible general fibre. In the case where we have $f\colon S \to B$, with S a surface and B a curve of genus b, we shall say that f is minimal if there is no (-1) curve contained in a fibre, and we will denote by

g the arithmetic genus of a(ny) fibre F. Notice that, for $g \geq 1$, f is minimal iff K_S is f-nef, i.e., K_S has intersection number ≥ 0 with each component of a fibre. In the case $g \geq 1$ it follows then easily that the minimal model of f is unique. F will also be called a genus b pencil of curves of genus g, and a higher genus pencil if $b > 1$ (cf. [Ca4]). The oldest result is:

THEOREM (of Zeuthen-Segre) 2.1. *Let $f\colon S \to B$ be a genus b pencil of curves of genus g: then we have the following inequality for the topological Euler-Poincaré characteristic of S: $e(S) \geq 4(g-1)(b-1)$. If $g \geq 2$, then equality holds if and only if f is a topological fibre bundle.*

THEOREM (of Arakelov) 2.2. *Let $f\colon S \to B$ be a minimal genus b pencil of curves of genus g: then we have the following inequality for the self intersection of the canonical divisor of S: $K_S^2 \geq 8(g-1)(b-1)$. If $g \geq 2$, then equality holds only if f has constant moduli, meaning that all the smooth fibres are isomorphic.*

The above two theorems were combined nicely by Beauville [Bea2] in order to yield a simple description, first conjectured by Castelnuovo, of the surfaces for which holds the inequality $p_g \leq 2(q-4)$ (see [Ros], [Com], [Jon]).

THEOREM (of Beauville) 2.3. *Let $f\colon S \to B$ be a minimal genus b pencil of curves of genus $g \geq 2$: then $\chi(S) \geq (b-1)(g-1)$, equality holding if and only if f is a holomorphic fibre bundle.*

Remarks 2.4.

(i) Let $f\colon S \to B$ be a *constant moduli* fibration: then f is *isotrivial*, in the sense that there is a Galois base change $C_2 \to B = C_2/G$ such that the pullback of f can be relatively blown down to a product projection $C_1 \times C_2 \to C_2$. We recall that the pullback is here the minimal resolution of singularities of the normalization of the fibre product $S \times_B C_2$. In this situation the smooth fibres are isomorphic to C_1 and if $g \geq 1$ we have a biregular action of G on $C_1 \times C_2$. We denote by Y the quotient $C_1 \times C_2/G$.

(ii) The case of a holomorphic bundle (also called *etale bundle*) is the one where G acts freely on the curve C_2.

(iii) The case of a *quasibundle* (according to the definition of Serrano this means that every singular fibre is a multiple of a smooth curve) gives rise to the notion of a *surface isogenous to a product* (cf. Section 3): this case is the one where G acts freely on the product.

(iv) Let $f\colon S \to B$ be more generally a constant moduli minimal fibration with $g \geq 1$: then the birational map $\varepsilon\colon S \to Y = (C_1 \times C_2)/G$ does not need to be a morphism (cf. e.g. the Comessatti's example analyzed in [C-C], pages 96–99). We can, however, say something to characterize the case where ε is not a morphism. Let $\pi\colon Z \to Y$ be a minimal resolution of singularities: then there is a birational map $\eta\colon S \to Z$ commuting with the projection onto B. Since f

is minimal, it follows that $Z \to B$ is not minimal. Assume E is a (-1) curve contained in a fibre: then by assumption E is not π-exceptional, whence E is the central curve, the unique curve mapping onto the curve E' in Y, quotient of a fibre $C_1 \times \{p\}$ by the stabilizer G_p of p.

Let $aE + \sum_{j=1}^{n} m_j D_j$ be the fibre corresponding to p, where a is the order of G_p. Then $E^2 = -1$ if and only if $E \sum_j m_j D_j = a$. Let us analyze the singularities of Y in the points of E': these correspond to a fixed point (p', p) for the action of G_p, thus we get a cyclic subgroup H of G_p of order n. H has a canonical generator given by the $(\frac{a}{n})^{\text{th}}$ power of the fixed generator of G_p which acts on the tangent space at p by the standard primitive a^{th} root of 1. In terms of the standard primitive n^{th} root ζ of 1, the local action on the product surface of our generator of H is a diagonal action with eigenvalues ζ^b, ζ, and according to [B-P-V], pages 80–85, the resolution of the cyclic quotient singularity yields a Hirzebruch-Jung string with exactly one curve D meeting E. Moreover, D meets E transversally at only one point, and its multiplicity in the fibre equals $(\frac{a}{n})q$, where $0 < q < n$, and q satisfies the congruence $qb \equiv 1 (\bmod n)$. Therefore, we have that E is a (-1) curve if and only if:

(i) the quotient of the action of G_p on C_1 yields $\mathbb{P}^1$,

(ii) the sum over the fixed points of the action of G_p on C_1 gives $\sum_{i=1}^{r} q_i (\frac{a}{n_i}) = a$.

Recall that the Hurwitz formula has to be satisfied:

$$2g - 2 = -2a + \sum_{i=1}^{r} (n_i - 1)\left(\frac{a}{n_i}\right).$$

We can conversely see that for each choice of such numbers q_i, n_i satisfying the above two equations we get a corresponding covering of $\mathbb{P}^1$. In fact, setting $q_i(\frac{a}{n_i}) = m_i$, it suffices to take the covering

$$\{(w, z) \mid w^a = \sum_{i=1}^{r} (z - z_i)^{m_i}\} \quad .$$

For instance, when a is prime, then $a = n_i$, thus $\sum_{i=1}^{r} q_i = a$, and for $q_i = 1$ we get the genus g of a plane curve of degree a (Fermat curve).

Remark 2.5. Let $f\colon S \to B$ be a constant moduli minimal fibration: then a converse to Arakelov's theorem does not hold, namely the equality $K_S^2 = 8(g-1)(b-1)$ does not need to hold. In fact what happens is that a small deformation of a constant moduli fibration does not need to be again a constant moduli fibration. The simplest example of this phenomenon is given by an isotrivial surface S which is a resolution of $Y = C_1 \times C_2/G$, with $G = \mathbb{Z}/2$. We can view Y as a singular double cover of $C_1' \times C_2'$ (where $C_i' = C_i/G$), with a branch curve B

which is a union of vertical and horizontal curves. Deforming B to a smooth divisor in the linear system $|B|$ (this is possible by Bertini's theorem), we get a deformation of S still having a map to $C_2' = C_2/G$, but which is no longer a constant moduli fibration. As a matter of fact, in this example, if the genus b of $C_2' = C_2/G$ is at least 2, then every deformation of S carries a fibration onto a curve of genus b, as implied by the following (cf. [Cat5], Thm. 1.10, p. 268).

ISOTROPIC SUBSPACE THEOREM 2.6. *Let X be a compact Kähler manifold. Then the correspondence which associates, to a fibration $f\colon X \to B$, where B is a curve of genus $b \geq 2$, the subspace $f^*(H^1(B,\mathbb{C}))$ induces a bijection between:*

(i) *isomorphism classes of fibrations $f\colon X \to B$ where B is a curve of genus $b \geq 2$ ($f \cong f'$ iff there is an isomorphism ψ such that $\psi f = f'$), and*

(ii) $2b$*-dimensional subspaces V which can be written as $U \oplus \overline{U}$, where U is a maximal isotropic subspace for the bilinear map $H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C})$ given by cup product.*

Remarks 2.7.

(a) Of course one also obtains another bijection by associating to f the subspace $f^*(H^1(B,\mathbb{R}))$ of $H^1(X,\mathbb{R})$ (respectively $f^*(H^1(B,\mathbb{Z}))$ of $H^1(X,\mathbb{Z})$).

(b) Moreover, any maximal isotropic subspace U of dimension $b \geq 2$ determines a fibration f, whence it is such that $U \cap \overline{U} = 0$.

It was shown in [Cat7] that the isotropic subspace theorem easily implies the following theorem ([Gro]).

GROMOV'S FEW RELATIONS THEOREM 2.8. *Let X be a compact Kähler manifold and assume there is a surjection $\pi_1(X) \to \Gamma$, where Γ has a presentation with n generators and m relations, where $n \geq m + 2$. Then there is a curve B of genus b, where $b \geq 2$ and $2b \geq n - m$, and a fibration $f\colon X \to B$ such that $f^*(H^1(B,\mathbb{C})) \supset H^1(\Gamma,\mathbb{C}) \subset H^1(X,\mathbb{C})$.*

Theorem 2.6 implies in particular that if two surfaces S, S' are homeomorphic under $\varphi\colon S' \to S$, and $f\colon S \to B$ is a genus b pencil with $b \geq 2$, then $V' = \varphi^*(f^*(H^1(B,\mathbb{C})))$ determines a genus b pencil $f'\colon S' \to B'$. In this situation, let g be the genus of the fibres of f, resp. g' the genus of the fibres of f'. Is it then true that $g = g'$? Using the Seiberg-Witten theory we can answer this question under a stronger assumption:

THEOREM ON THE DIFFERENTIABLE INVARIANCE OF THE FIBRE GENUS IN HIGHER GENUS PENCILS 2.9. *Let S, S' be compact Kähler minimal surfaces, $\varphi\colon S' \to S$ an orientation preserving diffeomorphism, and $f\colon S \to B$ a genus b pencil with $b \geq 2$. Then the genus b pencil $f'\colon S' \to B'$ determined by $V' = \varphi^*(f^*(H^1(B,\mathbb{C})))$ is such that the genus g' of the fibres equals the genus g of the fibres of f.*

Proof. We can without loss of generality identify S, S' as differentiable manifolds. Looking at the Jacobian variety $J(B)$ as a differentiable manifold J, we have two maps a, resp. a' of S into J. a is defined as the composition of f with the Albanese embedding of B into $J(B)$ and one moment's reflection shows that a is obtained (up to translation) by integration: $a(x) = \int_{x_o}^{x} \omega$, where ω is the vector ${}^t(\omega_1, \dots \omega_b)$, $\omega_1, \dots \omega_b$ being a basis of $f^* H^0(B, \Omega^1_B)$. Similarly, $a'(x) = \int_{x_o}^{x} \omega'$, with ω' the vector ${}^t(\omega'_1, \dots \omega'_b)$, $\omega'_1, \dots \omega'_b$ being a basis of $f'^* H^0(B', \Omega^1_{B'})$, and also a' factors, through f'.

Since the two isotropic subspaces correspond to each other, i.e., they generate the same real subspace of dimension $2b$ (spanned by integral classes), we can replace the forms ω'_i by suitable $\mathbb{C}$-combinations of the ω'_i's and of their complex conjugates, and after that we can clearly assume that the cohomology class of ω_i in $H^1(S, \mathbb{C})$ equals the one of ω'_i (the price is to lose the holomorphicity of the map a').

Whence, we can write $\omega' = \omega + d\psi$, with $\psi\colon\ S \to \mathbb{C}^b$ a differentiable function.

Step 1. a is homotopic to a' under $a(x,t) = \int_{x_o}^{x} \omega + [t\psi(x)]$, $[v]$ denoting the class in $J = H^0(B, \Omega^1_B)/H^1(B, \mathbb{Z}) = \mathbb{C}^b/\Lambda$ of a vector v of $\mathbb{C}^b$.

Claim 2. The respective fibres F of f, F' of f', yield cohomology classes which are either equal or opposite.

Proof of Claim 2. The cohomology class of F (i.e., Poincaré dual to F) is the pullback under a of the standard polarization λ in $H^2(B, \mathbb{Z}) \subset H^2(J, \mathbb{Z})$. Since a, a' are homotopic, to show that F, F' are cohomologous it suffices to show that if λ' is defined analogously, then $\lambda' = \lambda$. Now,

$$\begin{aligned} a^* H^2(B, \mathbb{Z}) &= f^* H^1(B, \mathbb{Z}) \wedge f^* H^1(B, \mathbb{Z}) = f'^* H^1(B, \mathbb{Z}) \wedge f'^* H^1(B, \mathbb{Z}) \\ &= a'^* H^2(B', \mathbb{Z}), \end{aligned}$$

thus λ' and λ are $\mathbb{Z}$ proportional, whence, since both of them are unimodular in $H^2(J, \mathbb{Z})$, $\lambda' = \pm\lambda$. □

From our assumption it follows that either S is ruled and $g = 0$, or S is elliptic and $g = 1$, or S is of general type and $g \geq 2$. The three cases are distinguished by the sign of $K^2, < 0, = 0, > 0$. Since K^2, by the Index formula and Noether's formula, is a topological invariant for orientation preserving homeomorphisms, we can directly reduce to consider the case where both S, S' are of general type. But then one of the main results of Seiberg-Witten theory (cf. [Mor], Cor. 7.4.3, p. 123), still identifying S and S' as differentiable manifolds, says that $K_{S'} = \pm K_S$. Since $2g - 2 = K_S F$, we get therefore $(2g' - 2) = \pm(2g - 2)$. But since as we noticed g', $g \geq 2$, we conclude that $g' = g$. □

Remarks 2.10.

(1) Under the hypotheses of Theorem 2.9 it follows that the fibres F, F' are diffeomorphic (being $g' = g$). This result is no longer true in higher dimensions, as remarked by Bogomolov and Kollar as a consequence of the s-cobordism theorem (cf. [Cat5], Prop. 1.6, p. 267; where, as pointed out by D. Kotschick, the h-cobordism theorem is incorrectly quoted instead of the s-cobordism theorem of [Maz]).

(2) If we only assume φ to be an orientation preserving homeomorphism, the answer to the question $g' = g$? is still unknown.

(3) Notice that the statement of Theorem 2.9 cannot be strengthened to require that there is a diffeomorphism $\psi\colon B' \to B$ such that $f\varphi = \psi f'$. In fact, the topological structure of the singular fibres is not invariant by deformation.

We end this section by quoting the analogues of Theorems 2.6 and 2.8 for the logarithmic case, obtained by I. Bauer in [Ba], (Theorems 2.1 and 3.1) and by D. Arapura in [Ara] (Corollary 1.9) i.e., for the case of a fibration $f\colon Y \to C$ where C is a curve and Y, C are only required to be quasiprojective (in this case the role of the genus of the base curve is taken by the logarithmic genus c of C, which can be defined as the difference between the first Betti number of C and the genus b of the compactification B of C). The following is a more detailed statement of Theorem 2.1 of [Ba].

THEOREM OF THE LOGARITHMIC ISOTROPIC SUBSPACE 2.11. *Let Y be a quasiprojective manifold $Y = X - D$, where X is smooth, projective, and D is a normal crossings divisor. Then every real maximal isotropic subspace V of $H^1(Y, \mathbb{R})$ either of dimension ≥ 3 or of dimension 2 but not coming from a nonisotropic subspace V' of $H^1(X, \mathbb{R})$ (in this case the theorem does not hold) determines a unique logarithmic irrational pencil $f\colon Y \to C$ onto a curve C with logarithmic genus at least 2. One has that C is projective iff $V \subset H^1(X, \mathbb{R})$, and is isotropic there, otherwise $V = f^*(H^1(C, \mathbb{R}))$ and one says that the pencil is strictly logarithmic.*

Proof. We indicate here how to modify the argument of Theorem 2.1 of [Ba]. One notices that we can assume that the elements $\alpha_1, \dots \alpha_l$ are $\mathbb{R}$- linearly independent in $H^1(Y, \mathbb{R})$ / $H^1(X, \mathbb{R})$, whence they are also $\mathbb{C}$-linearly independent in $H^0(X, \Omega^1_X(logD))$ / $H^0(X, \Omega^1_X)$. The case k = 0 works as in [loc. cit] (except that it is better to consider $m' = rank_{\mathbb{C}}(\eta_1, \dots \eta_l)$, , instead of $m = rank_{\mathbb{R}}(\eta_1, \dots \eta_l)$, and, in the case $m' = 1$, to replace α_2 by $\alpha_2 + i\eta_1$). The case k ≥ 1 is the one where it is difficult to apply the Castelnuovo de Franchis type Theorem, which requires two $\mathbb{C}$-linearly independent forms. This occurs precisely when $l = 0$ and the ω_i''s are dependent, i.e., in the case where the complex span of V is generated by a holomorphic 1-form ω and by its conjugate $\bar{\omega}$. This is exactly the case which is excluded by our assumption, since $\omega \wedge \bar{\omega}$ is certainly nonzero in the cohomology of X. The last assertion being easy to show, it remains to show an example where

$i\omega \wedge \bar{\omega}$ is zero in $H^2(Y, \mathbb{R})$ but Y has no holomorphic map to a curve of genus at least 1. Our example will be a principally polarized abelian surface X with Picard number $\rho = 3$, but not containing any elliptic curve. We take as period matrix in Siegel's upper half space the matrix $Z = iY$ with $Y_{11} = 1, Y_{12} = 2^{1/2}, Y_{22} = 5$. For this choice, using the notation of [L-B], exercise 4, page 43 and exercise 10, page 319, the Neron Severi Group corresponds to the integral antisymmetric matrices for which $f = b, a = 3c, e = 5d$, and there are no elliptic curves since the following Diophantine equation has no solutions: $3c^2 + 5d^2 = b^2$ (one verifies that this is the case by looking at the congruence $mod 5$).

Finally, one looks at the antisymmetric matrices which correspond to elements in the cohomology group $H^2(X, \mathbb{R})$, of the form $i\omega \wedge \bar{\omega}$: these are alternating forms which can be written as

$$-L(x) \wedge L(ix)$$

where L is an arbitrary $\mathbb{R}$-linear form on the vector space underlying X. As in [Ba], 2.6, $i\omega \wedge \bar{\omega}$ is zero in a suitable Zariski open set Y of X if and only if $-L(x) \wedge L(ix)$ lies in the real span of the Neron-Severi group. A direct calculation shows that $a = 3c, e = 5d$ are automatically verified, whereas $f = b$ boils down to the equation

$$3l_1^2 + 5l_3^2 = 5l_2^2 + l_4^2,$$

for the four coefficients of L: but this equation clearly has real solutions. □

The following result is due to I. Bauer [Ba] under the restriction $n \geq m + 3$ and to D. Arapura [Ara] in the general case $n \geq m + 2$.

THEOREM OF THE LOGARITHMIC FEW RELATIONS 2.12. *Let Y be a quasiprojective manifold $Y = X - D$, and assume there is a surjection $\pi_1(Y) \to \Gamma$, where Γ has a presentation with n generators and m relations, where $n \geq m + 2$. Then there is an integer $\beta \geq n - m$, a quasiprojective curve C with first Betti number equal to β, and a fibration $f\colon\ Y \to C$.*

We finally mention a quite straightforward generalization of the Zeuthen-Segre inequality to the logarithmic case. It requires, firstly, the use of Borel-Moore homology (cf. [B-M], [B-H]), thus it is phrased in terms of the Borel-Moore homology Euler number $e_{BM}(Y)$; secondly, it also requires the appropriate

Definition 2.13. Let Y be a quasiprojective surface $Y = S - D$, where S is smooth, projective, and D is a normal crossings divisor. Consider a logarithmic pencil $f\colon\ Y \to C$, where f is surjective with connected fibres, and we can assume that f extends to a holomorphic map $f\colon\ S \to B \supset C$. f is said to be *very good* if moreover every component Δ of D which does not map to (the finite set) $B - C$

maps onto B (in other words, we remove from S either entire fibres, or horizontal curves).

THEOREM 2.14. *Let Y be a quasiprojective surface Y, and $f\colon Y \to C$ a very good logarithmic pencil. Then $e_{BM}(Y) \geq e_{BM}(F) \cdot e_{BM}(C)$, F being the general fibre of f.*

Proof. Let the notation be as in Definition 2.13. Let $Z \subset S$ be $f^{-1}(C)$. Then by the usual argument $e_{BM}(Z) \geq e_{BM}(\Phi) \cdot e_{BM}(C)$, where Φ is the complete fibre. Write $Y = Z - D'$: then D' is a finite (proper) degree d covering of C, whence $e_{BM}(D') \leq d\, e_{BM}(C)$. We conclude since $e_{BM}(Y) = e_{BM}(Z) - e_{BM}(D') \geq (e_{BM}(\Phi) - d) \cdot e_{BM}(C) = e_{BM}(F) \cdot e_{BM}(C)$. □

3. Surfaces isogenous to a product.

Definition 3.1. A surface S is said to be *isogenous to a product* if S admits a finite unramified covering which is isomorphic to a product of curves ($u\colon C_1 \times C_2 \to S$) of genera g_i = genus $(C_i) \geq 1$. In the case where each $g_i \geq 2$ we shall also say that S is isogenous to a *higher product*.

Remark 3.2. If in the above definition we would allow $g_i = 0$, then we would have the case where $S \to B$ is a particular type of $\mathbb{P}^1$-bundle (this case is well understood, and if one views the condition upside down, one wants the existence of an unramified cover $C \to B$ such that the pullback of the bundle is trivial; the condition is not satisfied, for a small deformation of $S \to B$, in the case where b = genus$(B) \geq 1$). Whereas in the case $g_1 = g_2 = 1$ we have a classical hyperelliptic surface or an algebraic torus. Observe that both g_i's are ≥ 2 iff S is of general type. Moreover, a surface isogenous to a product is always minimal. We shall denote for later use by Π_g the fundamental group of a compact Riemann surface of genus g.

We shall now give a first topological characterization of surfaces of general type isogenous to a higher product. Note that in the remaining cases S is an elliptic surface, and, by surface classification, a topological characterization is possible except in the case where S is an algebraic torus (since the tori isogenous to a product of elliptic curves form a (complex) codimension 1 set). In fact, e.g., from [B-P-V], table 10, page 188, follows easily

Remark 3.3. S is a hyperelliptic surface iff $e(S) = 0$, $b_1(S) = 2$, and $\pi_1(S)$ is not Abelian. S is isogenous to a product with $g_1 = 1$, $g_2 > 1$, iff $e(S) = 0$, and $\pi_1(S)$ admits a finite index subgroup isomorphic to $\Pi_1 \times \Pi_{g_2}$.

THEOREM 3.4. *A surface S is isogenous to a higher product if and only if $\pi_1(S)$ admits a finite index subgroup Γ isomorphic to $\Pi_{g_1} \times \Pi_{g_2}$, with $g_1, g_2 > 1$, and* *moreover:*

(e) *if d denotes the index of* Γ, *then* $e(S) = \frac{4(g_1-1)(g_2-1)}{d}$.

If S is minimal, hypothesis (e) can be replaced by

(k) $K_S^2 = \frac{8(g_1-1)(g_2-1)}{d}$, *or by*

(c) $\chi(\mathcal{O}_S) = \frac{(g_1-1)(g_2-1)}{d}$.

Setting $\Pi = \Pi_{g_1} \times \Pi_{g_2}$, *one can indeed relax the hypothesis* $\Gamma \cong \Pi$ *to the weaker assumption that* $\Gamma/\Gamma_3 \cong \Pi/\Pi_3$, *where, for a group* Γ, Γ_3 *is the third group in the nilpotent series (i.e.,* Γ_2 *is the commutator subgroup* $[\Gamma,\Gamma]$ *and* $\Gamma_3 = [\Gamma, \Gamma_2]$*).*

Proof. The "only if" part is obvious. For the other implication, assuming we are given such a subgroup Γ, we take the associated unramified covering. Therefore it suffices to prove the theorem in the special case $d = 1$.

The first step of the proof consists in showing that we have two distinct pencils on S, providing a surjective holomorphic map to a product of curves $f = (f_1 \times f_2)\colon\ S \to C_1 \times C_2$.

For the first step we have two proofs, the first exploiting the isomorphism $\pi_1(S) \cong \Pi_{g_1} \times \Pi_{g_2}$ which produces, by Gromov's Theorem 2.8, two fibrations $f_i\colon\ S \to B_i$ such that

$$f_i^*(H^1(B_i,\mathbb{C})) \supset H^1(\Pi_{g_i},\mathbb{C}) \subset H^1(S,\mathbb{C}) = H^1(\Pi_{g_1},\mathbb{C}) \oplus H^1(\Pi_{g_2},\mathbb{C}).$$

We claim then that the two pencils are distinct. Were they equal, we would get a fibration $f\colon\ S \to B$ where B has genus $g_1 + g_2$. Therefore, in the latter case we would get a corresponding surjection of fundamental groups $\Pi_{g_1} \times \Pi_{g_2} \to \Pi_{g_1+g_2}$, and a corresponding homomorphism of cohomology algebras inducing an isomorphism of first cohomology groups. Let us consider in particular the induced homomorphism

$$\Lambda^4(H^1(\Pi_{g_1+g_2},\mathbb{C})) \to H^4(\Pi_{g_1} \oplus \Pi_{g_2},\mathbb{C}).$$

We get a contradiction, since on the one side the above homomorphism must be nonzero, being that

$$\Lambda^4(H^1(\Pi_{g_1},\mathbb{C}) \oplus H^1(\Pi_{g_2},\mathbb{C})) \to H^4(\Pi_{g_1} \oplus \Pi_{g_2},\mathbb{C}))$$

has nonzero image; while on the other the above homomorphism must be zero, since it factors through

$$\Lambda^4(H^1(\Pi_{g_1+g_2},\mathbb{C})) \to H^4(\Pi_{g_1+g_2},\mathbb{C}) = 0.$$

The second proof uses the weaker assumption $\Gamma/\Gamma_3 \cong \Pi/\Pi_3$ and the fact that, for a Kähler manifold X, the kernel of the cup product map $H^1(X,\mathbb{C}) \wedge H^1(X,\mathbb{C}) \to$

$H^2(X, \mathbb{C})$ is dual to $(\Gamma_2/\Gamma_3) \otimes \mathbb{R}$ (cf. [ABCKT], Cor. 3.14, p. 33 and Prop. 3.25, p. 38).

Applying this observation to $X = S$ and $X = C_1 \times C_2$, we infer that $H^1(X, \mathbb{C})$ is the direct sum of two maximal isotropic subspaces, whence we obtain the two distinct pencils by the Isotropic Subspace Theorem 2.6.

Step II. Since the pencils are distinct, we obtain a surjective holomorphic map $f = (f_1 \times f_2)\colon S \to C_1 \times C_2$. Consider now the fibration f_2, which factors through f. The genus g_1' of the fibres of f_2 is therefore at least g_1. Since by assumption $4(g_1 - 1)(g_2 - 1) = e(S)$ which, by Theorem 2.1 (Zeuthen-Segre) is $\geq 4(g_1' - 1)(g_2 - 1) \geq 4(g_1 - 1)(g_2 - 1)$, equality holds throughout whence $g_1' = g_1$, and f_2 is a topological bundle. Since the general fibre of f_2 is isomorphic then to C_1, and f_2 is a topological bundle, then f yields the desired isomorphism $S \cong C_1 \times C_2$.

In the case where we assume $K_S^2 = 8(g_1 - 1)(g_2 - 1)$, we invoke Arakelov's Theorem (2.2), and we obtain therefore that f is birational. Since moreover we assume S minimal, then f yields an isomorphism. Case (c) follows immediately from Theorem 2.3. □

Remark 3.5. As we saw in Section 1 (Cor. 1.3), the property that a surface S is isogenous to a higher product cannot be detected solely by its fundamental group. From the geographical viewpoint, the previous theorem is rather interesting, because, in terms of the topological invariants of the minimal surface $S, x = \chi(\mathcal{O}_S)$, $y = K_S^2$, it tells that on the lines x = constant, y = constant, $12x - y$ = constant, such a fundamental group commands a geometric property. It is an interesting question whether the condition of lying on the watershed line $y = 8x$ (S has zero index) can replace in Theorem 3.4 the conditions (e), (k) or (c). One can further weaken the hypothesis on the fundamental group of S using the following:

PROPOSITION 3.6. *A fundamental group* $\pi_1(S)$ *is* commensurable to a (direct) product $\Pi_{g_1'} \times \Pi_{g_2'}$ *if and only if it contains a subgroup of finite index which is isomorphic to a group* $\Pi_{g_1} \times \Pi_{g_2}$.

Proof. Assume $\pi_1(S)$ contains a finite index subgroup Γ which is isomorphic to a subgroup of $\Pi_{g_1'} \times \Pi_{g_2'}$ (again of finite index). Then Γ contains Γ' normal and of finite index in $\Pi_{g_1'} \times \Pi_{g_2'}$. Set $\Gamma_1' = \Gamma' \cap \Pi_{g_1'}$: it is normal in $\Pi_{g_1'}$, of finite index, and isomorphic to a Π_{g_1}. Define similarly $\Gamma_2' \cong \Pi_{g_2}$. Then $\Pi_{g_1} \times \Pi_{g_2}$ is of finite index in Γ, whence also in $\pi_1(S)$. □

In order to study the moduli space of surfaces isogenous to a product, we need to show that a stronger property is satisfied by these surfaces.

Definition 3.7. A surface S is said to be *strongly isogenous to a product* if S is the quotient of the action of a group G acting freely on a product of curves ($S = C_1 \times C_2/G$).

In order to study this stronger property, we need some auxiliary results, of which the following is well known.

RIGIDITY LEMMA 3.8. *Let $f\colon C_1 \times C_2 \to B_1 \times B_2$ be a surjective holomorphic map between products of curves. Assume that both B_1, B_2 have genus ≥ 2. Then, after possibly exchanging B_1 with B_2, there are holomorphic maps $f_i\colon C_i \to B_i$ such that $f(x,y) = (f_1(x), f_2(y))$.*

Sketch of proof. $\mathrm{Hol}(C_i, B_j)$ is a discrete (indeed finite) set, since the normal bundle to the graph of a $\varphi\colon C_i \to B_j$ is isomorphic to the pullback of the tangent bundle to B_j. Thus, if a component of f, $f_i(x,y)$ is not constant in x, it is constant in y and the statement follows then easily from the surjectivity of f. □

COROLLARY 3.9. *Assume that both C_1, C_2 are curves of genus ≥ 2. Then the inclusion $\mathrm{Aut}(C_1 \times C_2) \supset \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$ is an equality if C_1 is not isomorphic to C_2, whereas $\mathrm{Aut}(C \times C)$ is a semidirect product of $\mathrm{Aut}(C)^2$ with the $\mathbb{Z}/2$ given by the involution Φ exchanging the two coordinates.*

Remark 3.10. Assume that S can be represented as a quotient $S = C_1 \times C_2/G$ by a free action. Then we can assume that G, or the subgroup $G_o = G \cap \mathrm{Aut}(C)^2$ in the case $C_1 \cong C_2$, embeds in $\mathrm{Aut}(C_i)$ for $i = 1, 2$ (we shall then say that this quotient realization is minimal). Otherwise, e.g., the kernel $\mathcal{K}_1$ acts trivially on C_1 and freely on C_2; we can thus replace $C_1 \times C_2$ by $(C_1/\mathcal{K}_2) \times (C_2/\mathcal{K}_1)$.

PROPOSITION 3.11. *A surface S is strongly isogenous to a higher product if and only if it is isogenous to a higher product.*

Proof. We clearly only need to show that if there is a finite unramified covering $u\colon C_1 \times C_2 \to S$, then S can be represented as a quotient $S = C_1' \times C_2'/G$ by a free action. Let Z be the Galois closure of u, and G' the Galois group, so that we have another unramified covering $\psi\colon Z \to C_1 \times C_2$.

All the fibres of $Z \to C_2$ are unramified coverings of C_1, whence, taking the Stein factorization of $Z \to C_2$, we get a holomorphic bundle $Z \to C_2''$. Finally, the canonical unramified and Galois base change $C_2' \to C_2''$ associated to the kernel of the monodromy yields by pullback a product $C_1' \times C_2'$. To conclude, since S is a Galois unramified quotient of Z, and Z is a Galois unramified quotient of $C_1' \times C_2'$, we need to show that every Galois automorphism τ of Z lifts to $C_1' \times C_2'$. Let $\upsilon\colon C_1' \times C_2' \to Z$ be the quotient map. Then, by the rigidity lemma, either $\psi\tau\upsilon$, or possibly $\Phi\psi\tau\upsilon$ is a product map of respective maps $C_i' \to C_i$. Therefore either τ preserves the fibres of the map $p_2''\colon Z \to C_2''$, or sends them to the fibres of $p_1''\colon Z \to C_1''$. In both cases there is an isomorphism $\tau''\colon C_2'' \to C_i''$ such that $\tau'' p_2'' = p_i'' \tau$.

Clearly then τ'' leaves invariant the kernel of the monodromy, whence we get a lift $\tau'\colon C_2' \to C_i'$, and, by taking the fibre product with $\tau\colon Z \to Z$, we obtain the desired lifting of τ to $C_1' \times C_2'$. □

We can sharpen the previous result

COROLLARY 3.12. *Every minimal product cover of a surface S strongly isogenous to a higher product is Galois.*

Proof. In the notation of Proposition 3.11, we have $S = C_1' \times C_2'/G$ and a factorization $C_1' \times C_2' \to C_1 \times C_2 \to S$, so that there is a subgroup G' of G such that $C_1 \times C_2 = C_1' \times C_2'/G'$. But since $G' = \pi_1(C_1 \times C_2)/\pi_1(C_1' \times C_2')$ and by the rigidity lemma we have a product map, G' is a direct product $G_1' \times G_2'$. Then $G_1' \subset \mathcal{K}_2'$, $G_2' \subset \mathcal{K}_1'$ (cf. Remark 3.10), thus $C_1 \times C_2$ dominates $(C_1'/\mathcal{K}_2') \times (C_2'/\mathcal{K}_1')$. Finally, by the minimality assumption, we must have equality, whence $C_1 \times C_2 \to S$ is Galois. □

We shall now show that a minimal realization is unique: by the above it follows that any minimal product covering is Galois and unique.

PROPOSITION 3.13. *If S is isogenous to a higher product, then a minimal realization $S = C_1 \times C_2/G$ is unique.*

Proof. Assume that we have two such minimal realizations $S = C_1 \times C_2/G$, $S = B_1 \times B_2/\Gamma$. Then we can dominate both products $C_1 \times C_2$, $B_1 \times B_2$ with a component Z of their fibre product over S. By looking at the holomorphic bundle induced on Z by the projection onto B_2, we obtain a Galois unramified covering D_2 of B_2 and a product $D_1 \times D_2$ dominating Z compatibly with the projection onto B_2. By the rigidity lemma we know that $D_1 \times D_2$ dominates $C_1 \times C_2$ by a product map, and we can assume that it does with a map for which D_1 surjects onto C_1, (resp.: D_2 onto C_2). This shows that the two fibrations on S induced by the projections on C_2/G, resp. B_2/Γ, coincide. Since both products $C_1 \times C_2$, $B_1 \times B_2$, are obtained by the same canonical procedure (pullback under the Galois cover associated to the kernel of the monodromy), they are isomorphic. □

In most of the cases surfaces which are isogenous to a higher product are distinguished by the behavior of the Albanese map. Observe preliminarly:

Remark 3.14. A symmetric product $C^{(2)}$ of a curve C of genus $g \geq 2$ is never birational to a product of curves of respective genera $a \geq 2$, $b \geq 1$.

Proof. The invariants of $C^{(2)}$ are $q = g$, $p_g = \frac{g(g-1)}{2} \geq 1$, whereas the invariants of a product of curves of respective genera a, b are $q = a+b$, $p_g = ab$. If $g = a+b$, then $p_g = \frac{g(g-1)}{2} = ab + \frac{1}{2}(a^2 - a + b^2 - b)$ and this expression equals ab (which has to be ≥ 1) iff $a = b = 1$. In this case (excluded by our assumptions) then $g = 2$ and $C^{(2)}$ is the Jacobian of C, and it can indeed be isomorphic to a product of elliptic curves (cf. [L-B], pp. 313–316, and ex. 11, 12 on p. 319). □

PROPOSITION 3.15. *Let $S = C_1 \times C_2/G$ be the minimal realization of a surface S isogenous to a higher product, and let G^o be the subgroup of G given by the transformations which do not exchange the two factors C_1, C_2. We shall say that we have the* mixed *case (denoted by M) when $G \neq G^o$, and the* double *case otherwise (denoted by D). Denote by C_i' the quotient C_i/G when $G = G^o$, and let g_i' = genus of C_i'. In the mixed case, $G \neq G^o$, so we set $C' = C/G^o$ and we let g' = genus of C'. Then for the Albanese image $\alpha(S)$ of S, the following possibilities occur:*

(M^{++}) *when $G \neq G^o$, and $g' \geq 2$, then $\alpha(S)$ is birational to the symmetric product $C'^{(2)}$ of C'.*

(M^+) *when $G \neq G^o$, and $g' = 1$, then $\alpha(S)$ is an elliptic curve.*

(D^{++}) *when $G = G^o$, and $g_i' \geq 1$, then $\alpha(S)$ is isomorphic to the product $C_1' \times C_2'$.*

(GH) *(*Generalized Hyperelliptic*) is defined to be the case when $G = G^o$, G operates freely on C_1 and $g_2' = 0$: then $\alpha(S)$ is isomorphic to C_1', which has genus at least 2.*

In the case (D^+), when $g_1' \geq 1$, $g_2' = 0$, but G does not act freely on C_1, then $\alpha(S)$ is isomorphic to C_1'.

In the cases (M^0), when $g' = 0$, or (D^0) $(g_1' = g_2' = 0)$, then $\alpha(S)$ is a point (S has irregularity $q(S) = 0$).

Proof. Since $S = C_1 \times C_2/G$ we have $H^0(\Omega^1_S) = (H^0(\Omega^1_{C_1}) \oplus H^0(\Omega^1_{C_2}))^G \subset H^0(\Omega^1_{C_1})^{G^o} \oplus H^0(\Omega^1_{C_2})^{G^o} = H^0(\Omega^1_{C_1''}) \oplus H^0(\Omega^1_{C_2''})$. In case M the quotient $\mathbb{Z}/2 \cong G/G^o$ exchanges the two last addenda, whence $H^0(\Omega^1_S) \cong H^0(\Omega^1_{C''})$. Therefore $\alpha\colon S \to \alpha(S)$ factors as the projection to $C'^{(2)}$ composed with the Albanese map a of $C'^{(2)}$, which is just the Abel-Jacobi map on degree 2 divisors. Whence when $g' \geq 2$, a is birational to its image, and indeed an embedding unless C' is hyperelliptic. The cases $g' = 0, 1$ are obvious. In the nonmixed cases, α is the composition of the projection to $C_1' \times C_2'$ with the product of the respective Albanese maps. □

We have therefore divided the surfaces S isogenous to a higher product into 7 classes, (D^{++}), (D^+) and (D^0) = of double type, (GH) = Generalized Hyperelliptic surfaces, plus those of mixed type. We want to describe the latter in greater detail.

PROPOSITION 3.16. *Surfaces S isogenous to a higher product and of mixed type are obtained as follows. There is a (faithful) action of a finite group G^o on a curve C of genus at least 2 and a nonsplit extension $1 \to G^o \to G \to \mathbb{Z}/2 \to 1$, yielding* *a class* $[\varphi]$ *in* $\mathrm{Out}\,(G^o) = Aut(G^o)/Int(G^o)$*, which is of order* ≤ 2*. Once we fix a representative* φ *of the above class, there exists an element* $\tau = \lambda(\varphi)$ *in* G^o *such that*

(i) $\varphi\varphi(\gamma) = \tau\gamma\tau^{-1}$.

(ii) G^o *acts, under a suitable isomorphism of* C_1 *with* C_2*, by the formulae:* $\gamma(x, y) = (\gamma x, (\varphi\gamma)y)$ *for* γ *in* G^o*; whereas the lateral class of* G^o *consists of the transformations*

$$\tau'\gamma(x, y) = ((\varphi\gamma)y, \tau\gamma x).$$

Let Σ *be the subset of* G^o *consisting of the transformations having some fixed point. Then the condition that G acts freely amounts to:*

(A) $\Sigma \cap \varphi(\Sigma) = \{identity\}$

(B) *there is no* γ *in* G^o *such that* $\varphi(\gamma)\tau\gamma$ *is in* Σ*.*

Indeed if we pick another automorphism φ *of* G^o *in the given class* $[\varphi]$*, then the choice of* τ *has to vary according to the following cocycle-like condition:*

(iii) *if* $\hat{\varphi} = \varphi \circ \mathrm{Int}\,(\gamma'')$*, then* $\hat{\tau} = \varphi(\gamma'')\tau\gamma''$*.*

Proof. G^o is the subgroup of the transformations not exchanging the two factors of $C \times C$, while we let τ' be a transformation not in G^o. Then, after fixing some isomorphism $C_1 \times C_2 \cong C \times C$, $\tau'(x, y') = (\tau_2 y', \tau_1 x)$. We then choose $y = \tau_2 y'$ as a new coordinate on the second factor. Then $\tau'(x, y) = (y, \tau x)$, where $\tau = \tau_2\tau_1 x$. Since $\tau'\tau'(x, y) = (\tau x, \tau y)$, we see that τ belongs to G^o. Since the realization is minimal, we see G^o as $\subset \mathrm{Aut}\,(C)$ under the action on the first factor, and we write $\gamma(x, y) = (\gamma x, \gamma_2 y)$, so that $\gamma \to \gamma_2$ gives the other embedding of G^0 in $\mathrm{Aut}\,(C)$ provided by the action on the second factor. The condition that τ' normalizes G^o means that for every γ there is a γ'' such that $\tau'\gamma = \gamma''\tau'$.

More precisely this means: $(\gamma_2 y, \tau\gamma x) = (\gamma'' y, \gamma_2''\tau x)$. Thus $\gamma_2 = \gamma''$, $\gamma_2'' = \tau\gamma\tau^{-1}$. The first equality tells us that the image of G^o in $\mathrm{Aut}\,(C)$ is equal for the action on both factors, whence exactly that there is an automorphism φ of G^o such that $\gamma_2 = \varphi(\gamma)$; the second equality says exactly that $\varphi\varphi(\gamma) = \tau\gamma\tau^{-1}$.

There remains to ensure the condition that G acts freely: for G^o, this means that either γ or $(\varphi\gamma)$ acts freely on C, and this is just condition (A). For $\tau'\gamma$, it acts freely if there is no pair $(x, y) = (\varphi(\gamma)y, \tau\gamma x)$, i.e., $\varphi(\gamma)\tau\gamma$ (or, equivalently, $\tau\gamma\varphi(\gamma)$) has no fixed points on C.

Finally, our calculations above indeed show that, should we replace τ' by $\tau'' = \tau' \cdot \gamma''$, then we would have to take as new variable $y'' = \varphi(\gamma'')y$ so that $\tau''(x, y'') = (y'', \varphi(\gamma'')\tau\gamma'' x)$. Thus τ gets replaced by $\hat{\tau} = \varphi(\gamma'')\tau\gamma''$, and φ by $\hat{\varphi} = \varphi \circ \mathrm{Int}\,(\gamma'')$, where $\mathrm{Int}\,(\gamma'')(\gamma) = \gamma''\gamma\gamma''^{-1}$; and again $\hat{\varphi} \circ \hat{\varphi} = \mathrm{Int}\,(\hat{\tau})$. The extension $1 \to G^o \to G \to \mathbb{Z}/2 \to 1$ is nonsplit since $\tau\gamma\varphi(\gamma)$ is always different from the identity. Moreover such an extension gives (by conjugation of a lift of the generator of $\mathbb{Z}/2$) an element $[\varphi]$ of order 2 in $\mathrm{Out}\,(G^o) = \mathrm{Aut}\,(G^o)/\,\mathrm{Int}\,(G^o)$. Observe that such a $[\varphi]$, in general, only provides an extension

$$1 \to \mathrm{Int}\,(G^o) \to \Gamma \to \mathbb{Z}/2 \to 1. \qquad \square$$

Remarks 3.17.

(1) A trivial solution for finding surfaces of type (M) is to take an unramified Abelian Galois covering C of C', of order divisible by 2, let φ be the identity, and take τ an element in the Galois group G^o which is not a square.

(2) The easiest way to construct surfaces of the other types is to take a quotient $C_2 \to C_2/G$, and then for any curve C_1' of genus ≥ 2, take a surjection $\pi_1(C_1') \to G$.

We shall return more amply to these issues in the next section. We end this section by giving different characterizations of Generalized Hyperelliptic surfaces, again using the standard inequalities which we recalled in Section 2.

THEOREM 3.18. *Let S be a surface such that:*

(o) $K_S^2 = 8\chi(S) > 0$ *(S has index 0 but* $e(S) > 0$*).*

(i) *S has irregularity* $q \geq 2$ *and the Albanese map is a pencil (i.e.,* $\Lambda^3(H^1(S, \mathbb{C})) \to H^3(S, \mathbb{C})$ *has zero image).*

Then, letting g be the genus of the Albanese fibres, we have: $(g-1) \leq \frac{\chi(S)}{(q-1)}$.

A surface S is a Generalized Hyperelliptic surface if and only if (o), (i) hold and $g = 1 + \frac{\chi(S)}{(q-1)}$.

In particular, every S satisfying (o), (i), and with $p_g = 2q-2$ *is a Generalized Hyperelliptic surface where G is a group of automorphisms of a curve* C_2 *of genus 2 with* $C_2/G \cong \mathbb{P}^1$.

Proof. If (o) holds and S is fibred over a curve B of genus b, note that the inequalities of Zeuthen-Segre, Arakelov, Beauville, are all equivalent to each other. Since by (i) we have a pencil with base curve B of genus q, the first assertions are simply a restatement of the aforementioned theorems; for instance, if equality holds, we have that the Albanese map α yields a holomorphic bundle, whence S is a GH surface. Finally, $p_g = 2q - 2$, since $\chi(S) = 1 - q + p_g$, is equivalent to $\chi(S) = (q-1)$. Thus $g \leq 2$ and since $\chi(S) > 0$, $g = 2$, therefore we again have a holomorphic bundle and S is a GH surface. □

Remark 3.19. The GH surfaces with $g = 2$ can be explicitly classified. In fact, all the groups G acting effectively as a group of automorphisms of a curve C_2 of genus 2 with $C_2/G \cong \mathbb{P}^1$, are explicitly classified by Bolza ([Bo], cf. also the Pisa 1995 Ph.D. thesis by F. Zucconi). One can easily verify that for every such group G there is a surjection $\Pi_2 \to G$. It is a trivial remark that, for each $k \geq 2$, there is a surjection $\Pi_k \to \Pi_2$. We obtain therefore the following:

COROLLARY 3.20. *Assume* $\chi(S) = q-1$, $K_S^2 = 8\chi(S) > 0$, *and that the Albanese map is a pencil. Then S is a Generalized Hyperelliptic surface with Albanese fibre of genus* 2. *These exist for each integral value of* $\chi(S)$.

We end this section by describing surfaces in the classes (D^0) and (D^+), following some examples given by Beauville ([Bea1], exercise 4, p. 159).

LEMMA 3.21. *Let $G = (\mathbb{Z}/n)^2$ and $\Sigma = \{(a,b) \mid a = 0, or\ b = 0, or\ a = b\}$. Then there is an automorphism φ of G with $\varphi(\Sigma) \cap \Sigma = \{0\}$ if and only if 6 and n are relatively prime.*

Proof. If m divides n, then $(\mathbb{Z}/m)^2 \subset G$ and is characteristic. Since such a φ clearly does not exist if $n = 2$ or $n = 3$ because in these cases $\operatorname{card}\Sigma - 1 = 3n - 3 \geq \frac{(\operatorname{card} G)}{2}$, it follows that $2, 3$ cannot divide n. But if $2, 3$ do not divide n, the map φ sending $e_1 = (1,0)$ to $(1,-2)$, and $e_2 = (0,1)$ to $(1,-1)$ has determinant 1, whence it is an isomorphism. φ sends $e_0 = (1,1)$ to $(2,-3)$ and all determinants formed with two vectors e_i, $\varphi(e_j)$ are invertible in $\mathbb{Z}/n$, whence the assertion follows. □

Beauville's examples 3.22. Let C be the Fermat curve of degree n in the projective plane. $C = \{(z_0, z_1, z_2) \mid z_0^n + z_1^n + z_2^n = 0\}$ is a Galois cover of the line $\mathbb{P}^1 = \{(x_0, x_1, x_2) \mid x_0 + x_1 + x_2 = 0\}$ under the map $\psi\colon \mathbb{P}^2 \to \mathbb{P}^2$ sending (z_0, z_1, z_2) to (z_0^n, z_1^n, z_2^n). ψ is branched in 3 points, and is Galois with Galois group $G = (\mathbb{Z}/n)^2$. In this example the 3 stabilizers are the diagonal, and the two coordinate subgroups of $(\mathbb{Z}/n)^2$. Thus Lemma 3.21 applies if n is odd and not divisible by 3.

We let $S_n = C_n \times C_n/(\mathbb{Z}/n)^2$, where $G = (\mathbb{Z}/n)^2$ acts by $g(x,y) = (gx, \varphi(g)y)$. So the action is free and S_n is a surface of type (D^0) (i.e., isogenous to a product and with $q = 0$), with invariants $K^2 = 2(n-3)^2$, $\chi = \frac{(n-3)^2}{4}$. (The most famous of these examples occurs for $n = 5$, yielding $\chi = 1$.)

We also obtain surfaces of class (D^+), simply by letting $C_1 = C$, and C_2 be the fibre product of C with an elliptic curve E mapping to $\mathbb{P}^1$ with degree 2 and branched in 4 general points. Again by twisting the action on the second factor, we get a free action of G such that $S = C_1 \times C_2/G$ has invariants

$$K^2 = 4(n-3)(3n-3),\ \chi = \frac{(n-3)(3n-3)}{2},\ q(S) = 1.$$

The final remark we want to make is that, in view of the results we shall show in Section 4 (especially Thm. 4.13 and its proof) the above examples are strongly rigid; in the sense that any surface S with isomorphic fundamental group and the same Euler number as one of Beauville's examples is exactly one of those examples. This motivates the following:

Definition 3.23. A Beauville surface is a rigid surface which is isogenous to a product.

Classifying all the Beauville surfaces is then a problem in group theory. That is, we give a finite group G, representable as a quotient of a triangle group $\langle a, b, c \mid abc = 1 = a^p = b^q = c^r \rangle$, with p, q, r minimal. The question is to classify all those G for which there is an automorphism φ for which $\varphi(\Sigma) \cap \Sigma = \{1\}$, Σ being the union of the 3 subgroups generated by a, b, c, respectively.

4. The fundamental group of surfaces isogeneous to a product, and the irreducibility of their moduli space. Let S be a surface isogenous to a product, and let as usual $S = C_1 \times C_2/G$ be its minimal realization. Let us assume for the time being that S is not of mixed type. Therefore we have an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \pi_1(S) \to G \to 1, \tag{4.1}$$

where the two factors Π_{g_1}, Π_{g_2}, are both normal. We can therefore take the quotient ($i = 1, 2$ has to be understood as an element of $\mathbb{Z}/2$)

$$\Pi(i+1) := \pi_1(S)/\Pi_{g_i}, \tag{4.2}$$

which sits naturally into an exact sequence

$$1 \to \Pi_{g_i} \to \Pi(i) \to G \to 1. \tag{4.3}$$

If G operated freely, e.g., on C_1, we would then have a bundle $S \to C_1'$, with fibre C_2, and the homotopy exact sequence of the bundle would tell us immediately that $\Pi(1) \cong \pi_1(C_1')$; what we are going to see next is that (4.3-i) is the "orbifold" exact sequence associated to the possibly ramified Galois covering $C_i \to C_i'$, so that $\Pi(i)$ is the "orbifold fundamental group of the covering."

So, let us recall the notion of orbifold fundamental group.

Definition 4.4. Let X be a complex manifold and p: $X \to Y = X/G$ the quotient by a finite group of automorphisms. Then Y is a normal complex space, and we denote by B the branching locus of p ($B = p(\{x \mid x$ has a nontrivial stabilizer $G_x\})$). Let $B_1, \ldots B_r$ be the divisorial irreducible components of B. Let $\gamma_1, \ldots \gamma_r$ in $\pi_1(Y - B)$ be simple geometric loops around the components $B_1, \ldots B_r$. Then, since we have an exact sequence

$$1 \to \pi_1(X - p^{-1}(B)) \to \pi_1(Y - B) \to G \to 1,$$

we get that each γ_i maps to an element g_i, whose order will be denoted by m_i. Let m'' be the vector ${}^t(m_1, \ldots m_r)$. Then the *orbifold fundamental group* $\pi_1^{\text{orb}}(Y - B|m'')$ is defined to be the quotient of $\pi_1(Y - B)$ by the minimal normal subgroup containing the elements $(\gamma_i)^{m_i}$.

Notice that $p^{-1}(B_i)$ is a divisor $= m_i R_i$, where R_i (not necessarily connected) is a covering of B_i of degree $\frac{d}{m_i}$ (d being the order of G); $(\gamma_i)^{m_i}$ lifts to a simple geometric loop in $X - p^{-1}(B)$ around a component of R_i, whence it belongs to the kernel of $\pi_1(X - p^{-1}(B)) \to \pi_1(X)$. On the other hand, the above kernel is normally generated by a set of simple geometric loops around each component of each R_i. But, varying the components of R_i, we get such loops simply as conjugates of $(\gamma_i)^{m_i}$.

We have therefore shown:

PROPOSITION 4.5. *As in Definition* 4.4, *let X be a complex manifold and $p\colon X \to Y = X/G$ a finite quotient map. Then we have an exact sequence*

$$1 \to \pi_1(X) \to \pi_1^{\mathrm{orb}}(Y - B|m'') \to G \to 1.$$

PROPOSITION 4.6. *The exact sequence* (4.3-i)

$$1 \to \Pi_{g_i} \to \Pi(i) \to G \to 1,$$

is the orbifold exact sequence for the quotient map $C_i \to C_i'$.

Proof. Let C_2^* be equal to the inverse image of $C_2'^* = C_2' - B$, B being the branching divisor. Denoting by $S^* = (C_1 \times C_2^*)/G$, we get a covering space exact sequence, fitting into a diagram

(4.7)

$$\begin{array}{ccccccccc}
 & & & & \Pi_{g_1} & & & & \\
 & & & & \searrow & & & & \\
1 \to \Pi_{g_1} \times \pi_1(C_2^*) & \to & \pi_1(S^*) & \to & G & & \to & & 1 \\
\downarrow & & \downarrow & \searrow\!\!\!\!\swarrow & & & & & \\
1 \to \Pi_{g_1} \times \pi_1(C_2) & \to & \pi_1(S) & & \pi_1(C_2'^*) & \to & \pi_1^{\mathrm{orb}}(C_2'^*|m'') & \to & G,
\end{array}$$

where the vertical maps are surjective, and the oblique sequence is the exact homotopy sequence of the bundle $S^* \to C_2'^*$.

Notice that $S - S^*$ is the union of the fibres F_b for b in B, which are multiples of multiplicity m_b; thus the kernel of $\pi_1(S^*) \to \pi_1(S)$ is normally generated by simple geometric loops δ_b, mapping to $(\gamma_b)^{m_b}$ down in $\pi_1(C_2'^*)$. Therefore $\pi_1(S)$ surjects onto $\pi_1^{\mathrm{orb}}(C_2'^*|m'')$, and indeed the surjection $\pi_1(S) \to G$ factors through $\pi_1^{\mathrm{orb}}(C_2'^*|m'') \to G$.

It remains to show that Π_{g_1}, which is contained in the kernel, is indeed precisely the kernel of the above surjection. Please note that $\pi_1(C_2'^*) \to \pi_1^{\mathrm{orb}}(C_2'^*|m'') \to G$ is a sequence of surjections (it is not exact!).

After this remark, we observe that the kernel of $\pi_1(S^*) \to \pi_1^{\mathrm{orb}}(C_2'^*|m'')$ is normally generated by the δ_b's, and by Π_{g_1}, whence it is precisely the pullback of the normal subgroup Π_{g_1} of $\pi_1(S)$. The assertion for $\Pi(1)$ follows by identical proof. □

The proof of the following corollary is now obvious.

COROLLARY 4.7. *Let the surface S be isogenous to a product and not of mixed type. Then we have an exact sequence*

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \Pi(1) \times \Pi(2) \to G \times G \to 1$$

(where $\Pi(1) \times \Pi(2) \cong \pi_1^{\mathrm{orb}}(C_1'^{*}|m''(1)) \times \pi_1^{\mathrm{orb}}(C_2'^{*}|m''(2)))$ *such that* $\pi_1(S)$ *is the inverse image of G diagonally embedded in* $G \times G$*.*

Remark 4.8. Although Corollary 4.7 does not yet give a presentation of $\pi_1(S)$, it is possible to get one using the Reidemeister Schreier process. Let us give the classical example of the general hyperelliptic surface, where $G = \mathbb{Z}/2$. We get $1 \to \mathbb{Z}^4 \to \mathbb{Z}^2 \oplus G(2,2,2,2) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 1$. $G(2,2,2,2)$ has generators $a_1, \dots a_4$ such that $a_1 \cdot \dots \cdot a_4 = 1$, $a_i^2 = 1$, $\mathbb{Z}^2$ has generators E_1, E_2, $\mathbb{Z}^4$ has generators $e_1 = 2E_1$, $e_2 = E_2$, $e_3 = a_1 \cdot a_2$, $e_4 = a_2 \cdot a_3$, and each a_i maps to $(0,1)$, while E_1 maps to $(1,0)$, E_2 to $(0,0)$.

Let $a = (E_1, a_1)$: then $\pi_1(S)$ is generated by $e_1, \dots e_4, a$, where the e_i's commute, $a^2 = e_1$, and $ae_ja^{-1} = \delta_j e_j$, with $\delta_j = 1$ for $j = 1, 2$, $\delta_j = -1$ for $j = 3, 4$. Therefore $\pi_1(S)$ is a semidirect product of the normal subgroup $\mathbb{Z}^3$ generated by e_2, e_3, e_4, with the cyclic group $\mathbb{Z}$ generated by a.

Remarks 4.9.

(A) The exact sequence (4.3-i) is also important from the point of view of uniformization. Viewing $C_i = \mathcal{H}/\Pi_{g_i}$, where $\mathcal{H}$ is the upper half plane, one can extend the embedding of $\Pi_{g_i} \to PSL(2, \mathbb{R})$ to an embedding of $\Pi(i)$ such that $C_i' = \mathcal{H}/\Pi(i)$.

(B) The same exact sequence gives a homomorphism $\rho\colon\ G \to \mathrm{Out}(\Pi_{g_i})$ (to an element g of G we associate the action on Π_{g_i} obtained by conjugation with a lifting of g: this action is well defined then only up to inner automorphisms). ρ is injective because every complex automorphism which acts as the identity on the first homology group has to be the identity. Conversely, we claim that the above extension is completely classified by the injective homomorphism ρ. In fact, the inclusion $G \subset \mathrm{Out}(\Pi_{g_i})$ provides an exact sequence $1 \to \mathrm{Int}(\Pi_{g_i}) \to \Pi \to G \to 1$. Observe that the center of Π_{g_i} is trivial (two commuting hyperbolic transformations in $PSL(2, \mathbb{R})$ have the same fixed points, whence they lie in a maximal torus $\mathbb{R}^*$; thus, if they lie in a discrete subgroup, then they generate a cyclic subgroup). Therefore $\Pi_{g_i} \cong \mathrm{Int}(\Pi_{g_i})$, and we recover the extension (4.3-i).

What in fact Proposition 4.6 really tells us is that the geometry of the covering $C_i \to C_i'$ is encoded in the exact sequence (4.1). Indeed, (take e.g. $i = 2$) from it we recover firstly the orbifold exact sequence

$$1 \to \pi_1(C_2) = \Pi_{g_2} \to \pi_1^{\mathrm{orb}}(C_2'^*|m'') = \Pi(2) \to G \to 1.$$

Then, by Remark 4.9, it follows that every branch point b corresponds exactly to a $\Pi(2)$-conjugacy class of a maximal cyclic subgroup of $\Pi(2)$ (every finite order element in $PSL(2,\mathbb{R})$ is elliptic, and has precisely one fixed point in $\mathcal{H}$), whose order is the multiplicity m_b. The ramification points above correspond instead to the Π_{g_2} conjugacy classes of such cyclic subgroups. Moreover, since the epimorphism $\pi_1^{\mathrm{orb}}(C_2'^*|m'') \to \pi_1(C_2')$ has a kernel which is generated precisely by the elements of finite order, we likewise recover $\pi_1(C_2')$.

The irreducibility of the moduli space of surfaces isogenous to a product with given Euler number and given fundamental group would follow at once from a generalization of the classical results of Fricke-Teichmüller-theory, namely that the embeddings of $\Pi(i)$ in $PSL(2,\mathbb{R})$ form an irreducible variety. But we are not aware of such results in the literature, therefore we shall resort in the general case to the solution of the Nielsen realization problem (cf. [Ker], [Wol], [Tro]). In fact, the classical theory suffices when the action of G is free on both factors.

PROPOSITION 4.10. *The moduli space $\mathfrak{M}^{is}(\Gamma)$ of the surfaces S isogenous to a higher product not of mixed type, with the same Euler number $e(S)$ and with fundamental group $\pi_1(S)$ isomorphic to a fixed group Γ, is irreducible in the case where G acts freely on both factors.*

Remark 4.11. Γ being isomorphic to $\pi_1(S)$ it contains, by Proposition 3.13, a unique maximal subgroup Γ'' of finite index isomorphic to $\Pi_{g_1} \times \Pi_{g_2}$, such that $e(S) \cdot (\mathrm{index}(\Gamma'')) = 4(g_1 - 1)(g_2 - 1)$. The property that S is of mixed type can be read off Γ and the Euler number $e(S)$ of S. In fact, S is of mixed type exactly when—although $\Pi_{g_1} \times \Pi_{g_2}$ is, as we know, normal—the individual factors Π_{g_1} and Π_{g_2} are not normal. Also the property that G acts freely on both factors can be read off Γ, since it means that the quotients $\Gamma(i)$ sitting in the exact sequences $1 \to \Pi_{g_i} \to \Gamma(i) \to G \to 1$ have no element of finite order.

Proof of Proposition 4.10. The minimal realization $S = C_1 \times C_2/G$ is given by an action of G on C_1, and another action on C_2. Notice that we obtain a deformation of S by deforming C_1, C_2 together with the action of G. Consider now an isomorphism φ of $\pi_1(S)$ with Γ, and notice that Γ comes equipped with an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \Gamma \to G \to 1,$$

inducing

$$1 \to \Pi_{g_i} \to \Gamma(i) \to G \to 1.$$

From now on we will fix $\Gamma(1), \Gamma(2)$ as the fundamental groups of curves of respective genera g'_1, g'_2. Notice that φ then induces isomorphisms $\pi_1(C'_i) \cong \Gamma(i)$. Given another such surface $\hat{S}$, and another isomorphism $\hat{\varphi}$ of $\pi_1(\hat{S})$ with Γ, using the connectedness of the Teichmüller space for C'_2, we can find a real analytic path deforming $\hat{C}'_2$ to C'_2, and preserving the isomorphism with $\Gamma(2)$, and we can similarly deform $\hat{C}'_1$ to C'_1 preserving the isomorphism with $\Gamma(1)$. Since the action of G on a curve C_i is given by the surjection of $\pi_1(C'_i)$ onto G obtained by composing the isomorphism $\pi_1(C'_i) \cong \Gamma(i)$ with the given surjection of $\Gamma(i)$ onto G, our assertion is proven. □

In the general case, observe that the orbifold fundamental group extension

$$1 \to \pi_1(C_i) = \Pi_{g_i} \to \pi_1^{\text{orb}}(C_i'^{*}|m''(i)) \to G \to 1$$

gives an action on C_i which is topologically classified by the associated injective homomorphism $G \to \mathrm{Out}(\Pi_{g_i})$ mentioned earlier. More precisely, $\mathrm{Out}(\Pi_g) = \pi_0(\mathcal{D}\mathrm{iff}(M_g))$, M_g being a standard differentiable model for Riemann surfaces of genus g. The modular group $\mathrm{Mod}_g = \pi_0(\mathcal{D}\mathrm{iff}(M_g))$ acts properly discontinuously on the Teichmüller space T_g parametrizing the complex structures on M_g (modulo diffeomorphisms isotopic to the identity). Once a finite subgroup G of $\pi_0(\mathcal{D}\mathrm{iff}(M_g))$ is given, the fixed locus of the action on T_g corresponds to the isomorphism classes of curves of genus g with a holomorphic action by G which is differentially equivalent to the given one on M_g.

The following is a slight generalization of one of the solutions [Tro] of the Nielsen realization problem.

LEMMA 4.12. *Given a finite subgroup G of $\pi_0(\mathcal{D}\mathrm{iff}(M_g))$, the fixed locus Fix$(G)$ of G on T_g is nonempty and connected (indeed, diffeomorphic to an Euclidean space).*

Proof. Teichmüller space T_g is diffeomorphic to an Euclidean space of dimension $6g-6$, and it admits a Riemannian metric, the Weil-Petersson metric. Wolpert and Tromba proved:

(i) there exists a C^2-function f on T_g which is proper, G-invariant, nonnegative ($f \geq 0$), and finally f is strictly convex for the given metric (i.e., strictly convex along the W-P geodesics). We remark that, G being a finite group, its action can be linearized at the fixed points, in particular:

(ii) fix(G) is a smooth (non a priori connected) submanifold.

SUBLEMMA. *There is a unique critical point x_o for f, which is an absolute minimum on T_g. Every connected component M of Fix(G) contains x_o.*

Proof of the sublemma. Let M' be either the submanifold M, or T_g. Since f is proper and ≥ 0, there is a constant c such that $f^{-1}([0,c]) \neq \emptyset$, and clearly compact.

Thus there is an absolute minimum x_o on T_g, and an absolute minimum y_o for $f|_M$. Since f is strictly convex on T_g, every critical point for f is isolated and a local minimum. f behaves like a Morse function, therefore, T_g being homeomorphic to an Euclidean space $\mathbb{R}^{6g-6}$, it follows that x_o is the only critical point of f on T_g. It suffices therefore to show that y_o is a critical point for f on T_g, because then $y_o = x_o$ and M contains x_o.

Let us look at the tangent space V to T_g at y_o: this is a G- representation and the subspace of invariants V^G is just the tangent space to M at y_o. We look at the differential of f evaluated at y_o: this gives a linear form df in $V^\vee$ which is invariant since f is G-invariant. We can write $V = V^G \oplus W$, where W is a sum of irreducible nontrivial representations. Therefore $V^\vee = (V^G)^\vee \oplus W^\vee$, and $(V^\vee)^G = (V^G)^\vee$. Since df belongs to $(V^\vee)^G$, and it vanishes on (V^G), df is $= 0$, therefore we have shown that y_0 is a critical point on T_g. The same argument gives that there is exactly one critical point for $f|_M$, whence M is homeomorphic to an Euclidean space. □

We can easily conclude the proof of the lemma: x_0, being the unique minimum, belongs to Fix(G). Moreover, every connected component of Fix(G) contains x_0, thus Fix(G) is a nonempty connected submanifold. □

From the above, we also derive an existence theorem for moduli spaces.

THEOREM 4.13. *Let* Γ *be a group fitting into an exact sequence* $1 \to \Pi_{g_1} \times \Pi_{g_2} \to \Gamma \to G \to 1$ *where moreover:*

(a) *G is a finite group,*

(b) *the individual factors Π_{g_1}, Π_{g_2} are normal in Γ,*

(c) *the induced quotients* $1 \to \Pi_{g_i} \to \Gamma(i) \to G \to 1$ *are such that*

(c1) *there is no element of Γ mapping in each $\Gamma(i)$ to an element of finite order,*

(c′) *by the above exact sequences G embeds in* Out(Π_{g_i}).

Let d be the order of the group G. Then the moduli space of the surfaces S with $\pi_1(S) \cong \Gamma$, and with Euler number $e(S) = \frac{4(g_1-1)(g_2-1)}{d}$ is nonempty and irreducible.

Remark. As in Theorem 3.4, the assumption on $e(S)$ can be replaced by "S minimal and $K_S^2 = \frac{8(g_1-1)(g_2-1)}{d}$, or $\chi(\mathcal{O}_S) = \frac{(g_1-1)(g_2-1)}{d}$".

Proof of Theorem 4.13. The proof is entirely analogous to the one of Proposition 4.10, except that the sequence $1 \to \Pi_{g_2} \to \Gamma(2) \to G \to 1$ determines a fixed topological type of action for G on the curve C_2. According to Lemma 4.11, the pairs (curve C_2, action of G on C_2 with the fixed topological type) are parametrized by an irreducible and nonempty moduli space. The same holds for (curve C_1, action of G on C_1 with the fixed topological type). The conclusion is then the same as in Proposition 4.10, and we have shown that the moduli space is nonempty and irreducible. □

There finally remains the case of the surfaces of mixed type, for which we shall make use of the description given in Corollary 3.16.

THEOREM 4.14. *The moduli space $\mathfrak{M}^{is}(\Gamma)$ of the surfaces S isogenous to a higher product of mixed type, with the same Euler number $e(S)$ and with fundamental group $\pi_1(S)$ isomorphic to a fixed group Γ, is irreducible. In the case where G^o operates freely on C, it is indeed a quotient of the Teichmüller space $T_{g'}$ by a subgroup of* $\mathrm{Mod}_{g'}$*, whence a finite covering of the moduli space $\mathfrak{M}_{g'}$ of curves of genus g'.*

Proof. Γ comes equipped with a unique maximal subgroup isomorphic to $\Pi_g \times \Pi_g$, of index $\frac{4(g-1)^2}{e(S)}$. We get an exact sequence

$$1 \to \Pi_g \times \Pi_g \to \Gamma \to G \to 1,$$

and Γ contains an index 2 subgroup Γ^o containing both subgroups Π_g as normal subgroups. The exact sequence $1 \to \Pi_g \times \Pi_g \to \Gamma^o \to G^o \to 1$ provides two embeddings ε_i of $\Gamma^o(i)$ in $\mathrm{Aut}(\Pi_g)$ (where $1 \to \Pi_{g_i} \to \Gamma^o(i) \to G^o \to 1$). Let t' be an element in $\Gamma - \Gamma^o$, so that conjugating by t' yields an automorphism of $\Pi_g \times \Pi_g$ which exchanges the two factors. Therefore we can write $t'(\sigma_1, \sigma_2)t'^{-1} = (\vartheta\sigma_2, \psi\sigma_1)$. We change the isomorphism of the second factor with Π_g by setting $\vartheta\sigma_1 = \sigma_2'$. Applying the square of the transformation, we get an element of Γ^o, $\tau'' = \vartheta\psi$. In fact, $t'^2(\sigma_1, \sigma_2')t'^{-2} = (\vartheta\psi\sigma_1, \vartheta\psi\sigma_2')$. For γ in Γ^o, which we identify with the element of $\mathrm{Aut}(\Pi_g \times \Pi_g)$ given by inner conjugation, we associate an element $\varphi''(\gamma)$ simply by the rule $\mathrm{Int}(t') \circ \mathrm{Int}(\gamma) = \mathrm{Int}(\varphi''(\gamma)) \circ \mathrm{Int}(t')$. Thus we obtain an automorphism φ'' of Γ^o, and an easy calculation yields $\varphi'' \cdot \varphi'' = \tau''$.

Our proof is over: since again using the Teichmüller space of the curves C' with $\pi_1(C') \cong \Gamma^o(1)$ we can find an irreducibile parametrization of all the curves C with an action of G induced by the surjective homomorphism $\nu\colon \Gamma^o(1) \to G \to 1$. Then the data $([\varphi], \tau)$ are clearly recovered by the classes induced by τ'', φ''. Finally, when the action is free, we have to see when another different point of Teichmüller space gives an isomorphic surface to the one associated to $(C', \beta\colon \pi_1(C') \cong \Gamma^o(1))$. The condition is that we get an isomorphic curve, and that under this isomorphism the given subgroups (pullback of $\ker\nu$) correspond to each other. This means that we get the orbit of the subgroup of the Teichmüller modular group, $\mathrm{Mod}_{g'}$, consisting of the automorphisms μ which leave the subgroup $(\ker\nu)$ invariant. □

Remark 4.15. One can then ask the natural question: when is the above moduli space a variety of general type? (Cf. e.g. [Har].)

We will end this section by showing how to recover the "branching data" (e.g., the isotropy subgroups, ...) for the covering $C = C_i \to C' = C'_i$ (in terms of the orbifold fundamental group extension $1 \to \pi_1(C_i) = \Pi_{g_i} \to \pi_1^{\mathrm{orb}}(C_i'^*|m''(i)) = \Pi(i) \to G \to 1$) in a more computable way. Recall that, for x in C, the subgroup $\mathrm{Stab}\,(x)$ is cyclic, and therefore we will restrict ourselves to considering cyclic subgroups H of G.

Definition 4.16. Given a cyclic subgroup $H = \langle g \rangle$ of G, define $\sigma(H) = \mathrm{card}\{x \mid H = \mathrm{Stab}\,(x)\}$, $s(H) = \mathrm{card}\{x \mid H \subset \mathrm{Stab}\,(x)\} = \mathrm{card}\,(\mathrm{Fix}\,(g))$.

Our goal is to determine the function σ, since then, working out the action of the finite group G, via inner automorphisms, on the set of its cyclic subgroups, we shall recover branching indices and a picture of the covering (of course, once we know $\sigma(H)$, to identify explicitly the corresponding $\sigma(H)$ ramification points, it will in general be necessary to look at liftings $H \to \Pi(i)$ of the inclusion $H \subset G$). To this end, we shall first determine the function s. We already noticed, in fact, that we have an injective homomorphism of G to $\mathrm{Out}\,(\pi_1(C_i)) = \mathrm{Out}\,(\Pi_{g_i})$, inducing an injective homomorphism $G \to \mathrm{Aut}\,(H_1(C_i, \mathbb{Z}))$. By Lefschetz' fixed point formula we then get:

$$s(\langle g \rangle) = \mathrm{card}\,(\mathrm{Fix}\,(g)) = 2 - \mathrm{Tr}\,(g^*) \tag{4.17}$$

$$(g^* \text{ being the action of } g \text{ on } H^1(C_i, \mathbb{Z})).$$

We observe that

$$\sigma(H) = s(H) \text{ if } H \text{ is a maximal cyclic subgroup of } G. \tag{4.18}$$

PROPOSITION 4.19. *The function $\sigma(H)$ is effectively computable from the function $s(H)$.*

Proof. Define the coheight of a cyclic subgroup to be k if $k = \max\{m \mid$ there exists a proper chain of length m of cyclic subgroups $H = H_m \subset H_{m-1} \subset H_{m-2} \cdots \subset H_0\}$. Thus, the maximal cyclic subgroups are exactly those with coheight 0.

We can thus proceed by induction on the coheight k of H. In fact, $\{x \mid H = \mathrm{Stab}\,(x)\} \subset \{x \mid H \subset \mathrm{Stab}\,(x)\} = (\mathrm{Fix}\,(H))$; moreover, for each y in $(\mathrm{Fix}\,(H)) - \{x \mid H = \mathrm{Stab}\,(x)\}$; $\mathrm{Stab}\,(y)$ is a cyclic subgroup $H' \supset H$, and different from H. Obviously such a subgroup H' has coheight $\leq (k-1)$.

We finally observe that $\sigma(H) = s(H) - \sum_{\substack{H' \neq H \\ H' \supset H}} \sigma(H')$.

A closed "inversion" formula to recover σ from s involves the geometry of the directed graph whose vertices are the cyclic subgroups H of G. □

5. A topological characterization of isotrivially fibred surfaces. Let f: $S \to B$ be a constant moduli (relatively) minimal fibration. Taking the canonical Galois cover associated to the kernel of the monodromy of the associated holomorphic bundle f^*: $S^* \to B^*$, we obtain a minimal birational realization of S as a quotient of a product of two curves, ε: $S \cong Y = C_1 \times C_2/G$ ($B = C_1/G$). Recall (cf. Remarks 2.4) that ε is not necessarily a morphism, whence the other fibration $S \to C_2/G$ can only be a rational map, when $C_2' = C_2/G$ has genus 0. This motivates the asymmetry of the following:

Definition 5.1. Let f: $S \to B$ be a constant moduli minimal fibration. Then the good locus U of S is the complement of the inverse image under the rational map ε: $S \to Y = C_1 \times C_2/G$ of the singular locus of Y.

Note that the good locus U maps isomorphically to an open set Y^o in Y and that, in view of 2.4, when ε is not a morphism, the full fibres corresponding to the points where ε is not defined are being removed, both in S and in Y.

The good locus Y^o coincides, when ε is a morphism, with the locus where $C_1 \times C_2 \to Y = C_1 \times C_2/G$ is unramified, so we get an unramified covering of Y^o given by $(C_1 \times C_2) - \Sigma$, Σ being the finite set of points which have a nontrivial stabilizer. When ε is not a morphism, we are also removing from $C_1 \times C_2$ a finite (G-invariant) union of fibres.

Thus Y^o has an unramified covering of finite degree which has the fundamental group of a product of projective curves (case: ε a morphism) or the product of a projective curve with a quasiprojective one. In order to generalize to constant moduli fibrations the topological characterization of surfaces isogenous to a product, the most natural way is to recall that for a smooth complete surface S the Euler number $e(S) = c_2(\Omega^1_S)$. Then, for a quasiprojective manifold $U = X - D$ (X is smooth, proj., and D is a normal crossings divisor), the natural object to consider is $\Omega^1_X(logD)$ in the place of Ω^1_X; for the top Chern class $c_n(\Omega^1_X(\log D))$, which we identify with the number it gives by integration on X,

$$c_n(\Omega^1_X(\log D)) = e_{BM}(U) = e(X) - e(D) \tag{5.2}$$

holds, where $e_{BM}(U)$ is the Borel-Moore homology Euler number (cf. [B-M], [B-H]). If we have an unramified covering $Z \to U$ of degree d, then

$$e_{BM}(Z) = d\ e_{BM}(U). \tag{5.3}$$

Since removing points makes $e_{BM}(U)$ become smaller, going back to the proof of Theorem 3.4, we see immediately that the results are bound to be a little weaker in the quasiprojective case. We have two such generalizations of Theorem 3.4.

THEOREM 5.4. *Let $U = S - D$ be a quasiprojective surface, and assume that U is proper homotopically equivalent to the good locus Y^o of a constant* *moduli fibration. Then S carries a constant moduli fibration with the same invariants as the relatively minimal fibration associated to the projection $Y \to C_1/G$. Moreover, all such surfaces S form an irreducible subvariety of the moduli space.*

Proof. We proceed as in Theorem 3.4, replacing S by U, using Arapura and Bauer's logarithmic versions (Theorems 2.11 and 2.12) of the fibration theorems. We thus get, for a suitable Galois unramified cover U' of U, a surjective holomorphic map f to a product of logarithmic curves $C_1^* \times C_2^*$. We get a holomorphic extension $f\colon\ S' \to C_1 \times C_2$ by choosing S' to be a suitable completion of U'. It remains to show that f has degree 1. This amounts to $f^*\colon\ H^4(C_1 \times C_2, \mathbb{Z}) \to H^4(S', \mathbb{Z})$ being an isomorphism. But, on the other hand, $H^4(C_1 \times C_2, \mathbb{Z}) = \Lambda^4 H^1(C_1 \times C_2, \mathbb{Z})$, so we only need to show that there are 4 elements $\eta_1, \eta_2, \eta_3, \eta_4$ in $f^*(H^1(C_1 \times C_2, \mathbb{Z}))$ such that $\int_{S'} (\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4) = 1$ (integration here stands for evaluation on the fundamental class). We want to transform this calculation into a calculation on U', and if the hypothesis would be of a diffeomorphism between Y^o and U, then we could represent the η_i's by differential forms and use that $\int_{S'} = \int_{U'}$. We can accomplish the same purpose by using the fundamental class of U' in Borel-Moore homology $H_4^{BM}(U', \mathbb{Z})$, and its natural pairing with cohomology with compact supports.

In our situation we have natural isomorphisms $H_4^{BM}(U', \mathbb{Z}) \cong H_4(S', \mathbb{Z})$, $H^4(S', \mathbb{Z}) \cong H_c^4(U', \mathbb{Z})$, compatible with the 2 natural pairings into $\mathbb{Z}$. We have natural maps $H_c^i(U', \mathbb{Z}) \to H^i(S', \mathbb{Z}) \to H^i(U', \mathbb{Z})$ and we observe that the classes η_j in $H^i(U', \mathbb{Z})$ come from classes in $H_c^i(U', \mathbb{Z})$ which we shall denote by η_j'. More precisely, the exact sequence for Borel-Moore homology (which coincides for compact spaces with singular homology) yields

$$H_i(S' - U', \mathbb{Z}) \to H_i(S', \mathbb{Z}) \to H_i^{BM}(U', \mathbb{Z}) \to H_{i-1}(S' - U', \mathbb{Z}),$$

thus $H_i(S', \mathbb{Z}) \cong H_i^{BM}(U', \mathbb{Z})$ for $i = 3, 4$, and by duality we thus get $H_c^i(U', \mathbb{Z}) \cong H^i(S', \mathbb{Z})$ for $i = 0, 1$. Thus

$$\int_{S'} (\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4) = \langle [U'], (\eta_1' \wedge \eta_2' \wedge \eta_3' \wedge \eta_4') \rangle.$$

The last integer can now be calculated on the corresponding unramified cover $Y^{o'}$ of Y^o, using that the groups $H_i^{BM}(U', \mathbb{Z})$ are invariant by proper homotopy equivalences. Reversing the chain of arguments, since $Y^{o'}$ is a Zariski open set in a product of two curves, we get that

$$\langle [U'], (\eta_1' \wedge \eta_2' \wedge \eta_3' \wedge \eta_4') \rangle = 1,$$

as we wanted. The last assertion concerning the irreducibility of the family of such surfaces follows with the same proof as in Remark 4.11 and Lemma 4.12. □

Remark 5.5. From the proof of Theorem 5.4 we see immediately that an entirely analogous result holds for quasiprojective varieties of higher dimension. Moreover, one can clearly weaken the assumption on U to: we have an isomorphism of fundamental groups $\pi_1(U) \cong \pi_1(Y^0)$ and an isomorphism of cohomology algebras with compact supports $H_c^\star(U', \mathbb{Z}) \cong H_c^\star(Y^{0'}, \mathbb{Z})$.

We use now the logarithmic version (Theorem 2.14) of the Zeuthen-Segre inequality to obtain a sharper result in the 2-dimensional case. But to this end we must recall some notation:

Remarks 5.6.

(i) A manifold U is said to have only one end if there exists an increasing sequence (K_n) of compacts whose union is U, and such that $U - K_n$ is connected. In the particular case of a quasiprojective surface $U = S - D$, U has only one end iff D is connected.

(ii) Recall that $\pi_1^\infty(U)$, for a manifold with one end, is the limit of the fundamental groups $\pi_1(U - K_n)$ where K_n is a sequence of compacts as in (i).

(iii) If $U = S - D$, then $\pi_1^\infty(U)$ is the fundamental group of the boundary Σ of a good tubular neighborhood T of D (or, equivalently, of $T - D$).

(iv) If Y has several ends (in our case, they correspond to the connected components of D), we can define $\pi_1^\infty(U)$ as the disjoint union of the corresponding groups.

THEOREM 5.7. *Let $U = S - D$ be a quasiprojective surface. Assume that $\pi_1(U) \cong \pi_1(Y^0)$, where Y^0 is the good locus of a relatively minimal constant moduli fibration and that $\pi_1^\infty(U) \cong \pi_1^\infty(Y^0)$, compatibly with the natural homomorphisms $\pi_1^\infty \to \pi_1$. Assume further that $e(U) = e(Y^0)$ (by Poincare's duality this is equivalent to $e_{BM}(U) = e_{BM}(Y^0)$). Then S also carries a constant moduli fibration which, in case S is minimal, is a deformation of the given one.*

Proof. Let $\Gamma = \Gamma' \times \Gamma''$ be the subgroup of $\pi_1(Y^0)$ corresponding to the unramified covering birational to a product, and where Γ' is the factor which corresponds to the fibration. Observe that the pullback of Γ within the groups of $\pi_1^\infty(Y)$ determines $\pi_1^\infty(U')$ for the corresponding covering U' of U (by assumption, the number of ends and their fundamental groups at infinity are the same for both surfaces). We have that each group in $\pi_1^\infty(U')$ is trivial if Γ' is not a free group. And that, in case Γ' is free, there are as many nontrivial ones as there are ends of the target quasiprojective curve, each of them mapping onto an infinite cyclic subgroup of Γ'. By Theorem 2.12, the unramified covering U' associated to Γ admits a mapping to a product of two curves $B_1 \times B_2$, where B_2 is compact, and $B_1 = C_1 - Z$, where Z has cardinality k = number of the nontrivial groups in $\pi_1^\infty(U')$'s.

By our assumption, every connected component D'' of $S' - U'$ maps to a point in B_1. Moreover, either D" is a full fibre and then we have a corresponding nontrivial group in $\pi_1^\infty(U')$, or D" is not a full fibre and, by the well-known Zariski's lemma, the intersection matrix for the irreducible components of D" is negative definite. We can then apply Mumford's Main Result of [Mum], and conclude that the trivial groups in $\pi_1^\infty(U')$ correspond to the case where D'' is a point (here, S' is a minimal completion, whence D" need not be a normal crossings divisor). By adding these, we obtain a very good fibration. In fact, the remaining components D'' are contained in a fibre over Z, and we claim that they must be a full fibre. In fact, these fibres are missing in U': since their number is the same(= k) as the number of nontrivial groups in $\pi_1^\infty(U')$, the only missing curves are full fibres. We can thus conclude as in the proof of Theorem 3.4, in view of Theorem 2.14. □

Remark 5.8. In fact, one sees from the above proof that a weaker assumption is really sufficient on $\pi_1^\infty(U)$. Namely, that Γ is such to determine on U' the right number $a+k$ of ends, k whose corresponding group in $\pi_1^\infty(U')$ maps onto a cyclic nontrivial subgroup of Γ', and a yielding a trivial group of $\pi_1^\infty(U')$.

6. The fundamental group and moduli spaces. We have seen in Section 4 that surfaces isogenous to a product yield an irreducible moduli space, once one fixes, e.g., the Euler number and the fundamental group. On the other hand, there are several examples (cf. [Cat2], [Man1]) of surfaces of general type which are simply connected and for which the moduli space, even if we fix the oriented topological type (and also the canonical class, see [Man2]), still has many connected components.

The idea is to marry these two types of moduli spaces by using a small variation of the construction of Lemma 1.1 (the construction of Lemma 1.1, when applied to irregular surfaces, gives rise to deformations which are not complete intersections any longer, but just zero loci of a section of a rank 2 bundle). Unfortunately, we are not yet able to say whether we get in this way different connected components of moduli spaces. Nor can we in this way, for each group Γ which is a fundamental group of an algebraic surface, produce everywhere nonreduced moduli spaces of surfaces having Γ as fundamental group (using e.g. the examples of [Cat4]).

As in Section 1, we shall take $S \subset X \times Y = Z$ as a smooth complete intersection of two hypersurfaces where:

6.1. (i) X, Y are smooth complete surfaces of general type with ample canonical bundle.

6.1. (ii) S is a complete intersection of two divisors in Z of type $(A+B)$ and $(2A'+2B')$, where we let A, A' vary among the divisors algebraically equivalent to rK_X, with $r \geq 2$, and likewise we let B, B' vary among the divisors algebraically equivalent to rK_Y.

6.2. Therefore (cf. [Bo]), all the cohomology groups $H^i(X, mA) = 0$ except for $i = 0$, $m \geq 0$, $i = 2$, $m \leq 0$, and likewise for the cohomology groups $H^i(X, mA')$, $H^i(X, mB)$, $H^i(Y, mB')$.

We have the following exact sequence on S:

$$0 \to T_S \to T_X \oplus T_Y \mid_S \to \mathcal{O}_S(A+B) \oplus \mathcal{O}_S(2A'+2B') \to 0, \tag{6.3}$$

where we identify T_X, resp. T_Y, to their pullbacks to Z. Since X has ample canonical bundle we can take $r \geq 12$ (cf. [Cor], Lemma 4.1, p. 120) so that

$$H^i(X, T_X(-mA)) = 0 \quad \text{for} \quad i \leq 1,\ m > 0, \tag{6.4}$$

and similarly with Y in the place of X, B or B' instead of A. (Note that the above vanishing is indeed false for large r if X contains (-2)-curves.) It follows then that

$$H^i(Z, T_X \oplus T_Y(-mA - nA' - mB - nB')) = 0 \\ \text{for} \quad i < 4,\ m+n > 0, \tag{6.5}$$

and similarly $H^i(Z, \mathcal{O}_Z(-mA - nA' - mB - nB')) = 0$.

We finally have the Koszul sequence

$$0 \to \mathcal{O}_Z(-A - 2A' - B - 2B') \to \\ \mathcal{O}_Z(-A - B) \oplus \mathcal{O}_Z(-2A' - 2B') \to \mathcal{I}_S \to 0 \tag{6.6}$$

and the standard sequence

$$0 \to \mathcal{I}_S \to \mathcal{O}_Z \to \mathcal{O}_S \to 0, \tag{6.7}$$

both of which we will tensor by vector bundles on Z.

LEMMA 6.8. *$H^i(Z, T_X \oplus T_Y) \cong H^i(S, T_X \oplus T_Y|_S)$ for $i = 0, 1$. Moreover $H^1(Z, T_X \oplus T_Y) \cong H^1(X, T_X) \oplus H^1(Y, T_Y)$ (for $i = 0$ we get 0).*

Proof. Tensor (6.6) and (6.7) by $T_X \oplus T_Y$; use (6.5). The rest is straightforward. □

We obtain therefore an exact sequence

$$(6.9)\qquad H^0(\mathcal{O}_S(A+B)) \oplus H^0(\mathcal{O}_S(2A'+2B')) \to H^1(S,T_S) \to \\ \to H^1(X,T_X) \oplus H^1(Y,T_Y).$$

Moreover, from (6.2), (6.6), and (6.7) we also have that there is an exact sequence

$$(6.10)\qquad \begin{aligned} H^0(\mathcal{O}_Z(A+B)) \oplus H^0(\mathcal{O}_Z(2A'+2B')) &\to \\ H^0(\mathcal{O}_S(A+B)) \oplus H^0(\mathcal{O}_S(2A'+2B') &\to \\ H^1(\mathcal{O}_Z) \oplus H^1(\mathcal{O}_Z) &= \\ [H^1(\mathcal{O}_X) \oplus H^1(\mathcal{O}_Y)] \oplus [H^1(\mathcal{O}_X) \oplus H^1(\mathcal{O}_Y)] &\to 0. \end{aligned}$$

The last two sequences tell us about the surjectivity of the Kodaira-Spencer map.

PROPOSITION 6.11. *Assume that X, Y belong to respective irreducible families $\mathcal{X} \times \mathcal{Y}$, with surjective Kodaira Spencer maps. Take r so large that (6.4) holds for each X in $\mathcal{X}$, Y in $\mathcal{Y}$ (e.g. $r \geq 12$). Then the family of smooth surfaces S obtained as above as the complete intersection of two divisors in $X \times Y = Z$ of type $(A+B)$ and $(2A'+2B')$ (with A, A' any divisors algebraically equivalent to rK_X, B, B' any divisors algebraically equivalent to rK_Y) is irreducible with surjective Kodaira-Spencer map.*

Proof. Since $r \geq 2$, the family of effective divisors A algebraically equivalent to rK_X forms a projective bundle over the relative Picard scheme of $\mathcal{X}$, $\mathrm{Pic}^0(\mathcal{X})$, and similarly for $2A', B, 2B'$. Since, e.g., $H^0(Z, \mathcal{O}_Z(A+B)) = H^0(X, \mathcal{O}_X(A)) \otimes H^0(Y, \mathcal{O}_Y(B))$, we conclude likewise that the family of effective divisors on Z algebraically equivalent to $A+B$ forms a projective bundle over $\mathrm{Pic}^0(\mathcal{X}) \times \mathrm{Pic}^0(\mathcal{Y})$. Whence, by taking a fibre product we obtain an irreducible family whose Kodaira-Spencer map is clearly surjective at each point. □

7. Results in the higher dimensional case. As already remarked in the introduction, Mok [Mok1,2] extended to the case of a compact quotient $\mathcal{H}^n/\Gamma$ of a polydisc ($\mathcal{H}$ being the unit disc ≅ Poincaré's upper half plane) the result of Jost and Yau (including strong rigidity for the case where Γ is irreducible). We can (partially) further generalize the result of Jost and Yau (cf. Theorem 3.4) to higher dimensions:

THEOREM 7.1. *Let X be a compact Kähler manifold of dimension n, and assume that*

(i) $\Gamma = \pi_1(X)$ *admits a finite index subgroup* Γ' *isomorphic to* $\Pi_{g_1} \times \Pi_{g_2} \times \cdots \times \Pi_{g_n}$, *with* $g_1, \dots g_n > 1$, *and that moreover*

(ii) $H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$ *has index* d = *index* (Γ') *inside the image of* $H^{2n}(\Gamma', \mathbb{Z})$ *in* $H^{2n}(X', \mathbb{Z})$ *(*X' *being the covering associated to* Γ'*).*

Then X is a blowup of a variety isogenous to a higher product.

Proof. The beginning of the proof runs exactly as in Theorem 3.4. That is, we have a degree d unramified covering $\pi\colon X' \to X$, and therefore $H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$ has index d in $H^{2n}(X', \mathbb{Z}) \cong \mathbb{Z}$. Now, X' admits a holomorphic map f to a product $W = C_{g_1} \times C_{g_2} \times \cdots \times C_{g_n}$ of curves of respective genera $g_1, \dots g_n > 1$. By our assumption $f^* H^{2n}(W, \mathbb{Z}) \neq 0$, and f is surjective. Moreover, $H^{2n}(\Gamma', \mathbb{Z}) \cong f^* H^{2n}(W, \mathbb{Z}) \subset H^{2n}(X', \mathbb{Z}) \cong \mathbb{Z}$ is therefore a chain of equalities, and therefore $\deg(f) = 1$, so f is a birational morphism.

If f is not an isomorphism, then there exists a f-exceptional curve C' which has negative intersection with $K_{X'}$, whence also $\pi^{-1}(\pi(C))$ is f-exceptional, since $K_{X'} = \pi^*(K_X)$. We conclude, by induction on the number of elementary contractions, that the f-exceptional locus is π-invariant, whence also X is a blowup of a manifold Y having $C_{g_1} \times \cdots \times C_{g_n}$ as an unramified covering. □

Example 7.2. In the previous theorem one cannot replace condition (ii) by the analogous condition of condition (e) of Theorem 3.4:

$$e(X) = 2^n (g_1 - 1)(g_2 - 1) \dots (g_n - 1)/d.$$

In fact, let $W = C_{g_1} \times C_{g_2} \times C_{g_3}$ and let T be a double cover branched on a smooth ample surface $B \subset W$. Arguing, e.g., as in [Cat1], Section 1, we see that $\pi_1(T) \cong \pi_1(W)$. Whereas, again a Borel-Moore homology calculation shows

$$e(T) = 2e(W) - e(B) < 0 \tag{7.3}$$

(indeed both terms are negative, the second term by Castelnuovo's theorem (cf. [Bo]) since B is of general type). We obtain X by blowing up $1/2(e(W) - e(B))$ points; thus, $\pi_1(X) \cong \pi_1(W)$ and $e(X) = e(W)$, but the classifying map $f\colon X \to W$ has degree 2.

The above example shows that some minimality assumption is needed in dimension 3, when one can increase the Euler number by blowing up points, and decrease it by blowing up curves of genus ≥ 2.

Remark 7.4. It is rather clear that condition (ii) is also crucial in Theorem 7.1 in order to have the surjectivity of f, which, in dimension $n = 2$, follows automatically from the fact that the pencils are distinct. In higher dimensions, by Lefschetz, we can always take a surface S with given fundamental group Γ and cross it with any simply connected manifold: this shows that condition (ii) can be weakened, but not completely removed, e.g., we also have:

THEOREM 7.5. *Let X be a compact complex manifold of dimension n with ample canonical bundle and assume that:*

(i) $\Gamma = \pi_1(X)$ *admits a finite index subgroup* Γ' *isomorphic to*

$$\Pi_{g_1} \times \Pi_{g_2} \times \cdots \times \Pi_{g_n},$$

with $g_1, \ldots g_n > 1$,

(i) *the image of* $H^{2n}(\Gamma', \mathbb{Z})$ *in* $H^{2n}(X', \mathbb{Z})$ *is nonzero,*

(k) $K_X^n = n!\, 2^n (g_1 - 1)(g_2 - 1) \ldots (g_n - 1)/d.$

Then X is isogenous to a higher product, and the moduli space of such varieties is irreducible (weak rigidity).

Proof. As in Theorem 7.1 we get $f\colon X' \to W$, where W is a product of curves, and we need to show that f is an isomorphism. Without loss of generality let us assume $X = X'$ ($d = 1$). Then $K_X^n = K_W^n$ by our assumption.

On the other hand, if R is the ramification divisor, then $K_X = f^*K_W + R$. By a usual trick we calculate

$$K_X^n = (f^*K_W + R)^n = K_X^{n-1} \cdot (f^*K_W + R) \geq K_X^{n-1} \cdot f^*K_W,$$

equality holding, since K_X is ample, if and only if $\dim R < n - 1$. Then $K_X^{n-1} \cdot f^*K_W = K_X^{n-2} \cdot (f^*K_W + R) \cdot f^*K_W \geq K_X^{n-2} \cdot (f^*K_W^2)$, since $(K_X^{n-2} \cdot R) f^*K_W$ by the projection formula equals $f_*(K_X^{n-2} \cdot R) \cdot K_W$ and K_W is ample. Continuing in this way, $K_X^n \geq (f^*K_W)^n = deg(f) K_W^n$. Therefore $\deg(f) = 1$, and $\dim(R) < n - 1$. Therefore, by normality, f is an isomorphism.

Remark 7.6. It is rather clear that the classification of surfaces isogenous to a product which we carried through in Section 3 carries over without any major alteration for higher dimensional varieties isogenous to a product (except of course the more complicated algebraic treatment of the varieties of mixed type). We remark now that the same ideas lead to a minor improvement of the results of Mok [Mok1,2].

THEOREM 7.7. *Let X be a compact manifold of dimension n with ample canonical bundle, and assume that*

(i′) $\pi_1(X) \cong \Gamma$, *where* Γ *is a cocompact torsion free subgroup of* $\mathrm{Aut}(\mathcal{H}^n)$ ($\Gamma \cong \pi_1(W')$, *where* $\mathcal{H}^n/\Gamma = W'$ *is a compact manifold),*

(ii′) $H^{2n}(\Gamma, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})$ *is an isomorphism, or*

(ii″) $H^{2n}(\Gamma, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})$ *is nonzero and* $K_X^n = K_{W'}^n$.

Then also X is biholomorphic to a quotient $W = \mathcal{H}^n/\Gamma$.

Idea of proof. By topology, since W' is a $K(\pi, 1)$, we get a mapping $f'': X \to \mathcal{H}^n/\Gamma = W'$ which can be deformed to a harmonic map by the theorem of Eells and Sampson [E-L]. Conditions (ii) or (ii″) ensure that f is surjective. That the universal covering $X^\wedge$ of X has a holomorphic mapping $f^\wedge$ to $\mathcal{H}^n$ which is Γ-equivariant was proven by Mok in [Mok2]. Then the final argument goes exactly as in Theorems 7.1 or 7.5. □

Remark 7.8. In the 2-dimensional case, the hypothesis $K_X^2 = K_{W'}^2$ can be replaced by $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{W'})$ or by $e(X) = e(W')$, without then assuming X to be minimal. In fact, assume in Theorems 7.5 and 7.7, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{W'})$. Since $K_{W'}^2 = 8\,\chi(\mathcal{O}_{W'}) = 8\chi(\mathcal{O}_X)$, if f is not birational then $K_X^2 \geq 2K_W^2 = 16\chi(\mathcal{O}_X)$, contradicting the Miyaoka-Yau inequality [Ya1,2], [Miy1,2]. Similarly if $e(X) = e(W')$, and S is the minimal model of X, then $K_S^2 \geq 2K_W^2 = 4e(X) \geq 4e(S)$ again contradicts the same inequality. Then f is birational and an isomorphism since $e(X) = e(W)$.

We conjecture that the hypothesis on $\chi(\mathcal{O}_X)$ (and of minimality) should also be sufficient in higher dimension (cf. the work of Green-Lazarsfeld [G-L1,2] and Ein-Lazarsfeld [E-L]) and one can ask whether in Theorem 7.5 the hypothesis $e(X) = 2n(g_1 - 1)(g_2 - 1) \cdots (g_n - 1)/d$ would be sufficient, assuming the ampleness of K_X. Cf. also recent work of Kollar [Kol1,2] where instead more emphasis is set on the role of the plurigenera of X.

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