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Journal of the Institute of Mathematics of Jussieu / Volume 2 / Issue 02 / April 2003, pp 169 - 233
DOI: 10.1017/S1474748003000070, Published online: 13 May 2003

Journal of the Inst. of Math. Jussieu (2003) **2**(2), 169–233 © Cambridge University Press 169
DOI:10.1017/S1474748003000070 Printed in the United Kingdom

REAL HYPERELLIPTIC SURFACES AND THE ORBIFOLD FUNDAMENTAL GROUP

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(Received 19 December 2001; accepted 27 May 2002)

Abstract In this paper we finish the topological classification of real algebraic surfaces of Kodaira dimension zero and we make a step towards the Enriques classification of real algebraic surfaces, by describing in detail the structure of the moduli space of real hyperelliptic surfaces.

Moreover, we point out the relevance in real geometry of the notion of the orbifold fundamental group of a real variety, and we discuss related questions on real varieties (X, σ) whose underlying complex manifold X is a $K(\pi, 1)$.

Our first result is that if (S, σ) is a real hyperelliptic surface, then the differentiable type of the pair (S, σ) is completely determined by the orbifold fundamental group exact sequence.

This result allows us to determine all the possible topological types of (S, σ), and to prove that they are exactly 78.

It follows also as a corollary that there are exactly eleven cases for the topological type of the real part of S.

Finally, we show that once we fix the topological type of (S, σ) corresponding to a real hyperelliptic surface, the corresponding moduli space is irreducible (and connected).

We also give, through a series of tables, explicit analytic representations of the 78 components of the moduli space.

Keywords: real surfaces; orbifold fundamental group; hyperelliptic surfaces; real varieties

AMS 2000 *Mathematics subject classification:* Primary 14P99; 14P25; 14J15; 32Q57

1. Introduction

The purpose of this paper is twofold: on the one side, we finish the topological classification of real algebraic surfaces of Kodaira dimension zero and we make a step towards the Enriques classification of real algebraic surfaces, by describing in detail the structure of the moduli space of real hyperelliptic surfaces; on the other hand, we point out the relevance in real geometry of the notion of the orbifold fundamental group.

In order to illustrate the latter concept, let us begin by answering the reader's question: what is a real variety?

A smooth real variety is a pair (X, σ), consisting of the data of a smooth complex manifold X of complex dimension n and of an anti-holomorphic involution $\sigma : X \to X$ (an involution σ is a map whose square is the identity).

The quickest explanation of what anti-holomorphic means goes as follows: the smooth complex manifold X is determined by the differentiable manifold M underlying X and by a complex structure J on the complexification of the real tangent bundle of M.

If instead we consider the same manifold M together with the complex structure $-J$, we obtain a complex manifold which is called the conjugate of X and denoted by $\bar{X}$.

The involution σ is now said to be anti-holomorphic if it provides an isomorphism between the complex manifolds X and $\bar{X}$ (and then (X, σ) and $(\bar{X}, \sigma)$ are also isomorphic as pairs).

What are the main problems concerning real varieties? (We may restrict ourselves to the case where X is compact.)

(i) Describe the isomorphism classes of such pairs (X, σ).

(ii) Or, at least describe the possible topological or differentiable types of the pairs (X, σ).

(iii) At least describe the possible topological types for the real parts $X' := X(\boldsymbol{R}) = \mathrm{Fix}(\sigma)$.

Remark 1.1. Recall that Hilbert's 16th problem is a special case of the last question but for the more general case of a pair of real varieties $(Z \subset X, \sigma)$.

For a smooth real variety, we have the quotient double covering $\pi : X \to Y = X/\langle\sigma\rangle$, and the quotient Y is called the Klein variety of (X, σ).

In dimension $n = 1$ the datum of the Klein variety is equivalent to the datum of the pair (X, σ), but this is no longer true in higher dimension, where we will need also to specify the covering π.

The covering π is ramified on the so-called real part of X, namely, $X' := X(\mathbb{R}) = \mathrm{Fix}(\sigma)$, which is either empty, or a real submanifold of real dimension n.

If $X' := X(\mathbb{R}) = \mathrm{Fix}(\sigma)$ is empty, the orbifold fundamental group of Y is just defined as the fundamental group of Y.

If instead $X' \neq \emptyset$, we may take a fixed point $x_0 \in \mathrm{Fix}(\sigma)$ and observe that σ acts on the fundamental group $\pi_1(X, x_0)$: we can therefore define the orbifold fundamental group as the semidirect product of the normal subgroup $\pi_1(X, x_0)$ with the cyclic subgroup of order two generated by σ. It is easy to verify then that changing the base point does not alter the isomorphism class of the following exact sequence, yielding the orbifold fundamental group as an extension

$$1 \to \pi_1(X) \to \pi_1^{\mathrm{orb}}(Y) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

(changing the base point only affects the choice of a splitting of the above sequence).

We claim that the orbifold fundamental group exact sequence is a powerful topological invariant of the pair (X, σ) in the case where X has large fundamental group.

To illustrate a concrete issue of such a statement, let us consider the case of a $K(\pi, 1)$, i.e. the case where the universal cover of X is contractible: this case includes the case of complex tori, of hyperelliptic surfaces and their generalizations, as well as the case of quotients of the complex n-ball and of polydisks.

Then the homotopy type of X is determined by the fundamental group π and some interesting quite general open questions are the following ones (where, by abuse of language, we shall talk about orbifold fundamental group of a real variety, to refer to the above orbifold fundamental group exact sequence).

(i) If X is a compact complex manifold which is a $K(\pi, 1)$, to what extent does π also determine the differentiable type?

(ii) Same assumptions as above and fix the group π: when is the moduli space of those manifolds X irreducible or connected?

About this question, a vast literature already exists (cf. [**27**,**37**,**42**,**47**], cf. also [**12**]), especially concerning the rigidity property that a manifold homeomorphic to a given one should be either biholomorphic or anti-holomorphic to it, but only recently, in [**29**] and [**13**], were concrete examples given of compact complex surfaces which are $K(\pi, 1)$, and whose moduli spaces consist of two connected components exchanged by complex conjugation. This shows that even if we consider Kähler manifolds, we can find moduli spaces which are not irreducible. The series of examples in [**13**] are not rigid, and are quotients X of products of curves which not only do not admit any real structure but are also such that X and its conjugate $\bar{X}$ are not deformation equivalent.

(iii) Assume that we consider real varieties, (X, σ) where X is as above a $K(\pi, 1)$: does the orbifold fundamental group determine the differentiable type of the real variety?

(iv) Fixing the orbifold fundamental group, when do we get a connected moduli space?

Our purpose here is thus to give an issue where the classification of real varieties can be given in terms of the orbifold fundamental group.

Concerning the previous questions, in [**14**] it is shown that the complex manifolds isomorphic to a complex torus of dimension greater than or equal to 3 form a connected component of their 'moduli space', which admits other components corresponding to older examples given by Blanchard and Calabi in [**6**–**8**], and rediscovered via a different approach by Sommese [**43**]. These other examples provide the underlying differentiable manifold with a non-Kähler complex structure without trivial canonical bundle.

Our first result, concerning the topological type of real hyperelliptic surfaces, can now be briefly stated as follows.

Theorem 1.2. *Let (S, σ) be a real hyperelliptic surface. Then the differentiable type of the pair (S, σ) is completely determined by the orbifold fundamental group exact sequence.*

In a sequel to this paper, we plan to show other issues (e.g. in the Kodaira classification of non-algebraic real surfaces) where the topology of the pair (S, σ) is determined by the orbifold fundamental group exact sequence.

Returning to the case of real hyperelliptic surfaces, the previous theorem allows us to easily determine completely the possible topological types of (S, σ), and in particular we have the following two results.

Theorem 1.3. *Real hyperelliptic surfaces fall into exactly* 78 *topological types.*

Corollary 1.4. *Let (S, σ) be a real hyperelliptic surface. Then the real part $S(\mathbb{R})$ is either*

(i) *a disjoint union of t tori, where $0 \leqslant t \leqslant 4$;*

(ii) *a disjoint union of b Klein bottles, where $1 \leqslant b \leqslant 4$;*

(iii) *the disjoint union of one torus and one Klein bottle;*

(iv) *the disjoint union of one torus and two Klein bottles.*

As the reader may guess, the above results are too complicated to be described in detail here in the introduction: therefore we will limit ourselves to illustrate the underlying philosophy by describing it in the much simpler case of the real elliptic curves.

Classically (cf. [**1**] for a modern account) real elliptic curves have been classified according to the number ν of connected components (these are circles) of their real part: ν can only attain the values 0, 1 and 2 and completely determines the differentiable type of the involution.

The orbifold fundamental group explains easily this result: if there is a fixed point for σ, the orbifold fundamental group sequence splits and the action of σ on the elliptic curve C is completely determined by the action s of σ on $H_1(C, \mathbb{Z})$.

This situation gives rise to only two cases: s is diagonalizable, and $C(\mathbb{R})$ consists of two circles, or s is not diagonalizable, and $C(\mathbb{R})$ consists of only one circle.

If instead there are no fixed points, an easy linear algebra argument (cf. Lemma 5.5) shows that s is diagonalizable, and the translation vector of the affine transformation inducing σ can be chosen to be $\frac{1}{2}$ of the $+1$-eigenvector e_1 of s.

In fact, σ is represented by an affine transformation $(x, y) \to s(x, y) + (a, b)$, and s is not diagonalizable if and only if $s(x, y) = (y, x)$ for a suitable choice of two basis vectors. The identity map, which equals the square of σ, is the transformation $(x, y) \to (x, y) + (a + b, a + b)$, thus $a + b$ is an integer, and therefore the points $(x, x - a)$ yield a fixed S^1 on the elliptic curve.

The complete description of the moduli space of real hyperelliptic surfaces is too long to be reproduced in the introduction, we want here only to mention the following main result, which confirms a conjecture by Kharlamov, that more generally for all real Kähler surfaces of Kodaira dimension at most 1 the differentiable type of the pair (S, σ) should determine the deformation type (this is false already for complex surfaces if the Kodaira dimension equals 2, cf. [**9**, **11**, **13**, **29**, **33**]).

Theorem 1.5. *Fix the topological type of* (S, σ) *corresponding to a real hyperelliptic surface. Then the moduli space of the real surfaces* (S', σ') *with the given topological type is irreducible* (*and connected*).

Again, we wish to give the flavour of the argument by outlining it in the much simpler case of the elliptic curves. Assume for instance that our involution σ acts as follows: $(x, y) \to (y, x)$. We look then for a translation invariant complex structure J which makes σ anti-holomorphic, i.e. we seek for the matrices J with $J^2 = -1$ and with $Js = -sJ$.

The latter condition singles out the matrices

$$\begin{pmatrix} a & b \\ -b & -a \end{pmatrix},$$

while the first condition is equivalent to requiring that the characteristic polynomial be equal to $\lambda^2 + 1$, whence, it is equivalent to the equation $b^2 - a^2 = 1$.

We get therefore a hyperbola with two branches which are exchanged under the involution $J \to -J$, but, as we already remarked, J and $-J$ yield isomorphic real elliptic curves, thus the moduli space consists of just one branch of the hyperbola.

This example also serves the scope of explaining the statement in the above theorem that the moduli space is irreducible (and connected): the hyperbola is an irreducible algebraic variety, but not an irreducible analytic space, since it is not connected (in general, moduli spaces of real varieties will be semi-analytic spaces or semi-algebraic real spaces).

We want to emphasize once more that an interesting question is to determine, in the realm of the real varieties whose topological type is determined by the orbifold fundamental group, those for which the corresponding moduli spaces are irreducible (respectively, connected).

It is now however time to recall what the hyperelliptic surfaces are, why they have this name and, last but not least, point out how crucial is the role of the hyperelliptic surfaces in the Enriques classification of algebraic surfaces.

As elliptic curves are exactly the curves such that the homogeneous coordinates of their points cannot be uniformized by polynomials, yet they can be uniformized by entire holomorphic functions on $\mathbb{C}$, hyperelliptic varieties of dimension n were generally defined by Humbert and Picard through the entirely analogous property that the coordinates of their points, although not uniformizable by rational functions, could be uniformized by entire meromorphic functions on $\mathbb{C}^n$.

Among these varieties are clearly (nowadays) the abelian varieties, and the classification of such varieties in dimension two was achieved by Bagnera and de Franchis who got the Bordin Prize in 1909 for their important result (the classification by Enriques and Severi, who got the same prize for it the year before, had serious gaps which were corrected only later on).

The missing surfaces, which are now called hyperelliptic, were described as quotients of the product of two elliptic curves by the action of a finite group G. For this reason, some authors call these surfaces bi-elliptic surfaces (cf. [**5**]).

The classification is in the end very simple and produces a list of seven cases where the Bagnera–de Franchis group G and its action is explicitly written down.

The reason to recall all this is that, as a matter of fact, an important ingredient in the proof of our theorems is to rerun the arguments of the proof of the Classification Theorem, which characterizes the hyperelliptic surfaces as the algebraic surfaces with nef canonical divisor K, with $K^2 = p_g = 0$, $q = 1$, and moreover with Kodaira dimension equal to 0. This is done in § 2.

To keep close in spirit to the beautiful result of Bagnera and de Franchis we felt compelled to produce tables exhibiting simple and explicit actions for the 78 types of the real hyperelliptic surfaces: these are contained in the last section, and they summarize a lot of information that we could not give in a more expanded form.

Concerning now the Enriques classification of real algebraic surfaces, it has been focused up to now mostly on the classification of the topology of the real parts, the topological classification of real rational surfaces going back to Comessatti [**16**, **17**, **20**], as well as the classification of real abelian varieties [**20**] (see also [**40**, **41**]).

In the case of real $K3$-surfaces we have the classification by Nikulin and Kharlamov [**28**, **38**], for the real Enriques surfaces the one by Degtyarev and Kharlamov [**21**, **22**].

Finally, partial results on real ruled and elliptic surfaces have been obtained by Silhol [**41**] and by Mangolte [**35**].

The paper is organized as follows. In § 2 we recall the description given by Bagnera and de Franchis of the hyperelliptic surfaces as quotients of a product of two elliptic curves $E \times F$ by the product action of a finite group G acting on E as a group of translations and on F via an action whose quotient is $\mathbb{P}^1$.

G is called the Bagnera–de Franchis group (or symmetry group) and is a quotient of the fundamental group of the surfaces S.

In § 3 we observe that the orbifold fundamental group has a finite quotient $\hat{G}$ which contains G as a normal subgroup of index 2: $\hat{G}$ is called the extended Bagnera–de Franchis group and its structure will be investigated in detail in § 5.

In the rest of § 3 we show that isomorphisms of real hyperelliptic surfaces lift to isomorphisms of the respective products of elliptic curves, compatibly with the identifications of the respective extended Bagnera–de Franchis groups.

Section 4 is devoted first to showing that the representation of the orbifold fundamental group as a group of affine transformations of $\mathbb{Q}^4$ is uniquely determined, up to isomorphism, by the abstract structure of the group, which proves Theorem 1.2 (by the way, we show in the course of the proof a fact hardly mentioned in the literature, namely, that the differentiable structure of a hyperelliptic surface is determined by the fundamental group).

Second, after recalling quite briefly the notion of moduli spaces for real varieties, we show that, once this affine representation is fixed, the moduli space for the compatible complex structures is irreducible and connected.

Section 5 recalls some known facts about anti-holomorphic maps of elliptic curves and applies these results to the determination of the possible extended Bagnera–de Franchis group which do in effect occur.

Section 6 determines the analytical actions of these groups on the two factors under the condition that the exact sequence

$$1 \to G \to \hat{G} \to \mathbb{Z}/2\mathbb{Z} \to 1$$

splits, while § 7 deals with the simpler case where there is no splitting.

Section 8 gives the recipe to identify the real part $S(\mathbb{R})$ of our surfaces as a disjoint union of Klein bottles and tori.

Finally, § 9 applies the results developed insofar and achieves the classification of the 78 components of the moduli space, for which explicit analytical representations, describing the action of the extended Bagnera–de Franchis group, are given through a series of tables.

2. Basics on hyperelliptic surfaces

In this section we recall (see [**2**, **3**]; see also [**5**, Chapter VI, pp. 91–115], [**4**, pp. 147–149]) the definition of hyperelliptic surfaces and their characterization in the realm of the Enriques classification of complex algebraic surfaces. We shall also briefly recall the main lines of the proof of the Bagnera–de Franchis Classification Theorem, since we shall repeatedly need modified or sharper versions of the arguments used therein.

Definition 2.1. A complex surface S is said to be hyperelliptic if $S \cong (E \times F)/G$, where E and F are elliptic curves and G is a finite group of translations of E with a faithful action on F such that $F/G \cong \mathbb{P}^1$.

$G \subset \mathrm{Aut}(F)$, so $G = T \rtimes G'$ (semidirect product), where T is a group of translations and $G' \subset \mathrm{Aut}(F)$ consists of group automorphisms. Since $F/G \cong \mathbb{P}^1$, then $G' \neq 0$, hence $G' \cong \mathbb{Z}/m$, with $m = 2, 3, 4, 6$, by the following well-known result.

Fact 2.2. *Let F be an elliptic curve. Every automorphism of F is the composite of a translation and a group automorphism. The non-trivial group automorphisms are the symmetry $x \mapsto -x$ and also*

(i) *for the curve $F_{\mathrm{i}} = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \mathrm{i})$, $x \mapsto \pm \mathrm{i}x$;*

(ii) *for the curve $F_\rho = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \rho)$, where $\rho^3 = 1 \neq \rho$, $x \mapsto \pm\rho x$ and $x \mapsto \pm\rho^2 x$.*

Since G is abelian, as a group of translations of E, the product $T \rtimes G'$ must be direct. We have the following result.

Theorem 2.3 (Bagnera–de Franchis). *Every hyperelliptic surface is one of the following, where E, F are elliptic curves and G is a group of translations of E acting on F as specified:*

(1) *$(E \times F)/G$, $G = \mathbb{Z}/2\mathbb{Z}$ acting on F by symmetry.*

(2) *$(E \times F)/G$, $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ acting on F by $x \mapsto -x$, $x \mapsto x + \epsilon$, where ϵ belongs to the group F_2 of points of F of order 2.*

(3) $(E \times F_{\mathrm{i}})/G$, $G = \mathbb{Z}/4\mathbb{Z}$ *acting on* F_{i} *by* $x \mapsto \mathrm{i}x$.

(4) $(E \times F_{\mathrm{i}})/G$, $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *acting on* F_{i} *by* $x \mapsto \mathrm{i}x$, $x \mapsto x + \frac{1}{2}(1+\mathrm{i})$.

(5) $(E \times F_\rho)/G$, $G = \mathbb{Z}/3\mathbb{Z}$ *acting on* F_ρ *by* $x \mapsto \rho x$.

(6) $(E \times F_\rho)/G$, $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ *acting on* F_ρ *by* $x \mapsto \rho x$, $x \mapsto x + \frac{1}{3}(1-\rho)$.

(7) $(E \times F_\rho)/G$, $G = \mathbb{Z}/6\mathbb{Z}$ *acting on* F_ρ *by* $x \mapsto -\rho x$.

Hyperelliptic surfaces are algebraic surfaces with $p_g = 0$, $q = 1$, $K^2 = 0$, K nef.

We have the following basic result of the Enriques classification of surfaces (see [**30**]) (Kodaira indeed proved that the same result holds more generally for compact complex surfaces provided one replaces the hypothesis $q = 1$ by $b_1 = 2$).

Theorem 2.4. *The complex surfaces S with K nef, $K^2 = 0$, $p_g = 0$, and such that either S is algebraic with $q = 1$, or more generally $b_1 = 2$, are hyperelliptic surfaces if and only if* $\mathrm{kod}(S) = 0$ (*this is equivalent to requiring that all the fibres of the Albanese map be smooth of genus 1*).

Proof. Let $\alpha : S \to A$ be the Albanese map, $q = 1$, so that A is a curve of genus 1. Let $\pi : \mathbb{C} \to A$ be the universal covering and let us consider the pull-back diagram:

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & S \\ \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} \\ \mathbb{C} & \xrightarrow{\pi} & A \end{array}$$

All the fibres of α are smooth; therefore, the fibres of $\tilde{\alpha}$ are also smooth. So α and $\tilde{\alpha}$ are $\mathcal{C}^\infty$ bundles, by Ehresmann's Theorem. $\mathbb{C}$ is contractible, so $\tilde{S}$ is diffeomorphic to the product $\mathbb{C} \times F$. Thus we obtain a holomorphic map (since we have in fact a locally liftable holomorphic map $f : \mathbb{C} \to \mathcal{H}/PSL(2,\mathbb{Z})$, and $\mathbb{C}$ is simply connected),

$$f : \mathbb{C} \to \mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}, \quad t \mapsto \tau,$$

where

$$\tilde{\alpha}^{-1}(t) \cong \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}).$$

By Liouville's Theorem, we see that f is constant; therefore,

$$\tilde{S} \cong \mathbb{C} \times F.$$

If $A = \mathbb{C}/\Lambda$, then $S \cong (\mathbb{C} \times F)/\Lambda$, where Λ acts on $\mathbb{C}$ by translations and on F by a map $\mu : \Lambda \to \mathrm{Aut}(F)$.

Let $\Gamma := \mathrm{Aut}(F)/\mathrm{Aut}^0(F)$, where $\mathrm{Aut}^0(F)$ are the automorphisms which are homotopic to the identity, and let $\nu : \Lambda \to \Gamma$ be the induced map. We set $\Lambda' := \ker \nu$,

$\Gamma' := \nu(\Lambda)$. Then we get the following pull-back diagram:

$$\begin{CD} S' := \tilde{S}/\Lambda' @>{\psi}>> S = S'/\Gamma' \\ @V{\alpha'}VV @VV{\alpha}V \\ A' := \mathbb{C}/\Lambda' @>>> A = A'/\Gamma' \end{CD}$$

Λ' acts by translations on F, so it acts as the identity on $H^0(\Omega^1_F)$. Then there exist $\eta, \eta' \in H^0(\Omega^1_{S'})$ such that $\eta \wedge \eta' \not\equiv 0$, thus $K_{S'} \equiv 0$ and $q(S') = 2$, so S' is a complex torus. The map $\psi : S' \to S$ is an unramified covering of degree m, where $m \in \{2, 3, 4, 6\}$. Then $mK_S = \psi_* \psi^* K_S = \psi_* K_{S'} = 0$. In particular, $12K_S \equiv 0$. The map $\alpha' : S' \to A'$ is a fibre bundle on an elliptic curve and S' is a complex torus of dimension 2. Since $b_1 = 2$, S is algebraic, and hence S' is algebraic too and, by Poincaré's Reducibility Theorem, there exists a finite unramified covering $A'' \to A'$ yielding a product structure on the pull-back S'' of S':

$$\begin{CD} S'' := A'' \times_{A'} S' \cong A'' \times F @>>> S' \\ @V{\alpha''}VV @VV{\alpha'}V \\ A'' =: \mathbb{C}/\Lambda'' @>>> A' = \mathbb{C}/\Lambda' \end{CD}$$

Thus we have found that $S \cong (A'' \times F)/G$, where $G = \Lambda/\Lambda''$.

Moreover, by choosing the covering $A'' \to A'$ minimal with the above property, one sees that $E := A''$, F, G are as in the list by Bagnera–de Franchis. □

3. Real conjugations on hyperelliptic surfaces

Let us now suppose that S is a real hyperelliptic surface, i.e. there is an anti-holomorphic involution $\sigma : S \to S$, and we consider the isomorphism class of the pair (S, σ).

Since, by definition of the Albanese map α, once we fix a point $x_0 \in S$, $\alpha(x) = \int_{x_0}^{x}$, we obtain

$$\alpha(\sigma(x)) = \int_{x_0}^{\sigma(x)} = \int_{x_0}^{\sigma(x_0)} + \int_{\sigma(x_0)}^{\sigma(x)} .$$

If we define

$$\bar{\sigma}(\gamma) := \int_{x_0}^{\sigma(x_0)} + \sigma_*(\gamma),$$

we get an induced anti-holomorphic map on the Albanese variety $\bar{\sigma} : A \to A$ with the property that the following diagram commutes:

$$\begin{CD} S @>{\sigma}>> S \\ @V{\alpha}VV @VV{\alpha}V \\ A = \mathbb{C}/\Lambda @>{\bar{\sigma}}>> A = \mathbb{C}/\Lambda \end{CD}$$

A direct calculation, or the remark that $\alpha(S)$ generates A and $\bar{\sigma}^2$ is the identity on $\alpha(S)$, assures that $\bar{\sigma}$ is an anti-holomorphic involution on A.

Notice that, if $S(\mathbb{R}) \neq \emptyset$, we may choose a point x_0 with $\sigma(x_0) = x_0$, and then $\bar{\sigma}$ will be a group homomorphism.

We want to prove that σ lifts to a map $\tilde{\sigma} : A'' \times F \to A'' \times F$, where A'' and F are as in the proof of the previous theorem.

Observe that since we have the pull-back diagram

$$\begin{array}{ccc} S'' \cong A'' \times F & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \alpha} \\ A'' = \mathbb{C}/\Lambda'' & \longrightarrow & A = \mathbb{C}/\Lambda \end{array}$$

it suffices to prove that the involution $\bar{\sigma}$ on A lifts to A''.

In fact, σ then lifts to S'' as a fibre product and so we have an induced action on $A'' \times F$, preserving α''.

We need the following result.

Lemma 3.1. *Let $\pi : Y \to X$ be a connected covering space, and let g be a homeomorphism of X. Choose $x_0 \in X$, $y_0 \in Y$ with $\pi(y_0) = x_0$, and let $z_0 = g(x_0)$, $w_0 \in Y$ with $\pi(w_0) = z_0$.*

Then there exists a lifting $\tilde{g}$ of g with $\tilde{g}(z_0) = w_0$ if and only if, given a path $\tilde{\delta}$ from y_0 to w_0 and setting $\delta = \pi \circ \tilde{\delta}$, and considering the isomorphism $\Delta : \pi_1(X, z_0) \xrightarrow{\cong} \pi_1(X, x_0)$ such that $\Delta(\gamma) = \delta\gamma\delta^{-1}$, and similarly $\tilde{\Delta}$, we have

$$\Delta g_*(H) = H, \quad \text{where } H = H_{y_0} = \pi_*(\pi_1(Y, y_0)).$$

Proof. Consider the diagram of pointed spaces:

$$\begin{array}{ccc} (Y, y_0) & & (Y, w_0) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ (X, x_0) & \xrightarrow{g} & (X, z_0) \end{array}$$

Then $\tilde{g}$ exists and it is unique if and only if $g_*(H_{y_0}) = H_{w_0}$ or, applying Δ, if and only if

$$\Delta g_*(H_{y_0}) = \pi_*(\tilde{\Delta}\pi_1(Y, w_0)).$$

But $\tilde{\Delta}\pi_1(Y, w_0) = \pi_1(Y, y_0)$, and thus $\tilde{g}$ exists and is unique if and only if $\Delta g_*(H) = H$. □

Corollary 3.2. *Under the above notation, there exists a lifting $\tilde{g}$ of g if and only if $\Delta g_*(H)$ is a conjugate of H.*

Proof. There exists $\tilde{g}$ if and only if there exists $w_0' \in \pi^{-1}(z_0)$ such that $\Delta' g_*(H) = H$. This is equivalent to say that $\Delta g_*(H)$ is a conjugate of H (we conjugate by $\delta\delta'^{-1}$, where $\delta' = \pi \circ \tilde{\delta}'$ and $\tilde{\delta}'$ is a path from y_0 to w_0'). □

Corollary 3.3. *There exists a lifting $\tilde{g}$ with a fixed point if and only if there exists $x_0 \in \mathrm{Fix}(g)$ and a conjugate subgroup H' of H such that $g_*(H') = H'$.*

Furthermore, if g has order n, then $\tilde{g}$ also has order n (since $\tilde{g}^n$ is a lifting of the identity and has a fixed point).

Now we return to our situation.

Since the fundamental groups of S' (respectively, S'') give rise to subgroups of $\pi_1(S)$ which are the pull-backs of $\pi_1(A')$ (respectively, $\pi_1(A'')$) under the bundle homotopy exact sequence

$$1 \to \pi_1(F) \to \pi_1(S) \to \pi_1(A) \to 1,$$

we obtain that they are preserved under σ_* if and only if the corresponding subgroups of $\pi_1(A)$ are preserved under $\bar{\sigma}_*$.

In the latter case, all the fundamental groups are abelian and we need to prove that $\bar{\sigma}$ lifts to A'', or equivalently that Λ'' is $\bar{\sigma}$-invariant.

We observe that Λ' is invariant by $\bar{\sigma}$, since Λ' is the kernel of the topological monodromy $\nu : \Lambda \to \Gamma = \mathrm{Aut}(F)/\mathrm{Aut}^0(F)$, which is induced by α_*, and hence it is $\bar{\sigma}$-equivariant.

We observe that while Λ' is canonically defined as the kernel of the topological monodromy of α, it is *a priori* not clear that Λ'' can be canonically defined (we shall indeed prove later that Λ'' is the centre of the fundamental group of S).

However, using the list of Bagnera–de Franchis, case by case, we are going to see that Λ'' is a characteristic subgroup of Λ and therefore that it is $\bar{\sigma}$-invariant.

In fact, in the Cases 1, 3, 5, 7 of the list of Bagnera–de Franchis (see 2.3), there is nothing to prove, since we have $\Lambda'' = \Lambda'$.

In Case 2, we find $\Lambda'' = 2\Lambda$. In fact, we have $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, acting on $A'' \times F$ as

$$(x_1, x_2) \mapsto (x_1 + \eta, -x_2),$$
$$(x_1, x_2) \mapsto (x_1 + \eta', x_2 + \epsilon).$$

Analogously in Case 6, $\Lambda'' = 3\Lambda$; in Case 4, $\Lambda'' = 2\Lambda \cap \Lambda'$. Therefore, we always find that $\sigma(\Lambda'') = \Lambda''$ and thus we get a lifting

$$\begin{array}{ccc} A'' \times F & \xrightarrow{\tilde{\sigma}} & A'' \times F \\ \downarrow & & \downarrow \\ S & \xrightarrow{\sigma} & S \end{array}$$

Definition 3.4. The extended symmetry group $\hat{G}$ is the group generated by G and $\tilde{\sigma}$.

$\hat{G}$ is the group of homeomorphisms of S'' which lift the group $\{1, \sigma\}$. Hence we have the following extension that will be studied in the next section:

$$0 \to G \xrightarrow{\mathrm{i}} \hat{G} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 1. \qquad (3.1)$$

With the same arguments as above. we obtain the following result.

Theorem 3.5. *Let (S_1, σ_1), (S_2, σ_2) be isomorphic real hyperelliptic surfaces (i.e. there exists $\psi : S_1 \xrightarrow{\cong} S_2$ such that $\psi^{-1}\sigma_2\psi = \sigma_1$). Then the respective extended symmetry groups $\hat{G}$ are the same for S_1 and S_2. Moreover, let $S_1 = (E_1 \times F_1)/G$, $S_2 = (E_2 \times F_2)/G$ be two Bagnera–de Franchis realizations. Then there exists an isomorphism $\Psi : E_1 \times F_1 \to E_2 \times F_2$, of product type (i.e. $\Psi = \Psi_1 \times \Psi_2$), commuting with the action of $\hat{G}$, and inducing the given isomorphism $\psi : S_1 \xrightarrow{\cong} S_2$.*

Proof. ψ induces an isomorphism ψ_* of the Albanese varieties, which is compatible with the anti-holomorphic involutions $\bar{\sigma}_1$ and $\bar{\sigma}_2$ and such that we have a commutative diagram:

$$\begin{array}{ccc} S_1 & \xrightarrow{\psi} & S_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ A_1 & \xrightarrow{\psi_*} & A_2 \end{array}$$

Whence, by taking the coverings associated to the subgroups and points corresponding under ψ and ψ_*, we obtain isomorphisms $\tilde{\psi}$, ψ_1 and a commutative diagram:

$$\begin{array}{ccc} S_1'' & \xrightarrow{\tilde{\psi}} & S_2'' \\ \downarrow{\scriptstyle \alpha_1''} & & \downarrow{\scriptstyle \alpha_2''} \\ A_1'' & \xrightarrow{\psi_1} & A_2'' \end{array}$$

We observe that if $(E_1 \times F_1)/G$ is a Bagnera–de Franchis realization of S_1, then $E_1 \cong A_1''$ and there is an isomorphism of $(S_1'' \to A_1'')$ with $(E_1 \times F_1 \to E_1)$, commuting with the action of G.

We obtain therefore a commutative diagram

$$\begin{array}{ccc} E_1 \times F_1 & \xrightarrow{\tilde{\psi}} & E_2 \times F_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\psi_1} & E_2 \end{array}$$

where, moreover, $\tilde{\psi}$ and ψ_1 are real.

Since $\tilde{\psi}$ preserves the fibres of the two projections, we have

$$\tilde{\psi}(e, f) = (\psi_1(e), \psi_2(e, f)).$$

Let us fix an origin $0 \in E_1$. Then, since ψ_2 is an affine map, we can write

$$\psi_2(e, f) = \psi_2(0, f) + r(e),$$

where r is a holomorphic homomorphism $r : E_1 \to \mathrm{Pic}^0(F_2)$ ($\mathrm{Pic}^0(F_2)$ is the group of translations of F_2).

By G-equivariance, for all $g \in G \subset \mathrm{Pic}^0(E_1)$, we have

$$\psi_2(e+g, g(f)) = g(\psi_2(0,f) + r(e)). \tag{3.2}$$

But the left-hand side of (3.2) equals

$$\psi_2(0, g(f)) + r(e+g) = \psi_2(0, g(f)) + r(e) + r(g),$$

thus if we let g_* be the linear part of the affine transformation $g : F_2 \to F_2$, and we compare the linear parts of both sides of (3.2) with respect to e, we obtain

$$g_*(r(e)) \equiv r(e).$$

Since there is a $g \in G$ such that $g_* \neq \mathrm{Id}_{F_2}$, we infer that $r(e)$ is constant, or equivalently that $\psi_2(e,f) = \psi_2(f)$. □

Proposition 3.6. *Let $(S = (E \times F)/G, \sigma)$ be a real hyperelliptic surface, and let $\tilde{\sigma} : E \times F \to E \times F$ be a lifting of σ. Then the anti-holomorphic map $\tilde{\sigma}$ is of product type.*

Proof. As in the proof of the previous theorem, we have

$$\tilde{\sigma}(e,f) = (\sigma_1(e), \sigma_2(e,f)),$$

since $\tilde{\sigma}$ preserves the fibration onto E. Then, after choosing an origin $0 \in E$, we have

$$\sigma_2(e,f) = \sigma_2(0,f) + r(e),$$

where $r : E \to \mathrm{Pic}^0(F)$ is an anti-holomorphic homomorphism.

We know that $\tilde{\sigma}$ normalizes the group G. In particular, if we take some element $g \in G$, such that $g_* \neq \mathrm{Id}_F$, $\tilde{\sigma}$ has a matrix

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

while

$$g_* = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix},$$

with $\xi \neq 1$. We must have

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{\xi} \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix},$$

whence $b = 0$. □

Now we would like to see when the lifting $\tilde{\sigma}$ is an involution. Let us denote by $\phi : A'' \times F \to S$ the map in the diagram.

Remark 3.7. Let $\tilde{\sigma}$ be a lifting of σ, we have two different cases.

(1) $\exists z \in S$ such that $\sigma(z) = z$. Then, since the covering is Galois, for all $z' \in \phi^{-1}(z)$, there exists a lifting $\tilde{\sigma}$ of σ such that $\tilde{\sigma}(z') = z'$. But then $\tilde{\sigma}^2$ lifts the identity and has a fixed point; therefore, it is the identity.

(2) $\mathrm{Fix}(S) = \emptyset$. Let $z, w \in S$, $\forall z' \in \phi^{-1}(z)$, $\forall w' \in \phi^{-1}(w)$, $\exists!$ lifting $\tilde{\sigma}$, with $\tilde{\sigma}(z') = w'$.

4. Orbifold fundamental groups and the topological type of a real hyperelliptic surface

Let (X, σ) be a smooth real variety of dimension n (i.e. X is a smooth complex manifold of complex dimension n given together with an anti-holomorphic involution σ). Then we have a double covering $\pi : X \to Y = X/\langle\sigma\rangle$ ramified on $X' = X(\mathbb{R}) = \mathrm{Fix}(\sigma)$. Set $Y' := \pi(X')$.

We will define the orbifold fundamental group exact sequence of (X, σ) as the isomorphism class of a given extension

$$1 \to \pi_1(X, x_0) \to \pi_1^{\mathrm{orb}}(Y, y_0) \to \mathbb{Z}/2\mathbb{Z} \to 1. \tag{4.1}$$

The choice of a base point will, however, create some technical difficulties.

Definition 4.1.

(1) If $X' = \emptyset$, then we define $\pi_1^{\mathrm{orb}}(Y, y_0) = \pi_1(Y, y_0)$.

(2) If $X' \neq \emptyset$ and $x_0 \in X'$, $y_0 = \pi(x_0)$, σ acts on $\pi_1(X, x_0)$ and we define $\pi_1^{\mathrm{orb}}(Y, y_0)$ to be the semidirect product of the normal subgroup $\pi_1(X, x_0)$ with the cyclic group of order 2 generated by an element which will be denoted by $\tilde{\sigma}_0$ and whose action on $\pi_1(X, x_0)$ by conjugation is the one of σ.

(3) If $n \geqslant 3$, $Y' \neq \emptyset$ and $x_0 \notin X'$, define the orbifold fundamental group of Y based on y_0 as $\pi_1(Y - Y', y_0)$.

(4) Assume $x_0 \notin X'$, $X' \neq \emptyset$ and $\dim_{\mathbb{C}} X = 2$. Then the orbifold fundamental group of Y with base point $y_0 = \pi(x_0)$ is defined to be the quotient of $\pi_1(Y - Y', y_0)$ by the subgroup normally generated by $\gamma_1^2, \ldots, \gamma_m^2$, where $Y_1', \ldots, Y_m'$ are the connected components of Y' and γ_{i} is a simple loop around Y_{i}'.

Since in all cases we have a well-defined exact sequence

$$1 \to \pi_1(X, x_0) \to \pi_1^{\mathrm{orb}}(Y, y_0) \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

this will be called the orbifold fundamental group exact sequence.

Proposition 4.2. *The isomorphism class of the fundamental group exact sequence is independent of the choice of* y_0.

Proof. This is well known when comparing Cases 1, 3 and 4, which are mutually exclusive.

In Case 2 ($X' \neq \emptyset$) we claim that $\pi_1^{\text{orb}}(Y, y_0)$ is independent of the choice of $x_0 \in X'$. In fact, let δ be a path connecting x_0 with x_1. Then the map

$$\gamma \mapsto \delta^{-1}\gamma\delta$$

yields an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$. The action of σ on $\pi_1(X, x_1)$ reads out on $\pi_1(X, x_0)$ as the composition

$$\gamma \mapsto \delta^{-1}\gamma\delta \mapsto \sigma(\delta)^{-1}\sigma(\gamma)\sigma(\delta) \mapsto \delta\sigma(\delta)^{-1}\sigma(\gamma)\sigma(\delta)\delta^{-1}.$$

But this action is precisely the conjugation by $\tilde{\sigma}_1 := \delta\sigma(\delta)^{-1}\tilde{\sigma}_0$.

Since $\tilde{\sigma}_1$ is an element of order 2, we obtain that the split extensions

$$1 \to \pi_1(X, x_0) \to \pi_1^{\text{orb}}(Y, y_0) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

and

$$1 \to \pi_1(X, x_1) \to \pi_1^{\text{orb}}(Y, y_1) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

are isomorphic.

To relate Case 2 with the other two, it suffices, once $x_0 \in X'$ and x_1 are given, to choose a splitting of the extension

$$1 \to \pi_1(X, x_1) \to \pi_1^{\text{orb}}(Y, y_1) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

simply by taking γ_j with j such that $y_0 \in Y'_j$. □

Remark 4.3. Let us explain the definition of $\pi_1^{\text{orb}}(Y, y_0)$ in the case where $n \geqslant 2$ and $x_0 \notin X'$.

X' is a real submanifold of codimension n, and hence the map

$$\pi_1(X - X', x_0) \to \pi_1(X, x_0)$$

is surjective if $n \geqslant 2$ and it is an isomorphism for $n \geqslant 3$. The singularities of Y are contained in $Y' = \pi(X')$ and there we have a local model $\mathbb{R}^n \times (\mathbb{R}^n/(-1))$. Therefore, Y is smooth for $n = 2$ and topologically singular for $n \geqslant 3$. The local punctured fundamental group $\pi_1(Y - Y')_{\text{loc}}$ is isomorphic to $\mathbb{Z}$ for $n = 2$, while it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 3$. This means that the kernel of the surjection

$$\pi_1(Y - Y', y_0) \to \pi_1(Y, y_0)$$

is normally generated by loops γ around the components of Y'. If $n \geqslant 3$, then we automatically have $\gamma^2 = 1$.

Let $\tilde{X}$ be the universal covering of X, so that $X = \tilde{X}/\pi_1(X)$. The exact sequence (4.1) defines a group which is the group of liftings of the action of $\mathbb{Z}/2\mathbb{Z} \cong \{\text{Id}_X, \sigma\}$ to $\tilde{X}$, so that $Y = \tilde{X}/\pi_1^{\text{orb}}(Y)$.

Remark 4.4. If $X' \neq \emptyset$, then the exact sequence

$$1 \to \pi_1(X) \to \pi_1^{\mathrm{orb}}(Y) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

always splits, as follows by the definition.

Remark 4.5. The basic topological invariants of (X, σ) that we will use in this paper are the topological invariants of $X(\mathbb{R})$, the topological invariants of Y and $\pi_1^{\mathrm{orb}}(Y)$.

We have the following result.

Theorem 4.6. *Let (S, σ) be a real hyperelliptic surface. Then the topological and also the differentiable type of the pair (S, σ) is completely determined by the orbifold fundamental group exact sequence.*

Proof. We want first of all to show how $\Pi := \pi_1(S)$ determines the topological (actually differentiable) type of the hyperelliptic surface S.

Consider the exact homotopy sequence associated with the covering $\psi : S' \to S$ described in the proof of Theorem 2.4.

$$1 \to \Omega' = \pi_1(S') \cong \mathbb{Z}^4 \to \Pi \to G' \to 1, \tag{4.2}$$

where $G' = G/T$ and T is the subgroup of G acting by translations on $E \times F$. Let us look at the exact sequence

$$1 \to \Gamma = [\Pi, \Pi] \to \Pi \to \Lambda = \Pi/[\Pi, \Pi] \to 1. \tag{4.3}$$

Since the homomorphism $\Pi \to \Lambda$ is the map of the fundamental groups associated to the Albanese map (which is a fibre bundle), it follows that $\Gamma \cong \mathbb{Z}^2$ is the fundamental group of the fibre of α.

Ω' is then the kernel of the action of Π on Γ by conjugation. In fact, $\Gamma \subset \Omega'$ and Ω' is abelian. Moreover, by definition, the linear action of G' on the tangent space to F (which is $\Gamma \otimes \mathbb{R}$) is faithful.

Now Ω' is a representation of G' and Γ is a subrepresentation. An easy calculation (using, for example, the Bagnera–de Franchis list) shows that $(\Omega')^{G'} = \Lambda''$, which yields a direct sum $\Omega = \Lambda'' \oplus \Gamma$, such that $\Pi/\Omega = G$. Notice that since Ω' is abelian and by (4.2), we know that $(\Omega')^{G'}$ is the centre of Π, so Λ'' is the centre of Π.

Since $\Omega \subset \Omega'$, the universal cover $\tilde{X} \cong \mathbb{R}^4$ of S is homeomorphic to $\Omega' \otimes \mathbb{R}$ on which Ω' acts freely by translations. We have $S \cong (\Omega' \otimes \mathbb{R})/\Pi$, thus it suffices to show that the exact sequence (4.2) determines the action of Π on the universal covering $\Omega' \otimes \mathbb{R}$.

The action is given by a group of affine transformations of $\mathbb{R}^4$ and since Ω' is the subgroup of translations, the action of G' on the torus $\Omega' \otimes \mathbb{R}/\Omega'$ has a linear part which is determined by the conjugation action of G' on Ω'.

Since G' is cyclic, we may find a lifting $\lambda(g)$ to Π of its generator g. Now, for all $g' \in G'$, we have a lifting $\lambda(g')$ in Π, and if $m = |G'|$, we have $\lambda(g)^m \in \Omega'$. Every element $\gamma \in \Pi$ can be uniquely written as $\omega' \lambda(g')$, with $\omega' \in \Omega'$, $g' \in G'$.

Since $\lambda(g)^m \in \Omega'$ and it is invariant by conjugation by $\lambda(g)$, we have $\lambda(g)^m \in (\Omega')^{G'} = \Lambda''$. Therefore, we let g act on $(\Lambda'' \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ by

$$(e, f) \mapsto \left(e + \frac{1}{m}\lambda(g)^m, g(f)\right),$$

where g acts on Γ by conjugation.

It is immediate that the above action is precisely the one yielding S as $(E \times F)/G = (\Lambda'' \otimes \mathbb{R} \oplus \Gamma \otimes \mathbb{R})/\Pi$ and that the way we described it is completely dictated by Π as an abstract group.

Thus we have proven that Π determines the topological type of S.

Let us now consider the ramified covering $E \times F \to E \times F/\hat{G}$, where $\hat{G}$ is the extended symmetry group defined in (3.4). Then we have the following exact sequences:

$$\begin{array}{ccccccccc} 1 & \to & \Omega & \to & \Pi & \to & G & \to & 1, \\ & & \| & & \cap & & \cap & & \\ 1 & \to & \Omega & \to & \hat{\Pi} & \to & \hat{G} & \to & 1, \end{array}$$

where $\hat{\Pi}$ is the orbifold fundamental group

$$\begin{array}{ccccccccc} 1 & \to & \Omega' & \to & \Pi & \to & G' & \to & 1, \\ & & \| & & \cap & & \cap & & \\ 1 & \to & \Omega' & \to & \hat{\Pi} & \to & \hat{G}' & \to & 1. \end{array}$$

We want to describe how $\hat{\Pi}$ acts on $\Omega' \otimes \mathbb{R}$. In order to understand the action of $\hat{\Pi}$, it suffices to describe the action of a suitable element σ in $\hat{\Pi} - \Pi$. σ acts on $\Omega' \otimes \mathbb{R}$ by an affine transformation, $v \mapsto Av + b$, of which we know the linear part A, which is determined by the conjugation action of σ on Ω'.

On the other hand, we have $\sigma^2 \in \Pi$. Therefore, we know the affine map σ^2. Since we have

$$\sigma^2(v) = A^2 v + Ab + b =: A^2 v + b',$$

we are able to determine b uniquely from A and b', in the case in which $(A + I)$ is invertible.

We also know that σ is of product type and that $A^2 = A_1^2 \oplus A_2^2$ has the property that $A_1^2 = I$. Thus $W = \mathbb{R}_1^2 = \Lambda'' \otimes \mathbb{R}$ splits as a direct sum of eigenspaces $W^+ \oplus W^-$ and we can recover the translation vector b_1^+ by what we have remarked above. Whereas on $W^- = \operatorname{Im}(A_1 - I)$ we can change coordinates by a translation in such a way that the action of σ is linear on W^-.

Let us now determine the second component A_2. If (-1) is not an eigenvalue of A_2, we argue as before. Since σ is anti-holomorphic, (-1) is an eigenvalue of A_2 if and only if $A_2^2 = I$. Therefore, it remains to treat the case in which $A_2^2 = I$ and $(A_2 g)^2 = I$, $\forall g \in G$.

In this case, the second component of σ is given by $\sigma_2(x) = A_2 x + c$. Let us consider another element in $\hat{\Pi} - \Pi$, $s = g \circ \sigma$, where $g \in G$ is not a translation. We know the action of $\sigma \circ s$ and of $s \circ \sigma$, since they are in Π,

$$\sigma_2 \circ s_2(x) = A_2(gA_2x + c') + c,$$
$$s_2 \circ \sigma_2(x) = gA_2(A_2x + c) + c' = gx + gA_2c + c'.$$

Looking at the translation terms, we get knowledge of $gA_2c + c'$ and $A_2c' + c$.

We argue by looking at the rank of the matrix

$$\begin{pmatrix} gA_2 & I \\ I & A_2 \end{pmatrix},$$

which is applied to the vector

$$\begin{pmatrix} c \\ c' \end{pmatrix}.$$

It has the same rank as

$$\operatorname{rank}\begin{pmatrix} g & I \\ A_2 & A_2 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} g-I & I \\ 0 & A_2 \end{pmatrix} = 2 + \operatorname{rank}(g - I).$$

Then we are done, since $(g - I)$ is invertible, because g is holomorphic and it is not a translation, thus it does not have 1 as an eigenvalue. So the proof of the theorem is concluded, since we have shown that the exact sequence of the orbifold fundamental group completely determines the action of $\hat{\Pi}$ on $\Omega \otimes \mathbb{R}$. □

Corollary 4.7. *The topological type of a real hyperelliptic surface (S, σ) determines the following.*

(1) *The Bagnera–de Franchis group G.*

(2) *The extension $1 \to G \to \hat{G} \to \mathbb{Z}/2\mathbb{Z} \to 1$.*

(3) *The topological type of the action of $(\hat{G} \supset G)$ on E and F, or equivalently the affine equivalence class of the representation of $\hat{\Pi}/\Gamma \supset \Pi/\Gamma$ on $\Lambda'' \otimes \mathbb{Q}$ and of $\hat{\Pi}/\Lambda'' \supset \Pi/\Lambda''$ on $\Gamma \otimes \mathbb{Q}$. In particular, it determines the topological type of the real elliptic curve $(E/G, \hat{G}/G)$.*

Remark 4.8. Notice that the representation ρ of $\hat{\Pi}$ in the group of the affine transformations of $\Omega \otimes \mathbb{Q}$ takes values in $A(2, \mathbb{Z}, \mathbb{Q}) \times A(2, \mathbb{Z}, \mathbb{Q})$, where

$$A(2, \mathbb{Z}, \mathbb{Q}) = \{v \mapsto Bv + \beta \mid B \in GL(2, \mathbb{Z}),\ \beta \in \mathbb{Q}^2\}.$$

We want now to discuss, in some greater generality than needed for our present purposes, the notion of moduli space of real varieties. Since we will only assume that X is a complex manifold, we have to adopt the point of view of Kodaira, Spencer and Kuranishi (cf. [**10, 31, 32**]).

For a general complex manifold X, we have the Kuranishi family of deformations of X, $\phi : \mathcal{X} \to \mathcal{B}$. Here, the base $\mathcal{B}$ of the Kuranishi family is a complex analytic subset of the vector space $H^1(X, \Theta_X)$, corresponding to the complex structures J, which satisfy Kuranishi's integrability equation.

If (X, σ) is real, we want to see when a neighbouring complex structure J is such that the differentiable map σ remains anti-holomorphic (thus we get a deformation X_t of the complex manifold X such that the new pair (X_t, σ) is still real).

The corresponding equation is simply

$$\sigma_* J = -J\sigma_*,$$

or, equivalently,

$$-\sigma_* J \sigma_* = J.$$

We see immediately that this condition means that J lies in the fixed locus of the involution induced by σ on the Kuranishi family.

Interpreting J as a harmonic representative of a Dolbeault cohomology class $\theta \in H^1(X, \Theta_X)$, we are going to show, more precisely, that σ induces a complex anti-linear involution σ^* on the vector space $H^1(X, \Theta_X)$, such that the real part $\mathcal{B}(\mathbb{R})$ consists of the deformations of X for which σ remains real.

Recall that the complex structure J_0 of X induces a splitting of the complexified real tangent bundle of X, $T_X \otimes \mathbb{C} = T^{0,1} \oplus T^{1,0}$, where $T^{0,1} = \bar{T}^{1,0}$.

The form $\theta = \Sigma_{i\bar{j}} \theta_{i\bar{j}} (\partial / \partial z_{\mathrm{i}} \otimes \mathrm{d}\bar{z}_j)$ can be interpreted as yielding a linear map $\theta' : T^{0,1} \to T^{1,0}$ and, accordingly a new subbundle,

$$T^{1,0}_\theta = \{(u, v) \in T^{0,1} \oplus T^{1,0} \mid v = \theta'(u)\}.$$

θ' is completely determined by the matrix $\phi = \theta_{i\bar{j}}$ and then the subspace

$$T^{0,1}_\theta = \{(u, v) \in T^{0,1} \oplus T^{1,0} \mid u = \bar{\theta}'(v)\},$$

where $\bar{\theta}'$ is determined by the matrix $\bar{\phi} = \bar{\theta}_{\bar{i}j}$ in the chosen coordinate basis.

Saying that σ is anti-holomorphic amounts to saying that its differential induces complex linear isomorphisms $\sigma_* : T^{0,1} \to T^{1,0}$ and $\sigma_* : T^{1,0} \to T^{0,1}$, which are inverses and conjugate to each other.

Thus, if A is the matrix of $\sigma_* : T^{0,1} \to T^{1,0}$, $A^{-1} = \bar{A}$ is the matrix of $\sigma_* : T^{1,0} \to T^{0,1}$.

We want now to write down the condition that σ be anti-holomorphic for the new complex structure induced by the form θ.

Again, this means that $\sigma_* : T^{1,0}_\theta \to T^{0,1}_\theta$, i.e. that, for each vector u, the image

$$\sigma_*(u, \theta'(u)) = (\bar{A}\phi(u), A(u))$$

satisfies the equation of $T^{0,1}_\theta$, namely, we have

$$\bar{A}\phi(u) = \bar{\phi} A(u).$$

Again, we can equivalently define

$$\sigma^*(\phi) := A\bar{\phi}A.$$

Write the above (since it must hold for each vector u) as

$$\phi = \sigma^*(\phi).$$

It is now obvious that σ^* is complex anti-linear, and it is an involution since

$$\sigma^{*2}(\phi) = A\bar{A}\phi\bar{A}A = \phi.$$

By what we have shown above, it follows right away that $\mathcal{B}(\mathbb{R})$, being the intersection of a complex analytic space with the fixed part of a complex anti-linear involution, is a real analytic space.

In general, by a result of Wavrik [**46**], if the function $h : \mathcal{B} \to \mathbb{Z}$ given by

$$h(t) = h^0(X_t, \Theta_{X_t})$$

is constant on the germ $\mathcal{B}$, then the quotient

$$\mathcal{B}/\operatorname{Aut}(X)$$

is a local moduli space.

The group $\operatorname{Aut}(X)$ does not, however, act on the set $\mathcal{B}(\mathbb{R})$. Therefore, we take the smaller subgroup

$$\operatorname{Aut}^{\sigma}(X) := \{\phi \in \operatorname{Aut}(X) \mid \phi^*(\sigma) = \sigma\},$$

where

$$\phi^*(\sigma) := \phi\sigma\phi^{-1}.$$

We can finally define the real local moduli space as follows.

Definition 4.9. Under the assumption that the function $h^0(X_t, \Theta_{X_t})$ is constant on the base of the Kuranishi family, the real local moduli space of (X, σ) is defined as the quotient

$$\mathcal{B}(\mathbb{R})/\operatorname{Aut}^{\sigma}(X).$$

Remark 4.10. Therefore, the real local moduli space is just a real semi-algebraic space, and it maps neither surjectively nor one-to-one to the real part of the complex local moduli space.

Remark 4.11. Once we have local moduli spaces for the varieties or manifolds under consideration, the standard procedure is to consider the global moduli space as the set of isomorphism classes of such varieties and to use the local moduli spaces as giving local charts. For instance, if we have local moduli spaces for a certain class of complex manifolds, these charts provided by the local moduli spaces yield, for our global moduli space, the structure of a complex analytic space, possibly non-Hausdorff.

Likewise, for real algebraic varieties such that Wavrik's condition holds, the global moduli space is a semi-analytic space.

Now, one can give different definitions for polarized algebraic varieties, and again one has to distinguish between the real part of the quotient of a Hilbert scheme, and the quotient of its real part. However, the approach via the Kuranishi family is particularly suitable for the case of hyperelliptic surfaces, as we are going now to see.

Theorem 4.12. *The moduli space of real hyperelliptic surfaces of a given topological type is irreducible.*

Proof. Let us fix an orbifold fundamental group $\hat{\Pi}$. Then Theorem 4.6 and Corollary 4.7 tell us that $\hat{\Pi}$ determine the representation $\rho : \hat{\Pi} \to A(2, \mathbb{Z}, \mathbb{Q})^2$. Such a representation induces a representation of $\hat{G}'$ into $GL(2, \mathbb{Z})^2$,

$$\hat{G}' \xrightarrow{(\lambda_1, \lambda_2)} GL(2, \mathbb{Z})^2,$$

where $\hat{G}'$ is the group fitting into the exact sequence

$$1 \to G' \to \hat{G}' \to \mathbb{Z}/2\mathbb{Z} \to 1. \tag{4.4}$$

We observe that giving a hyperelliptic surface with the given topological type amounts to giving two complex structures J_1, J_2 on $\mathbb{Z}^2 \otimes \mathbb{Q}$ such that $\lambda_i(G')$ consists of $\mathbb{C}$-linear maps, whereas $\lambda_i(\hat{G}' - G')$ consists of $\mathbb{C}$-anti-linear maps $(i = 1, 2)$.

Therefore, we have to solve the following problem. Given a representation $\lambda : \hat{G}' \to GL(2, \mathbb{Z})$, find all the complex structures J such that, for a generator g' of G', we have

$$\lambda(g')J = J\lambda(g'),$$

and for a $\sigma \notin G'$, we have

$$\lambda(\sigma)J = -J\lambda(\sigma).$$

We have, of course, to keep in mind that if M is a fixed differentiable manifold with an involution $\sigma : M \to M$, and we look for the complex structures J which make σ an anti-holomorphic involution, for each solution J we shall also find the solution $-J$.

But, as we already remarked in § 1, σ provides an isomorphism between the complex manifolds (M, J) and $(M, -J)$, and clearly it conjugates the involution σ to itself.

Therefore, if we shall see that the parameter space for our complex structures J will consist of exactly two irreducible components, exchanged by the involution $J \to -J$, it will follow that the moduli space is irreducible.

We shall prove in the next section that the extension (4.4) always splits (see Corollary 5.3); therefore, we may assume that $\sigma^2 = 1$. In particular, we can find a basis in such a way that either

$$\zeta := \lambda(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\zeta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we now set $B := \lambda(g')$, since $G' \cong \mathbb{Z}/q\mathbb{Z}$ for $q = 2, 3, 4, 6$, we have either $\zeta B \zeta = B$ or $\zeta B \zeta = B^{-1}$. We have two different cases.

Case I. We have

$$\zeta := \lambda(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, if we set

$$B = \lambda(g') = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

we have again two cases:

(1) $\zeta B \zeta = B$; and

(2) $\zeta B \zeta = B^{-1}$.

In case 1, we find

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{pmatrix},$$

whence, since $\det(B) = 1$, we have $B = \pm \mathrm{Id}$. But we shall see in Lemma 5.8 that this case occurs if and only if $q = 2$ and $B = -\mathrm{Id}$.

In Case 2, we find

$$\begin{pmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix},$$

whence $b_{11} = b_{22} =: b$, $b^2 - b_{12}b_{21} = 1$, the characteristic polynomial is $\lambda^2 - 2b\lambda + 1$. But since either $B = -\mathrm{Id}$ or B has no real eigenvalues, $b^2 - 1 < 0$. Thus $b = 0$, whence

$$B = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and we may assume

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

by changing the choice of a generator of G'.

Case II. We have

$$\zeta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We then again have the two cases 1 and 2.

Case II.1. We find

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{22} & b_{21} \\ b_{12} & b_{11} \end{pmatrix},$$

thus

$$B = \begin{pmatrix} b & c \\ c & b \end{pmatrix},$$

and since $\det(B) = b^2 - c^2 = 1$, we have $b + c = \pm 1$, $b - c = \pm 1$, whence $c = 0$ and $B = -\mathrm{Id}$.

Case II.2. We have

$$\begin{pmatrix} b_{22} & b_{21} \\ b_{12} & b_{11} \end{pmatrix} = \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix},$$

which yields

$$B = \begin{pmatrix} b_{11} & c \\ -c & b_{22} \end{pmatrix}.$$

But $b_{11}b_{22} + c^2 = 1$, thus either $c = 0$ and $B = -\mathrm{Id}$, or $b_{11}b_{22} \leqslant 0$.

If $c = \pm 1$, $b_{11}b_{22} = 0$. By exchanging the two basis vectors e_1 and e_2, we may assume $c = -1$ and, by replacing B with B^{-1}, we can assume $b_{22} = 0$.

Thus

$$B = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}$$

and either $b = \pm 1$ if $G' = \mathbb{Z}/6\mathbb{Z}$ or $G' = \mathbb{Z}/3\mathbb{Z}$, or $b = 0$,

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $G' = \mathbb{Z}/4\mathbb{Z}$.

Assume then that $|c| \geqslant 2$. Since $B^q = 1$ ($q = 2, 3, 4, 6$), we have $|b_{11} + b_{22}| < 2$, else $B = \pm\mathrm{Id}$, which implies $c = 0$; absurd.

Thus we have two possible cases,

$$b_{11} + b_{22} = 0,$$
$$b_{11} + b_{22} = \pm 1.$$

In the first case, we find

$$B = \begin{pmatrix} b & c \\ -c & -b \end{pmatrix}$$

and $-b^2 + c^2 = 1$, thus $b = 0$, $c = \pm 1$; a contradiction.

In the second case, the matrix B has the form

$$\begin{pmatrix} b & c \\ -c & \pm 1 - b \end{pmatrix},$$

we may assume $b > 0$, and we have $\pm b - b^2 + c^2 = 1$.

The equation $\pm b - b^2 + c^2 = 1$ is equivalent to $c^2 = 1 + b^2 \mp b = (b \mp 1)^2 \pm b$, whence $\pm b = (c - b \pm 1)(c + b \mp 1)$. Taking absolute values in the last equation, we see that we cannot have $|c| \geqslant 2$.

We have therefore obtained the following possibilities for ζ and B.

Case I.1: $\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = -\mathrm{Id}$.

Case I.2: $\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = -\mathrm{Id}$ or $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Case II.1: $\zeta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = -\mathrm{Id}$.

Case II.2: $\zeta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = -\mathrm{Id}$ or $B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

or $B = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

or $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We now look for a complex structure

$$J = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

such that

$$\zeta J \zeta = -J,$$
$$BJ = JB.$$

Case I. The first equation reads

$$\begin{pmatrix} d_{11} & -d_{12} \\ -d_{21} & d_{22} \end{pmatrix} = -J.$$

Thus $d_{11} = d_{22} = 0$, and since the characteristic polynomial for J is $\lambda^2 + 1$, we have $\det(J) = 1$, equivalently, $d_{12}d_{21} = -1$, whence J has the form

$$\begin{pmatrix} 0 & -d \\ 1/d & 0 \end{pmatrix}.$$

Whence, as promised, the parameter space consists of the two branches of a hyperbola, which are exchanged by multiplication by -1.

Case II. The equation $\zeta J \zeta = -J$ reads

$$\begin{pmatrix} d_{22} & d_{21} \\ d_{12} & d_{11} \end{pmatrix} = -J.$$

Thus we obtain

$$J = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix},$$

with $a^2 - b^2 = -1$. Whence we can write $a = \frac{1}{2}(c - 1/c)$, $b = \frac{1}{2}(c + 1/c)$ and the conclusion is exactly as before.

Let us now consider the commutation relation $BJ = JB$.

If $B = -\operatorname{Id}$, it is obviously verified.

If

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have Cases I and II.

In Case I, the commutation relation yields

$$\begin{pmatrix} -d & 0 \\ 0 & -1/d \end{pmatrix} = \begin{pmatrix} -1/d & 0 \\ 0 & -d \end{pmatrix},$$

whence $d^2 = 1$, i.e. $d = \pm 1$,

$$J = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In Case II, the commutation relation yields

$$\begin{pmatrix} b & -a \\ -a & b \end{pmatrix} = \begin{pmatrix} b & a \\ a & b \end{pmatrix},$$

whence $a = 0$ and $b^2 = 1$, i.e. $b = \pm 1$,

$$J = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Assume now that

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we are in Case II and the commutation relation reads

$$\begin{pmatrix} a+b & a+b \\ a & b \end{pmatrix} = \begin{pmatrix} a+b & -a \\ -a-b & b \end{pmatrix},$$

whence $b = -2a$ and since $b^2 - a^2 = 1$, we have $a = \pm 1/\sqrt{3}$.

We observe that if J commutes with B, it also commutes with B^2. Thus, up to a base change, we have also solved the case $G' = \mathbb{Z}/3\mathbb{Z}$. $\square$

5. The extended symmetry group

In this section we want to describe the extended Bagnera–de Franchis group, fitting into the exact sequence (3.1),

$$0 \to G \xrightarrow{\mathrm{i}} \hat{G} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 1.$$

Since G is a normal abelian subgroup of $\hat{G}$, conjugation induces an action of $\hat{G}/G = \mathbb{Z}/2\mathbb{Z}$ on G.

Recall that once such an action is specified, the equivalence classes of such extensions are in bijective correspondence with the elements of

$$H^2(\mathbb{Z}/2\mathbb{Z}, G) = \frac{\ker(\sigma - 1)}{(1+\sigma)G} = \frac{G^{\mathbb{Z}/2}}{(1+\sigma)G},$$

where $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$.

Let us consider the anti-holomorphic elements of the extended Bagnera–de Franchis group, which will be denoted by $\tilde{\sigma}$ (they are the liftings of σ in $\hat{G}$). We showed in Proposition 3.6 that their action is also of product type, whence we can restrict our preliminary investigation to the question: which extended Bagnera–de Franchis groups act as a group of diholomorphic automorphisms of an elliptic curve?

Definition 5.1. A diholomorphic action of an extended Bagnera–de Franchis group $\hat{G}$ on an elliptic curve is an action such that G is precisely the subgroup of holomorphic automorphisms in $\hat{G}$, while the elements in $\hat{G} - G$ act as anti-holomorphic automorphisms.

Lemma 5.2. *Consider a di-holomorphic action of an extended Bagnera–de Franchis group $\hat{G}$ on an elliptic curve. Then the square of an anti-holomorphic map $\tilde{\sigma}$ in $\hat{G}$ is a translation.*

Proof. Let $\tilde{\sigma}^2 = g \in G$. Passing to the universal cover, we can write

$$\tilde{\sigma}(z) = a\bar{z} + b.$$

We then have

$$\tilde{\sigma}^2(z) = a\bar{a}z + a\bar{b} + b = g(z).$$

We know that $g(z) = \xi z + c$ with $\xi^n = 1$, $n = |G|$.

Since $\xi = |a|^2 \in \mathbb{R}_{>0}$, we obtain $\xi = 1$. □

Corollary 5.3. *If the group G is cyclic, then the extension (3.1) always splits.*

Proof. Since $G = G'$, for any lifting $\tilde{\sigma}$ of σ, we must have

$$\tilde{\sigma}^2 = \mathrm{Id} \in G,$$

so (3.1) splits. □

We want now to recall the description of the anti-holomorphic maps σ acting on an elliptic curve C, whose square σ^2 is a translation.

Write

$$C = \mathbb{C}/\Gamma,$$
$$\sigma(z) = a\bar{z} + b,$$
$$\sigma^2(z) = a\bar{a}z + a\bar{b} + b.$$

Whence

$$|a|^2 = 1. \tag{5.1}$$

The following condition must be verified,

$$a\bar{\Gamma} = \Gamma,$$

which is clearly equivalent to the existence of integers m, n, m', n' such that

$$a = m + n\tau \in \Gamma, \tag{5.2}$$
$$a\bar{\tau} = m' + n'\tau, \quad m', n' \in \mathbb{Z}, \quad \text{with } mn' - nm' = -1. \tag{5.3}$$

We may rewrite (5.1) as

$$(m + n\tau)(m + n\bar{\tau}) = m^2 + n^2|\tau|^2 + mn(\tau + \bar{\tau}) = 1. \tag{5.4}$$

We may assume that τ lies in the modular triangle, i.e. that

$$|\tau| \geqslant 1, \quad |\operatorname{Re}\tau| \leqslant \tfrac{1}{2}.$$

Then $\operatorname{Im}(\tau) \geqslant \frac{1}{2}\sqrt{3}$, and since $a = m + n\tau$, $\operatorname{Im}(a) = n\operatorname{Im}(\tau) \leqslant 1$, and we conclude that $|n| \leqslant 1$.

Remark 5.4. Exactly the following cases are the cases which occur.

(i) $n = 0$. Then, since $|a| = 1$,

$$m = a = -n' = \pm 1.$$

Moreover, equation (5.3) tells us that $2(\operatorname{Re})\tau = -n'm'$, whence we either have

$$\operatorname{Re}(\tau) = 0$$

or

$$\operatorname{Re}(\tau) = -\tfrac{1}{2}.$$

(ii) Observing that if $|n| = 1$, by (5.4), we infer that

$$m^2 + 1 \leqslant m^2 + |\tau|^2 = 1 \pm (\tau + \bar{\tau}) \leqslant 1 + |m|,$$

and thus $|m| \leqslant 1$, $|\tau| = 1$, giving rise to the following two cases.

(1) $|n| = 1$, $m = 0$, $|\tau| = 1$ (thus $a = \pm\tau$).

(2) $|n| = 1$, $|m| = 1$, $|\tau| = 1$ and since $1 + |\tau|^2 = 1 - mn(\tau + \bar{\tau})$, we may also assume that

$$\mathrm{Re}(\tau) = -\tfrac{1}{2},$$

and thus $m = n$, $a = m(1 + \tau)$.

Lemma 5.5. *Assume now that σ is an anti-holomorphism of an elliptic curve whose square is a translation of finite order d. Then we may choose the origin in the universal cover $\mathbb{C}$ in such a way that $\sigma(z) = a\bar{z} + b$, with a as above,*

$$\begin{aligned}
&b \in (1/d)\mathbb{Z} && \text{for } a \neq \pm 1,\\
&b \in \left(\frac{1}{2d}\right)\mathbb{Z} && \text{if } a = 1,\\
&b \in \left(\frac{1}{2d}\right)\mathbb{Z}(\tau) && \text{for } a = -1,\ \mathrm{Re}\,\tau = 0,\\
&b \in \left(\frac{1}{2d}\right)\mathbb{Z}(2\tau + 1) && \text{for } a = -1,\ \mathrm{Re}\,\tau = -\tfrac{1}{2}.
\end{aligned}$$

Assume further that $d = 1$, i.e. that σ is an involution. Then, obviously, we may get $b = 0$ if and only if $\mathrm{Fix}(\sigma) \neq \emptyset$.

$\mathrm{Fix}(\sigma) = \emptyset$ if and only if $\mathrm{Re}(\tau) = 0$ and we may choose the origin in such a way that $b = \frac{1}{2}$ for $a = 1$, or $b = \frac{1}{2}\tau$ for $a = -1$.

Moreover, σ normalizes a finite group of translations T if and only if, identifying T with a subgroup of C, $a\overline{T} = T$.

Proof. We first look for a vector w such that

$$a\bar{w} - w + b := \beta$$

be either a real vector, or an imaginary vector.

We therefore look at the image of the linear map

$$w \to a\bar{w} - w.$$

Its complexification $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ has a matrix

$$\begin{pmatrix} -1 & a \\ \bar{a} & -1 \end{pmatrix},$$

whose determinant is zero, whence the image of the above linear map equals $\mathbb{R}(a-1)$ for $a \neq 1$, and $\mathbb{R}\mathrm{i}(\mathrm{Im}\,\tau)$ for $a = 1$. We can therefore achieve that β is a real vector, unless a is real, $a \neq 1$, i.e. $a = -1$, in which case, we can achieve that β is an imaginary vector.

Since

$$\sigma^2(z) = z + a\bar{\beta} + \beta,$$

if β is real, we get

$$2\beta \in (1/d)\mathbb{Z} \quad \text{if } a = 1.$$

If $a \neq \pm 1$ and β is real, then we have

$$(1 \pm \tau)\beta \in (1/d)\mathit{\Gamma},$$

so

$$\beta \in (1/d)\mathbb{Z}.$$

If $a = -1$ and β is imaginary, then

$$2\beta \in (1/d)\mathit{\Gamma}$$

and the first assertion follows.

If $d = 1$ and $a \neq \pm 1$, then $\beta \in \mathbb{Z} \subset \mathit{\Gamma}$, thus there is a fixed point. Otherwise, we observe that if $\mathrm{Re}(\tau) = -\frac{1}{2}$, the involutions

$$z \mapsto \bar{z} + \tfrac{1}{2},$$
$$z \mapsto -\bar{z} + \tfrac{1}{2} + \tau = -\bar{z} + \mathrm{i}\,\mathrm{Im}(\tau)$$

have, respectively, $\frac{1}{2}\mathrm{i}\,\mathrm{Im}(\tau)$, $\frac{1}{4}$ as fixed points, thus in both cases we can assume $\beta = 0$.

It is immediate to verify that if $\mathrm{Re}(\tau) = 0$ and $a = 1$, $b = \frac{1}{2}$, or $a = -1$, $b = \frac{1}{2}\tau$, there are no fixed points.

The third assertion follows from the fact that

$$\sigma^{-1}(z) = a\bar{z} - a\bar{b},$$

whence, for a translation $z \to z + c$, conjugation by σ yields $z \to z + a\bar{c}$. □

Remark 5.6. Assume now that σ is an anti-holomorphic involution of an elliptic curve C, as in the previous lemma. Then there are only three possible topological types for the action of σ on C.

(i) $\mathrm{Fix}(\sigma) = \emptyset$.

(ii) $\mathrm{Fix}(\sigma)$ is homeomorphic to $\boldsymbol{S}^1$. This occurs if

$$|\tau| = 1, \quad a = \pm\tau$$

or if

$$\mathrm{Re}(\tau) = -\tfrac{1}{2}, \quad a = \pm 1, \quad b = 0.$$

(iii) $\mathrm{Fix}(\sigma)$ has two connected components homeomorphic to $\boldsymbol{S}^1$. This occurs if

$$\mathrm{Re}(\tau) = 0, \quad a = \pm 1, \quad b = 0.$$

Proof. Fix$(\sigma) = \emptyset$. By Lemma 5.5, there are only two cases. They are obviously topological equivalent (exchange the two basis vectors of Γ, 1 and τ).

If Fix$(\sigma) \neq \emptyset$, then we may assume $b = 0$, and the topological type is completely determined by the integral conjugacy class of the matrix

$$A = \begin{pmatrix} m & m' \\ n & n' \end{pmatrix},$$

whose square is the identity and which has 1 and -1 as eigenvalues.

If A is diagonalizable, then Fix(σ) has two connected components homeomorphic to $\boldsymbol{S}^1$, otherwise A is conjugated to the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and Fix(σ) is homeomorphic to $\boldsymbol{S}^1$. □

Remark 5.7. Let G be a Bagnera–de Franchis group. Then G has a first incarnation as a group of translations of E, and a second one as a direct product

$$G = G' \times T,$$

where T is a group of translation and G' is cyclic and a subgroup of the multiplicative group. If $\hat{G}$ is an extended Bagnera–de Franchis group, we let σ be an element in $\hat{G} - G$: it conjugates G, sending T to itself. In the first incarnation, group conjugation is given, as we saw in the previous lemma, by complex conjugation followed by multiplication by a (if σ acts by $\sigma(z) = a\bar{z} + b$).

It follows that the extension

$$0 \to G \to \hat{G} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

splits if and only if, c' being the translation vector of σ^2, then c' lies in the image of the endomorphism of G given by $s + \mathrm{Id}$, s being the action of σ. In the second incarnation, let

$$z \to \xi z$$

be a generator g' of G' and let $\sigma(z) = a\bar{z} + b$. Then σ conjugates g' to the transformation

$$z \to \bar{\xi} z + a(\bar{\xi} - 1)\bar{b}.$$

Whence, if the action of σ on G' by conjugation is trivial, then

$$G' \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad 2b \in \Gamma.$$

Proof. We only need to remark that $a\bar{c} \in \Gamma$ holds if and only if $c \in \Gamma$. □

We can now give the list of all the possible groups $\hat{G}$.

Lemma 5.8. *Let us consider the extension (3.1),*

$$0 \to G \to \hat{G} \to \mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle \to 0.$$

We have the following possibilities for the action of σ on G.

(1) *If $G = \mathbb{Z}/2\mathbb{Z}$, then*

$$\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

(2) *If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then either σ acts as the identity on G and*

$$\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

if (3.1) splits,

$$\hat{G} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

if (3.1) does not split, and in this latter case the square of σ is the generator of T. Or σ acts as

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the sequence splits,

$$\hat{G} = D_4,$$

the dihedral group, and again the square of the generator of $\mathbb{Z}/4\mathbb{Z}$ is the generator of T.

(3) *If $G = \mathbb{Z}/4\mathbb{Z}$, then σ acts as $-\mathrm{Id}$ on G and*

$$\hat{G} = D_4.$$

(4) *If $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then either*

$$\hat{G} = T \times D_4 \cong \mathbb{Z}/2\mathbb{Z} \times D_4$$

or $\hat{G}$ is isomorphic to the group

$$G_1 := \langle \sigma, g, t \mid \sigma^2 = 1,\ g^4 = 1,\ t^2 = 1,\ t\sigma = \sigma t,\ tg = gt,\ \sigma g = g^{-1} t \sigma \rangle,$$

and its action on the second elliptic curve F is generated by the following transformations:

$$\sigma(z) = \bar{z} + \tfrac{1}{2}, \qquad g(z) = \mathrm{i}z, \qquad t(z) = z + \tfrac{1}{2}(1 + \mathrm{i}).$$

*The group G_1 is classically denoted by c_1 (cf. [**26**, p. 39]).*

In particular, in both cases, sequence (3.1) splits.

(5) *If $G = \mathbb{Z}/3\mathbb{Z}$, then σ acts as $-\mathrm{Id}$ on G and*

$$\hat{G} = \mathcal{S}_3.$$

(6) *If* $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, *then we may choose* G' *so that* σ *acts as* $-\mathrm{Id} \times \mathrm{Id}$ *on* $G = G' \times T$ *and*

$$\hat{G} = \mathcal{S}_3 \times \mathbb{Z}/3\mathbb{Z}.$$

(7) *If* $G = \mathbb{Z}/6\mathbb{Z}$, *then* σ *acts as* $-\mathrm{Id}$ *on* G *and*

$$\hat{G} = D_6.$$

Proof. Thanks to (5.3), we know that in the Cases 1, 3, 5, 7, the sequence (3.1) splits.

Case 1. If $G = \mathbb{Z}/2\mathbb{Z}$, then σ acts as the identity on G, whence $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Cases 3, 5, 7. From the previous remark, we can immediately determine $\hat{G}$ in these cases where G is, respectively, equal to $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$. In fact, $|\xi| = 1$, whence $\bar{\xi} = \xi^{-1}$. Thus σ acts as $-\mathrm{Id}$ on $G = G'$, the extension splits and $\hat{G}$ is a dihedral group, respectively, D_4, $D_3 = \mathcal{S}_3$, D_6.

Case 6. ($G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.) Any element σ of order 2 makes (3.1) split.

Since $T = \mathbb{Z}/3\mathbb{Z}$ is invariant and σ acts as -1 on G/T, we can take eigenspaces T and G' such that σ acts on G' as $-\mathrm{Id}$ and as $+\mathrm{Id}$ or $-\mathrm{Id}$ on T. The second case is not possible, since, looking at the first incarnation, we obtain that the two eigenvalues of the action of σ on the lattice Γ of E (either equal to 1 or -1) reduce to -1 modulo 3, thus they equal -1 and σ acts holomorphically; a contradiction.

Case 2. ($G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.) σ acts on G and trivially on $T = \mathbb{Z}/2\mathbb{Z}$, whence either the action is the identity, or is given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In the latter case, since, as we noticed,

$$H^2(\mathbb{Z}/2\mathbb{Z}, G) = \frac{\ker(\sigma - 1)}{(1+\sigma)G},$$

then

$$H^2(\mathbb{Z}/2\mathbb{Z}, G) = 0,$$

so the extension splits and $\hat{G} = D_4$ (and an element of order 4 is given by σe_1, whose square is indeed e_2, the generator of T).

If σ acts as the identity on G, $\hat{G}$ is abelian, and the exact sequence splits if and only if there is no element of order 4. Hence the only possibilities for $\hat{G}$ are the following:

(i) $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the sequence splits;

(ii) $\hat{G} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In the latter case, we recall that by Lemma 5.2 the square of σ is the generator of T.

Case 4. (G is, non-canonically, isomorphic to $\mathbb{Z}/4\mathbb{Z} \times T$, with $T \cong \mathbb{Z}/2\mathbb{Z}$.) σ^2 belongs to T. Moreover, σ acts as Id on T and as $-\mathrm{Id}$ on G/T.

Case 4.I. There is an element g such that $\sigma g \sigma^{-1} = g^{-1}$. In this case, for each $g \in G$, we get

$$(\sigma g)^2 = \sigma g \sigma^{-1} g (\sigma)^2 = (\sigma)^2$$

because $\mathrm{Id} + s = 0$.

Whence, we distinguish two cases.

Case 4.I.a. ($(\sigma)^2 = 1$.) Here, it follows right away that

$$\hat{G} \cong T \oplus D_4.$$

Case 4.I.b. ($(\sigma)^2 = t$, t being the generator of T.) This case can be excluded as follows. First of all, notice that for the elliptic curve F we have $\tau = \mathrm{i}$, and t equal to the translation by the half-period $\frac{1}{2}(1 + \mathrm{i})$.

We have $\sigma g \sigma^{-1} = g^{-1}$ for each element of order 4, in particular, for the multiplication by i. Furthermore, up to multiplying σ by a power of the above element, we may assume (cf. the discussion preceding Lemma 5.5) that $a = 1$, i.e. that $\sigma(z) = \bar{z} + b$.

Then our conjugation relation reads out as follows,

$$a(-1 - \mathrm{i})\bar{b} \equiv 0 \pmod{\mathit{\Gamma}},$$

or, more simply, as

$$(1 + \mathrm{i})\bar{b} \equiv 0 \pmod{\mathit{\Gamma}}.$$

Whence b is a half-period $b = \frac{1}{2}(x + \mathrm{i}y)$ ($x, y \in \mathbb{Z}$) and the above condition means that $x + y \equiv 0 \pmod 2$.

But a previous calculation shows that the square of σ is a translation by

$$a\bar{b} + b = x \equiv 0 \pmod{\mathit{\Gamma}},$$

contradicting $(\sigma)^2 = t$.

There remains the following case.

Case 4.II. ($\sigma g \sigma^{-1} = g^{-1} t$.) In this case, we can always choose σ such that $(\sigma)^2 = 1$. In fact, if $(\sigma)^2 = t$, then

$$(\sigma g)^2 = \sigma g \sigma^{-1} g (\sigma)^2 = g^{-1} t g (\sigma)^2 = g^{-1} t g t = 1.$$

Again, here, $\sigma g \sigma^{-1} = g^{-1} t$ for each element of order 4 in G. As in Case 4.I.b, we choose g as the element given (on F) by multiplication by i, and we have that t is the translation by the half-period $\frac{1}{2}(1 + \mathrm{i})$.

In this case, however, we can only multiply σ by the square of g, whence we may only assume $a = 1$ or $a = \mathrm{i}$.

Table 1.

case	a
$\mathrm{Re}(\tau)=0$, $\lvert\tau\rvert>1$	± 1
$\tau=\mathrm{i}$	$1\equiv -1$
$\tau=\mathrm{i}$	$\mathrm{i}\equiv -\mathrm{i}$
$\lvert\tau\rvert=1$, $-\frac{1}{2}<\mathrm{Re}(\tau)<0$	$\pm\tau$
$\tau=\rho$	$1\equiv\rho\equiv\rho^2$
$\tau=\rho$	$-1\equiv-\rho\equiv-\rho^2$
$\mathrm{Re}(\tau)=-\frac{1}{2}$, $\lvert\tau\rvert>1$	± 1

Since we are assuming $(\sigma)^2 = 1$, we get that $a\bar{b}+b\equiv 0 \pmod{\Gamma}$. Furthermore, $\sigma g\sigma^{-1}=g^{-1}t$ reads, if $\sigma(z)=a\bar{z}+b$, as follows,

$$(**)\quad -a(1+\mathrm{i})\bar{b}\equiv 1/2(1+\mathrm{i}) \pmod{\Gamma},$$

i.e.

$$(1+\mathrm{i})[a\bar{b}+\tfrac{1}{2}]\in\Gamma,$$

whence, as before, b is a half-period $b=\frac{1}{2}(x+\mathrm{i}y)$ $(x,y\in\mathbb{Z})$, and $(**)$ simply means $x+y\equiv 1 \pmod 2$.

Up to exchanging σ with σt, we may assume $x=1$, $y=0$ and either

$$\sigma(z)=\bar{z}+\tfrac{1}{2}$$

or

$$\sigma(z)=\mathrm{i}\bar{z}+\tfrac{1}{2}.$$

However, the condition $a\bar{b}+b\equiv 0 \pmod{\Gamma}$ is only verified in the former case.

Our group $\hat{G}$ is generated by elements t, σ of order 2, g of order 4, t, g^2 are in the centre. Moreover, the relation $\sigma g\sigma^{-1}=g^{-1}t$ holds. $\hat{G}$ is thus isomorphic to the group G_1. □

6. The split case

In the next two sections we shall collect a number of auxiliary results (for instance, showing that certain cases cannot occur) upon which the classification tables in § 9 are based.

In this section we first treat the case in which the extension (3.1)

$$0\to G\xrightarrow{\mathrm{i}}\hat{G}\xrightarrow{\pi}\mathbb{Z}/2\mathbb{Z}\to 1,$$

that we studied in the last section, splits.

Let σ be an anti-holomorphic involution on an elliptic curve C.

For the reader's convenience, we reproduce once more (see Remark 5.4) the list of the possible linear parts of the anti-holomorphic maps σ with σ^2 equal to a translation.

In Table 1 we have used the fact that if $\tau = \mathrm{i}$, by conjugating with the automorphism of C given by multiplication by i, we can reduce the case $a = -1$ to the case $a = 1$, and the case $a = \mathrm{i}$ to the case $a = -\mathrm{i}$.

Analogously, if $\tau = \rho$, by conjugating with the automorphisms of C given by multiplication by ρ, or by ρ^2, we can reduce the cases $a = \rho, \rho^2$ to the case $a = 1$, the cases $a = -\rho, -\rho^2$ to the case $a = -1$.

First of all, we consider the action of $\hat{G}$ on the curve E. Let $\sigma(z) = a\bar{z} + b$ be any anti-holomorphic involution of $\hat{G}$ acting on E. By Lemma 5.5, we know that we have the following possibilities for b: if $\mathrm{Re}(\tau) = 0$, $a = 1$, we have $b = 0, \frac{1}{2}$; if $\mathrm{Re}(\tau) = 0$, $a = -1$, we have $b = 0, \frac{1}{2}\tau$; if $|\tau| = 1$, or $\mathrm{Re}(\tau) = -\frac{1}{2}$, we can assume $b = 0$.

Definition 6.1. We call Cases 1, 3, 5, 7 of Lemma 5.8 the simple dihedral cases: they are characterized by the property that $G = \mathbb{Z}/q\mathbb{Z}$, $\hat{G} = D_q$ (where we have, respectively, $q = 2, 4, 3, 6$).

Remark 6.2. In a dihedral case with $G = \mathbb{Z}/q\mathbb{Z}$, $\hat{G} = D_q$, $\hat{G}$ is generated by elements g and σ, where g acts on E by $g(z) = z + c$, c being a torsion element in $\mathrm{Pic}^0(E)$ of order precisely q and such that $a\bar{c} \equiv -c \pmod{\Gamma}$.

Proof. We choose generators σ and g such that $\sigma \circ g \equiv g^{-1} \circ \sigma$, equivalently $a\bar{c} \equiv -c \pmod{\Gamma}$. □

Lemma 6.3. *The subgroup H_q of $\mathrm{Pic}^0(E)_q$ on which σ acts as $-\mathrm{Id}$ is isomorphic to $\mathbb{Z}/q\mathbb{Z}$ if and only if either*

$$|\tau| = 1, \qquad a = \pm\tau, \qquad q = 2, 4, 3, 6,$$

or

$$\mathrm{Re}(\tau) = -\tfrac{1}{2}, \qquad a = \pm 1, \qquad q = 2, 4, 3, 6,$$

or

$$\mathrm{Re}(\tau) = 0, \qquad a = \pm 1, \qquad q = 3.$$

If $\mathrm{Re}(\tau) = 0$, $a = \pm 1$, $q = 2, 4, 6$, then H_q is isomorphic to $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. We know that the map $z \mapsto a\bar{z}$ is represented by a matrix of the form

$$\begin{pmatrix} m & m' \\ n & n' \end{pmatrix}$$

on $\mathbb{Z} \oplus \mathbb{Z}\tau$, hence the action on $\mathrm{Pic}^0(E)$ with basis $1/q$, τ/q is given by the reduction modulo q of the above integral matrix.

We have to solve the equation $a\bar{c} + c \equiv 0 \pmod{\Gamma}$, with $c \in \mathrm{Pic}^0(E)_q$, so we consider the kernel of the reduction of the matrix

$$M = \begin{pmatrix} m+1 & m' \\ n & n'+1 \end{pmatrix}$$

modulo q.

For $|\tau| = 1$, $a = \pm\tau$, we have

$$m = n' = 0, \qquad n = m' = \pm 1,$$

while for $\mathrm{Re}(\tau) = -\frac{1}{2}$, $a = \pm 1$, we have

$$n = 0, \qquad m = \pm 1, \qquad n' = m' = -m,$$

therefore, the kernel of the linear map $a\bar{z} + z$ on $\mathrm{Pic}^0(E)_q$ is isomorphic to $\mathbb{Z}/q\mathbb{Z}$.

It remains to consider the case $\mathrm{Re}(\tau) = 0$, $a = \pm 1$. We have

$$n = m' = 0, \qquad m = \pm 1, \qquad n' = -m.$$

So, if $a = 1$,

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

and if $a = -1$,

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

This tells us that the kernel of M is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ if $q = 3$, while it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ if $q = 2, 4, 6$. □

Lemma 6.4. *If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we must have*

$$\mathrm{Re}(\tau) = 0, \qquad a = \pm 1,$$

and $b = 0, \frac{1}{2}$ for the case $a = 1$, $b = 0, \frac{1}{2}\tau$ for the case $a = -1$.

If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\hat{G} = D_4$, we have either

$$|\tau| = 1, \qquad a = \pm\tau,$$

or

$$\mathrm{Re}(\tau) = -\tfrac{1}{2}, \qquad a = \pm 1.$$

Proof. If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, since $\hat{G}$ is abelian, the map $d \mapsto a\bar{d}$ is the identity on the points of two torsion and one can easily see (cf. the proof of the previous lemma) that this can happen if and only if $\mathrm{Re}(\tau) = 0$ and $a = \pm 1$.

If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\hat{G} = D_4$, we know from Lemma 5.8 that σ acts as

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

on the points of two torsion of E. Then we have either $|\tau| = 1$, $a = \pm\tau$ or $\mathrm{Re}(\tau) = -\frac{1}{2}$, $a = \pm 1$. □

Lemma 6.5. *If $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then again we must have*

$$\mathrm{Re}(\tau) = 0, \qquad a = \pm 1,$$

and $b = 0, \frac{1}{2}$ for the case $a = 1$, $b = 0, \frac{1}{2}\tau$ for the case $a = -1$.

For the case

$$\hat{G} \cong T \times D_4 \cong \mathbb{Z}/2\mathbb{Z} \times D_4,$$

the subgroup $G \subset \mathrm{Pic}^0(E)_4$ consists of the points c of four torsion satisfying the equation $a\bar{c} \equiv -c$.

For the case $\hat{G} = G_1$, we can choose generators of $\hat{G}$,

$$\sigma(z) = a\bar{z} + b, \qquad g(z) = z + \eta, \qquad t(z) = z + \epsilon,$$

with η of order 4, ϵ of order 2, such that

(i) *if $a = 1$, $\epsilon = \frac{1}{2}$ and $\eta = \frac{1}{4} + \frac{1}{4}\tau$, $b = 0, \frac{1}{2}$;*

(ii) *if $a = -1$, $\epsilon = \frac{1}{2}\tau$, $\eta = \frac{1}{4} + \frac{1}{4}\tau$, $b = 0, \frac{1}{2}\tau$.*

Proof. In both cases, the action of σ on $\mathrm{Pic}^0(E)_2 \subset G$ is trivial, whence we get the same conditions upon a, b, τ as in the previous lemma.

Let g be any element in G of order 4. Then the following relations hold,

$$\sigma \circ g = g^{-1} \circ \sigma$$

for the case $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times D_4$,

$$\sigma \circ g = g^{-1} t \circ \sigma$$

for the case $\hat{G} = G_1$ (where t is the generator of T).

If $\hat{G} \cong \mathbb{Z}/2\mathbb{Z} \times D_4$, then the above equation shows that G is contained in the subgroup of $\mathrm{Pic}^0(E)_4$, where σ acts as $-\mathrm{Id}$. But we have seen in the proof of Lemma 6.3 that this subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ if $\mathrm{Re}(\tau) = 0$, $a = \pm 1$, whence it equals G.

If $\hat{G} = G_1$, then the relation

$$a\bar{\eta} + \eta \equiv \epsilon \pmod{\Gamma}$$

gives

$$2\,\mathrm{Re}(\eta) \equiv \epsilon \pmod{\Gamma}$$

if $a = 1$,

$$2\mathrm{i}\,\mathrm{Im}(\eta) \equiv \epsilon \pmod{\Gamma}$$

if $a = -1$; therefore, $\epsilon \equiv \frac{1}{2}$ if $a = 1$, $\epsilon \equiv \frac{1}{2}\tau$ if $a = -1$.

If $a = 1$, it follows that $\mathrm{Re}(\eta) = \pm\frac{1}{4}$, and since 2η is of two torsion but distinct from ϵ, we get that $\eta = \pm\frac{1}{4} + \pm\frac{1}{4}\tau$. Up to replacing g with its inverse, or up to composing with t, we achieve $\eta = \frac{1}{4} + \frac{1}{4}\tau$, and an entirely similar argument works in the case $a = -1$. $\square$

Lemma 6.6. *If $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\hat{G} = \mathcal{S}_3 \times \mathbb{Z}/3\mathbb{Z}$, then G is isomorphic to the group $\mathrm{Pic}^0(E)_3$ and the action of any anti-holomorphic involution σ on $\mathrm{Pic}^0(E)_3$ has 1 and -1 as eigenvalues.*

Therefore, the datum of a $\hat{G}$-action is equivalent to the datum of an isomorphism class of an anti-holomorphic involution.

We thus have the usual following possibilities for σ:

(i) $\mathrm{Re}(\tau) = 0$, $a = 1$, $b = 0, \frac{1}{2}$;

(ii) $\mathrm{Re}(\tau) = 0$, $a = -1$, $b = 0, \frac{1}{2}\tau$;

(iii) $|\tau| = 1$, $a = \pm\tau$;

(iv) $\mathrm{Re}(\tau) = -\frac{1}{2}$, $a = \pm 1$.

Proof. Since σ is an anti-holomorphic involution, the eigenvalues of the action of σ on the lattice Γ of E are 1 and -1, thus the same holds for their reduction modulo 3, therefore we have all the possible values of a and b (see Lemma 5.5). □

Let us now consider the action of $\hat{G}$ on F.

Lemma 6.7. *Consider a cyclic group G' of automorphisms on an elliptic curve F, generated by a transformation g having a fixed point 0.*

Then the order d of g equals 2, 4, 3, 6 and $\mathrm{Fix}(g)$ is as follows.

(i) *If $d = 2$, $\mathrm{Fix}(g)$ is the subgroup F_2 of the 2-torsion points of F.*

(ii) *If $d = 4$, $\mathrm{Fix}(g)$ is the subgroup of F_2 isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by $\frac{1}{2}(1 + \mathrm{i})$.*

(iii) *If $d = 3$, $\mathrm{Fix}(g)$ is the subgroup of F_3 isomorphic to $\mathbb{Z}/3\mathbb{Z}$ generated by $\frac{1}{3}(1 - \rho)$.*

(iv) *If $d = 6$, $\mathrm{Fix}(g)$ is reduced to the origin.*

Proof. The proof is a simple computation. We notice that if $d = 3$, or $d = 6$, $\Gamma \cong \mathbb{Z}[\rho]/(\rho^2 + \rho + 1)$. □

Remark 6.8. Let σ be an anti-holomorphic involution in $\hat{G}$ acting on F.

If $g \in G$ and $\sigma \circ g = g^{-1} \circ \sigma$, then σ leaves the sets $\mathrm{Fix}(g)$, $\mathrm{Fix}(g^2) - \mathrm{Fix}(g)$ and $\mathrm{Fix}(g^3) - \mathrm{Fix}(g)$ invariant.

Moreover, $\mathrm{Fix}(g)$ is a subgroup of F and σ acts on $\mathrm{Fix}(g)$ by an affine transformation.

Notice that g acts trivially on $\mathrm{Fix}(g)$, whence the action on $\mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$ is independent of the choice of σ in the simple dihedral case, when we take for g a generator of the subgroup G. We thus get, in this case, a topological invariant of the real hyperelliptic surface.

Corollary 6.9. *In the Cases 5, 7 of Lemma 5.8, we may pick as the origin in F a common fixed point of g and σ. Therefore, if $G = \mathbb{Z}/3\mathbb{Z}$, we may assume $\sigma(z) = \pm\bar{z}$; if $G = \mathbb{Z}/6\mathbb{Z}$, we may assume $\sigma(z) = \bar{z}$.*

Proof. We have already observed that we may choose $a = \pm 1$. If $G = \mathbb{Z}/6\mathbb{Z}$, by exchanging σ with $g^n \circ \sigma$, we see that we can assume $a = 1$.

If $G = \mathbb{Z}/3\mathbb{Z}$, $g(z) = \rho z$, $\mathrm{Fix}(g^2) - \mathrm{Fix}(g)$ is reduced to the origin, and σ must act on $\mathrm{Fix}(g^2) - \mathrm{Fix}(g)$ by the previous remark.

If $G = \mathbb{Z}/6\mathbb{Z}$, $g(z) = -\rho z$, $\mathrm{Fix}(g)$ is reduced to the origin and σ must act on $\mathrm{Fix}(g)$ by the previous remark. □

Lemma 6.10. *If $G = \mathbb{Z}/4\mathbb{Z}$, $\hat{G} = D_4$, we may assume $a = 1$, and either*

$$\sigma(z) = \bar{z}$$

or

$$\sigma(z) = \bar{z} + \tfrac{1}{2}(1 + \mathrm{i}).$$

Proof. We know that $a = \pm 1, \pm \mathrm{i}$, but, by exchanging σ with $g^n \circ \sigma$, we can assume $a = 1$.

By Lemma 6.7 and Remark 6.8, we know that σ acts on the subgroup of F_2 generated by $\frac{1}{2}(1 + \mathrm{i})$, which concludes the proof. □

Lemma 6.11. *If $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\hat{G} = D_3 \times \mathbb{Z}/3\mathbb{Z}$, we may choose generators of $\hat{G}$ acting on F given by*

$$\sigma(z) = -\bar{z}, \qquad g(z) = \rho z, \qquad t(z) = z + \tfrac{1}{3}(1 - \rho).$$

Proof. By Lemma 5.8, we know that we may assume that $\hat{G}$ is generated by

$$\sigma(z) = a\bar{z} + b, \qquad g(z) = \rho z, \qquad t(z) = z + \tfrac{1}{3}(1 - \rho)$$

such that $\sigma \circ g = g^{-1} \circ \sigma$, $\sigma \circ t = t \circ \sigma$. We have already remarked that we may choose $a = \pm 1$, and one can easily see that the relation $\sigma \circ t = t \circ \sigma$ holds only for $a = -1$.

Since $\sigma \circ g = g^{-1} \circ \sigma$, we can assume $b = 0$, by Remark 6.8, as in the case $G = \mathbb{Z}/3\mathbb{Z}$. □

Lemma 6.12. *If $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we have the following cases.*

(i) *If $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times D_4$, we can choose the following set of generators of $\hat{G}$ acting on F:*

$$g(z) = \mathrm{i}z, \qquad t(z) = z + \tfrac{1}{2}(1 + \mathrm{i}), \qquad \sigma(z) = \bar{z}.$$

(ii) *If $\hat{G} = G_1$, we can choose the following set of generators of $\hat{G}$ acting on F:*

$$g(z) = \mathrm{i}z, \qquad t(z) = z + \tfrac{1}{2}(1 + \mathrm{i}), \qquad \sigma(z) = \bar{z} + \tfrac{1}{2}.$$

Proof. For the case $\hat{G} = G_1$, see Lemma 5.8.

Assume $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times D_4$, then we have already observed in Lemma 5.8 that we may assume that $\hat{G}$ is generated by the transformations

$$g(z) = \mathrm{i}z, \qquad t(z) = z + \tfrac{1}{2}(1 + \mathrm{i}), \qquad \sigma(z) = a\bar{z} + b,$$

such that $\sigma \circ g = g^{-1} \circ \sigma$, $\sigma \circ t = t \circ \sigma$.

Therefore, by composing σ with a power of g, we may assume $a = 1$, and the relation $\sigma \circ g = g^{-1} \circ \sigma$ gives, by Remark 6.8, either $b = 0$ or $b = \frac{1}{2}(1 + \mathrm{i})$. The second case can be excluded by exchanging σ with $t \circ \sigma$. □

Lemma 6.13. *If $G = \mathbb{Z}/2\mathbb{Z}$, $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by*

$$\sigma(z) = a\bar{z} + b, \qquad g(z) = -z,$$

we have the following possibilities for F, a and b:

(i) $\mathrm{Re}(\tau) = 0$, $a = 1$, *b an element in F_2;*

(ii) $|\tau| = 1$, $a = \tau$, $b = 0$;

(ii) $\mathrm{Re}(\tau) = -\frac{1}{2}$, $a = 1$, $b = 0$.

Moreover, there are three different topological types of the action of the affine transformation σ on $\mathrm{Fix}(g)$, where g is a generator of G (see Remark 6.8), namely:

(i) *if* $\mathrm{Re}(\tau) = 0$, $a = 1$, $b = 0$, *then σ acts as the identity on* $\mathrm{Fix}(g)$*;*

(ii) *if* $\mathrm{Re}(\tau) = 0$, $a = 1$, $b \in \mathrm{Pic}^0(E)_2$ *of order 2, then σ acts as a translation on* $\mathrm{Fix}(g)$*;*

(iii) *if* $|\tau| = 1$, $a = \tau$, *or* $\mathrm{Re}(\tau) = -\frac{1}{2}$, $a = 1$, *then σ acts as a linear map with matrix*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on $\mathrm{Fix}(g)$.

Proof. Since $\hat{G}$ is abelian, we have the condition $2b \equiv 0 \pmod{\Gamma}$, while the condition $\sigma^2 = 1$ reads $a\bar{b} + b \equiv 0 \pmod{\Gamma}$.

By exchanging σ with $-\sigma$, a gets multiplied by -1, so we obtain the statement on the possible values of a.

Assume now that $\mathrm{Re}(\tau) = 0$, $a = 1$.

The conditions

$$2b \equiv 0 \pmod{\Gamma}, \qquad a\bar{b} + b = \bar{b} + b = 2\,\mathrm{Re}(b) \equiv 0 \pmod{\Gamma}$$

give us the following possibilities for b:

$$b = 0,\ \tfrac{1}{2},\ \tfrac{1}{2}\tau,\ \tfrac{1}{2}(1 + \tau).$$

Assume now that $|\tau| = 1$, $a = \tau$. The condition $2b \equiv 0 \pmod{\Gamma}$, allows us to write $b = \frac{1}{2}x + \tau\frac{1}{2}y$, with $x, y \in \{0, 1\}$.

The condition

$$a\bar{b} + b = \tau\bar{b} + b \equiv 0 \pmod{\Gamma}$$

implies $x \equiv y \pmod 2$, so either $b = 0$ or $b = \frac{1}{2}(1+\tau)$, but, by conjugation with the translation $\phi(z) = z + \frac{1}{2}$, we can assume

$$b = 0, \qquad \sigma(z) = \tau\bar{z}.$$

Assume now that $\mathrm{Re}(\tau) = -\frac{1}{2}$.

We may assume $a = 1$, and the condition $2b \equiv 0 \pmod \Gamma$ allows us to write $b = \frac{1}{2}x + \tau\frac{1}{2}y$, with $x, y \in \{0, 1\}$, while the condition $a\bar{b} + b \equiv 0 \pmod \Gamma$ reads $2\,\mathrm{Re}(b) = x - \frac{1}{2}y = n \in \mathbb{Z}$. Thus $y \equiv 0 \pmod 2$ and either $b = 0$ or $b = \frac{1}{2}$, but, by translating by $\phi(z) = z + \frac{1}{2}\tau$, we can assume

$$b = 0, \qquad \sigma(z) = \bar{z}.$$

Finally, we know from Remark 6.8 that σ acts as an affine transformation on $\mathrm{Fix}(g) = \mathrm{Pic}^0(E)_2$, that this action is a topological invariant, and we observe that there are the following four different types of affine transformations acting on $\mathrm{Pic}^0(E)_2$: the identity; a translation; the map

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

and the map A composed with a translation.

Then the statement follows by an easy computation. □

There remains to treat the case $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 6.14. *Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then either*

$$\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

or

$$\hat{G} = D_4,$$

and we can choose generators of $\hat{G}$ as follows,

$$g(z) = -z, \qquad t(z) = z + \epsilon,$$

ϵ of order 2,

$$\sigma(z) = a\bar{z} + b,$$

where the following hold.

(i) *If $\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we have the following cases.* $\mathrm{Re}(\tau) = 0$, $a = 1$*: if $\epsilon = \frac{1}{2}$, $b = 0, \frac{1}{2}\tau$; if $\epsilon = \frac{1}{2}\tau$ or $\epsilon = \frac{1}{2} + \frac{1}{2}\tau$, we can choose $b = 0, \frac{1}{2}$,*

$$\begin{array}{llll} |\tau| = 1, & a = \tau, & \epsilon = \frac{1}{2}(1+\tau), & b = 0, \\ \mathrm{Re}(\tau) = -\frac{1}{2}, & a = 1, & \epsilon = \frac{1}{2}, & b = 0. \end{array}$$

(ii) *If* $\hat{G} = D_4$, *we have the following cases.* $\mathrm{Re}(\tau) = 0$*: if* $a = 1$,

$$b = \tfrac{1}{4}\tau,\ \tfrac{1}{4}\tau + \tfrac{1}{2}, \qquad \epsilon = \tfrac{1}{2}\tau;$$

if $a = -1$,

$$b = -\tfrac{1}{4},\ -\tfrac{1}{4} + \tfrac{1}{2}\tau, \qquad \epsilon = \tfrac{1}{2}.$$

$|\tau| = 1$*: if* $a = \tau$,

$$b = \tfrac{1}{4} - \tfrac{1}{4}\tau, \qquad \epsilon = \tfrac{1}{2} + \tfrac{1}{2}\tau;$$

if $a = -\tau$,

$$b = \tfrac{1}{4} + \tfrac{1}{4}\tau, \qquad \epsilon = \tfrac{1}{2} + \tfrac{1}{2}\tau.$$

$\mathrm{Re}(\tau) = -\frac{1}{2}$*: if* $a = 1$,

$$b = \tfrac{1}{4} + \tfrac{1}{2}\tau, \qquad \epsilon = \tfrac{1}{2};$$

if $a = -1$,

$$b = -\tfrac{1}{4}, \qquad \epsilon = \tfrac{1}{2}.$$

Proof. Assume, first of all, that

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \qquad \hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

By exchanging σ with $-\sigma$, a gets multiplied by -1, thus we have the statement for a.

Since $\hat{G}$ is abelian, we have the condition $2b \equiv 0 \pmod{\Gamma}$, while the condition $\sigma^2 = 1$ reads $a\bar{b} + b \equiv 0 \pmod{\Gamma}$.

Assume, first of all, that $\mathrm{Re}(\tau) = 0$, $a = 1$. Then, as in the case $G = \mathbb{Z}/2\mathbb{Z}$, the conditions $2b \equiv 0 \pmod{\Gamma}$ and $a\bar{b} + b \equiv 0 \pmod{\Gamma}$ give us the following possibilities for b:

$$b = 0,\ \tfrac{1}{2},\ \tfrac{1}{2}\tau,\ \tfrac{1}{2}(1+\tau).$$

Now, by composing σ with t, we obtain the statement.

If $|\tau| = 1$, $a = \tau$, as in the case $G = \mathbb{Z}/2\mathbb{Z}$, we find $\sigma(z) = \tau\bar{z}$. Since σ must commute with t, we must have $a\bar{\epsilon} = \tau\bar{\epsilon} = \epsilon$, which yields $\epsilon = \frac{1}{2}(1+\tau)$.

If $\mathrm{Re}(\tau) = -\frac{1}{2}$, we can assume $a = 1$, and, as in the case $G = \mathbb{Z}/2\mathbb{Z}$, we find $\sigma(z) = \bar{z}$. The condition $a\bar{\epsilon} = \bar{\epsilon} = \epsilon$ yields $\epsilon = \frac{1}{2}$.

If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\hat{G} = D_4$, we know by Lemma 5.8 that we may choose generators g, t, σ such that $\sigma(-z) = -\sigma(z) + \epsilon$ or, equivalently, $2b \equiv \epsilon \in \frac{1}{2}\Gamma - \Gamma$, $\sigma(z+\epsilon) = \sigma(z) + \epsilon$ or, equivalently, $a\bar{\epsilon} \equiv \epsilon \pmod{\Gamma}$, while $\sigma^2 = 1$ yields $a\bar{b} + b \in \Gamma$. An element of order 4 in $\hat{G}$ is $\sigma \circ g$, whose square is t.

Assume $\mathrm{Re}(\tau) = 0$, $a = 1$. Then the conditions $2b \in \frac{1}{2}\Gamma - \Gamma$ and $2\,\mathrm{Re}(b) \equiv 0 \pmod{\Gamma}$ imply $2b \equiv \frac{1}{2}\tau \pmod{\Gamma}$. Therefore, we can choose

$$b = \pm\tfrac{1}{4}\tau,$$

or

$$b = \pm\tfrac{1}{4}\tau + \tfrac{1}{2}, \qquad \epsilon = \tfrac{1}{2}\tau.$$

But, by composing σ with t, we can assume

$$b = \tfrac{1}{4}\tau,$$

or

$$b = \tfrac{1}{4}\tau + \tfrac{1}{2}.$$

A similar computation gives the result for $a = -1$.

If $|\tau| = 1$, $a = \tau$, then the conditions $2b \in \frac{1}{2}\Gamma - \Gamma$ and $\tau\bar{b} + b \equiv 0 \pmod{\Gamma}$ tell us that we can choose

$$b = \tfrac{1}{4} - \tfrac{1}{4}\tau, \qquad \epsilon = \tfrac{1}{2}(1+\tau).$$

In fact, by the first condition, we can write $2b = \frac{1}{2}n + \frac{1}{2}m\tau$, with $n, m \in \{0, 1\}$. We have $2(a\bar{b} + b) = 2(\tau\bar{b} + b) = \frac{1}{2}(n+m) + \frac{1}{2}\tau(n+m) \in 2\Gamma$, thus $n + m \equiv 0 \pmod{4}$, and we obtain $b \equiv \frac{1}{4} - \frac{1}{4}\tau \pmod{\Gamma}$. A similar computation gives the result for $a = -\tau$.

If $\mathrm{Re}(\tau) = -\frac{1}{2}$, $a = 1$, the conditions $2b \in \frac{1}{2}\Gamma - \Gamma$ and $\bar{b} + b = 2\,\mathrm{Re}(b) \equiv 0 \pmod{\Gamma}$ tell us that we can choose

$$b \equiv \tfrac{1}{4} + \tfrac{1}{2}\tau, \qquad \epsilon = \tfrac{1}{2}.$$

In fact, we can write $2b = \frac{1}{2}n + \frac{1}{2}m\tau$, with $n, m \in \{0, 1\}$, Then we have $2\,\mathrm{Re}(b) = \frac{1}{2}n - \frac{1}{4}m \equiv 0 \pmod{\Gamma}$ and $m \equiv 2n \pmod{4}$, whence we may take

$$b \equiv \tfrac{1}{4} + \tfrac{1}{2}\tau,$$

and therefore

$$\epsilon = \tfrac{1}{2}.$$

A similar computation gives the result for $a = -1$. □

7. The non-split case

In this section we want to treat the case where the exact sequence (3.1) does not split. We have

$$G = T \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

$$0 \to T \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{j} \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 1$$

and

$$\sigma^2 = t,$$

where t is the generator of T, a translation.

We get as generators σ and g, where g acts on F by multiplication by -1.

We consider first the action of $\hat{G}$ on F.

Let $\sigma(z) = a\bar{z} + b$. Then b is a half-period for the following reasons: σ, g commute, a condition that is equivalent to $2b \equiv 0 \pmod{\Gamma}$, and, moreover, $b \neq 0$, else σ^2 is the identity.

If b is a half-period, then necessarily σ^2 has order at most two, and it has order exactly two if and only if $(a\bar{b} + b) \notin \Gamma$.

Lemma 7.1. *The case* $\mathrm{Re}(\tau) = 0$, $|\tau| > 1$ *is impossible for the curve* F*. In the case* $\tau = \mathrm{i}$*, for the curve* F *we can assume* $a = \mathrm{i}$.

Proof. In fact, assume $a = \pm 1$. Then, since $(a\bar{b} + b) \in \frac{1}{2}\Gamma - \Gamma$, either

$$2\,\mathrm{Re}(b) \in \tfrac{1}{2}\Gamma - \Gamma$$

or

$$2\mathrm{i}\,\mathrm{Im}(b) \in \tfrac{1}{2}\Gamma - \Gamma,$$

contradicting $2b \in \Gamma$, which in this case is equivalent to $2\,\mathrm{Re}(b) \in \Gamma$ and $2\mathrm{i}\,\mathrm{Im}(b) \in \Gamma$.

Finally, if $\tau = \mathrm{i}$, the case $a = -\mathrm{i}$ can be reduced to the case $a = \mathrm{i}$ modulo changing coordinates via the automorphism of F given by multiplication by i. ☐

Proposition 7.2. *In the non-split case, let* $|\tau| = 1$ *for* F*. Then we can choose an element* σ *of* $\hat{G}$ *of order 4 such that*

$$\sigma(z) = \tau\bar{z} + \tfrac{1}{2}.$$

In this case, σ^2 *is the translation by the vector* $t = \frac{1}{2}(1 + \tau)$.

If $\mathrm{Re}(\tau) = -\frac{1}{2}$*, we can choose an element* σ *of* $\hat{G}$ *of order 4 such that*

$$\sigma(z) = \bar{z} + \tfrac{1}{2}\tau.$$

In this case, σ^2 *is the translation by the vector* $t = \frac{1}{2}$.

If $\tau = \rho$*, the case* $a = \rho$ *can be reduced to the case* $a = 1$*, by conjugating with the automorphism of* F *given by multiplication by* ρ*. Thus we can always assume*

$$\sigma(z) = \bar{z} + \tfrac{1}{2}\tau, \qquad \sigma^2(z) = z + \tfrac{1}{2}.$$

Proof. We observe first of all that by exchanging σ with σg, a gets multiplied by -1, while conjugating σ with an automorphism of F given by multiplication by λ, a gets multiplied by λ^2.

Therefore, we may assume $a = \tau$ if $|\tau| = 1$ and $a = 1$ if $\mathrm{Re}(\tau) = -\frac{1}{2}$.

Furthermore, by exchanging σ with σ^{-1}, we can substitute b with $-a\bar{b}$.

Case 1. ($|\tau| = 1$.) Since $2b \in \Gamma$, we may write $b = \frac{1}{2}x + \frac{1}{2}y\tau$, with $x, y \in \{0, 1\}$.

The condition

$$a\bar{b} + b = \tau\bar{b} + b \in \tfrac{1}{2}\Gamma - \Gamma$$

yields $x + y \equiv 1 \pmod 2$, so either $b = \frac{1}{2}$ or $b = \frac{1}{2}\tau$. But, by exchanging b with $-\tau\bar{b}$, we can assume $b = \frac{1}{2}$.

Whence we have

$$\sigma(z) = \tau\bar{z} + \tfrac{1}{2},$$

and therefore

$$\sigma^2(z) = z + \tfrac{1}{2} + \tfrac{1}{2}\tau.$$

Case 2. ($\mathrm{Re}(\tau) = -\frac{1}{2}$.) The condition

$$a\bar{b} + b = \bar{b} + b \in \tfrac{1}{2}\Gamma - \Gamma$$

is equivalent to the condition $2\,\mathrm{Re}(b) \in \frac{1}{2}\Gamma - \Gamma$, which implies $\mathrm{Re}(b) \equiv \pm\frac{1}{4} \pmod{\Gamma}$.

Since $2b \in \Gamma$, we easily see that b is either congruent to $\frac{1}{2}\tau$ or to $\frac{1}{2} + \frac{1}{2}\tau$. But, by exchanging b with $-a\bar{b} = -\bar{b}$, we can assume $b = \frac{1}{2}\tau$.

Thus we may assume

$$\sigma(z) = \bar{z} + \tfrac{1}{2}\tau,$$

and

$$\sigma^2(z) = z - \tfrac{1}{2} \equiv z + \tfrac{1}{2} \pmod{\Gamma}.$$

In the case $\tau = \rho$, we observe that the two maps $z \mapsto \tau\bar{z} + \frac{1}{2}$ and $z \mapsto \bar{z} + \frac{1}{2}\tau$ are conjugated by the automorphism given by multiplication by ρ. □

Proposition 7.3. *In the non-split case for the curve E, only the case* $\mathrm{Re}(\tau) = 0$, $|\tau| \geqslant 1$, $a = \pm 1$ *occurs.*

If $a = 1$, $\sigma(z) = \bar{z} + b$, we have $\sigma^2(z) = z + \frac{1}{2}$, $b \equiv \frac{1}{4}$.

If $a = -1$, we have $\sigma(z) = -\bar{z} + b$, $\sigma^2(z) = \frac{1}{2}\tau$, $b \equiv \frac{1}{4}\tau$.

Proof. The fact that $\hat{G}$ is abelian implies that for every $d \in \frac{1}{2}\Gamma - \Gamma$, we must have $a\bar{d} \equiv d \pmod{\Gamma}$, whence the map $d \mapsto a\bar{d}$ is the identity on the points of two torsion of E.

If $|\tau| = 1$, we know that $a = \pm\tau$, but then, for $d = \frac{1}{2}$, we have

$$a\bar{d} = \pm\tfrac{1}{2}\tau \not\equiv \tfrac{1}{2} \pmod{\Gamma};$$

a contradiction.

If $\mathrm{Re}(\tau) = -\frac{1}{2}$, then $a = \pm 1$ and if we take $d = \frac{1}{2}\tau$,

$$a\bar{d} = \pm\tfrac{1}{2}\bar{\tau} = \mp\tfrac{1}{2}(\tau + 1) \not\equiv \tfrac{1}{2}\tau \pmod{\Gamma};$$

again a contradiction.

Therefore, we can assume $\mathrm{Re}(\tau) = 0$, $a = \pm 1$. In this case, the map $d \mapsto a\bar{d}$ is the identity on the points of two torsion of E.

If $a = 1$, the condition

$$\sigma^2(z) - z = a\bar{b} + b = \bar{b} + b = 2\,\mathrm{Re}(b) \in \tfrac{1}{2}\Gamma - \Gamma$$

implies $\mathrm{Re}(b) \equiv \pm\frac{1}{4}$ and $\sigma^2(z) = z + \frac{1}{2}$. By Lemma 5.5, we can assume $b \equiv \pm\frac{1}{4}$ and by exchanging σ with σ^{-1} we can assume $b = \frac{1}{4}$.

If $a = -1$, the condition

$$\sigma^2(z) - z = a\bar{b} + b = -\bar{b} + b = 2\mathrm{i}\,\mathrm{Im}(b) \in \tfrac{1}{2}\Gamma - \Gamma$$

implies $\mathrm{Im}(b) \equiv \pm\frac{1}{4}\,\mathrm{Im}(\tau)$ and $\sigma^2(z) = z + \frac{1}{2}\tau$. Again, we conclude by Lemma 5.5 and by substituting σ with σ^{-1}. □

Remark 7.4. Observe that, by Remark 3.7 (1), in the non-split case we must have $\mathrm{Fix}(\sigma) = S(\mathbb{R}) = \emptyset$.

8. Topology of the real part of S

In this section we want to describe the topology of the fixed-point locus of the involution σ acting on a real hyperelliptic surface S.

Remark 8.1. The fixed-point locus of σ can only be a disjoint union of tori and Klein bottles.

In fact, we have the Albanese map

$$\begin{array}{ccc} S & \xrightarrow{\ \sigma\ } & S \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ A = \mathbb{C}/\Lambda & \xrightarrow{\ \bar{\sigma}\ } & A = \mathbb{C}/\Lambda \end{array}$$

and each component of the real locus is a S^1 bundle on S^1.

We have the following result.

Lemma 8.2. *Let σ be as above with $\mathrm{Fix}(\sigma) \neq \emptyset$ and assume that the group G is of odd order, i.e. either $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Then the connected components of $\mathrm{Fix}(\sigma)$ are homeomorphic to $S^1 \times S^1$.*

Proof. Let $\pi : E \times F \to (E \times F)/G$ be the projection. We notice (see Remark 3.7) that if $z \in \mathrm{Fix}(\sigma)$, then, for every $\tilde{z} \in \pi^{-1}(z)$, there exists $\tilde{\sigma} : E \times F \to E \times F$ such that $\tilde{\sigma}(\tilde{z}) = \tilde{z}$. We have the following commutative diagram,

$$\begin{array}{ccc} E \times F & \xrightarrow{\ \tilde{\sigma}\ } & E \times F \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow{\ \sigma\ } & S \end{array}$$

where $\tilde{\sigma}$ is an anti-holomorphic involution on $E \times F$.

Let C be the connected component of $\mathrm{Fix}(\sigma)$ that contains z and let $\tilde{C}$ be the connected component of $\pi^{-1}(C)$ that contains $\tilde{z}$.

Clearly, $\tilde{\sigma}(\tilde{C}) = (\tilde{C})$. Moreover, $\tilde{\sigma}$ is a lifting of the identity of C and with $\tilde{\sigma}(\tilde{z}) = (\tilde{z})$, whence $\tilde{\sigma}$ is the identity on $\tilde{C}$.

We know that the components of $\mathrm{Fix}(\sigma)$ can be either tori or Klein bottles.

Since $\tilde{C} \cong S^1 \times S^1$, in the latter case, $\tilde{C}$ would be an oriented covering of odd degree (equal to $|G|$) of a Klein bottle. But this is impossible since every oriented covering of a Klein bottle factors through the orientation covering which has degree 2. $\square$

Using the Harnack–Thom–Krasnov inequality (cf. [**41**, p. 16] or [**24**, Chapter 1]), we are able to bound the number of connected components of the real part of S in all the cases given by the list of Bagnera–de Franchis. We will show later that for all G we can find real hyperelliptic surfaces S that reach these bounds (i.e. we can find M-surfaces).

Remark 8.3.

(1) If $G = \mathbb{Z}/2\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 4$.

(2) If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 3$.

(3) If $G = \mathbb{Z}/4\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 3$.

(4) If $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 2$.

(5) If $G = \mathbb{Z}/3\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 2$.

(6) If $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 2$.

(7) If $G = \mathbb{Z}/6\mathbb{Z}$, then $h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant 2$.

Proof. We recall the Harnack–Thom–Krasnov inequality,

$$\sum_m (\dim_{\mathbb{Z}/2} H^m(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})) \leqslant \sum_m (\dim_{\mathbb{Z}/2} H^m(S, \mathbb{Z}/2\mathbb{Z}) - 2\lambda_m),$$

where

$$\lambda_m := \dim_{\mathbb{Z}/2}(1+\sigma)H^m(S, \mathbb{Z}/2\mathbb{Z}).$$

Now we have

$$\sum_m (\dim_{\mathbb{Z}/2} H^m(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})) = 2h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) + h^1(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}),$$

and since the components of $S(\mathbb{R})$ are either tori or Klein bottles, we have

$$h^1(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 2h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}).$$

Thus we find

$$4h^0(S(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leqslant \sum_m (\dim_{\mathbb{Z}/2} H^m(S, \mathbb{Z}/2\mathbb{Z})). \tag{8.1}$$

We notice that $b_2(S) = 2$, and we have the following list for $H_1(S, \mathbb{Z})$ (see, for example, [**44**]):

(i) $G = \mathbb{Z}/2\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$;

(ii) $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$;

(iii) $G = \mathbb{Z}/4\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$;

(iv) $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2$;

(v) $G = \mathbb{Z}/3\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$;

(vi) $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2$;

(vii) $G = \mathbb{Z}/6\mathbb{Z}$, $H_1(S, \mathbb{Z}) = \mathbb{Z}^2$.

The universal coefficients theorem and the Poincaré duality allow us to compute all the Betti numbers of S with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Then, by (8.1), we obtain the given bounds on the number of connected components of $S(\mathbb{R})$. □

We want now to show how one can compute the number k of Klein bottles, respectively, the number t of two-dimensional tori in the real part $S(\mathbb{R})$ of a real hyperelliptic surface.

For every connected component V of $\mathrm{Fix}(\sigma) = S(\mathbb{R})$, the inverse image $\pi^{-1}(V)$ splits as the G-orbit of any of its connected components; let W be one such.

We have already observed that there is a lifting $\tilde{\sigma}$ of σ such that W is in the fixed locus of $\tilde{\sigma}$.

If $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ are two distinct anti-holomorphic involutions, then

$$\pi^{-1}(V) \cap \mathrm{Fix}(\tilde{\sigma}_1) \cap \mathrm{Fix}(\tilde{\sigma}_2) = \emptyset.$$

In fact, otherwise, there would exist a component W_h such that $\tilde{\sigma}_{1|W_h} = \tilde{\sigma}_{2|W_h} = \mathrm{Id}$, and thus $\tilde{\sigma}_1\tilde{\sigma}_2 = \mathrm{Id}$; a contradiction.

We conclude from the above argument that the connected components of $\mathrm{Fix}(\sigma)$ correspond bijectively to the set $\mathcal{C}$ obtained as follows. Consider all the liftings $\tilde{\sigma}$ of σ which are involutions and pick one representative $\tilde{\sigma}_{\mathrm{i}}$ for each conjugacy class.

Then we let $\mathcal{C}$ be the set of equivalence classes of connected components of $\cup\,\mathrm{Fix}(\tilde{\sigma}_{\mathrm{i}})$, where two components A, A' of $\mathrm{Fix}(\tilde{\sigma}_{\mathrm{i}})$ are equivalent if and only if there exists an element $g \in G$ such that $g(A) = A'$.

Let $\tilde{\sigma}$ be an anti-holomorphic involution which is a lifting of σ and such that $\mathrm{Fix}(\tilde{\sigma}) \neq \emptyset$.

Since $\tilde{\sigma}$ is of product type, $\mathrm{Fix}(\tilde{\sigma})$ is a disjoint union of $2^{a_1+a_2}$ copies of $\boldsymbol{S}^1 \times \boldsymbol{S}^1$, where $a_{\mathrm{i}} \in \{0, 1\}$.

In fact, if an anti-holomorphic involution $\hat{\sigma}$ on an elliptic curve C has fixed points, then $\mathrm{Fix}(\hat{\sigma})$ is a disjoint union of 2^a copies of $\boldsymbol{S}^1$, where $a = 1$ if the matrix of the action of $\hat{\sigma}$ on $H_1(C, \mathbb{Z})$ is diagonalizable, else $a = 0$.

The above considerations allow us to compute the number of such components V: in order to determine their nature, observe that $V = W/H$, where W is as before and $H \subset G$ is the subgroup such that $HW = W$.

Since the action of G on the first curve E is by translations, the action on the first $\boldsymbol{S}^1$ is always orientation preserving, whence V is a Klein bottle if and only if H acts on the second $\boldsymbol{S}^1$ by some orientation-reversing map, or equivalently H has some fixed point on the second $\boldsymbol{S}^1$.

Let $h \in H$ be a transformation having a fixed point on the second $\boldsymbol{S}^1$. Since the direction of this $\boldsymbol{S}^1$ is an eigenvector for the tangent action of h, it follows that the tangent action is given by multiplication by -1.

The existence of such an element h for the second $\boldsymbol{S}^1$ is obvious if $a_1 = a_2 = 0$ and there exists $h \in H$ which commutes with $\tilde{\sigma}$ (this always occurs except in the last case of Lemma 5.8 (2)). In this case, we get one Klein bottle.

In the other cases, one needs a more delicate analysis, the details of which we omit here.

In the next section we shall give a precise description of all the isomorphism classes of real hyperelliptic surfaces and we shall also determine the real part $S(\mathbb{R})$ for each isomorphism class.

We end this section by showing, with a simple example, the kind of computations that we make in order to find the real part of a given hyperelliptic surface.

Assume that

$$S = (E \times F)/G,$$

where

$$E = \mathbb{C}/(\mathbb{Z} + \mathrm{i}\mathbb{Z}), \qquad F = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

with $\mathrm{Re}(\tau) = 0$, $G = \mathbb{Z}/2\mathbb{Z}$ acting on $E \times F$ by

$$(z_1, z_2) \mapsto (z_1 + \tfrac{1}{2}(1 + \mathrm{i}), -z_2).$$

Assume that the anti-holomorphic involution σ acting on S has a lifting $\tilde{\sigma} = \sigma_1 \times \sigma_2$, where

$$\sigma_1(z_1) = \mathrm{i}\bar{z}_1,$$
$$\sigma_2(z_2) = \bar{z}_1.$$

If g is the generator of G, $g(z_1, z_2) = (z_1 + \frac{1}{2}(1 + \mathrm{i}), -z_2)$, then the other lifting of σ is given by

$$\tilde{\sigma}' := \tilde{\sigma} \circ g(z_1, z_2) = g \circ \tilde{\sigma}(z_1, z_2) = (\mathrm{i}\bar{z}_1 + \tfrac{1}{2}(1 + \mathrm{i}), -\bar{z}_2).$$

The fixed-point locus of $\tilde{\sigma}$ is

$$(\varGamma_1 \times \varGamma_2) \cup (\varGamma_1 \times \varLambda_2),$$

where

$$\varGamma_1 = \{z_1 \in \mathbb{C}/(\mathbb{Z} + \mathrm{i}\mathbb{Z}) \mid \mathrm{Re}(z_1) = \mathrm{Im}(z_1)\},$$
$$\varGamma_2 = \{z_2 \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \mid \mathrm{Im}(z_2) = \tfrac{1}{2}\},$$
$$\varLambda_2 = \{z_2 \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \mid \mathrm{Im}(z_2) = 0\}.$$

The fixed-point locus of $\tilde{\sigma}'$ is

$$(\varGamma_1' \times \varGamma_2') \cup (\varGamma_1' \times \varLambda_2'),$$

where

$$\Gamma_1' = \{z_1 \in \mathbb{C}/(\mathbb{Z}+\mathrm{i}\mathbb{Z}) \mid \mathrm{Re}(z_1) = \mathrm{Im}(z_1) \pm \tfrac{1}{2}\},$$
$$\Gamma_2' = \{z_2 \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \mid \mathrm{Re}(z_2) = \tfrac{1}{2}\},$$
$$\Lambda_2' = \{z_2 \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \mid \mathrm{Re}(z_2) = 0\}.$$

Now, if

$$(z_1, z_2) \in \Gamma_1 \times \Gamma_2,$$

then

$$(z_1 + \tfrac{1}{2}(1+\mathrm{i}), -z_2) \in \Gamma_1 \times \Gamma_2$$

also, so, if we denote by $\pi : E \times F \to S$ the projection, we obtain that $\pi(\Gamma_1 \times \Gamma_2)$ is a Klein bottle.

If

$$(z_1, z_2) \in \Gamma_1 \times \Lambda_2,$$

then

$$(z_1 + \tfrac{1}{2}(1+\mathrm{i}), -z_2) \in \Gamma_1 \times \Lambda_2$$

also, and again $\pi(\Gamma_1 \times \Lambda_2)$ is a Klein bottle.

Analogously, one verifies that

$$g(\Gamma_1' \times \Gamma_2') = (\Gamma_1' \times \Gamma_2')$$

and

$$g(\Gamma_1' \times \Lambda_2') = (\Gamma_1' \times \Lambda_2'),$$

thus also $\pi(\Gamma_1' \times \Gamma_2')$ and $\pi(\Gamma_1' \times \Lambda_2')$ are disjoint Klein bottles and the real part $S(\mathbb{R})$ of S is a disjoint union of four Klein bottles.

We observe that the example that we have just computed is a special case of the second case in Table 2.

9. Moduli space of real hyperelliptic surfaces

In this section we describe all the isomorphism classes of real hyperelliptic surfaces. Each slot in the following tables corresponds to a fixed topological type.

In the column of the values of σ_1, we have the occurrence of two possible values corresponding to two anti-holomorphic maps which are topologically equivalent, but not necessarily analytically equivalent. As explained in § 1, they are in the same connected component of the moduli space of real elliptic curves. The key observation is that the curve with $\tau_1 = \mathrm{i}$ has an isomorphism given by the multiplication by i that conjugates the two different anti-holomorphic maps; hence this isomorphism class is a ramification point for the map between the moduli space of real curves and the real part of the moduli space of complex curves.

Assume first of all that we are in the split case for the extension (3.1).
We set

$$\begin{aligned} E &= \mathbb{C}/(\mathbb{Z}+\tau_1\mathbb{Z}),\\ F &= \mathbb{C}/(\mathbb{Z}+\tau_2\mathbb{Z}),\\ \tau_j &= x_j + \mathrm{i}y_j,\\ \tilde{\sigma} &= (\sigma_1,\sigma_2) : E\times F \to E\times F. \end{aligned}$$

The action of G is given as in the Bagnera–de Franchis list.

The first case that we will consider is the one with

$$G = \mathbb{Z}/2\mathbb{Z}, \qquad \hat{G} = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

Let g be a generator of G: its action on E is a translation by an element $\eta \in \mathrm{Pic}^0(E)_2$, thus

$$g(z_1,z_2) = (z_1+\eta, -z_2) \quad \forall (z_1,z_2)\in E\times F.$$

Let us now give a list of the topological invariants that distinguish the different cases presented in Table 2.

(Recall that, for an anti-holomorphic involution σ on an elliptic curve C, the number $\nu(\sigma)$ of connected components of $\mathrm{Fix}(\sigma)$, which is either 0 or 1 or 2, is a topological invariant of the involution.)

(1) In our case, given an anti-holomorphic involution on a hyperelliptic surface S, since we have exactly two liftings of σ on $E\times F$,

$$\tilde{\sigma} = (\sigma_1,\sigma_2), \qquad \tilde{\sigma}\circ g,$$

the sets

$$\{\nu(\sigma_1), \nu(\sigma_1\circ g)\}, \tag{1a}$$
$$\{\nu(\sigma_2), \nu(\sigma_2\circ g)\}, \tag{1b}$$

are topological invariants of σ.

Since G acts on E by translation and on F by multiplication by -1, the parity of $\nu(\sigma_{\mathrm{i}})$ (determined by the alternative; the action on the first homology group is diagonalizable or not) is independent of the choice of the lifting σ_{i}, whence it provides a topological invariant.

Other topological invariants are

(2) the homology of $S(\mathbb{R})$;

(3) the action of the involution σ_2 on the fixed-point locus of g in F (see Remark 6.8);

Table 2.
($G = \mathbb{Z}/2\mathbb{Z}$, $\hat{G} = (\mathbb{Z}/2\mathbb{Z})^2$.)

τ_1	τ_2	$\sigma_1(z_1)$, η	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}$ or $-\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$	$\bar{z}_2$	$4K$
$\|\tau_1\| = 1$, $\cup x_1 = -\frac{1}{2}$	$x_2 = 0$	$\pm\tau_1\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ if $\|\tau_1\| = 1$, $\pm\bar{z}_1, \eta = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2$	$4K$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}$ or $-\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}\tau_2$	$2T$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}\tau_2$	$2T$
$x_1 = 0$	$\|\tau_2\| = 1$ or $x_2 = -\frac{1}{2}$	$\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{2}$	$\tau_2\bar{z}_2$ if $\|\tau_2\| = 1$, $\bar{z}_2$ if $x_2 = -\frac{1}{2}$	$2T$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2$	$2T$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2 + \frac{1}{2}\tau_2$	$2T$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}$ or $-\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}$ or $-\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}(1+\tau_2)$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$ or$-\bar{z}_1$, $\eta = \frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}(1+\tau_2)$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2 + \frac{1}{2}$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2 + \frac{1}{2}(1+\tau_2)$	$\emptyset$
$\|\tau_1\| = 1$, $\cup x_1 = -\frac{1}{2}$, $\|\tau_1\| \geqslant 1$	$x_2 = 0$,	$\pm\tau_1\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ if $\|\tau_1\| = 1$, $\pm\bar{z}_1$, $\eta = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}(1+\tau_2)$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{2}$	$\bar{z}_2$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}\tau_2$	$\emptyset$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}(1+\tau_2)$	$\emptyset$
$x_1 = 0$	$\|\tau_2\| = 1$, $\cup x_2 = -\frac{1}{2}$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{2}$	$\tau_2\bar{z}_2$ if $\|\tau_2\| = 1$, $\bar{z}_2$ if $x_2 = -\frac{1}{2}$	$\emptyset$
$x_1 = 0$	$\|\tau_2\| = 1$, $\cup x_2 = -\frac{1}{2}$	$\bar{z}_1$, $\eta = \frac{1}{2}$ or $-\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$	$\tau_2\bar{z}_2$ if $\|\tau_2\| = 1$, $\bar{z}_2$ if $x_2 = -\frac{1}{2}$	$2K$
$\|\tau_1\| = 1$, $\cup x_1 = -\frac{1}{2}$	$\|\tau_2\| = 1$, $\cup x_2 = -\frac{1}{2}$	$\pm\tau_1\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ if $\|\tau_1\| = 1$, $\pm\bar{z}_1$, $\eta = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\tau_2\bar{z}_2$ if $\|\tau_2\| = 1$, $\bar{z}_2$ if $x_2 = -\frac{1}{2}$	$2K$
$x_1 = 0$	$x_2 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{2}$	$\bar{z}_2$	$4T$
$x_1 = 0$	$\|\tau_2\| = 1$, $\cup x_2 = -\frac{1}{2}$	$\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$	$\tau_2\bar{z}_2$ if $\|\tau_2\| = 1$, $\bar{z}_2$ if $x_2 = -\frac{1}{2}$	T
$\|\tau_1\| = 1$, $\cup x_1 = -\frac{1}{2}$	$x_2 = 0$	$\pm\tau_1\bar{z}_1$, $\eta = \frac{1}{2}(1+\tau_1)$ if $\|\tau_1\| = 1$, $\pm\bar{z}_1$, $\eta = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}\tau_2$	T

(4) the action of $\hat{G}'$ on $\Lambda'' \otimes \mathbb{R}$, where $\Lambda'' = \pi_1(E) \cong \mathbb{Z}^2$, given by the orbifold fundamental group exact sequence

$$1 \to \Lambda'' \oplus \Gamma \to \hat{\Pi} \to \hat{G} \to 1,$$

where $\Gamma = \pi_1(F)$ (see § 4).

In particular, given a transformation in G, we have the condition whether its first action is representable by a translation in $\Lambda'' \otimes \mathbb{R}$ which is an eigenvector for σ_1.

Table 2 classifies the case with

$$\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We are in the split case, whence the results of § 6 apply. Thus we can give a complete list of normal forms for the action of $\hat{G}$.

Notice once more, however, that normal forms which are non-isomorphic for general choices of the parameters may turn out to be equivalent for special values.

Let us now explain how to distinguish the different cases listed in Table 2.

Case 1. ($S(\mathbb{R}) = 4K$.) The first two cases of the list are distinguished by the parity of $\nu(\sigma_1)$ ($\nu = 2$, respectively, $\nu = 1$).

Notice here that if $S(\mathbb{R})$ has four connected components, then necessarily the number of connected components of $\mathrm{Alb}(S)(\mathbb{R})$ is two. However, since E and $\mathrm{Alb}(S)$ are only isogenous, $\nu(\sigma_1)$ does not need to equal $\nu(\mathrm{Alb}(\sigma))$ (see the example at the end of § 8).

Case 2. ($S(\mathbb{R}) = 2T$.) The third case listed with $S(\mathbb{R}) = 2T$ is distinguished from all the others by the parity of $\nu(\sigma_2) = 1$.

The fourth case is distinguished from all the others by the invariant (3), or by ($1b$) (the set of values of $\nu(\sigma_2)$ equals $\{2, 2\}$).

The first is distinguished from the second by the invariant ($1a$) (the respective sets are $\{2, 0\}$, $\{2, 2\}$)) and from the fifth by the invariant (4); in fact, the action of g on $\Lambda'' \otimes \mathbb{R}$ is given by

$$z_1 \mapsto z_1 + \tfrac{1}{2}$$

in the first case, by

$$z_1 \mapsto z_1 + \tfrac{1}{2}(1 + \tau_1)$$

in the fifth case, and

$$\sigma_1(z_1) = \bar{z}_1$$

in both cases, but $\frac{1}{2}$ is an eigenvector of the action of σ_1 on Λ'', while

$$\tfrac{1}{2}(1 + \tau_1)$$

is not.

The second is distinguished from the fifth by the invariant ($1a$) (the respective sets are $\{2, 2\}$, $\{2, 0\}$)).

Case 3. ($S(\mathbb{R}) = \emptyset$.) The sixth is distinguished from all others by the parity of $\nu(\sigma_1) = 1$.

The last one is distinguished from all others by the parity of $\nu(\sigma_2) = 1$.

The seventh is distinguished from all others by the invariant (3), or by ($1b$) (the set of values of $\nu(\sigma_2)$ equals $\{2, 2\}$, while in all the other cases it equals either $\{0, 2\}$ or $\{0, 0\}$, or $\{1, 1\}$ in the last case).

The first case is distinguished from the second, the third, the fifth and the ninth case by the invariant ($1b$); in the first case the set of values of $\nu(\sigma_2)$ equals $\{0, 2\}$, in the others $\{0, 0\}$.

The first case is distinguished from the eighth case by the invariant ($1a$); in the first case the set of values of $\nu(\sigma_1)$ equals $\{2, 0\}$, in the eight case it equals $\{0, 0\}$.

The first case is distinguished from the fourth by the invariant (4); in fact, the action of g on $\Lambda'' \otimes \mathbb{R}$ is given by

$$z_1 \mapsto z_1 + \tfrac{1}{2}$$

in the first case, by

$$z_1 \mapsto z_1 + \tfrac{1}{2}(1 + \tau_1)$$

in the fourth case, and

$$\sigma_1(z_1) = \bar{z}_1$$

in both cases, but $\frac{1}{2}$ is an eigenvector of the action of σ_1 on Λ'', while $\frac{1}{2}(1 + \tau_1)$ is not.

The second case is distinguished from the third by the invariant (4); in fact, the action of g on $\Lambda'' \otimes \mathbb{R}$ is given by

$$z_1 \mapsto z_1 + \tfrac{1}{2}$$

in the second case, by

$$z_1 \mapsto z_1 + \tfrac{1}{2}\tau_1$$

in the third case, and

$$\sigma_1(z_1) = \bar{z}_1$$

in both cases, but $\frac{1}{2}$ is the $+1$ eigenvector of the action of σ_1 on Λ'', while $\frac{1}{2}\tau_1$ is the -1 eigenvector.

The second case is distinguished from the fourth and the eighth case by the invariant ($1b$); in the second case the set of values of $\nu(\sigma_2)$ equals $\{0, 0\}$, in the other cases $\{0, 2\}$.

The second case is distinguished from the fifth by the invariant (4); in fact, the action of g on $\Lambda'' \otimes \mathbb{R}$ is given by

$$z_1 \mapsto z_1 + \tfrac{1}{2}$$

in the second case, by

$$z_1 \mapsto z_1 + \tfrac{1}{2}(1 + \tau_1)$$

in the fifth case, and

$$\sigma_1(z_1) = \bar{z}_1$$

in both cases, but $\frac{1}{2}$ is an eigenvector of the action of σ_1 on Λ'', while $\frac{1}{2}(1+\tau_1)$ is not.

The second case is distinguished from the ninth case by the invariant (1a); in the second case the set of values of $\nu(\sigma_1)$ equals $\{2,0\}$, in the ninth case it equals $\{0,0\}$.

The third case is distinguished from the fourth by the invariant (1b); in the third case the set of values of $\nu(\sigma_2)$ equals $\{0,0\}$, in the fourth case $\{0,2\}$.

The third case is distinguished from the fifth by the invariant (4); in fact, the action of g on $\Lambda'' \otimes \mathbb{R}$ is given by

$$z_1 \mapsto z_1 + \tfrac{1}{2}\tau_1$$

in the third case, by

$$z_1 \mapsto z_1 + \tfrac{1}{2}(1+\tau_1)$$

in the fifth case, and

$$\sigma_1(z_1) = \bar{z}_1$$

in both cases, but $\frac{1}{2}\tau_1$ is an eigenvector of the action of σ_1 on Λ'', while $\frac{1}{2}(1+\tau_1)$ is not.

The third case is distinguished from the eighth and the ninth case by the invariant (1a); in the third case the set of values of $\nu(\sigma_1)$ equals $\{2,2\}$, in the eighth and the ninth case it equals $\{0,0\}$.

The fourth case is distinguished from the fifth and the ninth case by the invariant (1b); in the fourth case the set of values of $\nu(\sigma_2)$ equals $\{0,2\}$, in the other cases $\{0,0\}$.

The fourth case is distinguished from the eighth case by the invariant (1a); in the fourth case the set of values of $\nu(\sigma_1)$ equals $\{0,2\}$, in the eighth case it equals $\{0,0\}$.

The fifth case is distinguished from the eighth case by the invariant (1b); in the fifth case the set of values of $\nu(\sigma_2)$ equals $\{0,0\}$, in the eighth case $\{2,0\}$.

The fifth case is distinguished from the ninth case by the invariant (1a); in the fifth case the set of values of $\nu(\sigma_1)$ equals $\{2,0\}$, in the ninth case it equals $\{0,0\}$.

The eighth case is distinguished from the ninth case by the invariant (1b); in the eighth case the set of values of $\nu(\sigma_2)$ equals $\{2,0\}$, in the ninth case $\{0,0\}$.

Case 4. ($S(\mathbb{R}) = 2K$.) The parity of $\nu(\sigma_1)$ distinguishes the two cases.

Case 5. ($S(\mathbb{R}) = T$.) The parity of $\nu(\sigma_2)$ distinguishes the two cases.

Let us now consider the case $G = \mathbb{Z}/4\mathbb{Z}$ (Table 3). If g is a generator of G, we denote by

$$g(z_1, z_2) = (z_1 + \eta, \mathrm{i}z_2)$$

the action of g on $E \times F$.

The topological invariants that distinguish the different cases here are the above-mentioned invariant (2) and the analogous of the invariants (1a) and (1b) that we had in the case $G = \mathbb{Z}/2\mathbb{Z}$,

$$\{\nu(\sigma_1 \circ g^n) \text{ for } n = 0,1,2,3\}, \qquad (1a')$$

$$\{\nu(\sigma_2 \circ g^n) \text{ for } n = 0,1,2,3\}. \qquad (1b')$$

Table 3.
($G = \mathbb{Z}/4\mathbb{Z}$, $\hat{G} = D_4$; $\tau_2 = \mathrm{i}$.)

τ_1	$\sigma_1(z_1)$, η	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2} + \frac{1}{4}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{4} + \frac{1}{2}\tau_1$	$\bar{z}_2$	$2T$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{2} + \frac{1}{4}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{4} + \frac{1}{2}\tau_1$	$\bar{z}_2$	T
$x_1 = 0$,	$\bar{z}_1$, $\eta = \frac{1}{4}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{4}$	$\bar{z}_2 + \frac{1}{2}(1 + \mathrm{i})$	T
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{2} + \frac{1}{4}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{4} + \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}(1 + \mathrm{i})$	T
$x_1 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2} + \frac{1}{4}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{4} + \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}(1 + \mathrm{i})$	$\emptyset$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{4}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{4}$	$\bar{z}_2$	$\emptyset$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{4}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{4}$	$\bar{z}_2 + \frac{1}{2}(1 + \mathrm{i})$	$\emptyset$
$x_1 = 0$	$\bar{z}_1$,$\eta = \frac{1}{4}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{4}$	$\bar{z}_2$	$3T$
$\lvert\tau_1\rvert = 1$, $\cup x_1 = -\frac{1}{2}$	$\tau_1\bar{z}_1$, $\eta = \frac{1}{4}(1 - \tau_1)$ or $-\tau_1\bar{z}_1$, $\eta = \frac{1}{4}(1 + \tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta = \frac{1}{4} + \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{4}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2$	$3K$
$\lvert\tau_1\rvert = 1$, $\cup x_1 = -\frac{1}{2}$	$\tau_1\bar{z}_1$, $\eta = \frac{1}{4}(1 - \tau_1)$ or $-\tau_1\bar{z}_1$, $\eta = \frac{1}{4}(1 + \tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta = \frac{1}{4} + \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{4}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}(1 + \mathrm{i})$	K

In the list, if $S(\mathbb{R}) = T$, the second and the third cases are distinguished by invariant $(1a')$; in the second case the set of values of $\nu(\sigma_1)$ equals $\{2, 2, 2, 2\}$, in the third it equals $\{0, 2, 0, 2\}$.

The first case is distinguished from the others by the invariant $(1b')$; in the first case $\{\nu(\sigma_2 \circ g^n) \text{ for } n = 0, 1, 2, 3\} = \{2, 1, 2, 1\}$, while in the other cases $\{\nu(\sigma_2 \circ g^n) \text{ for } n = 0, 1, 2, 3\} = \{0, 1, 0, 1\}$.

If $S(\mathbb{R}) = \emptyset$, the first case is distinguished from the third case by the invariant $(1a')$; in the first case the set of values of $\nu(\sigma_1)$ equals $\{2, 0, 2, 0\}$, in the third it equals $\{0, 0, 0, 0\}$.

The second case is distinguished from the other cases by the invariant $(1b')$; in the second case the set of values of $\nu(\sigma_2)$ equals $\{2, 1, 2, 1\}$, in the other cases it equals $\{0, 1, 0, 1\}$.

If $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Tables 4 and 5), we take generators g, t such that, on F,

$$g(z_2) = \mathrm{i}z_2, \qquad t(z_2) = z_2 + \tfrac{1}{2}(1 + \mathrm{i}).$$

If $\hat{G} \cong G_1$ (Table 4), we see that the topological type of $S(\mathbb{R})$ distinguishes the cases. The same holds if

$$\hat{G} \cong D_4 \times \mathbb{Z}/2\mathbb{Z}.$$

If $G = \mathbb{Z}/3\mathbb{Z}$ (Table 6), we denote by

$$g(z_1, z_2) = (z_1 + \eta, \rho z_2)$$

the action of a generator g of G on $E \times F$.

Table 4.
($G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\hat{G} = G_1$; $\tau_2 = \mathrm{i}$.)

τ_1	$\sigma_1(z_1)$	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$\bar{z}_1$, $G = \langle \frac{1}{4}(1+\tau_1)\rangle \times \langle \frac{1}{2}\rangle$ or $-\bar{z}_1$, $G = \langle \frac{1}{4}(-1+\tau_1)\rangle \times \langle \frac{1}{2}\tau_1\rangle$	$\bar{z}_2 + \frac{1}{2}$, $\epsilon = \frac{1}{2}(1+\mathrm{i})$	$\emptyset$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $G = \langle \frac{1}{4}(1+\tau_1)\rangle \times \langle \frac{1}{2}\rangle$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $G = \langle \frac{1}{4}(-1+\tau_1)\rangle \times \langle \frac{1}{2}\tau_1\rangle$	$\bar{z}_2 + \frac{1}{2}$, $\epsilon = \frac{1}{2}(1+\mathrm{i})$	T

Table 5.
($G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\hat{G} = D_4 \times \mathbb{Z}/2\mathbb{Z}$; $\tau_2 = \mathrm{i}$.)

τ_1	$\sigma_1(z_1)$	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$\bar{z}_1$, $G = \langle \frac{1}{4}\tau_1\rangle \times \langle \frac{1}{2}\rangle$ or $-\bar{z}_1$, $G = \langle \frac{1}{4}\rangle \times \langle \frac{1}{2}\tau_1\rangle$	$\bar{z}_2$, $\epsilon = \frac{1}{2}(1+\mathrm{i})$	$2T$
$x_1 = 0$	$\bar{z}_1$, $G = \langle \frac{1}{4}\tau_1\rangle \times \langle \frac{1}{2}(1+\tau_1)\rangle$ or $-\bar{z}_1$, $G = \langle \frac{1}{4}\rangle \times \langle \frac{1}{2}(1+\tau_1)\rangle$	$\bar{z}_2$, $\epsilon = \frac{1}{2}(1+\mathrm{i})$	$K \sqcup T$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $G = \langle \frac{1}{4}\tau_1\rangle \times \langle \frac{1}{2}\rangle$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $G = \langle \frac{1}{4}\rangle \times \langle \frac{1}{2}\tau_1\rangle$	$\bar{z}_2$, $\epsilon = \frac{1}{2}(1+\mathrm{i})$	T
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $G = \langle \frac{1}{4}\tau_1\rangle \times \langle \frac{1}{2}(1+\tau_1)\rangle$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $G = \langle \frac{1}{4}\rangle \times \langle \frac{1}{2}(1+\tau_1)\rangle$	$\bar{z}_2$, $\epsilon = \frac{1}{2}(1+\mathrm{i})$	K

We notice that

$$\mathrm{Fix}(g) = \mathbb{Z}(1-\rho)/3\mathbb{Z}(1-\rho)$$

and the action of $\sigma_2(z_2) = \bar{z}_2$ on $\mathrm{Fix}(g)$ is $-\mathrm{Id}$, while the action of $\sigma_2(z_2) = -\bar{z}_2$ on $\mathrm{Fix}(g)$ is Id, so these two actions are topologically different.

This invariant and $S(\mathbb{R})$ are sufficient to distinguish all the cases.

If $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (Table 7), with generators g, t acting on F, by

$$g(z_2) = \rho z_2, \qquad t(z_2) = z_2 + \tfrac{1}{3}(1-\rho),$$

the homology of $S(\mathbb{R})$ is the only topological invariant needed in order to distinguish the three cases.

If $G = \mathbb{Z}/6\mathbb{Z}$ (Table 8), we take a generator g of G such that

$$g(z_1, z_2) = (z_1 + \eta, -\rho z_2),$$

and also here the homology of $S(\mathbb{R})$ is the only topological invariant needed to distinguish the four cases.

Let us now consider the case where

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Table 6.
($G = \mathbb{Z}/3\mathbb{Z}$, $\hat{G} = \mathcal{S}_3$; $\tau_2 = \rho$.)

τ_1	$\sigma_1(z_1)$, η	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$\bar{z}_1$, $\eta = \frac{1}{3}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{3}$	$\bar{z}_2$	$2T$
$x_1 = 0$	$\bar{z}_1$, $\eta = \frac{1}{3}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{3}$	$-\bar{z}_2$	$2T$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{3}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{3}$	$\bar{z}_2$	$\emptyset$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{3}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{3}$	$-\bar{z}_2$	$\emptyset$
$\lvert\tau_1\rvert = 1$, $\cup x_1 = -\frac{1}{2}$	$\tau_1\bar{z}_1$, $\eta = \frac{1}{3}(1-\tau_1)$ or $-\tau_1\bar{z}_1$, $\eta = \frac{1}{3}(1+\tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta = \frac{1}{3}(1-\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{3}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2$	T
$\lvert\tau_1\rvert = 1$, $\cup x_1 = -\frac{1}{2}$	$\tau_1\bar{z}_1$, $\eta = \frac{1}{3}(1-\tau_1)$ or $-\tau_1\bar{z}_1$, $\eta = \frac{1}{3}(1+\tau_1)$ if $\lvert\tau_1\rvert = 1$ $\bar{z}_1$, $\eta = \frac{1}{3}(1-\tau_1)$ or $-\bar{z}_1$, $\eta = \frac{1}{3}$ if $x_1 = -\frac{1}{2}$	$-\bar{z}_2$	T

Table 7.
($G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\hat{G} = \mathcal{S}_3 \times \mathbb{Z}/3\mathbb{Z}$; $\tau_2 = \rho$.)

τ_1	$\sigma_1(z_1)$	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$\bar{z}_1$, $G = \langle\frac{1}{3}\tau_1\rangle \times \langle\frac{1}{3}\rangle$ or $-\bar{z}_1$, $G = \langle\frac{1}{3}\rangle \times \langle\frac{1}{3}\tau_1\rangle$	$-\bar{z}_2$, $\epsilon = \frac{1}{3}(1-\rho)$	$2T$
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $G = \langle\frac{1}{3}\tau_1\rangle \times \langle\frac{1}{3}\rangle$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $G = \langle\frac{1}{3}\rangle \times \langle\frac{1}{3}\tau_1\rangle$	$-\bar{z}_2$, $\epsilon = \frac{1}{3}(1-\rho)$	$\emptyset$
$\lvert\tau_1\rvert = 1$,	$\tau_1\bar{z}_1$, $G = \langle\frac{1}{3}(1-\tau_1)\rangle \times \langle\frac{1}{3}(1+\tau_1)\rangle$ or $-\tau_1\bar{z}_1$, $G = \langle\frac{1}{3}(1+\tau_1)\rangle \times \langle\frac{1}{3}(1-\tau_1)\rangle$ if $\lvert\tau_1\rvert = 1$,	$-\bar{z}_2$, $\epsilon = \frac{1}{3}(1-\rho)$	T
$\cup x_1 = -\frac{1}{2}$	$\bar{z}_1$, $G = \langle\frac{1}{3}(1-\tau_1)\rangle \times \langle\frac{1}{3}\rangle$ or $-\bar{z}_1$, $G = \langle\frac{1}{3}\rangle \times \langle\frac{1}{3}(1-\tau_1)\rangle$ if $x_1 = -\frac{1}{2}$		

Table 8.
($G = \mathbb{Z}/6\mathbb{Z}$, $\hat{G} = D_6$; $\tau_2 = \rho$.)

τ_1	$\sigma_1(z_1)$, η	$\sigma_2(z_2)$	$S(\mathbb{R})$
$x_1 = 0$	$\bar{z}_1$, $\eta = \frac{1}{6}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{6}$	$\bar{z}_2$	$2T$
$x_1 = 0$	$\bar{z}_1$, $\eta = \frac{1}{2} + \frac{1}{6}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{2}\tau_1 + \frac{1}{6}$	$\bar{z}_2$	T
$x_1 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta = \frac{1}{6}\tau_1$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta = \frac{1}{6}$	$\bar{z}_2$	$\emptyset$
$\lvert\tau_1\rvert = 1$, $\cup x_1 = -\frac{1}{2}$	$\tau_1\bar{z}_1$, $\eta = \frac{1}{6}(1-\tau_1)$ or $-\tau_1\bar{z}_1$, $\eta = \frac{1}{6}(1+\tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta = \frac{1}{6} + \frac{1}{3}\tau_1$ or $-\bar{z}_1$, $\eta = \frac{1}{6}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2$	$2K$

(Table 9). Here, we may choose as generators of G the generator t of T and another element g, which is not canonically defined.

Let η_1, ϵ_1, ϵ_2 be such that

$$g(z_1, z_2) = (z_1 + \eta_1, -z_2), t(z_1, z_2) = (z_1 + \epsilon_1, z_2 + \epsilon_2).$$

Here we have the same topological invariants (1), (2) as in the case $G = \mathbb{Z}/2\mathbb{Z}$.

Furthermore, the normal subgroup T of G is of order 2. Let, as usual, t be a generator. We may consider all the possible liftings of t to a vector t' in the lattice

$$\Omega' = \Lambda' \oplus \Gamma.$$

The condition whether there exists such a t' whose two components are eigenvectors for the action of $\tilde{\sigma}$ on Ω' is a topological invariant of the real hyperelliptic surface (notice that the two possible choices for σ_2 differ just up to multiplication by -1).

Let us now explain how to distinguish the different cases listed in Table 9. Assume first of all that

$$\hat{G} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Case 1. ($S(\mathbb{R}) = 2K$.) The first case is distinguished from the second and the fourth since in the first case $\epsilon_2 = \frac{1}{2}(1 + \tau_2)$, therefore any lifting of t to a vector $t' \in \Omega'$ cannot be an eigenvector for the action of σ_2, while in the second and the fourth case $\epsilon_2 = \frac{1}{2}$, respectively, $\frac{1}{2}\tau_2$, which are eigenvectors for the action of σ_2.

The first case is distinguished from the third and the sixth by the parity of $\nu(\sigma_2)$. In fact, $\nu(\sigma_2) = 2$ in the first case, while in the third and the sixth case we have $\nu(\sigma_2) = 1$.

The first case is distinguished from the fifth since in the first case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the fifth case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1.

The second case is distinguished from the third and the sixth by the parity of $\nu(\sigma_2)$. In fact, $\nu(\sigma_2) = 2$ in the second case, while in the third and the sixth case we have $\nu(\sigma_2) = 1$.

The second case is distinguished from the fourth since in the second case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the fourth case $\epsilon_1 = \frac{1}{2}(1 + \tau_1)$, which is not an eigenvector for the action of σ_1.

The second case is distinguished from the fifth since in the second case ϵ_2 is an eigenvector for the action of σ_2 on Γ, while in the fifth case $\epsilon_2 = \frac{1}{2}(1 + \tau_2)$, which is not an eigenvector for the action of σ_2.

The third case is distinguished from the fourth and the fifth by the parity of $\nu(\sigma_2)$. In fact, $\nu(\sigma_2) = 1$ in the third case, while in the other two cases we have $\nu(\sigma_2) = 2$.

The third case is distinguished from the sixth since in the third case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the sixth case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1

The fourth and the fifth cases are distinguished from the sixth by the parity of $\nu(\sigma_2)$. In fact, $\nu(\sigma_2) = 1$ in the sixth case, while in the other two cases we have $\nu(\sigma_2) = 2$.

The fourth case is distinguished from the fifth since in the fourth case ϵ_2 is an eigenvector for the action of σ_2 on Γ, while in the fifth case $\epsilon_2 = \frac{1}{2}(1 + \tau_2)$, which is not an eigenvector for the action of σ_2.

Table 9.
($G = (\mathbb{Z}/2\mathbb{Z})^2$, $\hat{G} = (\mathbb{Z}/2\mathbb{Z})^3$; τ_1: $x_1 = 0$.)

τ_2	$\sigma_1(z_1)$, η_1, ϵ_1	$\sigma_2(z_2)$, ϵ_2	$S(\mathbb{R})$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	$2K$
$x_2 = 0$	$\bar{z}_1, \eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}$	$2K$
$\lvert\tau_2\rvert = 1$ or $x_2 = -\frac{1}{2}$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$	$\tau_2\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$ if $\lvert\tau_2\rvert = 1$, $\bar{z}_2$, $\epsilon_2 = \frac{1}{2}$ if $x_2 = -\frac{1}{2}$	$2K$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}\tau_2$	$2K$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	$2K$
$\lvert\tau_2\rvert = 1$ or $x_2 = -\frac{1}{2}$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\tau_2\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$ if $\lvert\tau_2\rvert = 1$, $\bar{z}_2$, $\epsilon_2 = \frac{1}{2}$ if $x_2 = -\frac{1}{2}$	$2K$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}\tau_2$, $\epsilon_2 = \frac{1}{2}$	T
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2 + \frac{1}{2}\tau_2$, $\epsilon_2 = \frac{1}{2}$	T
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}\tau_2$	T
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	T
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	T
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$	$\bar{z}_2 + \frac{1}{2}$, $\epsilon_2 = \frac{1}{2}\tau_2$	$\emptyset$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2 + \frac{1}{2}$, $\epsilon_2 = \frac{1}{2}\tau_2$	$\emptyset$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}$, $\epsilon_2 = \frac{1}{2}\tau_2$	$\emptyset$
$x_2 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	$\emptyset$
$x_2 = 0$	$\bar{z}_1 + \frac{1}{2}$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1 + \frac{1}{2}\tau_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	$\emptyset$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}\tau_2$	$3T$

Table 9. (*Cont.*)
($G = (\mathbb{Z}/2\mathbb{Z})^2$, $\hat{G} = (\mathbb{Z}/2\mathbb{Z})^3$; τ_1: $x_1 = 0$.)

τ_2	$\sigma_1(z_1)$, η_1, ϵ_1	$\sigma_2(z_2)$, ϵ_2	$S(\mathbb{R})$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	$2T$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\bar{z}_2 + \frac{1}{2}\tau_2$, $\epsilon_2 = \frac{1}{2}$	$2T$
$\lvert\tau_2\rvert = 1$ or $x_2 = -\frac{1}{2}$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$	$\tau_2\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$ if $\lvert\tau_2\rvert = 1$, $\bar{z}_2$, $\epsilon_2 = \frac{1}{2}$ if $x_2 = -\frac{1}{2}$	$2T$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$	$2T$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}\tau_1$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}\tau_2$	$2K \sqcup T$
$x_2 = 0$	$\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$	$\bar{z}_2$, $\epsilon_2 = \frac{1}{2}$	$2K \sqcup T$

Table 9. (*Cont.*)
($G = (\mathbb{Z}/2\mathbb{Z})^2$, $\hat{G} = D_4$; τ_1: $\lvert\tau_1\rvert = 1$ or $x_1 = -\frac{1}{2}$; τ_2: $x_2 = 0$.)

$\sigma_1(z_1)$, η_1, ϵ_1	$\sigma_2(z_2)$, ϵ_2	$S(\mathbb{R})$
$\tau_1\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\tau_1\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2 + \frac{1}{4}\tau_2$, $\epsilon_2 = \frac{1}{2}\tau_2$ or $-\bar{z}_2 - \frac{1}{4}$, $\epsilon_2 = \frac{1}{2}$	$2T$
$\tau_1\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\tau_1\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\tau_2\bar{z}_2 + \frac{1}{4}(1-\tau_2)$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$ or $-\tau_2\bar{z}_2 + \frac{1}{4}(1+\tau_2)$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$ or $\bar{z}_2 + \frac{1}{4} + \frac{1}{2}\tau_2$, $\epsilon_2 = \frac{1}{2}$ or $-\bar{z}_2 - \frac{1}{4}$, $\epsilon_2 = \frac{1}{2}$ if $x_2 = -\frac{1}{2}$	$2T$
$\tau_1\bar{z}_1$, $\eta_1 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ or $-\tau_1\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}(1+\tau_1)$ if $\lvert\tau_1\rvert = 1$, $\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ or $-\bar{z}_1$, $\eta_1 = \frac{1}{2}\tau_1$, $\epsilon_1 = \frac{1}{2}$ if $x_1 = -\frac{1}{2}$	$\bar{z}_2 + \frac{1}{2} + \frac{1}{4}\tau_2$, $\epsilon_2 = \frac{1}{2}\tau_2$ or $-\bar{z}_2 - \frac{1}{4} + \frac{1}{2}\tau_2$, $\epsilon_2 = \frac{1}{2}$	$\emptyset$

Case 2. ($S(\mathbb{R}) = T$.) The first two cases are distinguished from the last three cases by the invariant (1*b*); in the first two cases the set of values of $\nu(\sigma_2)$ equals $\{2, 0\}$, in the last three cases it equals $\{2, 2\}$.

The first case is distinguished from the second since in the first case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the second case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1.

The third case is distinguished from the fourth and the fifth since in the third case ϵ_2 is an eigenvector for the action of σ_2 on Γ, while in the other two cases $\epsilon_2 = \frac{1}{2}(1+\tau_2)$, which is not an eigenvector for the action of σ_2.

The fourth case is distinguished from the fifth since in the fourth case ϵ_1 is the $+1$ eigenvector for the action of σ_1 on Λ', while in the fifth case ϵ_1 is the -1 eigenvector for the action of σ_1.

Case 3. ($S(\mathbb{R}) = \emptyset$.) The first three cases are distinguished from the last two cases by the invariant $(1b)$; in the first three cases the set of values of $\nu(\sigma_2)$ equals $\{0, 2\}$, in the last two cases it equals $\{2, 2\}$.

The second case is distinguished from the first and the third since in the first and the third case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the second case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1.

The first case is distinguished from the third since in the first case ϵ_1 is the -1 eigenvector for the action of σ_1 on Λ', while in the third case ϵ_1 is the $+1$ eigenvector for the action of σ_1.

The fourth case is distinguished from the fifth since in the fifth case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the fourth case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1.

Case 4. ($S(\mathbb{R}) = 2T$.) The third case is distinguished from all the others by the parity of $\nu(\sigma_2) = 1$.

The second case is distinguished from all the others by the invariant $(1b)$. In fact, the set of values of $\nu(\sigma_2)$ is $\{2, 0\}$ in the second case, while in the other cases it is either $\{2, 2\}$, or $\{1, 1\}$ (only in the third case).

The first case is distinguished from the last one since in the first case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the last case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1.

Case 5. ($S(\mathbb{R}) = 2K \sqcup T$.) The first case is distinguished from the second one since in the first case ϵ_1 is an eigenvector for the action of σ_1 on Λ', while in the second case $\epsilon_1 = \frac{1}{2}(1+\tau_1)$, which is not an eigenvector for the action of σ_1.

If $G = D_4$, the cases with $S(\mathbb{R}) = 2T$ are distinguished by the parity of $\nu(\sigma_2)$, which is equal to 2 in the first case, and equal to 1 in the second case.

We finally give the table for the non-split case, where

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

is generated by elements g, t such that

$$g(z_1, z_2) = (z_1 + \eta_1, -z_2), t(z_1, z_2) = (z_1 + \epsilon_1, z_2 + \epsilon_2).$$

Recall the by now standard notation $\tau_j = x_j + \mathrm{i} y_j$.

Table 10.
($G = (\mathbb{Z}/2\mathbb{Z})^2$, $\hat{G} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; $S(\mathbb{R}) = \emptyset$.)

τ_1	τ_2	$\tilde{\sigma}_1$, η_1, ϵ_1	$\tilde{\sigma}_2$, ϵ_2
$x_1 = 0$	$\lvert\tau_2\rvert = 1$ or $x_2 = -\frac{1}{2}$	$\bar{z}_1 + \frac{1}{4}$, $\epsilon_1 = \frac{1}{2}$, $\eta_1 = \frac{1}{2}(1+\tau_1)$ or $-\bar{z}_1 + \frac{1}{4}\tau$, $\epsilon_1 = \frac{1}{2}\tau_1$, $\eta_1 = \frac{1}{2}(1+\tau_1)$	$\tau_2\bar{z}_2 + \frac{1}{2}$, $\epsilon_2 = \frac{1}{2}(1+\tau_2)$ if $\lvert\tau_2\rvert = 1$, $\bar{z}_2 + \frac{1}{2}\tau$, $\epsilon_2 = \frac{1}{2}$ if $x_2 = -\frac{1}{2}$

Acknowledgements. We thank V. Kharlamov for an interesting conversation concerning the status of the Enriques classification of algebraic surfaces, and for communicating his conjecture to us. Also, S. T. Yau and A. Sommese pointed out some literature (see [**8**]) leading (cf. [**14**]) to examples of complex manifolds which are $K(\pi, 1)$s and where the moduli space of diffeomorphic complex manifolds can be disconnected.

The present research was performed in the realm of the DFG Schwerpunkt Program 'Globale Methode in der komplexen Geometrie', and of the EU Project EAGER, and was partially supported by the Italian MURST 40/100 Program 'Geometria Algebrica'.

Note added in proof. In a recent preprint 'Real structures on minimal ruled surfaces' (AG/0201158), J. Y. Welschinger showed that the deformation classes of minimal ruled surfaces are determined by the topology of the real structure.

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