

Moduli spaces of surfaces and real structures

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This article is dedicated to the memory of Boris Moisezon

Abstract

We give infinite series of groups Γ and of compact complex surfaces of general type S with fundamental group Γ such that

- 1) Any surface S' with the same Euler number as S , and fundamental group Γ , is diffeomorphic to S .
- 2) The moduli space of S consists of exactly two connected components, exchanged by complex conjugation.

Whence,

- i) On the one hand we give simple counterexamples to the DEF = DIFF question whether deformation type and diffeomorphism type coincide for algebraic surfaces.
- ii) On the other hand we get examples of moduli spaces without real points.
- iii) Another interesting corollary is the existence of complex surfaces S whose fundamental group Γ cannot be the fundamental group of a real surface.

Our surfaces are surfaces isogenous to a product; i.e., they are quotients $(C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G .

They resemble the classical hyperelliptic surfaces, in that G operates freely on C_1 , while the second curve is a *triangle curve*, meaning that $C_2/G \cong \mathbf{P}^1$ and the covering is branched in exactly three points.

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1. Introduction

Let S be a minimal surface of general type; then to S we attach two positive integers $x = \chi(\mathcal{O}_S)$, $y = K_S^2$ which are invariants of the oriented topological type of S .

The moduli space of the surfaces with invariants (x, y) is a quasi-projective variety defined over the integers, in particular it is a real variety (similarly for the Hilbert scheme of 5-canonical embedded canonical models, of which the moduli space is a quotient; cf. [Bo], [Gie]).

For fixed (x, y) we have several possible topological types, but (by the result of [F]) indeed at most two if moreover the surface S is assumed to be simply connected (actually by [Don1,2], related results hold more generally for the topological types of simply connected compact oriented differentiable 4-manifolds; cf. [Don4,5] for a precise statement, the so-called 11/8 conjecture).

These two cases are distinguished as follows:

- S is EVEN, i.e., its intersection form is even: then S is a connected sum of copies of $\mathbf{P}_{\mathbb{C}}^1 \times \mathbf{P}_{\mathbb{C}}^1$ and of a $K3$ surface if the signature is negative, and of copies of $\mathbf{P}_{\mathbb{C}}^1 \times \mathbf{P}_{\mathbb{C}}^1$ and of a $K3$ surface with reversed orientation if the signature is positive.
- S is ODD: then S is a connected sum of copies of $\mathbf{P}_{\mathbb{C}}^2$ and $\mathbf{P}_{\mathbb{C}}^{2\text{ opp}}$.

Remark 1.1. $\mathbf{P}_{\mathbb{C}}^{2\text{ opp}}$ stands for the same manifold as $\mathbf{P}_{\mathbb{C}}^2$, but with reversed orientation. Beware that some authors use the symbol $\mathbf{P}_{\mathbb{C}}^2$ for $\mathbf{P}_{\mathbb{C}}^{2\text{ opp}}$, whereas for us the notation \bar{X} will denote the conjugate of a complex manifold X (\bar{X} is just the same differentiable manifold, but with complex structure $-J$ instead of J). Observe that, if X has odd dimension, then \bar{X} acquires the opposite orientation of X , but if X has even dimension, then X and \bar{X} are orientedly diffeomorphic.

Recall moreover:

Definition 1.2. A real structure σ on a complex manifold X is the datum of an isomorphism $\sigma : X \rightarrow \bar{X}$ such that $\sigma^2 = \text{Identity}$. One moment's reflection shows then that σ yields an isomorphism between the pairs (X, σ) and (\bar{X}, σ) .

In general, the fundamental group is a powerful topological invariant. Invariants of the differentiable structure have been found by Donaldson, by Seiberg-Witten and several other authors (cf. [Don3], [D-K], [Witten], [F-M3], [Mor]) and it is well known that on a connected component of the moduli space the differentiable structure remains fixed (we use for this result the slogan $\text{DEF} \Rightarrow \text{DIFF}$).

Actually, if two surfaces S, S' are deformation equivalent then there exists a diffeomorphism carrying the canonical class $K_S \in H^2(S, \mathbf{Z})$ of S to $K_{S'}$; moreover, for minimal surfaces of general type it was proven (cf. [Witten], or [Mor, Cor. 7.4.2, p. 123]) that any diffeomorphism between S and S' carries K_S either to $K_{S'}$ or to $-K_{S'}$.

Up to recently, the question $\text{DEF} = \text{DIFF} ?$ was open.

The converse question $\text{DIFF} \Rightarrow \text{DEF}$, asks whether the existence of an orientation preserving diffeomorphism between algebraic surfaces S, S' would imply that S, S' would be deformation equivalent (i.e., in the same connected component of the moduli space). This question was a "speculation" by Friedman and Morgan [F-M1, p. 12] (in the words of the authors, *ibidem* page 8, "those questions which we have called speculations... seem to require completely new ideas").

The speculation was inspired by the successes of gauge theory, and reading the question I thought the answer should be negative, but would not be easy to find.

Recently, ([Man4]) Manetti was able to find counterexamples of surfaces with first Betti number equal to 0 (but not simply connected).

His result on the one side uses methods and results developed in a long sequel of papers ([Cat1,2,3], [Man1,2,3]), on the other, it uses a rather elaborate construction.

About the same time of this paper Kharlamov and Kulikov ([K-K]) gave a counterexample for rigid surfaces, in the spirit of the work of Jost and Yau ([J-Y1,2]). Here, we have found the following rather simple series of examples:

THEOREM 1.3. *Let S be a surface isogenous to a product, i.e., a quotient $S = (C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G . Then any surface with the same fundamental group as S and the same Euler number of S is diffeomorphic to S . The corresponding moduli space $M_S^{\text{top}} = M_S^{\text{diff}}$ is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation. There are infinitely many examples of the latter case, and moreover these moduli spaces are almost all of general type.*

Remark 1.4. The last statement is a direct consequence of the results of Harris-Mumford ([H-M]).

COROLLARY 1.5. 1) $\text{DEF} \neq \text{DIFF}$.

2) *There are moduli spaces without real points.*

The more prudent question of asking whether moduli spaces with several connected components studied in the previously cited papers of ours and Manetti would yield diffeomorphic 4-manifolds was raised again by Donaldson

(in [Don5, pp. 65–68]), who also illustrated the important role played by the symplectic structure of an algebraic surface. The referee of this paper points out an important fact: the standard diffeomorphism between S and \bar{S} carries the canonical class K_S to $-K_{\bar{S}}$, and moreover one could summarize the philosophy of our topological proof as asserting that there exists no orientation-preserving self-homeomorphism of S , or homotopy equivalence, carrying K_S to $-K_S$. He then proposes that one could sharpen the Friedman-Morgan conjecture by asking whether the existence of a diffeomorphism carrying the canonical class to the canonical class would suffice to imply deformation equivalence.

Unfortunately, this question also has a negative answer, as we show in a sequel to this paper ([Cat7], [C-W]), whose methods are completely different from the ones of the present paper.

In [Cat7] we give a criterion in order to establish the symplectomorphism of two algebraic surfaces which are not deformation equivalent, and show that the examples of Manetti give a counterexample to the refined conjecture. Since, however, these examples are not simply connected, we also discuss some simply connected examples which are not deformation equivalent: in [C-W] we then show their symplectic equivalence.

Returning to the examples shown in the present paper, we deduce moreover, as a byproduct of our arguments, the following:

THEOREM 1.6. *There are infinite series of groups Γ which are fundamental groups of complex surfaces but which cannot be fundamental groups of a real surface.*

One word about the construction of our examples: We imitate the hyperelliptic surfaces, in the sense that we take $S = (C_1 \times C_2)/G$ where G acts freely on C_1 , whereas the quotient C_2/G is $\mathbf{P}_{\mathbf{C}}^1$. Moreover, we assume that the projection $\phi : C_2 \rightarrow \mathbf{P}_{\mathbf{C}}^1$ is branched in only three points, namely, we have a so-called *triangle curve*.

It follows that if two surfaces of this sort were antiholomorphic, then there would be an antiholomorphism of the second triangle curve (which is rigid).

Now, giving such a branched cover ϕ amounts to viewing the group G as a quotient of the free group with two elements. Let a, c be the images of the two generators, and set $abc = 1$.

We find such a G with the properties that the respective orders of a, b, c are distinct, whence we show that an antiholomorphism of the triangle curve would be a lift of the standard complex conjugation if the three branch points are chosen to be real, e.g. $-1, 0$ and $+1$.

But such a lifting exists if and only if the group G admits an automorphism τ such that $\tau(a) = a^{-1}, \tau(c) = c^{-1}$.

Appropriate semidirect products do the game for us.

Remark 1.7. It would be interesting to classify the rigid surfaces, isogenous to a product, which are not real. Examples due to Beauville ([Bea], [Cat6]) yield real surfaces.

2. A nonreal triangle curve

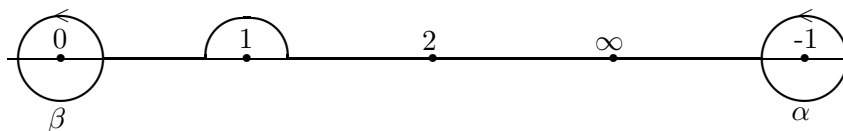
Consider the set $B \subset \mathbf{P}_{\mathbf{C}}^1$ consisting of three real points $B := \{-1, 0, 1\}$.

We choose 2 as a base point in $\mathbf{P}_{\mathbf{C}}^1 - B$, and take the following generators α, β, γ of $\pi_1(\mathbf{P}_{\mathbf{C}}^1 - B, 2)$:

- α goes from 2 to $-1 - \varepsilon$ along the real line, passing through $+\infty$, then makes a full turn counterclockwise around the circumference with centre -1 and radius ε , then goes back to 2 along the same way on the real line.
- γ goes from 2 to $1 + \varepsilon$ along the real line, then makes a full turn counterclockwise around the circumference with centre $+1$ and radius ε , then goes back to 2 along the same way on the real line.
- β goes from 2 to $1 + \varepsilon$ along the real line, makes a half turn counterclockwise around the circumference with centre $+1$ and radius ε , reaching $1 - \varepsilon$, then proceeds along the real line reaching $+\varepsilon$, makes a full turn counterclockwise around the circumference with centre 0 and radius ε , goes back to $1 - \varepsilon$ along the same way on the real line, makes again a half turn clockwise around the circumference with centre $+1$ and radius ε , reaching $1 + \varepsilon$; finally it proceeds along the real line returning to 2.

An easy picture shows that α, γ are free generators of $\pi_1(\mathbf{P}_{\mathbf{C}}^1 - B, 2)$ and

$$\alpha\beta\gamma = 1.$$



With this choice of basis, we have provided an isomorphism of $\pi_1(\mathbf{P}_{\mathbf{C}}^1 - B, 2)$ with the group

$$T_{\infty} := \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma = 1 \rangle.$$

For each finite group G generated by two elements a, b , passing from Greek to italic letters we obtain a tautological surjection

$$\pi : T_{\infty} \rightarrow G.$$

That is, we set $\pi(\alpha) = a, \pi(\beta) = b$ and we define $\pi(\gamma) := c$. (then $abc = 1$).

Definition 2.1. We let the triangle curve C associated to π be the *Galois covering* $f : C \rightarrow \mathbf{P}_{\mathbf{C}}^1$, branched on B and with group G determined by the chosen isomorphism $\pi_1(\mathbf{P}_{\mathbf{C}}^1 - B, 2) \cong T_{\infty}$ and by the group epimorphism π .

Remark 2.2. Under the above notation, we set m, n, p the periods of the respective elements a, b, c of G (these are the branching multiplicities of the covering f). Composing f with a projectivity we can assume that $m \leq n \leq p$.

Notice that the Fermat curve $C := \{(x_0, x_1, x_2) \in \mathbf{P}_{\mathbf{C}}^2 \mid x_0^n + x_1^n + x_2^n = 0\}$ is in two ways a triangle curve, since we can take the quotient of C by the group $G := (\mathbf{Z}/n)^2$ of diagonal projectivities with entries n -th roots of unity, but also by the full group $A = \text{Aut}(C)$ of automorphisms, which is a semidirect product of the normal subgroup G by the symmetric group exchanging the three coordinates. For G the three branching multiplicities are all equal to n , whereas for A they are equal to $(2, 3, 2n)$.

Another interesting example is provided by the *Accola curve* (cf. [ACC1,2]), the curve Y_g birational to the affine curve of equation

$$y^2 = x^{2g+2} - 1.$$

If we take the group $G \cong \mathbf{Z}/2 \times \mathbf{Z}/(2g+2)$ which acts multiplying y by -1 , respectively x by a primitive $2g+2$ -root of 1, we realize Y_g as a triangle curve with branching multiplicities $(2, 2g+2, 2g+2)$. However, G is not the full automorphism group; in fact if we add the transformation sending x to $1/x$ and y to iy/x^{g+1} , then we get a nonsplit extension of G by $\mathbf{Z}/2$ (which is indeed the full group of automorphisms of Y_g as is well known and as also follows from the next lemma), a group which represents Y_g as a triangle curve with branching multiplicities $(2, 4, 2g+2)$.

One can get many more examples by taking unramified coverings of the above curves (associated to characteristic subgroups of the fundamental group).

The following natural question arises then: which are the curves which admit more than one realization as triangle curves?

We are not aware whether the answer is already known in the literature, but (although this is not strictly needed for our purposes) we will show in the next lemma that this situation is rather exceptional if the branching multiplicities are all distinct:

LEMMA 2.3. *Let $f : C \rightarrow \mathbf{P}_{\mathbf{C}}^1 = C/G$ be a triangle covering where the branching multiplicities m, n, p are all distinct (with the assumption that $m < n < p$). The group G equals the full group A of automorphisms of C if the triple is not $(3, m_1, 3m_1)$ or $(2, m_1, 2m_1)$.*

Proof. I. By Hurwitz's formula the cardinality of G is in general given by the formula

$$|G| = 2(g-1)(1 - 1/m - 1/n - 1/p)^{-1}.$$

II. Assume that $A \neq G$ and let $F : \mathbf{P}_{\mathbf{C}}^1 = C/G \rightarrow \mathbf{P}_{\mathbf{C}}^1 = C/A$ be the induced map. Then $f' : C \rightarrow \mathbf{P}_{\mathbf{C}}^1 = C/A$ is again a triangle covering, otherwise the number of branch points would be ≥ 4 and we would have a nontrivial family of such Galois covers with group A (the cross ratios of the branch points would provide locally nonconstant holomorphic functions on the corresponding subspace of the moduli space). Whence, also a nontrivial family of G -covers, a contradiction.

III. Observe that, given two points y, z of C , $f'(y) = f'(z)$ if and only if $z \in Ay$ and then the branching indices of y, z for f' are the same. On the other hand, the branching index of y for f' is the product of the branching index of y for f times the one of $f(y)$ for F .

IV. We claim now that the three branch points of f cannot have distinct images through F : otherwise the branching multiplicities $m' \leq n' \leq p'$ for f' would be not less than the respective multiplicities for f , and by the analogue of formula I for $|A|$ we would obtain $|A| \leq |G|$, a contradiction.

V. Note that if the branching multiplicities m, n, p are all distinct, then G is equal to its normalizer in A , because if $\phi \in A$, $G = \phi G \phi^{-1}$, then ϕ induces an automorphism of $\mathbf{P}_{\mathbf{C}}^1$, fixing B , and moreover such that it sends each branch point to a branch point of the same order. Since the three orders are distinct, this automorphism must be the identity on $\mathbf{P}_{\mathbf{C}}^1$, whence $\phi \in G$.

VI. Let x_1, x_2, x_3 be the branch points of f of respective multiplicities m_1, m_2, m_3 (that is, we consider again the three integers m, n, p , but allow another ordering). Suppose now that $F(x_1) = F(x_2) \neq F(x_3)$: we may clearly assume $m_1 < m_2$. Thus the branching multiplicities for f' are n_0, n_2, n_3 , where n_2, n_3 are the respective multiplicities of $F(x_2) \neq F(x_3)$. Thus n_2 is a common multiple of m_1, m_2 , $n_2 = \nu_1 m_1 = \nu_2 m_2$, n_0 is greater or equal to 2, $n_3 = m_3 \nu_3$, whence $m_2 \leq n_2, n_2 \geq 2m_1$.

We obtain

$$\begin{aligned} |A|/|G| &\leq (1 - 1/m_3 - 3/n_2)(1 - 1/2 - 1/m_3 - 1/n_2)^{-1} \\ &= 2 \frac{n_2 m_3 - n_2 - 3m_3}{n_2 m_3 - 2n_2 - 2m_3} = 2 + \frac{2n_2 - 2m_3}{n_2(m_3 - 2) - 2m_3}. \end{aligned}$$

Thus $|A|/|G| \leq 2$ if $m_3 \geq 5$, $|A|/|G| \leq 3$ if $m_3 = 4$.

VII. However, if $|A|/|G| \leq 2$ then G is normal in A ; thus, by our assumption and by V, $G = A$. Now, we need only to take care of the possibility $|A|/|G| \geq 3$.

VIII. Under the hypothesis of VI, we get $d := \deg(F) = |A|/|G| = k_0 n_0$. Since $n_0 \geq 2$, if $d = 3$ we get $n_0 = 3$. We also have

$$(i) \quad d = \nu_3(1 + k_3 m_3), \quad (ii) \quad d = \nu_1 + \nu_2 + k_2 n_2 \quad (k_2, k_3 \geq 0).$$

Now, if $m_3 = 4$ we get $d = 3 = n_0 = \nu_3$; but then F cannot have further ramification points, contradicting $\nu_1 \geq 2$.

If instead $m_3 = 3$ the above inequality yields $d = |A|/|G| \leq 3 + n_2/(n_2 - 6)$. But $n_2 = \nu_1 m_1 \geq 8$ (this is obvious if $m_1 \geq 4$, else $m_1 = 2$ but then $m_2 \geq 8$).

Next, $n_2 \geq 8$ implies $d \leq 7$. From (ii) and $n_2 \geq 8$ follows then $k_2 = 0$, whence $d = \nu_1 + \nu_2$.

Then the previous inequality yields

$$d \leq 2 \frac{2n_2 - 3d}{n_2 - 6};$$

i.e., $d(n_2 - 6) \leq 4n_2 - 6d$, whence $d \leq 4$.

If $d = 3$ we get the same contradiction from $d = n_0 = \nu_3$. Else, $d = 4$ and equality holds, whence $\nu_3 = 1, n_0 = 2$, and $\nu_1 = 3, \nu_2 = 1$. In this case we get $d = |A|/|G| = 4$, $m_3 = n_3 = 3$, $n_0 = 2$, $n_2 = 3m_1 = m_2 \geq 8$. Then the branching indices are

$$(3, m_1, 3m_1) \text{ for } G \text{ and } (2, 3, 3m_1) \text{ for } A.$$

Assume finally that $m_3 = 2$. If $n_3 = 2$, then $n_0 \geq 3$, thus the usual inequality gives

$$d \leq \left(1/2 - \frac{\nu_1 + \nu_2}{n_2}\right) \frac{6n_2}{n_2 - 6} = 3 \frac{n_2 - 2(\nu_1 + \nu_2)}{n_2 - 6} \leq 3.$$

But again $d = 3$ implies $n_0 = 3$, and $\nu_3 = 3$ yields the usual contradiction. Thus $\nu_3 = 1 = \nu_2$ and then $m_3 = n_3 = 2$, $\nu_1 = 2$, $n_0 = 3$, $n_2 = 2m_1 = m_2$ and we have therefore the case $d = 3$ and branching indices

$$(2, m_1, 2m_1) \text{ for } G \text{ and } (2, 3, 2m_1) \text{ for } A.$$

IX. There remains the case where $F(x_1) = F(x_2) = F(x_3)$. Then the branching order of f' in $F(x_i)$ is a common multiple ν of m, n, p , and we get the estimate

$$\begin{aligned} |A|/|G| &\leq (1 - 1/m - 1/n - 1/p)(1 - 1/2 - 1/3 - 1/\nu)^{-1} \\ &= (1 - 1/m - 1/n - 1/p) \frac{6\nu}{\nu - 6}. \end{aligned}$$

Now, if $p < \nu$, then $\nu \geq 2p, \nu \geq 3n, \nu \geq 4m$; thus $|A|/|G| \leq \frac{6(\nu-9)}{\nu-6} < 6$. However, looking at the inverse image of $F(x_i)$ under F , we obtain

$$(*) |A|/|G| \geq \nu/m + \nu/n + \nu/p,$$

whence $|A|/|G| \geq 9$, a contradiction.

Thus $p = \nu$, and then from this equality follow also the further inequalities $\nu \geq 2n, \nu \geq 3m$. We get $|A|/|G| \leq 6$ from the first inequality, and from (*) we derive that $|A|/|G| \geq 6$.

The only possibility is: $|A|/|G| = 6, p = 3m, p = 2n$.

In this case therefore the three local monodromies of F are given by three permutations in six elements, with cycle decompositions of respective types $(1, 2, 3), (n)^k, (n')^{k'}$, where $nk = n'k' = 6$. The Hurwitz formula for F ($\deg F = 6$) shows that the respective types must then be $(1, 2, 3), (2, 2, 2), (3, 3)$.

We will conclude then, deriving a contradiction by virtue of the following Lemma.

LEMMA 2.4. *Let τ, σ be permutations in six elements of respective types $(2, 2, 2), (3, 3)$. If their product $\sigma\tau$ has a fixed point, then it has a cycle decomposition of type $(1, 4, 1)$.*

Proof. We will prove the lemma by suitably labelling the six elements. Assume that 2 is the element fixed by $\sigma\tau$: then we label $1 := \tau(2)$. Since $\sigma(1) = 2$, we also label $3 := \sigma(2)$. Further we label $4 := \tau(3), 5 := \sigma(4)$, so that τ is a product of the three transpositions $(1, 2), (3, 4), (5, 6)$, while σ is the product of the two three-cycles $(1, 2, 3), (4, 5, 6)$.

An easy calculation shows that $\sigma\tau$ is the four-cycle $(1, 3, 5, 4)$. □

Remark 2.5. The above proof of lemma 2.3 provides explicitly a realization of $T := T(3, m_1, 3m_1)$ as a (nonnormal) index 4 subgroup of $T' := T(2, 3, 3m_1)$, resp. of $T := T(2, m_1, 2m_1)$ as a (nonnormal) index 3 subgroup of $T' := T(2, 3, 2m_1)$.

For every finite index normal subgroup K of T' , with $K \subset T$, we get $G := (T/K) \subset A := (T'/K)$ and corresponding triangle curves.

Thus the exceptions can be characterized.

We come now to our particular triangle curves. Let r, m be positive integers $r \geq 3, m \geq 4$ and set

$$p := r^m - 1, n := (r - 1)m .$$

Notice that the three integers $m < n < p$ are distinct.

Let G be the following semidirect product of \mathbf{Z}/p by \mathbf{Z}/m :

$$G := \langle a, c \mid a^m = 1, c^p = 1, aca^{-1} = c^r \rangle$$

The definition is well posed (i.e., the semidirect product of \mathbf{Z}/p by \mathbf{Z} given by $G'' := \langle a, c \mid c^p = 1, aca^{-1} = c^r \rangle$ descends to a semidirect product of \mathbf{Z}/p by \mathbf{Z}/m) since

$$a^i ca^{-i} = c^{r^i}$$

and, by very definition of $m, r^m \equiv 1 \pmod{p}$.

LEMMA 2.6. *Define $b \in G$ by the equation $abc = 1$. Then the period of b is exactly n .*

Proof. The elements of G can be uniquely written as $\{c^i a^j \mid 0 \leq i \leq p - 1, 0 \leq j \leq m - 1\}$. The period of b equals the one of its inverse, namely, ca . Now,

$$(ca)^i = c (aca^{-1}) (a^2 ca^{-2}) \cdots (a^{i-1} ca^{-(i-1)}) a^i = c^{1+r+\dots+r^{i-1}} a^i = c^{\frac{r^i-1}{r-1}} a^i.$$

Whence, $b^i = 1$ if and only if $m \mid i$ and $p \mid \frac{r^i-1}{r-1}$.

Let therefore mk be the period of b ; then k is the smallest integer with $\frac{r^{mk}-1}{r-1} \equiv 0 \pmod{r^m-1}$. Since $\frac{r^{mk}-1}{r-1} = \frac{r^{mk}-1}{r^m-1} \frac{r^m-1}{r-1}$, all we want is $\frac{r^{mk}-1}{r^m-1} \equiv 0 \pmod{r-1}$; however, $\frac{r^{mk}-1}{r^m-1} \equiv \sum_{j=0}^{k-1} r^{mj} \equiv k \pmod{r-1}$. Therefore $k = r - 1$ and the period of b equals n . \square

PROPOSITION 2.7. *The triangle curve C associated to π is not antiholomorphically equivalent to itself (i.e., it is not isomorphic to its conjugate).*

Proof. We shall derive a contradiction assuming the existence of an antiholomorphic automorphism σ of C .

Step I. $G = A$, where A is the group of holomorphic automorphisms of C , $A := \text{Bihol}(C, C)$.

Proof. This follows from the previous Lemma 2.3, since in this case we assumed $m \geq 4$, and since $p = r^m - 1, n = m(r - 1)$, obviously

$$p > (r - 1 + m)(r - 1)^{m-1} \geq (2 + m)9(r - 1) > 2n.$$

Step II. If σ exists, it must be a lift of complex conjugation.

Proof. In fact σ normalizes $\text{Aut}(C)$, whence it must induce an antiholomorphism of $\mathbf{P}^1_{\mathbf{C}}$ which is the identity on B , and therefore must be complex conjugation.

Step III. Complex conjugation does not lift.

Proof. This is purely an argument about covering spaces: complex conjugation acts on $\pi_1(\mathbf{P}^1_{\mathbf{C}} - B, 2) \cong T_{\infty}$, as is immediate with our choice of basis, by the automorphism τ sending α, γ to their respective inverses.

Thus, complex conjugation lifts if and only if τ preserves the normal subgroup $K := \ker(\pi)$. In turn, this means that there is an automorphism $\rho : G \rightarrow G$ with

$$\rho(a) = a^{-1}, \rho(c) = c^{-1}.$$

Recall now the relation $aca^{-1} = c^r$: Applying ρ , we would get $a^{-1}c^{-1}a = c^{-r}$, or, equivalently,

$$a^{-1}ca = c^r.$$

But then we would get $c = a^{-1}(aca^{-1})a = c^{n^2}$, which holds only if

$$r^2 \equiv 1 \pmod{p}.$$

This is the desired contradiction, because $r^2 - 1 < r^m - 1 = p$. □

Remark 2.8. What the above proposition says can be rephrased in the following terms (cf. [Cat6, pp. 29–31]).

Denote by Π_g the fundamental group of a compact curve of genus g .

The epimorphism $\pi : T_\infty \rightarrow G$ factors through an epimorphism π' of the triangle group $T(m, n, p) := \langle a, b, c \mid a^m = 1, b^n = 1, c^p = 1, abc = 1 \rangle$ onto G , and once an isomorphism $\ker \pi' \cong \Pi_g$ is fixed, the pair $(C, \ker \pi' \cong \Pi_g)$ yields a point in the Teichmüller space \mathcal{T}_g . This point is the only fixpoint for the action of G on \mathcal{T}_g induced by the natural homomorphism $G \rightarrow \text{Out}(\ker \pi' \cong \Pi_g)$. The pair $(\bar{C}, \ker \pi' \circ \tau')$ corresponds to the epimorphism $\pi' \circ \tau' : T_{m,n,p} \rightarrow G$. What we have shown is that \bar{C}, C yield different points in the moduli space $\mathcal{M}_g = \mathcal{T}_g / \text{Out}(\Pi_g)$. Thus, \bar{C} and C correspond to two topological actions of G which are conjugate by an orientation reversing homeomorphism, but not by an orientation-preserving one.

3. Theorems, and corrigenda

We begin this section by recalling some results of ([Cat6]), and we draw some consequences for real surfaces. For one of the theorems of ([Cat6]) we shall need to make a small correction which, although it amounts only to remembering that $(-1)^2 = 1$, will be completely crucial to our argument.

Recall [Cat6, 3.1–3.13]:

Definition 3.1. A projective surface S is said to be *isogenous* to a (higher) product if it admits a finite unramified covering by a product of curves of genus ≥ 2 . In this case, there exist Galois realizations $S = (C_1 \times C_2)/G$, and each such Galois realization dominates a uniquely determined minimal one. Note that S is said to be of nonmixed type if G acts via a product action of two respective actions on C_1, C_2 . Otherwise S is of mixed type and it has a canonical unramified double cover which is of unmixed type and $C_1 \cong C_2$ (see 3.16 of [Cat6] for more details on the realization of surfaces of mixed type). In the latter case the canonical double cover corresponds to a subgroup $G^0 \subset G$ of index 2.

PROPOSITION 3.2. *Let S, S' be surfaces isogenous to a higher product, and let $\sigma : S \rightarrow S'$ be an antiholomorphic isomorphism. Let moreover $S = (C_1 \times C_2)/G, S' = (C'_1 \times C'_2)/G'$ be the respective minimal Galois realizations. Then, up to a possible exchange of C'_1 with C'_2 , there exist antiholomorphic isomorphisms $\tilde{\sigma}_i, i = 1, 2$ such that $\tilde{\sigma} := \tilde{\sigma}_1 \times \tilde{\sigma}_2$ normalizes the action of G ; in particular $\tilde{\sigma}_i$ normalizes the action of G^0 on C_i .*

Proof. Let us view σ as yielding a complex isomorphism $\sigma : S \rightarrow \bar{S}'$. Consider the exact sequence corresponding to the minimal Galois realization $S = (C_1 \times C_2)/G$,

$$1 \rightarrow H := \Pi_{g_1} \times \Pi_{g_2} \rightarrow \pi_1(S) \rightarrow G \rightarrow 1.$$

Applying σ_* to it, we infer by Theorem 3.4 of ([Cat6]) that there is an exact sequence associated to a Galois realization of \bar{S}' . Since σ is an isomorphism, we get a minimal one, which is however unique. Whence, we get an isomorphism $\tilde{\sigma} : (C_1 \times C_2) \rightarrow (\bar{C}'_1 \times \bar{C}'_2)$, which is of product type by the rigidity lemma (e.g., Lemma 3.8 of [Cat6]). Moreover, this isomorphism must normalize the action of $G \cong G'$, which is exactly what we claim. \square

The following is the correction of Theorems 4.13, 4.14 of [Cat6]:

THEOREM 3.3. *Let S be a surface isogenous to a product, i.e., a quotient $S = (C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G . Then any surface S' with the same fundamental group as S and the same Euler number of S is diffeomorphic to S . The corresponding moduli space $M_S^{\text{top}} = M_S^{\text{diff}}$ is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

Proof. The only modification in the proof given in [Cat6] occurs on the last lines of page 30.

As in the previous proposition, an isomorphism between the fundamental groups of S , resp. S' yields a differentiable action of G on the product of curves $(C'_1 \times C'_2)$ yielding the minimal Galois realization of S' . In fact the above isomorphism of fundamental groups, by unicity of the Galois realization, yields an isomorphism of H with H' . This isomorphism yields an orientation-preserving diffeomorphism $(C_1 \times C_2) \rightarrow (C'_1 \times C'_2)$ which is of product type.

Now, the diffeomorphisms between the respective factors are either both orientation-preserving (this was the case we were considering in the argument in loc. cit.), or both orientation-reversing.

In the latter case, the topological action of G^0 on the product of the conjugate curves $(\bar{C}'_1 \times \bar{C}'_2)$, which is of product type, yields actions of G^0 on the respective factors \bar{C}'_1, \bar{C}'_2 which are of the same oriented topological type as the respective actions on C_1, C_2 (again here we might have to exchange the roles of C'_1, C'_2 if the genera g_1, g_2 are equal). Therefore we conclude in this case that the conjugate of the surface S' belongs to the irreducible subset of the moduli space containing S . \square

We are now going to explain the construction of our examples:

Let G be the semidirect product group constructed in Section 2, and let C_2 be the corresponding triangle curve.

By the formula of Riemann Hurwitz the genus of C_2 equals

$$g_2 = 1 + \frac{1}{2}[(m - 1)(r^m - 2) - 1 - \frac{r^m - 1}{r - 1}].$$

Let moreover g'_1 be any number greater than or equal to 2, and consider the canonical epimorphism ψ of $\Pi_{g'_1}$ onto a free group of rank g'_1 , such that in terms of the standard bases $a_1, b_1, \dots, a_{g'_1}, b_{g'_1}$, respectively $\gamma_1, \dots, \gamma_{g'_1}$, we have $\psi(a_i) = \psi(b_i) = \gamma_i$. Then compose ψ with any epimorphism of the free group onto G , e.g. it suffices to compose with any μ such that $\mu(\gamma_1) = a$, $\mu(\gamma_2) = b$ (and $\mu(\gamma_j)$ can be chosen to be whatever we want for $j \geq 3$).

For any point C'_1 in the Teichmüller space we obtain a canonical covering associated to the kernel of the epimorphism $\mu \circ \psi : \Pi_{g'_1} \rightarrow G$; call it C_1 .

Definition 3.4. Let S be the surface $S := (C_1 \times C_2)/G$ (S is smooth because G acts freely on the first factor).

THEOREM 3.5. *For any two choices $C'_1(I), C'_1(II)$ of C'_1 in the Teichmüller space there are surfaces $S(I), S(II)$ such that $S(I)$ is never isomorphic to $\bar{S}(II)$. When C'_1 is varied, there is a connected component of the moduli space, which has only one other connected component, given by the conjugate of the previous one.*

Proof. By Theorem 3.3 it suffices to show the first statement, because we know already, by the rigidity of the second triangle curve, that we get a connected component of the moduli space varying C'_1 . By Proposition 3.2, it follows the fact that if $S(I)$ were isomorphic to $\bar{S}(II)$, then there would be an antiholomorphism of C_2 to itself. This is however excluded by Proposition 2.3. □

We come now to the last result:

THEOREM 3.6. *Let S be a surface in one of the families constructed above. Assume moreover that X is another complex surface such that $\pi_1(X) \cong \pi_1(S)$. Then X does not admit any real structure.*

Proof. Observe that since S is a classifying space for the fundamental group of $\pi_1(S)$, then by the isotropic subspace theorem of [Cat3] the Albanese mapping of X maps onto a curve $C'(I)_2$ of the same genus as C'_2 .

Consider now the unramified covering \tilde{X} associated to the kernel of the epimorphism $\pi_1(X) \cong \pi_1(S) \rightarrow G$.

Again by the isotropic subspace theorem, there exists a holomorphic map $\tilde{X} \rightarrow C(I)_1 \times C(I)_2$, where moreover the action of G on \tilde{X} induces actions of G on both factors which either have the same oriented topological types as the actions of G on C_1 , resp. C_2 , or have both the oriented topological types of the actions on the respective conjugate curves.

By the rigidity of the triangle curve C_2 , in the former case $C(I)_2 \cong C_2$, in the latter $C(I)_2 \cong \bar{C}_2$.

Assume now that X has a real structure σ : then the same argument as in [C-F, §2] shows that σ induces a product antiholomorphic map $\tilde{\sigma} : C(I)_1 \times C(I)_2 \rightarrow C(I)_1 \times C(I)_2$. In particular, we get a nonconstant antiholomorphic map of C_2 to itself, contradicting Proposition 2.3 . \square

Finally, in [Cat6], motivated by some examples by Beauville ([Bea, p. 159]) we gave the following:

Definition 3.7. A *Beauville surface* is a rigid surface isogenous to a product.

However, in commenting in five lines on where the problem of the classification of such surfaces lies, I confused together the nonmixed type and the mixed type (which is more difficult to get).

Therefore, I would simply like here to comment that to obtain a Beauville surface of nonmixed type it is equivalent to give a finite group G together with two systems of generators $\{a, b\}$ and $\{a', b'\}$ which satisfy a further property, denoted by (*) in the sequel.

In fact, the choice of the two systems of generators yields two epimorphisms $\pi, \pi' : T_\infty \rightarrow G$ where we recall that $T_\infty := \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma = 1 \rangle$ is the fundamental group of the projective line minus three points.

We get corresponding curves C, C' with an action of G , and the product action of G on $C \times C'$ is free if and only if, defining c, c' by the properties $abc = a'b'c' = 1$, and letting Σ be the union of the conjugates of the cyclic subgroups generated by a, b, c respectively, and defining Σ' analogously, we have

$$(*) \quad \Sigma \cap \Sigma' = \{1_G\}.$$

In the mixed case, one requires instead that the two systems of generators be related by an automorphism ϕ of G which should satisfy the further conditions:

- i) ϕ^2 is an inner automorphism, i.e., $\phi^2 = \text{Int}_\tau$ for some $\tau \in G$
- (*): $\Sigma \cap \phi(\Sigma) = \{1_G\}$.
- There is no $g \in G$ such that $\phi(g)\tau g \in \Sigma$.

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trying in particular to construct nonreal Beauville surfaces. V. Kharlamov had
independently a similar idea, and we spent one afternoon together trying to
make it work with several examples. Later on Kharlamov and Kulikov found
the right examples ([K-K]), one of them a quotient of a Hirzebruch covering of
the plane.

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