CANONICAL PROJECTIONS OF IRREGULAR ALGEBRAIC SURFACES

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We dedicate this article to the memory of Paolo Francia.

1. Introduction

In Chapter VIII of his book "Le superficie algebriche" F. Enriques raised the question to describe concretely the canonical surfaces with $p_g = 4$ and discussed possible constructions of the regular ones of low degree.

A satisfactory answer to the existence question for degree ≤ 10 was given by Ciliberto [1981], and for the case of regular surfaces a satisfactory structure theorem for the equations of the image surface and its singular locus was achieved by the first author [1984b].

The main purpose of this paper is to extend those results to the case of irregular surfaces.

Irregular varieties are often easy to construct via transcendental methods, as is the case for elliptic curves or Abelian varieties. But the problem of explicitly describing the equations of their projective models has always been a challenge for algebraic geometry (cf. Enriques [1949] , Mumford [1966-67]).

From an algebraic point of view, we might say that irregularity is responsible for the failure of the Cohen-Macaulay condition for the canonical ring of a variety of general type.

In this context therefore the method of Hilbert resolutions must be replaced by another method, and we show in this paper that Beilinson's theorem [1978] allows us to give a suitable generalization of the structure theorem of the first author [1984b].

The first two sections are devoted to this extension, and the situation that we consider is the following: φ is a morphism to $\mathbb{P}^3$ given by four independent sections of the canonical bundle K_S of a minimal surface of general type, and we assume that the degree of φ is at most two. The following is the main theorem, concerning the case where deg $\varphi = 1$.

Theorem 2.9. *The datum* $\varphi : S \to Y \subset \mathbb{P}^3 = \mathbb{P}(V)$ *of a good birational canonical projection determines a homomorphism*

$$(\mathcal{O}_\mathbb{P} \oplus E)^*(-5) \xrightarrow{\alpha} (\mathcal{O}_\mathbb{P} \oplus E),$$

where $E = (K^2 - q + p_g - 9)\mathcal{O}_\mathbb{P}(-2) \oplus q\Omega^1_\mathbb{P}(-1) \oplus (p_g - 4)\Omega^2_\mathbb{P}$, *such that*

(i) α *is symmetric,*
(ii) *det α is an irreducible polynomial (defining Y),*
(iii) α *satisfies the ring condition (cf. 2.7), and,*
   - *defining $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-algebras given by the module coker (α) provided with the ring structure determined by α as in 2.7(1),*
(iv) Spec $\mathcal{F}$ *is a surface with at most rational double points as singularities.*

Conversely, given α satisfying i), ii), iii) and iv), $X = \operatorname{Spec} \mathcal{F}$ is the canonical model of a minimal surface S and

$$\varphi : S \to X \to Y \subset \mathbb{P}^3 = \mathbb{P}(V)$$

is a good birational canonical projection.

We recall (cf. 2.7) that the ring condition, or rank condition, on a matrix α is the condition that the ideal sheaf generated by the top minors of the matrix α' obtained by deleting the first row of α equals the ideal sheaf generated by the minors of α of the same size.

The following sections are more in the spirit of Enriques, and we discuss explicitly the irregular canonical surfaces with $p_g = 4$ of lowest degree, for instance $K^2 = 12$ in the case of irregularity $q = 1$.

We classify completely the above surfaces with $p_g = 4, q = 1$: they can be described as a genus three non hyperelliptic fibration over an elliptic base curve.

This classification shows that the corresponding moduli space has only one irreducible component (cf. Theorem 5.10), and we dwell over the geometry of a dense open set of the moduli space, corresponding to surfaces which we name "of the main stream".

Examples with $p_g = 4, K^2 = 12, q = 3$ are the polarizations of type $(1, 1, 2)$ on an Abelian threefold: a remarkable subfamily of surfaces for which the canonical map becomes a degree two covering of a canonical surface with $K^2 = 6$ is given by the "special" surfaces which are the pull backs, under a degree two isogeny, of the theta divisor of a principal polarization.

A further example, with $p_g = 4, q = 2$ and $K^2 = 18$ is provided by certain Abelian covers with Galois group $(\mathbb{Z}/2)^2$ of a principally polarized Abelian surface.

The results presented in this paper were announced in [Catanese, 1997] and very recently A. Canonaco [2002] was able to extend the method for canonical projections to a weighted 3-dimensional projective space.

There are still many questions which this paper leaves open:

   - A precise description of the structure theorem in the general degree two case, without the assumption that Y be normal
   - The extension of the structure theorem to the case of higher dimensional varieties (cf. however [Catanese 1985] in the "pluriregular case")
   - The complete classification of canonical surfaces of low degree (this is still open also in the regular case, as soon as $K^2 \geq 8$, see [Catanese1984b] for the case $K^2 = 7$)
   - The construction of new examples without transcendental methods but via computer algebra.

  • Decide when is the corresponding moduli space unirational and, in this case, give an explicit rationally parametrized family

NOTATION

$S: =$ the minimal model of a surface of general type
$\mathcal{R} = \mathcal{R}(S) : =$ the canonical ring of S
$X = \operatorname{Proj}(\mathcal{R})$ is the canonical model of S
$\pi : S \to X$ is the canonical morphism

K is a canonical divisor on X or on S (note: $\pi^*(K_X) = K_S$)
$\varphi : S \to Y \subset \mathbb{P}^3$ is a good canonical projection, i.e.,
φ is given by a base point free 4-dimensional subspace V^* of $\mathrm{H}^0(\mathcal{O}_S(K))$

(here V^* denotes the dual vector space of V, and $\mathbb{P}^3 = \mathbb{P}(V)$ is the projective space of 1-dimensional vector subspaces of V).
φ is said to be almost generic if it is good and yields a morphism which is either birational to Y or of degree 2 onto Y (in [Catanese 1984b] φ was said to be quasi-generic if moreover Y is normal in case $\deg(\varphi) = 2$).

$\varphi : S \to Y$ factors through π and a finite morphism $\psi : X \to Y$;

$\{y_0, y_1, y_2, y_3\}$: a basis of $V^* \subset \mathrm{H}^0(\mathcal{O}_S(K)) \cong \mathrm{H}^0(\mathcal{O}_X(K))$:
$V^* \oplus W^* \cong \mathrm{H}^0(\mathcal{O}_S(K)) \cong \mathrm{H}^0(\mathcal{O}_X(K))$, a fixed splitting;
$\mathcal{A} =$ is the graded polynomial ring $\mathbb{C}[y_0, y_1, y_2, y_3] = \operatorname{Sym}(V^*)$;
$M^\sim$: for a graded A-module M, denotes the associated sheaf on $\mathbb{P}^3$; therefore we shall consider

$$\mathcal{R}^\sim = \varphi_*\mathcal{O}_S = \psi_*\mathcal{O}_X = (\psi_*\omega_X)(-1).$$

$\mathcal{T}_Z$ will denote the tangent sheaf of a quasi-projective scheme Z

Given Cartier divisors D, D',

$D \equiv D'$ means that D is linearly equivalent to D', while

$D \sim D'$ means that D is algebraically equivalent to D'.

2. Determinantal presentations of canonical projections

By Beilinson's theorem [1978], for any coherent sheaf $\mathcal{F}$ on a projective space $\mathbb{P}$ there is a complex

$$\cdots \to \mathcal{C}^i(\mathcal{F}) = \bigoplus_{q-p=i} H^q(\mathcal{F}(-p)) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{P}}(p) \to \mathcal{C}^{i+1}(\mathcal{F}) \to \cdots$$

whose cohomology is concentrated in degree 0 and yields $\mathcal{F}$. For a new proof we also refer to the paper of Eisenbud, Fløystad and Schreyer [2001]. In particular, there is a spectral sequence with $E_1^{-p,q}$ equal to $H^q(\mathcal{F}(-p)) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{P}}(p)$, and with $d_1^{-p,q} : E_1^{-p,q} \to E_1^{-p+1,q}$ given by the identity tensor

$$id = (\Sigma_j y_j \otimes y_j^*) \in V^* \otimes_{\mathbb{C}} V = H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes_{\mathbb{C}} \operatorname{Hom}(\Omega^1_{\mathbb{P}}(1), \mathcal{O}_{\mathbb{P}})$$

which acts according to the tensor rule

$$(x \otimes e)(s \otimes \omega) = (xs) \otimes (\omega \neg e),$$

$\neg$ denoting contraction of a contravariant tensor with a vector .

From now on we let $\mathcal{F}(m)$ be $= \psi_* \mathcal{O}_X(m)$, with $m = 0, 2$ or 3 .

Later on, we shall simply denote $\mathcal{F}(0)$ by $\mathcal{F}$.

The Beilinson table in the particular case $m = 3$ reads out as:

0	0	0	0
$H^2(\mathcal{O}_X)$	$H^2(\mathcal{O}_X(K))$	0	0
$H^1(\mathcal{O}_X)$	$H^1(\mathcal{O}_X(K))$	0	0
$H^0(\mathcal{O}_X)$	$H^0(\mathcal{O}_X(K))$	$H^0(\mathcal{O}_X(2K))$	$H^0(\mathcal{O}_X(3K))$

since $H^3(\psi_* \mathcal{O}_X(i)) = 0$ for all i,

$$H^2(\psi_* \mathcal{O}_X(i)) = H^2(\mathcal{O}_X(iK)) = H^0(\mathcal{O}_X(-(i-1)K))^* = 0 \text{ for } i \geq 2,$$

and

$$H^1(\psi_* \mathcal{O}_X(i)) = H^1(\mathcal{O}_X(iK)) = H^1(\mathcal{O}_X(-(i-1)K))^* = 0 \text{ for } i \neq 0, 1.$$

From Beilinson's theorem we obtain with a simpler proof a stronger version of a result by Ciliberto

Theorem 2.1 (Ciliberto, Thm. 2.4,(iii) 1983)**.** *If $p_g \geq 4$ and $\mid K \mid$ has no base points, then $\mathcal{R}$ is generated in degrees ≤ 3 as an $\mathcal{A}$-module.*

Proof. The d_1-differential on the top row of the Beilinson spectral sequence for $\mathcal{F}(3) = \psi_* \mathcal{O}_X(3)$ has the form

$$H^2(\mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^3_{\mathbb{P}}(3) = (W \oplus V) \otimes_{\mathbb{C}} \Omega^3_{\mathbb{P}}(3) \to H^2(\mathcal{O}_X(K)) \otimes_{\mathbb{C}} \Omega^2_{\mathbb{P}}(2) = \Omega^2_{\mathbb{P}}(2)$$

where the first summand maps to 0, and the second maps according to the twisted Serre dual of the Euler sequence: therefore it is surjective with kernel $\mathcal{K}$ isomorphic to a direct sum $(W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}(-1)) \oplus \mathcal{O}_{\mathbb{P}}(-2) \cong [(p_g - 4)]\mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}(-2)$.

Hence not only $\mathcal{F}(3)$ is a quotient of $H^0(\mathcal{O}_X(3K)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}$, but $\mathcal{F}(3)$ has a locally free resolution by sheaves which are direct sums of sheaves $\mathcal{G}$ isomorphic to

either $\mathcal{O}_\mathbb{P}(-2)$, or $\Omega_\mathbb{P}^p(p)$. Such sheaves $\mathcal{G}$ have the property that $\mathrm{H}^j(\mathcal{G}(m)) = 0$ for $j > 0$ and $m \geq 0$.

Therefore, if we tensor this resolution by $\mathcal{O}_\mathbb{P}(m)$, $m \geq 0$, it remains exact on global sections, in particular we have that

$$\mathrm{H}^0(\mathcal{O}_X(3K)) \otimes_\mathbb{C} \mathrm{H}^0(\mathcal{O}_\mathbb{P}(m-3)) \to \mathrm{H}^0(\mathcal{O}_X(mK))$$

is surjective for $m \geq 3$.   Q.E.D.

In the case $m = 2$ the Beilinson table reads out as

0	0	0	0
$\mathrm{H}^2(\mathcal{O}_X(-K))$	$\mathrm{H}^2(\mathcal{O}_X)$	$\mathrm{H}^2(\mathcal{O}_X(K))$	0
0	$\mathrm{H}^1(\mathcal{O}_X)$	$\mathrm{H}^1(\mathcal{O}_X(K))$	0
0	$\mathrm{H}^0(\mathcal{O}_X)$	$\mathrm{H}^0(\mathcal{O}_X(K))$	$\mathrm{H}^0(\mathcal{O}_X(2K))$

and the symmetry with respect of the middle point of the second row from the bottom takes each vector space to its Serre dual and each d_1-differential to its Serre dual map.

Sublemma 2.2. *The d_1-differential from*

$$\mathrm{H}^0(\mathcal{O}_X) \otimes \Omega_\mathbb{P}^2(2) \to \mathrm{H}^0(\mathcal{O}_X(K)) \otimes \Omega_\mathbb{P}^1(1)$$

is an isomorphism onto a subbundle.

Proof. d_1 factors through the natural map of $\Omega_\mathbb{P}^2(2) \to V^* \otimes \Omega_\mathbb{P}^1(1)$ and the subbundle inclusion of $V^* \otimes \Omega_\mathbb{P}^1(1)$ inside $\mathrm{H}^0(\mathcal{O}_X(K)) \otimes \Omega_\mathbb{P}^1(1)$, hence it suffices to show that the first map is a subbundle inclusion. But this follows from the Beilinson complex for $\mathcal{O}_\mathbb{P}(2)$ which yields the exact sequence

$$(2.1) \quad 0 \to \Omega_\mathbb{P}^2(2) \to V^* \otimes \Omega_\mathbb{P}^1(1) \to \mathrm{Sym}^2(V^*) \otimes \mathcal{O}_\mathbb{P} \to \mathcal{O}_\mathbb{P}(2) \to 0$$

Q.E.D.

ψ is either birational onto Y or $2 : 1$ by assumption. In the second case one can treat separately the case where Y is a quadric surface (this leads to $K_S^2 = 4, q = 0$, cf. Enriques [1949] pages 270-271, i.e., to a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on a curve of type $(3,3)$).

Therefore we assume from now on that

Assumption 2.3. *Y is not a quadric.*

In algebraic terms, this means that $\mathrm{H}^0(\mathcal{O}_\mathbb{P}(2)) = \mathrm{Sym}^2(V^*)$ is a direct summand of $\mathrm{H}^0(\mathcal{O}_X(2K))$, so we can choose a splitting

$$(2.2) \qquad \mathrm{H}^0(\mathcal{O}_X(2K)) \cong \mathrm{Sym}^2(V^*) \oplus U^*.$$

Moreover, by (2.1), we can replace the Beilinson complex by a homotopic one, which gives a new diagram

0	0	0	0
$\mathcal{O}(-3) \oplus (U \otimes \mathcal{O}(-1))$	$W \otimes \Omega^2(2)$	0	0
0	$\mathrm{H}^1(\mathcal{O}_X) \otimes \Omega^2(2)$	$\mathrm{H}^1(\mathcal{O}_X(K)) \otimes \Omega^1(1)$	0
0	0	$W^* \otimes \Omega^1(1)$	$(U^* \otimes \mathcal{O}) \oplus \mathcal{O}(2)$

Again, here, there is a symmetry taking vector spaces and linear maps to their Serre duals.

Let us denote by E the vector bundle on $\mathbb{P}^3$ defined by

$$(2.3) \qquad E(2) = (U^* \otimes \mathcal{O}_\mathbb{P}) \oplus (\mathrm{H}^1(\mathcal{O}_X(K)) \otimes \Omega^1_\mathbb{P}(1)) \oplus (W \otimes \Omega^2_\mathbb{P}(2))$$

We have therefore concluded that $\mathcal{F} = \psi_* \mathcal{O}_X$ admits a locally free resolution of length 1 of the form

$$(2.4) \qquad 0 \to (\mathcal{O}_\mathbb{P} \oplus E)^*(-5) \to \mathcal{O}_\mathbb{P} \oplus E \to \mathcal{F} = \psi_* \mathcal{O}_X \to 0.$$

Remark that again here, for each twist $m \geq 2$, (2.4) is exact on global sections. Now, the two locally free terms of the resolution are dual to each other up to a twist, and we shall see that one can indeed achieve that the resolution itself is given by a symmetric map

$$\alpha : (\mathcal{O}_\mathbb{P} \oplus E)^*(-5) \to \mathcal{O}_\mathbb{P} \oplus E \quad (\text{that is, } \alpha = \alpha^*(-5)) .$$

For simplicity of notation we denote by α^t the map $\alpha^*(-5)$, and by F the vector bundle $\mathcal{O}_\mathbb{P} \oplus E$.

As a first step we state

Lemma 2.4. *Let $\mathcal{F}'$ be the cokernel of another homomorphism $\beta : F^*(-5) \to F$. Then every homomorphism of $\mathcal{F}$ to $\mathcal{F}'$ has a lift to a homomorphism of complexes. Furthermore, any lift of an isomorphism is an isomorphism of complexes.*

Proof. By the exact sequence

$$\mathrm{Hom}(F, F) \to \mathrm{Hom}(F, \mathcal{F}') \to \mathrm{Ext}^1(F, F^*(-5))$$

to get a lifting it suffices to have that

$$\mathrm{Ext}^1(F, F^*(-5)) = \mathrm{Ext}^1(F(2), F^*(-3)) = 0$$

This holds true since the summands for $F(2), F^*(-3)$ are either $\Omega^j(j)$'s or of rank one (and of degree 0 or 2 for $F(2)$): moreover, by Bott's formula $\mathrm{H}^1(\Omega^j(m)) = 0$, unless $j = 1$ and $m = 0$, and by Lemma 2 of Beilinson [1978], $\mathrm{Ext}^p(\Omega^j(j), \Omega^i(i)) = 0$ for $p \geq 1$. So every homomorphism of $\mathcal{F}$ to $\mathcal{F}'$ has a lift to a homomorphism $f : F \to F$. Moreover f restricted to $F^*(-5)$ factors through $g : F^*(-5) \to F^*(-5)$. This proves the first statement.

Since also every homomorphism of $\mathcal{F}'$ to $\mathcal{F}$ has a lift by the same argument, it suffices to prove the second statement for $\mathcal{F}' = \mathcal{F}$ and for the identity homomorphism of $\mathcal{F}$. One lift (f, g) is therefore the identity on the complex and this is an isomorphism. Every other lift (f_1, g_1) of the same automorphism differs by a homotopy, i.e. $f_1 = f + \alpha \circ h$ and $g_1 = g + h \circ \alpha$ for some homomorphism $h \in \mathrm{Hom}(F, F^*(-5))$.

Since $\mathrm{H}^0(\Omega^i_\mathbb{P}(m)) = 0$ for $i \geq 1, m \leq i$, and again by Beilinson's lemma, it follows that

$$\mathrm{Hom}(F, F^*(-5)) = \mathrm{Hom}(F(2), F^*(-3)) =$$
$$[\mathrm{Hom}(W, \mathrm{H}^1(\mathcal{O}_X)) \otimes \mathrm{End}(\Omega^2_\mathbb{P}(2))] \oplus$$
$$\oplus [\mathrm{Hom}(W, W^*) \otimes \mathrm{Hom}(\Omega^2_\mathbb{P}(2), \Omega^1_\mathbb{P}(1)))] \oplus$$
$$\oplus [\mathrm{Hom}(\mathrm{H}^1(\mathcal{O}_x(K), W^*) \otimes \mathrm{End}(\Omega^1_\mathbb{P}(1))]$$

On the other hand, if we look at the summands of α involving $\text{Hom}(\Omega^i_\mathbb{P}(i), \Omega^j_\mathbb{P}(j))$, with $i, j = 1$ or 2, the only non zero one is the term in $\text{Hom}(\text{H}^1(\mathcal{O}_X) \otimes \Omega^2_\mathbb{P}(2), \text{H}^1(\mathcal{O}_X(K)) \otimes \Omega^1_\mathbb{P}(1))$. Therefore the compositions $\alpha \circ h$ and $h \circ \alpha$ are nilpotent and f_1, g_1 are isomorphisms, since f, g are identity matrices. Q.E.D.

To obtain a symmetric resolution we apply $\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\quad, \omega_\mathbb{P}(-1))$ to the sequence (2.4) and get the exact sequence

(2.5) $$0 \to F^*(-5) \to F \to \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1)) \to 0$$

But $\mathcal{F}(1) = \psi_* \omega_X$, thus by relative duality for ψ, the last term is isomorphic to $\psi_* \mathcal{O}_X = \mathcal{F}$. Pick an isomorphism $\varepsilon : \mathcal{F} \to \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1))$. By Lemma 2.4 there is a lift

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F^*(-5) & \xrightarrow{\alpha} & F & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow g & & \downarrow f & & \downarrow \varepsilon & & \\
0 & \longrightarrow & F^*(-5) & \xrightarrow{\alpha^t} & F & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1)) & \longrightarrow & 0
\end{array}
$$

Applying once more $\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\quad, \omega_\mathbb{P}(-1))$ to the above diagram we get

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F^*(-5) & \xrightarrow{\alpha} & F & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1)), \omega_\mathbb{P}(-1)) & \longrightarrow & 0 \\
& & \downarrow f^t & & \downarrow g^t & & \downarrow \varepsilon' & & \\
0 & \longrightarrow & F^*(-5) & \xrightarrow{\alpha^t} & F & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1)) & \longrightarrow & 0
\end{array}
$$

and we obtain a canonical isomorphism

(2.6) $$\mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1)), \omega_\mathbb{P}(-1)) \cong \mathcal{F}$$

induced by the identity of F, since $(\alpha^t)^t = \alpha$. Under this identification $\varepsilon' = \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\varepsilon, \omega_\mathbb{P}(-1))$ is the isomorphism $\mathcal{F} \to \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \omega_\mathbb{P}(-1))$ induced by g^t. Let's assume now that φ is birational: then $\text{Hom}_{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \mathcal{F}) = \mathbb{C}$ and we have $\varepsilon' = \lambda \varepsilon$ for some $\lambda \in \mathbb{C}^*$. Moreover by (2.6)

$$
\begin{aligned}
\varepsilon &= \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\varepsilon', \omega_\mathbb{P}(-1)), \text{ since both are induced by f,} \\
&= \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\lambda \varepsilon, \omega_\mathbb{P}(-1)) = \lambda \, \mathcal{E}xt^1_{\mathcal{O}_\mathbb{P}}(\varepsilon, \omega_\mathbb{P}(-1)) \\
&= \lambda \, \varepsilon' = \lambda^2 \varepsilon,
\end{aligned}
$$

i.e. $\lambda = \pm 1$. (We will see in the end that actually $\lambda = 1$.) Thus f and λg^t cover the same isomorphism ε, and so does $(f + \lambda g^t)/2$. Furthermore $(f + \lambda g^t)/2$ is an isomorphism by Lemma 2.4. We claim that $\beta = ((f + \lambda g^t)/2) \circ \alpha$ is the desired symmetric matrix. Indeed β and α have isomorphic cokernels, hence both resolve $\mathcal{F} = \psi_* \mathcal{O}_X$ and

$$\beta^t = \alpha^t \circ ((f + \lambda g^t)/2)^t = (\alpha^t \circ f^t + \lambda \alpha^t \circ g)/2) = (g^t \circ \alpha + \lambda f \circ \alpha)/2 = \lambda \beta$$

is either symmetric or skew-symmetric depending on the value of λ. Since φ is birational $\det \beta$ gives the equation of Y. In particular $\det \beta$ is not a square. Hence β cannot be skew-symmetric, i.e. $\lambda = 1$.

Proposition 2.5. *If $\varphi \colon S \to Y$ is birational then there is a resolution*

$$0 \to (\mathcal{O}_\mathbb{P} \oplus E)^*(-5) \xrightarrow{\alpha} \mathcal{O}_\mathbb{P} \oplus E \to \mathcal{F} = \psi_* \mathcal{O}_X \to 0$$

given by a symmetric map α (that is, $\alpha = \alpha^*(-5)$) .

Proof. Take for α the matrix β as above.   Q.E.D.

Remark 2.6. *The matrix α is a block matrix with entries as indicated below:*

Summands	$\mathcal{O}(-3)$	$\oplus$ $U \otimes \mathcal{O}(-1)$	$\oplus$ $\mathrm{H}^{0,1} \otimes \Omega^2(2)$	$\oplus$ $W^* \otimes \Omega^1(1)$
$\mathcal{O}(2)$	$S_5 V^*$	$S_3 V^*$	$\mathrm{H}^0(\Lambda^2 \mathcal{T}_{\mathbb{P}})$	$\mathrm{H}^0(\mathcal{T}_{\mathbb{P}})$
$\oplus$				
$U^* \otimes \mathcal{O}$	$S_3 V^*$	$V^* \cong \Lambda^3 V$	$\Lambda^2 V$	V
$\oplus$				
$\mathrm{H}^{2,1} \otimes \Omega^1(1)$	$\mathrm{H}^0(\Lambda^2 \mathcal{T}_{\mathbb{P}})$	$\Lambda^2 V$	V	0
$\oplus$				
$W \otimes \Omega^2(2)$	$\mathrm{H}^0(\mathcal{T}_{\mathbb{P}})$	V	0	0

Note that the $h^{2,1} \times h^{0,1}$ block is actually skew-symmetric, since it is induced from wedge-product

$$\mathrm{H}^1(\mathcal{O}_X) \times \mathrm{H}^1(\mathcal{O}_X) \to \mathrm{H}^2(\mathcal{O}_X) = \mathrm{H}^0(\mathcal{O}_X(K))^*$$

composed with the projection

$$\mathrm{H}^0(\mathcal{O}_X(K))^* \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}}(1))^* = V$$

However each element of

$$V \cong \mathrm{Hom}(\Omega^2_{\mathbb{P}}(2), \Omega^1_{\mathbb{P}}(1) \cong \mathrm{Hom}((\Omega^1_{\mathbb{P}}(1))^*, (\Omega^2_{\mathbb{P}}(2))^*)$$

gives a skew-symmetric morphism of bundles. So the resulting morphism

$$\mathrm{H}^{0,1} \otimes \Omega^2_{\mathbb{P}}(2) \to \mathrm{H}^{2,1} \otimes \Omega^1_{\mathbb{P}}(1)$$

is symmetric again. For general sign patterns in Beilinson monads of symmetric or skew-symmetric sheaves see Eisenbud and Schreyer [2001].

For a morphism $\alpha : G \to F$ of vector bundles on a scheme Z we denote by $I_r(\alpha)$ the ideal sheaf of $r \times r$ minors of α, i.e. the image of $\Lambda^r G \otimes (\Lambda^r F)^* \to \mathcal{O} := \mathcal{O}_Z$ under the natural map induced by $\Lambda^r(\alpha)$. So $I_r(\alpha)$ is the $(\mathrm{rank}(F) - r)^{\mathrm{th}}$ Fitting ideal of $\mathrm{coker}(\alpha)$.

Theorem 2.7. *[Catanese [1984b], de Jong and van Straten [1990]] Let $\alpha = (\alpha_1, \alpha') : G \to \mathcal{O} \oplus E$ be a morphism of vector bundles with $r = \mathrm{rank}\, E = \mathrm{rank}\, G - 1$. Suppose $\det(\alpha)$ is a non zero-divisor and $((\det(\alpha)) = I_{r+1}(\alpha))$ $\mathrm{depth}(I_r(\alpha'), \mathcal{O}) \geq 2$. Let $Y \subset Z$ denote the subscheme defined by $\det(\alpha)$. Then the following are equivalent:*

(1) $\mathcal{F} = \mathrm{coker}(\alpha)$ *carries the structure of a sheaf of commutative $\mathcal{O}_Y$-algebras with $1 \in \mathcal{F}$ given by the image of $1 \in \Gamma(Z, \mathcal{O}) \subset \Gamma(Z, \mathcal{O} \oplus E)$,*

(R.C.) $I_r(\alpha) = I_r(\alpha')$.

Proof. Since $\det(\alpha)$ is a non-zero divisor,

$$0 \to G \xrightarrow{\alpha} F \to \mathcal{F} \to 0$$

with $F = \mathcal{O} \oplus E$ is exact. As an $\mathcal{O}_Y$-module $\mathcal{F}$ has an infinite periodic resolution

$$\ldots \to \mathcal{B}^2 \otimes F_Y \xrightarrow{\beta_Y} \mathcal{B} \otimes G_Y \xrightarrow{\alpha_Y} \mathcal{B} \otimes F_Y \xrightarrow{\beta_Y} G_Y \xrightarrow{\alpha_Y} F_Y \to \mathcal{F} \to 0$$

where $\mathcal{B} = \Lambda^{r+1}G \otimes (\Lambda^{r+1}F)^*$, $-_Y = - \otimes \mathcal{O}_Y$ denotes restriction to Y, and the map

$$\beta : \Lambda^{r+1}G \otimes \Lambda^{r+1}F^* \otimes F \to G$$

is induced by $\Lambda^r(\alpha)$, i.e. β is given by the matrix of cofactors of α. Exactness follows, since $\alpha \cdot \beta = \det(\alpha)\mathrm{id}_F$ and $\beta \cdot \alpha = \det(\alpha)\mathrm{id}_G$ and $\det(\alpha)$ is a non-zerodivisor, by Eisenbud [1980].

The above resolution is obtained by truncating the infinite exact periodic complex

$$\ldots \to \mathcal{B}^2 \otimes F_Y \xrightarrow{\beta_Y} \mathcal{B} \otimes G_Y \xrightarrow{\alpha_Y} \mathcal{B} \otimes F_Y \xrightarrow{\beta_Y} G_Y \xrightarrow{\alpha_Y} F_Y \xrightarrow{\beta_Y} \mathcal{B}^{-1} \otimes G_Y \xrightarrow{\alpha_Y} \mathcal{B}^{-1} \otimes F_Y \to \ldots$$

Similarly we have an exact infinite periodic complex

$$\ldots \longrightarrow \mathcal{B} \otimes F_Y^* \xrightarrow{\alpha_Y^t} \mathcal{B} \otimes G_Y^* \xrightarrow{\beta_Y^t} F_Y^* \xrightarrow{\alpha_Y^t} G_Y^* \xrightarrow{\beta_Y^t} \mathcal{B}^{-1} F_Y^* \longrightarrow \ldots$$

It follows then that $\mathcal{C} := \mathcal{H}om_Y(\mathcal{F}, \mathcal{O}_Y)$ satisfies $\mathcal{F} = \mathcal{H}om_Y(\mathcal{C}, \mathcal{O}_Y)$.

Recall that $F = \mathcal{O} \oplus E$ whence $\mathcal{O}_Y \subset \mathcal{F}$ by Cramer's rule. By the assumption $\mathrm{depth}(I_r(\alpha'), \mathcal{O}) \geq 2$ the quotient $\mathcal{F}/\mathcal{O}_Y$, which is annihilated by $I_r(\alpha')$, satisfies $\mathcal{H}om_Y(\mathcal{F}/\mathcal{O}_Y, \mathcal{O}_Y) = 0$. Therefore

$$
\begin{aligned}
\mathcal{O}_Y &= \mathcal{H}om_Y(\mathcal{O}_Y, \mathcal{O}_Y) \supset \mathcal{C} = \mathcal{H}om_Y(\mathcal{F}, \mathcal{O}_Y) \\
&\cong \ker(F_Y^* \to G_Y^*) \cong \mathrm{im}(\mathcal{B} \otimes G_Y^* \to F_Y^*),
\end{aligned}
$$

so $\mathcal{C} \subset \mathcal{O}_Y$ is the ideal sheaf

$$\mathrm{im}(\mathcal{B} \otimes G_Y^* \to \mathcal{O}_Y) = I_r(\alpha')/(\det(\alpha)).$$

Suppose $\mathcal{F}$ is a ring. Then $\mathcal{C} \subset \mathcal{O}_Y$ is called the conductor of $\mathcal{O}_Y \subset \mathcal{F}$ and $\mathcal{F} \subset \mathcal{H}om_Y(\mathcal{C}, \mathcal{C})$, i.e. the image of $\mathcal{C} \times \mathcal{F}$ in $\mathcal{O}_Y$ is contained in $\mathcal{C} \subset \mathcal{O}_Y$: in fact for each (c, m) in $\mathcal{C} \times \mathcal{F}$ the image $c(m) \in \mathcal{O}_Y$ carries $\mathcal{F}$ into $\mathcal{O}_Y$ since $c(m)\mathcal{F} = c(m\mathcal{F}) \subset \mathcal{O}_Y$. Thus

$$\mathcal{F} \subset \mathcal{H}om_Y(\mathcal{C}, \mathcal{C}) \subset \mathcal{H}om_Y(\mathcal{C}, \mathcal{O}_Y) = \mathcal{F}.$$

In particular $\mathcal{H}om_Y(\mathcal{C}, \mathcal{C}) = \mathcal{H}om_Y(\mathcal{C}, \mathcal{O}_Y)$. Now the last equality means that every $\varphi \in \mathcal{H}om_Y(\mathcal{C}, \mathcal{O}_Y) = \mathcal{F} = \mathrm{coker}(G_Y \to F_Y)$ has image in $\mathcal{C} = I_r(\alpha')/(\det(\alpha))$. That is the pairing induced by β_Y

$$F_Y \times (\mathcal{B} \otimes G_Y^*) \to \mathcal{O}_Y$$

has image in $I_r(\alpha')\mathcal{O}_Y$. Since β is the matrix of cofactors of α, $\mathcal{H}om_Y(\mathcal{C}, \mathcal{C}) = \mathcal{H}om_Y(\mathcal{C}, \mathcal{O}_Y)$ is equivalent to $I_r(\alpha) = I_r(\alpha')$.

Conversely, if (R.C.) holds, we have $\mathcal{F} = \mathcal{H}om_Y(\mathcal{C}, \mathcal{O}_Y) = \mathcal{H}om_Y(\mathcal{C}, \mathcal{C})$ is an $\mathcal{O}_Y$-algebra under composition. The structure is commutative, since $\mathcal{C}$ as $\mathcal{O}_Y$-module is invertible on $Y - V(I_r(\alpha'))$ and $\mathrm{depth}(I_r(\alpha'), \mathcal{F}) \geq 1$ by assumption.

<div align="right">Q.E.D.</div>

Remark 2.8. *We call the condition* (R.C.) $I_r(\alpha) = I_r(\alpha')$ *the Ring Condition (or Rank Condition). For a symmetric matrix*

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12}^t & \alpha'' \end{pmatrix} : (\mathcal{O} \oplus E)^*(-5) \to \mathcal{O} \oplus E$$

the ring condition implies the further rank condition

$$I_{r-1}(\alpha') = I_{r-1}(\alpha'')$$

for lower minors of $(\alpha')^t = (\alpha_{12}, \alpha'') : E^*(-5) \rightarrow \mathcal{O} \oplus E$ in case $depth(I_{r-1}(\alpha''), \mathcal{O}) = 3$ has the expected maximal value.

Cf. Prop. 4.1 of Mond and Pellikaan [1987], and Prop. 5.8 and 5.10 of Catanese [1984b].

Theorem 2.9. *The datum $\varphi : S \rightarrow Y \subset \mathbb{P}^3 = \mathbb{P}(V)$ of a good birational canonical projection determines a morphism*

$$(\mathcal{O}_\mathbb{P} \oplus E)^*(-5) \xrightarrow{\alpha} (\mathcal{O}_\mathbb{P} \oplus E),$$

where $E = (K^2 - q + p_g - 9)\mathcal{O}_\mathbb{P}(-2) \oplus q\Omega^1_\mathbb{P}(-1) \oplus (p_g - 4)\Omega^2_\mathbb{P}$, such that

  (i) *α is symmetric,*
  (ii) *$\det \alpha$ is an irreducible polynomial (defining Y),*
 (iii) *α satisfies the ring condition, and*
   - *defining $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-algebras given by the module coker (α) provided with the ring structure determined by α as in 2.7(1),*
  (iv) Spec *$\mathcal{F}$ is a surface with at most rational double points as singularities.*

Conversely, given α satisfying i), ii), iii) and iv), $X = \operatorname{Spec} \mathcal{F}$ is the canonical model of a minimal surface S and

$$\varphi : S \rightarrow X \rightarrow Y \subset \mathbb{P}^3 = \mathbb{P}(V)$$

is a good birational canonical projection.

Proof. The first statement follows by combining 2.7 and 2.5. Notice that, since $\det \alpha$ is irreducible (in one direction, this is a consequence of the birationality of φ, in the other direction, it holds by assumption), the ring condition $I_r(\alpha) = I_r(\alpha')$ implies $depth(I_r(\alpha'), \mathcal{O}_\mathbb{P}) \geq 2$, since otherwise all $r \times r$ minors of α would have a common irreducible factor, whose square would divide $\det(\alpha)$. So the assumptions of 2.7 are satisfied.

For the converse, we note that duality applied to

$$\psi : X = \operatorname{Spec} \mathcal{F} \rightarrow Y \subset \mathbb{P}^3$$

gives

$$\mathcal{F}(1) = \psi_* \omega_X,$$

since α is symmetric. So $\psi^* \mathcal{O}_\mathbb{P}(1) \cong \omega_X$, and, because X has only rational double points as singularities, $\varphi^* \mathcal{O}_\mathbb{P}(1) \cong \omega_S$ holds on the desingularization S of X. So $\varphi : S \rightarrow Y \subset \mathbb{P}^3$ is a quasi generic birational canonical projection.

$$\text{Q.E.D.}$$

Remark 2.10. *Given a symmetric matrix $\alpha : (\mathcal{O} \oplus E)^*(-5) \rightarrow \mathcal{O} \oplus E$ as in the theorem, we denote by $\alpha' : (\mathcal{O} \oplus E)^*(-5) \rightarrow E$ and $\alpha'' : E^*(-5) \rightarrow E$ the distinguished submatrices. Then $\det(\alpha'')$ defines the adjoint surface F of degree $K^2 - 5$. F intersects Y precisely in the non-normal locus Γ of Y, which is defined by $I_r(\alpha')$. F is singular at the points of the subscheme T defined by $I_{r-1}(\alpha'')$, Γ has embedding dimension 3 at T, and the points of T are at least triple for Y.*

Typically the points of T correspond to triple points of Y, and F has ordinary quadratic singularities in T. In terms of the invariants $d = K^2, q$ and p_g of S we have

$$\deg \Gamma = 1/2 \, d^2 - 5/2 \, d + 1$$

and

$$\deg T = 1/6 \, d^3 - 5/2 \, d^2 + 37/3 \, d - 4(1 - q + p_g).$$

Proposition 2.11. *Suppose $\varphi : S \to Y \subset \mathbb{P}^3 = \mathbb{P}(V)$ is a good canonical projection with Y not a quadric. If the Albanese image of S is a curve then*

$$K^2 \geq 4q + 4 + 2(p_g - 4) = 2p_g + 4q - 4.$$

If equality holds then the map is not birational.

Proof. Since $Y \subset \mathbb{P}^3$ is a surface we have the presentation 2.4 of $\phi_* \mathcal{O}_S = \psi_* \mathcal{O}_X$ given by a matrix α with entries as indicated in Remark 2.6 (with α perhaps not symmetric). If the Albanese dimension is one then

$$\mathrm{H}^0(\Omega_S^1) \times \mathrm{H}^0(\Omega_S^1) \to \mathrm{H}^0(\Omega_S^2)$$

is the zero map, and so is $\mathrm{H}^1(\mathcal{O}_S) \times \mathrm{H}^1(\mathcal{O}_S) \to \mathrm{H}^2(\mathcal{O}_S)$ by Hodge symmetry. Thus we have a large block of zeroes. On the other hand the determinant of α equals the equation of Y to the power $\deg(\varphi)$, in particular $\det \alpha \neq 0$. Thus the $(q + p_g - 4) \times (q + p_g - 4)$ block cannot be too big. More precisely,

$$1 + \dim U \geq 3(q + p_g - 4).$$

Since $1 + \dim U = 1 + K^2 + 1 - q + p_g - \dim S_2 V = K^2 - q + p_g - 8$ the desired inequality follows. Moreover in case of equality we have

$$\det \alpha = \lambda[\det(\mathcal{O}(-3) \oplus (U \otimes \mathcal{O}(-1)) \to (\mathrm{H}^{2,1} \otimes \Omega^1(1)) \oplus (W \otimes \Omega^2(2))]^2$$

for some scalar $\lambda \in \mathbb{C}$. Thus φ cannot be birational.   Q.E.D.

Remark 2.12. *The same argument, but without the assumption that the Albanese image of S be a curve, gives in general the inequality $K^2 \geq 2p_g - 2q - 4$ which is however weaker than Noether's inequality $K^2 \geq 2p_g - 4$. For $q \geq 1$ we may get $K^2 \geq 2p_g - 2q + 2$, which is still however weaker than the inequality given by Debarre [1982] for irregular surfaces, namely, $K^2 \geq 2p_g$.*

3. The case of double covers

Suppose the good canonical projection $\psi : S \to Y \subset \mathbb{P}^3$ is $2 : 1$ and that Y is not a quadric. Then $\mathcal{F} = \psi_* \mathcal{O}_X$ still has a locally free resolution of length 1 of the form (2.4)

$$0 \to (\mathcal{O}_\mathbb{P} \oplus E)^*(-5) \to \mathcal{O}_\mathbb{P} \oplus E \to \mathcal{F} = \psi_* \mathcal{O}_X \to 0.$$

with E the vector bundle on $\mathbb{P}^3$ defined by

$$E(2) = (U^* \otimes \mathcal{O}_\mathbb{P}) \oplus (\mathrm{H}^1(\mathcal{O}_X(K)) \otimes \Omega_\mathbb{P}^1(1)) \oplus (W \otimes \Omega_\mathbb{P}^2(2)).$$

However the proof that the resolution can be chosen symmetric needs further arguments. Let $\rho : Z \to Y$ denote the normalization of Y. Then $\psi : X \to Y$ factors over $\varepsilon : X \to Z$, where ε is $2 : 1$. The covering involution $\Phi : X \to X$ induces a decomposition of $\varepsilon_* \mathcal{O}_X$ into invariant and anti-invariants parts:

(3.1) $$\varepsilon_* \mathcal{O}_X = \mathcal{O}_Z \oplus \mathcal{H}.$$

On Y this induces the decomposition

$$(3.2) \qquad \mathcal{F} = \psi_* \mathcal{O}_X = \rho_* \mathcal{O}_Z \oplus \rho_* \mathcal{H}.$$

This in turn decomposes the Beilinson cohomology groups of $\mathcal{F}$ and (assuming that Y is not a quadric), this gives a decomposition of (2.4). There are two cases how the isomorphism $\mathcal{F} \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}(-1))$ can respect the summands (which are generically of rank 1 on Y). Either

(a) $\rho_* \mathcal{O}_Z \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\rho_* \mathcal{O}_Z, \omega_{\mathbb{P}}(-1))$

or

(b) $\rho_* \mathcal{O}_Z \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\rho_* \mathcal{H}, \omega_{\mathbb{P}}(-1))$

Case (a) occurs when y_0, y_1, y_2, y_3 pullback to Φ-invariant sections of $\mathrm{H}^0(\mathcal{O}_X(K))$ i.e. $V^* \subset \mathrm{H}^0(\mathcal{O}_X(K))^+$.

Case (b) occurs when $V^* \subset \mathrm{H}^0(\mathcal{O}_X(K))^-$.

The Φ-invariant case (a).

Since $V^* \subset \mathrm{H}^0(\mathcal{O}_X(K))^+$, the isomorphism $\psi_* \omega_X$ with $\psi_* \mathcal{O}_X(1)$ respects the invariant and anti-invariant summands. Therefore,

$$(\psi_* \omega_X)^+ = \rho_* \omega_Z = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\rho_* \mathcal{O}_Z, \omega_{\mathbb{P}})$$

is isomorphic to $\rho_* \mathcal{O}_Z(1)$ as asserted. Since moreover ρ is birational, $\rho^* \rho_* \mathcal{O}_Z(1) = \mathcal{O}_Z(1)$ and by the projection formula we infer that $\mathcal{O}_Z(1) \cong \omega_Z$, in particular, that Z is Gorenstein, whence the canonical model of a surface of general type.

Similarly, we see that $(\psi_* \omega_X)^- = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\rho_* \mathcal{H}, \omega_{\mathbb{P}}) = \rho_* \mathcal{H}(1)$.

Decomposing then

$$(3.3) \qquad E = E_+ \oplus E_-$$

into parts coming from $\rho_* \mathcal{O}_Z \oplus \rho_* \mathcal{H}$ we obtain in this case a decompositon of (2.4) into

$$(3.4) \qquad 0 \to (\mathcal{O}_{\mathbb{P}} \oplus E_+)^*(-5) \xrightarrow{\alpha_+} \mathcal{O}_{\mathbb{P}} \oplus E_+ \to \rho_* \mathcal{O}_Z \to 0$$

and

$$(3.5) \qquad 0 \to (E_-)^*(-5) \xrightarrow{\alpha_-} E_- \to \rho_* \mathcal{H} \to 0$$

The argument of 2.5 shows that both α_+ and α_- can be chosen to be symmetric matrices. We can also apply Theorem 2.9 to Z, since as we observed $Z = X/\Phi$ is a canonical model of a surface. (3.4 is the determinantal description ot the good birational canonical projection $\rho: Z \to Y \subset \mathbb{P}^3 = \mathbb{P}(V)$.

We leave aside for the time being the task of describing the $\mathcal{O}_Z$ module structure on $\rho_* \mathcal{H}$: we simply observe that the bilinear map

$$(E_-) \times (E_-) \to \mathcal{O}_{\mathbb{P}}$$

is given as in [Catanese 1981] by the adjoint matrix of α_-, which solves the problem in the very particular case where Y is normal.

The Φ-anti-invariant case (b).   Here (2.4) decomposes into

$$(3.6) \qquad 0 \to (E_-)^*(-5) \xrightarrow{\alpha_+} \mathcal{O}_{\mathbb{P}} \oplus E_+ \to \rho_* \mathcal{O}_Z \to 0$$

and

$$(3.7) \qquad 0 \to (\mathcal{O}_{\mathbb{P}} \oplus E_+)^*(-5) \xrightarrow{\alpha_-} E_- \to \rho_* \mathcal{H} \to 0$$

and we may choose $\alpha_- = (\alpha_+)^t$. Notice that α_+ satifies the ring condition (R.C.).

To recover X from (3.4) and (3.5) or from (3.6) we need in addition to describe the map

$$(3.8) \qquad S_2(\rho_* \mathcal{H}) \to \rho_* \mathcal{O}_Z.$$

4. Generalities on irregular surfaces with $p_g = 4$.

As explained in the introduction, one of the main purposes of our investigation is to understand the equations of the projections of irregular canonical surfaces in $\mathbb{P}^3$.

For this reason the most natural case to consider is the case where $p_g = 4$, and there is no choice whatsoever to make for the projection.

We recall, for the reader's benefit, some important inequalities for irregular surfaces

- Castelnuovo' s Theorem [1905] : $\chi(S) \geq 1$ if the surface S is not ruled.

From Castelnuovo's theorem follows that if $p_g = 4$, then the irregularity $q(S)$ is ≤ 4.

We have also the

- Inequality of Castelnuovo-Beauville (Beauville [1982]): $p_g \geq 2q-4$, equality holding if and only if S is a product of a curve of genus 2 with a curve of genus $g \geq 2$.

Whence follows that, if $p_g = q = 4$, then S is the product of two curves of genus 2 and its canonical map is a $(\mathbb{Z}/2)^2$-Galois covering of a smooth quadric.

Debarre has moreover shown [1982] that for an irregular surface one has the following

- Debarre's inequality: if S is irregular, then $K^2 \geq 2p_g$.

Therefore, in our case, the above inequality yields more than the more general inequalities by Castelnuovo [1891] and by Horikawa [1976b]], Reid [1979], Beauville [1979] and Debarre [1982]: $K^2 \geq 3p_g + q - 7$ if the canonical map is birational.

Our inequality $K^2 \geq 2p_g + 4q - 4$ if the Albanese image is a curve severely restricts the numerical possibilities if $p_g = 4, q = 3$, since by the Bogomolov-Miyaoka-Yau inequality we always have $K^2 \leq 9\chi$, thus $16 \leq K^2 \leq 18$ if the Albanese map is a pencil.

This case is completely solved by using the following inequalities for surfaces fibred over curves:

- Arakelov's inequality: let $f : S \to B$ be a fibration onto a curve B of genus b, with fibres curves of genus $g \geq 2$. Then $K_S^2 \geq 8(b-1)(g-1)$, equality holding only if the fibration has constant moduli.

- Beauville's inequality: let $f : S \to B$ be a fibration onto a curve B of genus b, with fibres curves of genus $g \geq 2$. Then $\chi(S) \geq (b-1)(g-1)$, equality holding if and only if the fibration is an etale bundle (there is an etale cover $B' \to B$ such that the pull back is a product $B' \times F$).

By Beauville's inequality follows that if $q = 3$ and the Albanese image is a curve, then (take as B the genus 3 curve which is the Albanese image, and f the Albanese map) the Albanese map is an etale bundle with fibre F of genus 2.

In particular, we have $K_S^2 = 16$ and all our surfaces are obtained as follows:

let G be a finite group acting faithfully on a curve F of genus 2 in such a way that $F/G \cong \mathbb{P}^1$, and take an etale G-cover $B' \to B$ of a curve B of genus 3: then our surfaces S with $p_g = 4$, $q = 3$, $K_S^2 = 16$ are exactly the quotients $(F \times B')/G$ of the product $(F \times B')$ by the diagonal action of G.

The groups G as above were classified by Bolza [1888] (cf. also [Zucconi1994]).

If instead the Albanese image is a curve of genus $q = 2$, then, since we assume $p_g = 4$, then $\chi(S) = 3 \geq (g-1)$, and the genus of the Albanese fibres is at most 4.

The case $g = 4$ gives again rise to an etale bundle with fibre F of genus 4, so that our surface S is a quotient $S = F \times B'/G$ where $F/G \cong \mathbb{P}^1$ and $B = B'/G$ is a curve of genus $b = 2$ (one must impose the condition that G operate freely on the product).

In particular, we have $K_S^2 = 24$. A concrete example is furnished by $G = \mathbb{Z}/2$, which operates freely on the genus 3 curve B'. Then $H^0(K_S) = H^0(K_{F \times B'})^+ = H^0(K_F)^- \otimes H^0(K_{B'})^-$, and since $H^0(K_{B'})^-$ is 1-dimensional, the canonical system is a pencil and the canonical image of S is the canonical image of F, namely, a twisted cubic curve in $\mathbb{P}^3$.

It would be also interesting to determine what is the minimal value of K^2 for an irregular surface with $p_g = 4$ such that the canonical map is birational.

In the next section we shall see that the answer to this question is: $K^2 = 12$ in the case $q = 1$, and in this case we shall give a complete classification of the surfaces for which the canonical map is birational.

On the other hand, the problem remains for $q = 2, 3$. In the forthcoming sections, for $q = 3$ we provide examples where $K^2 = 12$ (of course the Albanese image is a surface, as we already observed), while for $q = 2$ we show that there are examples with $K^2 = 18$ (this is not the maximum allowed by the B-M-Y inequality, yielding $K^2 \leq 9\chi = 27$, but still not so low).

For the case where the degree of the canonical map is 2 , we recall

Theorem 4.1. *(Debarre's theorem 4.8 [1982]) Assume that $1 \leq q \leq 3$ and that the canonical map has degree 2: then $K^2 \geq 2p_g + q - 1$.*

Finally, we recall the following inequality for fibred surfaces (cf. Xiao [1987], Konno [1993])

- Xiao-Konno inequality: if $f : S \to B$ is a fibration to a curve B of genus b, with fibres of genus g, and without constant moduli, then the slope $\lambda(f) = (K_S - f^*(K_B))^2 / \deg(f_*\omega_{S|B}) = \frac{K_S^2 - 8(b-1)(g-1)}{\chi(S) - (b-1)(g-1)}$ satisfies $4(g-1)/g \le \lambda(f) \le 12$, the first inequality being an inequality iff all the fibres are hyperelliptic.

It follows that $K_S^2 \le 12\chi(S) - 4(b-1)(g-1)$, and this implies that if the Albanese image is a curve of genus $q \ge 2$, then $K_S^2 \le 12p_g - 12(q-1) - 4(q-1)(g-1) = 12p_g - 4(q-1)(g+2)$.

Therefore, if $p_g = 4$ and if the Albanese image is a curve of genus $q \ge 2$, we obtain indeed $16 \le K_S^2 \le 48 - 4(q-1)(g+2)$. For $q = 3$ this confirms that we must have $K^2 = 16$ and $g = 2$, a case that we have already illustrated. While, if $q = 2$ we have $12 \le K_S^2 \le 48 - 4(g+2)$: whence, $g \le 5$ which is weaker than the inequality $g \le 4$ we have already obtained.

5. IRREGULAR SURFACES WITH $p_g = 4, q = 1$.

Horikawa [1981] proved that for an irregular surface with $K^2 < 3\chi$ ($3\chi = 12$ here), the Albanese map has a curve as image, and the fibres have either genus 2 or they have genus 3 and are hyperelliptic: moreover only the first possibility occurs if $K^2 < 8/3\chi$.

In this section we shall restrict our attention to the case $p_g = 4, q = 1$, thus $\chi = 4$ and $K^2 = 8$ is the smallest value for K^2, while $K^2 = 12$ is the smallest value for which we can have a birational canonical map (in view of the quoted result by Horikawa).

Let $a : S \to A$ be the Albanese map of S. By the above inequality for the slope, if $K^2 = 12$ then $g \le 4$, and if $g = 4$ all the fibres of a are hyperelliptic. But if the Albanese fibres are hyperelliptic, then the canonical map ϕ factors through the hyperelliptic involution i . We make therefore the following

Assumption 5.1. $p_g = 4, q = 1$, and the general Albanese fibre is non hyperelliptic of genus $g = 3$.

Under the above assumption, let $\mathcal{V}$ be the vector bundle on A defined by

$$(5.1) \qquad\qquad \mathcal{V} = a_*\omega_S,$$

which enjoys the base change property. $\mathcal{V}$ has rank $g = 3$, and $h^0(\mathcal{V}) = p_g = 4$. More generally, we can consider the vector bundles $\mathcal{V}_i = a_*(\omega_S^{\otimes i})$, which have zero H^1-cohomology groups and have degree

$\deg(\mathcal{V}_i) = \chi + \frac{i(i-1)}{2} K^2 = 4 + \frac{i(i-1)}{2} K^2$.

Under the assumption 5.1, we have an exact sequence

$$(5.2) \qquad\qquad 0 \to Sym^2(\mathcal{V}) \to \mathcal{V}_2 \to \mathcal{C} \to 0,$$

where $\mathcal{C}$ is a torsion sheaf .

Since the degree of $Sym^2(\mathcal{V})$ equals 16, we obtain

Proposition 5.2. *Under the assumption 5.1 we have $K^2 \ge 12$, equality holding iff there is no hyperelliptic fibre, that is, $Sym^2(\mathcal{V}) \cong \mathcal{V}_2$.*

By the theorem of Fujita [1978] $\mathcal{V}$ is semipositive, moreover by a corollary of a theorem of Simpson [1993] observed for instance in 2.1.7 of Zucconi [1994], there is in general a splitting

$$(5.3) \qquad \mathcal{V} = (\bigoplus_{\tau \in \mathrm{Pic}(A)_{tors} - \{0\}} L_\tau) \oplus (\bigoplus_i W_i)$$

where, as indicated, L_τ is a non trivial torsion line bundle, and instead W_i is an indecomposable bundle of strictly positive degree.

We are interested mostly in the case where the canonical map ϕ is birational; since $\mathcal{V}$ has rank 3, the hypothesis that ϕ be birational implies that $\mathcal{V}$ must be generically generated by global sections. Thus we make the following

Assumption 5.3. $\mathcal{V}$ *is generically generated by global sections .*

In particular, there are no summands of type L_τ in (5.3) .

Moreover, by Atiyah [1957], Lemma 15, page 430, setting $\deg(W_i) = d_i$ and $\mathrm{rank}(W_i) = r_i$, we have $h^0(W_i) = \deg(W_i) = d_i$, and therefore $d_i \geq r_i$. Conversely, by loc. cit. Theorem 6, page 433) and by induction follows

Proposition 5.4. *If W is an indecomposable vector bundle on an elliptic curve of degree $d \geq r = \mathrm{rank}(W)$, then W is generically generated by global sections.*

Therefore, since $\Sigma_i d_i = 4, \Sigma_i r_i = 3$, and the r_i's, d_i 's are > 0, we have only the following possibilities for the pairs (r_i, d_i) which are ordered by the slope d/r:

  (i) (3,4)
 (ii) (2,3) (1,1)
(iii) (1,2) (2,2)
(iv) (1,2) (1,1) (1,1).

The structure of these bundles is then clear by the quoted results of Atiyah: for each line bundle $\mathcal{O}_A(p)$ of degree one, where $p \in A = \mathrm{Pic}^1(A)$, there are a point $u \in A$ and line bundles $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^0(A)$ such that $\mathcal{V}' = \mathcal{V} \otimes \mathcal{O}_A(-p)$ is respectively equal to

$$\begin{array}{rl} \text{(i)} & E_3(u) \\ \text{(ii)} & E_2(u) \oplus \mathcal{L} \\ \text{(iii)} & \mathcal{O}_A(u) \oplus (\mathcal{L} \otimes F_2) \\ \text{(iv)} & \mathcal{O}_A(u) \oplus \mathcal{L} \oplus \mathcal{L}', \end{array}$$

(5.4)

where $E_1(u) := \mathcal{O}_A(u)$ and $E_i(u)$ is defined inductively as the unique non trivial extension

$$0 \to \mathcal{O}_A \to E_i(u) \to E_{i-1}(u) \to 0,$$

while $F_1 := \mathcal{O}_A$ and F_i is defined inductively as the unique non trivial extension

$$0 \to \mathcal{O}_A \to F_i \to F_{i-1} \to 0.$$

Lemma 5.5. *Assume that X is the canonical model of a surface with $p_g = 4, q = 1, K^2 = 12$, and non hyperelliptic Albanese fibres.*

Then the relative canonical map $\omega : X \to Proj(\mathcal{V}) = \mathbb{P}$ is an embedding. Moreover, there is a point $p \in A$ such that, setting $\mathcal{V}' = \mathcal{V} \otimes \mathcal{O}_A(-p)$, $(\det \mathcal{V}') \cong \mathcal{O}_A(p)$ and X is a divisor in the linear system $|4D|$, D being the tautological divisor of $Proj(\mathcal{V}')$. (Notice that p is defined only up to 4-torsion) .

Conversely, if $\mathcal{V}$ is as in (5.4), any divisor X in $|4D|$ with at most R.D.P.'s as singularities is the canonical model of a surface with $p_g = 4, q = 1, K^2 = 12$, and non hyperelliptic Albanese fibres.

Proof. By 5.2, the relative canonical map is an embedding of the canonical model X of S if and only if the relative bicanonical map is an embedding. Let therefore F be a fibre of the Albanese map $a : X \to A$. Since $\mathcal{V}_2$ enjoys the base change property, we are just asking whether $\mathcal{O}_F(2K_X)$ is very ample on F.

By Catanese and Franciosi [1996] or Catanese, Franciosi, Hulek and Reid [1999] we get that very ampleness holds provided that F is 2-connected, i.e., there is no decomposition $F = F_1 + F_2$ with F_1, F_2 effective and with $F_1 F_2 \leq 1$.

If such a decomposition would exist, we claim that we may then assume $F_1 F_2 = 1$.

Otherwise $F_1 F_2 \leq 0$ and, since also $F_i^2 \leq 0$, it follows by Zariski's Lemma that $F = 2F_1$. Since F_1 has genus 2 the exact sequence

$$\mathcal{O}_{F_1}(K_X - F_1) \to \mathcal{O}_F(K_X) \to \mathcal{O}_{F_1}(K_X) \to 0$$

shows easily that the hyperelliptic involution on F_1 extends to a hyperelliptic involution on F, a contradiction.

Since the genus of F equals 3, $K_X F = 4 = K_X F_1 + K_X F_2$. Since moreover $K_X F_i \equiv F_i^2 \ (mod\ 2) \equiv -F_1 F_2 \ (mod\ 2)$, we may assume w.l.o.g. that $K_X F_1 = 1$. So F_1 is an elliptic tail, while F_2 has genus 2.

More precisely, we have $h^0(\mathcal{O}_{F_1}(K_X)) = 1$, $h^0(\mathcal{O}_{F_1}(2K_X)) = 2$, contradicting the fact that $Sym^2(H^0(K_F)) = (H^0(2K_F))$, since in fact $h^1(\mathcal{O}_{F_2}(2K_X - F_1)) = 0$.

We proved now that X is embedded in $\mathbb{P}$, whence it follows that the surjection $Sym^4(\mathcal{V}) \to \mathcal{V}_4$ has as kernel an invertible sheaf L on the elliptic curve A. The exact sequence

$$0 \to L \to Sym^4(\mathcal{V}) \to \mathcal{V}_4 \to 0,$$

and the easy calculation: $\deg(Sym^4(\mathcal{V})) = 80, \deg(\mathcal{V}_4) = 76$ shows that $\deg(L) = 4$, so there is point $p \in A$ such that $L = \mathcal{O}_A(4p)$, and we have the following linear equivalence in $\mathbb{P}$: $X \equiv 4H - 4F$, where H is the tautological hyperplane divisor, and F is the fibre of a over p.

If we write $\mathbb{P}$ as $Proj(\mathcal{V}') = Proj(\mathcal{V}(-p))$, and let D be the corresponding hyperplane divisor, then X is a divisor in the linear system $|4D|$.

Conversely, since $K_\mathbb{P} \equiv -3H + \omega^*(\det \mathcal{V})$, if we choose a divisor $X \equiv 4H - \omega^*(\det \mathcal{V})$ we get that $K_X \equiv H$, so that $\mathcal{V}$ is the direct image of the canonical sheaf $\mathcal{O}_X(K_X)$. It is then clear that $L = \omega^*(\det \mathcal{V})$, and since we chose p such that $X \equiv 4H - 4F$, $K_X \equiv D + \omega^*(\det \mathcal{V}') \equiv D + F$, we have proven that $(\det \mathcal{V}') \cong \mathcal{O}_A(p)$.

Moreover, we have that $p_g(X) = h^0(\mathcal{V}) = 4$, $q = 1 + h^1(\mathcal{V}) = 1 + 0 = 1$, while $K_X^2 = (D+F)^2(4D)$. By the Leray-Hirsch formula we get $D^2 = FD$, moreover $D^2F = 1$, whence $K_X^2 = 4(1+2) = 12$.   Q.E.D.

Remark 5.6. *Observe that $det(\mathcal{V}) \cong \mathcal{O}_A(4p)$, whence in the notation of the previous lemma we have $det(\mathcal{V}') \cong \mathcal{O}_A(p)$. Thus $p \equiv u$ in case i), while $p \equiv u + \mathcal{L}$, $p \equiv u + 2\mathcal{L}$, $p \equiv u + \mathcal{L} + \mathcal{L}'$, in the respective cases ii), iii), iv).*

It follows that the pair $(A, \mathcal{V})$ has 1 modulus in case i), 2 moduli in case ii), while we are going to show next that the pair has 1 modulus in case iii), and 1 or 2 moduli in the last case iv).

There remains as a first problem the question about the existence of the surfaces under consideration, that is, whether the general element in the linear system $|4D|$ has only Rational Double Points as singularities. The result is a consequence of techniques developed in Catanese and Ciliberto [1993].

Proposition 5.7. *Let $\mathcal{V}$ be as in (5.4) a rank 3 bundle over an elliptic curve, and X as in Lemma 5.5 a general divisor in the linear system $|4D|$ on $\mathbb{P} = Proj(\mathcal{V})$. Then X is smooth in cases i) and ii). Instead, in case iii), the general element X has only Rational Double Points as singularities if and only if $\mathcal{L}^4 \cong \mathcal{O}_A$.*

Finally, in case iv), the general element X has Rational Double Points as singularities if and only if one of the bundles $\mathcal{L}^k \otimes \mathcal{L}'^{4-k}$ is trivial.

Proof. In case i), the linear system $|4D|$ is very ample on $\mathbb{P}$ by Theorem 1.21 of Catanese and Ciliberto [1993], so a general X is smooth by Bertini's Theorem.

Notice that to apply Bertini's theorem it suffices to show that the general element of the linear system $|4D|$ is smooth along the base locus of $|4D|$. To show that $|4D|$ is base point free is in turn sufficient the condition that the vector bundle $Sym^4(\mathcal{V}')$ be generated by global sections.

In case ii), we observe that $Sym^4(\mathcal{V}')$ is a direct sum

$$\bigoplus_{k=1,..4} Sym^k(E_2(v_k)) \bigoplus \mathcal{L}^4,$$

where the v_k's are suitable points on A. The symmetric powers with $k \geq 2$ are generated by global sections by virtue of Theorem 1.18 of Catanese and Ciliberto [1993].

Whereas a bundle of the form $E_2(v_k)$ has only one section, which is nowhere vanishing. Therefore, the base locus of $|4D|$ is contained in the section Δ of $\mathbb{P}$ dual to the subbundle $E_2(u)$, and there, if x, y are local equations for Δ, then the Taylor development of the equation of a divisor in $|4D|$ only fails at most to have a term of type x. Therefore in this case the general element X is smooth.

Let us then consider case iv): it is immediate to remark that the linear system $|4D|$ has as fixed part the projective $\mathbb{P}^1$-subbundle $\mathbb{P}'$ annihilated by $\mathcal{O}_A(u)$ in the case where no line bundle $\mathcal{L}^k \otimes \mathcal{L}'^{4-k}$ is trivial.

This cannot occur, so assume that one of such line bundles is trivial.

Case iv) -(I''): *$\mathcal{L}'^4$ and $\mathcal{L}^4$ are trivial.*

This is the easy case where $|4D|$ has no base points, whence Bertini's theorem applies.

Case iv) -(I'): $\mathcal{L}'^4$ *is not trivial but* $\mathcal{L}^4$ *is trivial.*

This is the case where the base locus of $|4D|$ is the section Δ annihilitated by $\mathcal{O}_A(u) \oplus \mathcal{L}$.

At each point of Δ exists then a term of order 1 in the Taylor expansion of the equation f of X, except at the point $t' = 0$, where $t' = 0$ lies over the unique point $v \in A$ such that $v \equiv u + 3\mathcal{L}'$.

In this point we have then local coordinates x, z, t' with $\Delta = \{z = x = 0\}$ and we surely get monomials z^2, zt', zx (zx corresponds to the fact that the unique section of $\mathcal{O}_A(u) \otimes \mathcal{L} \otimes \mathcal{L}'^2$ does not vanish in our point $t' = 0$, otherwise $\mathcal{L} \cong \mathcal{L}'$, contradicting our assumption (iv, I')). Since moreover we get the monomial x^4, we certainly obtain for general X at worst a Rational Double Point of type A_3.

Case iv) -(II): $\mathcal{L}'^4$ *and* $\mathcal{L}^4$ *are not trivial but* $\mathcal{L}^2 \otimes \mathcal{L}'^2$ *is trivial* .

In this case the base locus of $|4D|$ is given by the two sections Δ, Δ', where Δ' is annihilated by $\mathcal{O}_A(u) \oplus \mathcal{L}'$.

In this case, by symmetry, let us study the singularity of a general X along Δ.

We get, as in the previous case, a section zt', and sections z^2, x^2 , thus a singularity of type A_1 at worst.

Case iv) -(III): $\mathcal{L}'^4$, $\mathcal{L}^2 \otimes \mathcal{L}'^2$ *and* $\mathcal{L}^4$ *are not trivial but* $\mathcal{L}^3 \otimes \mathcal{L}'$ *is trivial.*

In this case again the base locus of $|4D|$ is given by the two sections Δ, Δ'. For Δ we get monomials of type x^3, z^2, zt', thus a singularity of type A_2 at worst, while for Δ' we get monomials of type x, z^2, zt', thus no singularity at all for general X along Δ'.

We can finally analyse case iii).

To this purpose, recall

Theorem 5.8. *(Atiyah's Theorem 9 in [57]) Let F_2 be the indecomposable bundle on an elliptic curve with trivial determinant and of rank* 2*: then* $Sym^k(F_2) \cong F_{k+1}$.

We observe then that the tensor product of F_r with a line bundle $\mathcal{M}$ is generated by global sections if $\deg(\mathcal{M}) \geq 2$, whereas, if $d := \deg(\mathcal{M}) \leq 1$, $F_r \otimes \mathcal{M}$ is generated by global sections outside a unique point where all sections vanish if $d = 1$, whereas for $d = 0$, $F_r \otimes \mathcal{M}$ has no global sections unless the line bundle $\mathcal{M}$ is trivial. In this last case, there is only one non zero section, which vanishes nowhere.

After these remarks, it is clear that, in case iii), if $\mathcal{L}^4$ is non trivial, then the fixed part of $|4D|$ contains the $\mathbb{P}^1$-bundle $\mathbb{P}'$ annihilated by $\mathcal{O}_A(u)$. Otherwise, the base locus consists of a section Δ and a fibre F_v of $\mathbb{P}'$.

At the intersection point of these two curves, we have local coordinates z, x, t such that $z = 0$ defines $\mathbb{P}'$, $z = x = 0$ defines Δ, $z = t = 0$ defines F_v.

In the Taylor expansion of the equation of X we get x^4, zt, z^2, therefore, by an argument we already used, in this point we get at worst a singularity of type A_3, while the other points of the base locus are smooth for general X.

<div align="right">Q.E.D.</div>

We derive from the previous result some preliminary information on the moduli space of the above surfaces.

We need however to slightly simplify our previous presentation. Observe therefore that we can exchange the roles of $\mathcal{L}$, $\mathcal{L}'$, and we can tensor $\mathcal{V}'$ by a line bundle of 4-torsion.

Therefore, in the last case iv), we may simplify our treatment to consider only the subcases

- (iv, I) $\mathcal{L} \cong \mathcal{O}_A$
- (iv, II) $\mathcal{L}' \cong \mathcal{L}^{-1}$, $\mathcal{L}$ not of 4-torsion
- (iv, III) $\mathcal{L}' \cong \mathcal{L}^{-3}$, $\mathcal{L}$ not of 4-torsion

(the reader may in fact observe that cases (iv, I') and (iv, I'') are just special subcases of (iv, I)).

Corollary 5.9. *Consider the open set $\mathcal{M}$ of the moduli space of the surfaces with $p_g = 4, q = 1, K^2 = 12$ such that assumptions 5.1 and 5.3 are verified.*

Then $\mathcal{M}$ consists of the following 10 locally closed subsets:

- *$\mathcal{M}(i)$, of dimension 20, corresponding to case i)*
- *$\mathcal{M}(ii, 0)$ corresponding to the case $\mathcal{L}^4 \cong \mathcal{O}_A$, and $\mathcal{M}(ii, 1)$ corresponding to the case $\mathcal{L}^4 \not\cong \mathcal{O}_A$, both of dimension 19*
- *$\mathcal{M}(iii)$ corresponding to case iii), of dimension 18*
- *$\mathcal{M}(iv, I)$ of dimension 18, corresponding to the case where $\mathcal{L} \cong \mathcal{O}_A$, but $\mathcal{L}'$ is neither of 3-torsion nor of 4-torsion*
- *$\mathcal{M}(iv, II)$ of dimension 19, corresponding to the case iv), II), ($\mathcal{L}$ not of 4-torsion)*
- *$\mathcal{M}(iv, III)$ of dimension 19, corresponding to the case iv), III), ($\mathcal{L}$ not of 4-torsion)*
- *$\mathcal{M}(iv, I, 1/4)$ of dimension 18, corresponding to case iv),I), where we may assume $\mathcal{L} \cong \mathcal{O}_A$ and $\mathcal{L}'$ of 4-torsion but not of 2-torsion*
- *$\mathcal{M}(iv, I, 1/2)$ of dimension 19, corresponding to case iv),I), where we may assume $\mathcal{L} \cong \mathcal{O}_A$ and $\mathcal{L}'$ of 2-torsion, but non trivial*
- *$\mathcal{M}(iv, I, 1/3)$ of dimension 18, corresponding to case iv),I), where we may assume $\mathcal{L} \cong \mathcal{O}_A$ and $\mathcal{L}'$ of 3-torsion and non trivial*
- *$\mathcal{M}(iv, I, 1)$ of dimension 19, corresponding to case iv),I), where we may assume $\mathcal{L} \cong \mathcal{O}_A$ and also $\mathcal{L}' \cong \mathcal{O}_A$.*

Proof. Observe that the moduli space of elliptic curves together with a torsion sheaf of torsion precisely n is irreducible of dimension 1.

Observe moreover that the hypersurface X (the canonical model of S) moves in a linear system $|4D|$ in $\mathbb{P}$, whose dimension is given by $h^0(Sym^4(\mathcal{V}')) - 1$. By Riemann-Roch, since $\mathcal{V}'$ has rank = 3 and degree = 1, we have that $h^0(Sym^4(\mathcal{V}')) = 20 + h^1(Sym^4(\mathcal{V}'))$.

Moreover we have that the dimension of each stratum, since each surface of general type has a finite automorphism group, equals $1 + h^0(Sym^4(\mathcal{V}')) - h^0(End(\mathcal{V}'))$. This justifies the assertion about the dimensions.

Furthermore, we may observe that the conditions that a vector bundle be indecomposable is an open one, while the condition that a line bundle be of n-torsion is a closed one.

<div align="right">Q.E.D.</div>

The previous corollary allows us to conclude that our moduli space is irreducible: we use for this purpose a lower bound for the dimension of the moduli space which is a consequence of a general principle stated by Ziv Ran [1995], and turned into a precise theorem by Herb Clemens [2000].

Theorem 5.10. *The open set of the moduli space of surfaces with $p_g = 4, q = 1, K^2 = 12$, with non hyperelliptic Albanese fibres, and with $\mathcal{V}$ as in (5.4), i.e., generically generated by global sections, is irreducible of dimension* 20.

Proof. In view of the previous corollary, our moduli space has a stratification by locally closed sets, of which one only, $\mathcal{M}(i)$, has dimension 20, while the others have strictly smaller dimension. Since $\mathcal{M}(i)$ is clearly irreducible, it suffices to show that the dimension of the moduli space is at least 20 in each point. Equivalently, since the germ of the moduli space at the point corresponding to the surface S is the quotient of the base of the Kuranishi family of S by the finite group of automorphisms of S, it suffices to show that the dimension of the Kuranishi family is at least 20.

Now, in any case, the dimension of the Kuranishi family is always at least

$$h^1(S, T_S) - \dim(\text{Obs}(S)),$$

but in this case the obstruction space $Obs(S)$ is not the full cohomology group $H^2(S, T_S)$. Because we have a natural Hodge bilinear map

$$\gamma : H^0(S, \Omega_S^1) \times H^0(S, \Omega_S^2) \to H^0(S, \Omega_S^1 \otimes \Omega_S^2),$$

and the natural subspace $H := \text{Im}(H^0(S, \Omega_S^1) \otimes H^0(S, \Omega_S^2)) \subset H^0(S, \Omega_S^1 \otimes \Omega_S^2)$ determines by Serre duality a quotient map $\gamma^\vee : H^2(S, T_S) \to H^\vee$. By Theorem 10.1 of Clemens [2000] we have that $\gamma^\vee(Obs(S)) = 0$.

Since in this case it is obvious that $\dim(H) = 4$, γ being non degenerate, it follows that the base of the Kuranishi family has dimension $\geq -\chi(S, T_S) + 4 = 10\chi(S) - 2K_S^2 + 4 = 20$.   Q.E.D.

Theorem 5.11. *Assume that X is the canonical model of a surface with $p_g = 4, q = 1, K^2 = 12$, and non hyperelliptic Albanese fibres, and $\mathcal{V}$ is as in (5.4), i.e., generically generated by global sections . Then in cases (i), (ii) the canonical map ϕ is always a birational morphism, whereas in the other cases ϕ is birational for a general choice of X in the given linear system. The case $\deg(\varphi) = 3$ never occurs.*

Proof. Since $\mathcal{V}$ is generically generated by global sections, and the general fibre F_a is a non hyperelliptic curve of genus 3, it follows that F_a maps isomorphically to a plane quartic curve Γ_a. Let H_a be the plane containing Γ_a: since $K_X \equiv D + F$, then the pull-back divisor of H_a splits as $F_a + D_{-a}$, where $D_{-a} \sim D$.

Since $12 \geq \deg \phi \cdot \deg \Sigma$, and there are plane sections H_a which intersect Σ in a curve containing Γ_a, $\deg \Sigma \geq 4$, and the only possibility to exclude is that $\deg \phi = 2$ or 3.

In the case where $\deg \phi = 2$ let $\imath : X \to X$ be the corresponding biregular involution.

$\imath$ acts also on the Albanese variety A, and in a non trivial way, since a general fibre F_a is embedded by the canonical map ϕ, and let us then denote $a' := \imath(a)$.

If $\imath$ had no fixpoints on A, then $X \to X/\imath := Y$ would be unramified, so that $K_Y^2 = 6$, $\chi(\mathcal{O}_Y) = 2$, $q(Y) = 1$, whence $p_g(Y) = 2$, contradicting the fact that ϕ factors through Y. We may therefore assume that $a' = -a$ for a suitable choice of the origin in A.

Therefore, the inverse image of H_a contains $F_a + F_{a'}$, and we can write a linear equivalence $K_X \equiv F_a + F_{a'} + C_a$, where C_a is effective and $C_a = C_{-a}$.

Observe that $K_X \cdot F = 4$, $K_X \cdot C_a = 4$, $C_a \cdot F = 4$ whence C_a is not vertical for the Albanese map. Moreover, $12 = K_X^2 = (2F + C_a)^2 = 16 + C_a^2$, whence $C_a^2 = -4$.

In particular the algebraic system C_a has a fixed part.

We obtain a contradiction as follows.

First of all, since $|K_X - F_a - F_{a'}| \neq \emptyset$, we get $H^0(A, \mathcal{V}(-a - a')) \neq 0$, and this leaves out only the cases (5.4) iii) and iv), and moreover with $a + a' \equiv u + p$ on A.

We saw that $A/\imath \cong \mathbb{P}^1$, so that all the curves C_a are linearly equivalent. Indeed, a closer look reveals that all the curves C_a are the intersection of X with a fixed $\mathbb{P}^1$-subbundle of $\mathbb{P}$, thus we may consider the curve $C = C_a, \forall a \in A$.

The curve C maps to a line L under the two dimensional linear system corresponding to $H^0(A, W)$, where we write $\mathcal{V} = \mathcal{O}_A(u + p) \oplus W$.

Before we further investigate the geometry of the situation, remark that $\imath$ acts equivariantly on X and A, therefore $\mathcal{V}$ is isomorphic to $\imath^*(\mathcal{V})$ and indeed we have an action of $\imath$ on $\mathcal{V}$.

This however implies that $\mathcal{L}$ is of 2-torsion in case iii), while in case iv) $\mathcal{L} \cong -\mathcal{L}'$. Once these conditions are satisfied, it is clear that we have an involution $\imath$ on $\mathbb{P}$ and that the system $|\mathcal{O}_{\mathbb{P}}(1)|$ is invariant, but it remains to be seen whether the hypersurface X is also $\imath$-invariant (notice that the involution is completely determined by the four fixed points O such that $2O \equiv u + p$).

It is easy to verify that for a general choice of X in $|4D|$, this does not hold.

CLAIM: $\deg(\phi) = 3$ **NEVER OCCURS.**

Consider in fact the possibility that $deg(\phi) = 3$: then Γ_a is a full hyperplane section of Σ, and K_X is base-point free (in general, if $|K_X| = |M| + \Psi$, with Ψ a non trivial fixed part, then $M^2 = K_X^2 - K_X \cdot \Psi - M \cdot \Psi < K_X^2$, if then $|M|$ has base points, then $M^2 > \deg(\phi) \cdot \deg(\Sigma)$: while here $K_X^2 = 12 = \deg(\phi) \cdot \deg(\Sigma))$.

Observe that the surface Σ is normal, since it has a smooth hyperplane section.

Let $\pi : \tilde{\Sigma} \to \Sigma$ be a minimal resolution of Σ and denote by $\tilde{X}$ a minimal resolution of the fibre product $\tilde{\Sigma} \times_\Sigma X$: since $\tilde{X}$ is birational to X, $R^1 p_* \mathcal{O}_{\tilde{X}} = 0$, $p : \tilde{X} \to \Sigma$ being the composite morphism. Whence it follows that $R^1 \pi_* \mathcal{O}_{\tilde{\Sigma}} = 0$,

i.e., Σ has only rational double points as singularities, and $\tilde{\Sigma}$ is a smooth $K3$-surface.

We will now consider the ramification formula for ϕ. Let B be the reduced branch divisor of ϕ, set $\phi^*(B) = R + R'$, R being the ramification divisor, and observe that $R' \geq 1/2R$. The fact that Σ is a $K3$ with R.D.P.'s implies that $R \equiv K_X$, i.e., there is a hyperplane divisor H with $R = \phi^*(H)$.

On the other hand, since $\deg(\phi) = 3$ it follows that $\phi_*(R_{red}) = B$, whence B is the reduced divisor of the plane section which pulls back to H: this is a contradiction since then $R = \phi^*(H) \geq \phi^*(B) = R + R'$, while $R' > 0$ (this follows since $R > 0$, and $R' \geq 1/2R$). $\hspace{2cm}$ Q.E.D.

Remark 5.12. *We saw that we have several strata of the above irreducible moduli space. The stratum of maximal dimension, such that the moduli space is just its closure will be called the 'Main Stratum', and we shall say that the surfaces which belong to this Main Stratum are of the "Main Stream".*

It is certainly, as we shall see , the one which is most interesting and related to the geometry of elliptic space curves of degree 4.

We should also remark that a detailed and more general study of surfaces with irregularity $q = 1$ and with $K^2 = 3\chi$ was undertaken in the 1996 Thesis of T. Takahashi. However his results are weaker than ours in the case where $p_g = 4$, so we could not use this reference.

We finally come to a discussion of the geometry of the surfaces of the "Main Stream" (case i)).

Let A be an elliptic curve of degree 4 in $\mathbb{P}^3$. Then, as it is well known, A is the complete intersection of 2 quadric surfaces Q, Q'.

We may indeed without loss of generality assume that the pencil of quadrics be Heisenberg invariant, in other words that:

$$A = \{(x)|x_0^2 + x_2^2 - \lambda^2 x_1 x_3 = 0, x_1^2 + x_3^2 - \lambda^2 x_0 x_2 = 0\}.$$

It is also well known (Atiyah [1957], Catanese and Ciliberto [1988]) that in case i) the projective bundle $\mathbb{P}$ is nothing else than the triple symmetric product of the elliptic curve A, $\mathbb{P} = A^{(3)}$.

In this context the canonical mapping of X is induced by a morphism $\phi : \mathbb{P} \to (\mathbb{P}^3)^\vee$ which can be explained without formulae as follows: consider a point P of $\mathbb{P}$, i.e., P is a divisor of degree 3 on A. Then there is a unique plane $\phi(P)$ containing this divisor.

This geometric explanation shows that the degree of ϕ is 4 (as it had to be, since F being a fibre of the Albanese map a, $(D+F)^3 = 4$ by the Leray-Hirsch formula which we already mentioned).

In fact the reason of the above is that

- the projection onto the elliptic curve (the Albanese map) associates to a divisor $P = P_1 + P_2 + P_3$ the sum of the three points P_1, P_2, P_3 in the elliptic curve A,
- the tautological divisor D on $\mathbb{P}$ consists of the divisors where P_1 is fixed (whence, $D^3 = 1$).

- Let $I = \mathrm{Proj}(\mathcal{T}_{\mathbb{P}^3})$ be the incidence correspondence, $I \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee$: then we claim that $I \cap (A \times (\mathbb{P}^3)^\vee) = A^{(3)}$.

  Proof. The isomorphism is given by $(x, h) \to \mathrm{div}_A(h) - x$. $\qquad \square$

  Observe only that the first projection does not correspond precisely to the Albanese map, but only to the composition of the Albanese map with an involution of A, since to a divisor $P = P_1 + P_2 + P_3$ corresponds the point x such that $x + P_1 + P_2 + P_3$ is linearly equivalent to the hyperplane divisor of A.

- We claim that the second projection is given by the linear system $|D + F|$.

  Proof. Any hyperplane in $(\mathbb{P}^3)^\vee$ is the hyperplane H_x dual to a point $x \in \mathbb{P}^3$. Let $x \in A$: then the inverse image of H_x is given by the divisors P' of degree 3 on A such that P'_1, P'_2, P'_3 span a plane containing x. Thus, we have two possibilities: either we take the divisors P such that $P + x$ is linearly equivalent to the hyperplane divisor on A, and thus we get a fibre F, or we simply take the divisors P' of degree 3 for which $P' \geq x$, i.e., we get a divisor of type D.

Set for convenience $W := \mathbb{P} = A^{(3)}$ and observe that the pull back H_2 of the hyperplane divisor in $(\mathbb{P}^3)^\vee$ is thus linearly equivalent to $D + F$. Observe also that the pull back of the hyperplane divisor in $(\mathbb{P}^3)$ is linearly equivalent to $4F$. Therefore, the desired canonical model $X \subset W$ is in the linear system $|4D| = |4H_2 - H_1|$.

We can perhaps summarize these observations as follows:

Proposition 5.13. *The canonical model of a surface with $p_g = 4, q = 1, K^2 = 12$ of the Main Stream, i.e., of type i), is a divisor of bidegree $(-1, 4)$ on the variety W given by the intersection of the incidence variety $I \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee$ (itself a divisor of bidegree $(1, 1)$) with the pull back of the elliptic curve A under the first projection.*

Thus W is a complete intersection of type $(1, 1), (2, 0)(2, 0)$, but X is not a complete intersection. The canonical divisor on X is induced by the divisor of bidegree $(0, 1)$ on $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$.

From this it is easy to produce equations of explicit examples of these surfaces via computer algebra. The method is based on the following

Remark 5.14. *Since W is a complete intersection in $M := \mathbb{P}^3 \times (\mathbb{P}^3)^\vee$ it follows easily that the restriction homomorphism $\mathrm{H}^0(\mathcal{O}_M(n, 4)) \to \mathrm{H}^0(\mathcal{O}_W(n, 4))$ is surjective as soon as $n \geq 2$.*

Fix therefore a divisor $\bar{B}$ in $|\mathcal{O}_M(3, 0)|$, for instance the pull back of the three planes $\{x | x_0 x_1 x_2 = 0\}$. Then the linear system $|4D|$ on W is the residual system $|\mathrm{H}^0(\mathcal{O}_W(3, 4)(-\bar{B}))|$.

6. Surfaces with $p_g = 4, q = 3$ and canonical map of degree 1 or 2

In this section we shall consider surfaces with $p_g = 4, q = 3, K^2 = 12$, contained in an Abelian 3-fold as a polarization of type $(1, 1, 2)$: we will first show that this family is stable by small deformation.

Later, we will show that for a general such surface the canonical map is a birational morphism onto a surface of degree twelve in $\mathbb{P}^3$, whereas, for all the surfaces which are the pull back of a theta divisor on a principally polarized Abelian 3-fold, then the canonical map is of degree 2 onto an interesting sextic surface.

More precisely, our situation will be as follows: we let J be a principally polarized Abelian variety of dimension 3, which is the Jacobian of a curve C of genus 3, and we let Θ be its principal polarization. We let $\pi : A \to J$ be an isogeny of degree 2 and S a smooth divisor in the complete linear system $|\pi^*\Theta|$ associated to the pull back $\pi^*\Theta$.

Since at some step we will also need theta functions, we represent the Jacobian variety J as $J = \mathbb{C}^3/\mathbb{Z}^3 + \Omega\mathbb{Z}^3 := \mathbb{C}^3/\Lambda(\Omega)$, with Ω in the Siegel upper half-space (we have thus already represented the theta divisor as a symmetric divisor with respect to the origin in J).

We set then $A = \mathbb{C}^3/\Lambda$ with $\Lambda \subset \Lambda(\Omega)$ of index 2 dual under the symplectic pairing to $\Lambda(\Omega) + \mathbb{Z}b$, with $b \in 1/2\mathbb{Z}^3$ (e.g., $b = 1/2e_3$).

Proposition 6.1. *A basis of* $\mathrm{H}^0(A, \mathcal{O}_A(\pi^*\Theta))$ *is given by even functions.*

Proof. Let $c \in \{0, b\}$, and consider the basis given by the following two elements:

$$\theta[0, c](z, \Omega) := \sum_{n \in \mathbb{Z}^3} exp(2\pi i \ (1/2 \ {}^t n\Omega n + {}^t n(z + c))).$$

An elementary calculation shows that

$$\begin{aligned}
\theta[0, c](-z, \Omega) &= \sum_{n \in \mathbb{Z}^3} exp(2\pi i \ (1/2 \ {}^t n\Omega n + {}^t n(-z + c))) \\
&= \sum_{m = -n \in \mathbb{Z}^3} exp(2\pi i \ (1/2 \ {}^t m\Omega m + {}^t m(z - c))) \\
&= \theta[0, c](z, \Omega)
\end{aligned}$$

since, $\forall m \in \mathbb{Z}^3$, $exp(2\pi i({}^t m(-2c))) = 1$.

<div align="right">Q.E.D.</div>

Proposition 6.2. *Let S be a smooth surface in a polarization of type $(1, 1, 2)$ in an Abelian 3-fold A. Then the invariants of S are $p_g = 4, q = 3, K^2 = 12$.*

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(S) \to \omega_S \to 0$$

and observe that $\mathrm{H}^i(\mathcal{O}_A(S)) = 0$ for $i = 1, 2$. Whence, $p_g = h^0(\omega_S) = 4$, and $q := h^1(\mathcal{O}_S)(= h^1(\omega_S)$ by Serre duality$) = 3$.

Moreover, we have $K_S^2 = S^3 = 12$. <div align="right">Q.E.D.</div>

Proposition 6.3. *Let S be a smooth surface in a polarization of type $(1, 1, 2)$ in an Abelian 3-fold A. Then any small deformation is a surface of the same kind.*

Proof. Since the canonical divisor of A is trivial, the normal bundle of S in A is $N_S = \omega_S$, whence its cohomology groups have respective dimensions $h^0(N_S) = 4, h^1(N_S) = 3, h^2(N_S) = 1$.

The tangent sheaf sequence reads out as follows:

$$0 \to \mathcal{T}_S \to \mathcal{T}_A \otimes \mathcal{O}_S \cong \mathcal{O}_S^3 \to N_S \to 0,$$

whose exact cohomology sequence is:

$$0 \to \mathrm{H}^0(N_S)/\mathrm{H}^0(\mathcal{O}_S^3) \cong \mathbb{C} \to \mathrm{H}^1(\mathcal{T}_S) \to \mathrm{H}^1(\mathcal{T}_A \otimes \mathcal{O}_S) \to \mathrm{H}^1(N_S) \to \mathrm{H}^2(\mathcal{T}_S) \to ...$$

We get a smooth 7-dimensional family by varying A in its 6-dimensional local moduli space (Siegel's upper half space), and S in the corresponding 1-dimensional linear system.

This family will be shown to coincide with the Kuranishi family once we prove that $\mathrm{H}^1(\mathcal{T}_S)$ has dimension 7, or, equivalently, (since $\mathrm{H}^1(\mathcal{T}_A \otimes \mathcal{O}_S) \cong \mathrm{H}^1(\mathcal{O}_S^3) \cong \mathbb{C}^9$) we show the surjectivity of $\mathrm{H}^1(\mathcal{T}_A \otimes \mathcal{O}_S) \to \mathrm{H}^1(N_S)$.

To understand this map, consider an element $\sum_{i=1,2,3} \xi_i \otimes \psi_i \in \mathrm{H}^1(\mathcal{T}_A \otimes \mathcal{O}_S)$, where ξ_1, ξ_2, ξ_3 yield a basis of $\mathrm{H}^0(\mathcal{T}_A)$, $\psi_i \in \mathrm{H}^1(\mathcal{O}_S) \cong \mathrm{H}^1(\mathcal{O}_A)$.

Let $\{U_\alpha\}$ be an open cover of A such that $S \cap U_\alpha = div(f_\alpha)$, and let $f_\alpha = g_{\alpha,\beta} f_\beta$ in $U_\alpha \cap U_\beta$:

then the image of $\sum_{i=1,2,3} \xi_i \otimes \psi_i$ is given by $\sum_{i=1,2,3} \xi_i(f_\alpha) \otimes \psi_i$.

We use moreover the isomorphism $\mathrm{H}^1(N_S) \cong \mathrm{H}^2(\mathcal{O}_A)$: since for a vector field ξ we have $\xi(f_\alpha) = g_{\alpha,\beta}\, \xi(f_\beta)(mod\, f_\beta)$, the image of $\sum_{i=1,2,3} \xi_i \otimes \psi_i$ into $\mathrm{H}^2(\mathcal{O}_A)$ is the cohomology class $\sum_{i=1,2,3} \xi_i(g_{\alpha,\beta}) \cup \psi_i$.

We are quickly done, since

- the map $\xi \in \mathrm{H}^0(\mathcal{T}_A) \to \xi(g_{\alpha,\beta}) \in \mathrm{H}^1(\mathcal{O}_A)$ is an isomorphism, being the tangent map at the origin of the isogeny $\tau : A \to \mathrm{Pic}(A)$ such that $\tau(x) = S - (S + x)$
- $\mathrm{H}^1(\mathcal{O}_A) \cup \mathrm{H}^1(\mathcal{O}_A) \to \mathrm{H}^2(\mathcal{O}_A)$ is onto.

$$\text{Q.E.D.}$$

Theorem 6.4. *Let S be a smooth divisor yielding a polarization of type $(1, 1, 2)$ on an Abelian 3-fold: then the canonical map of S is in general a birational morphism onto a surface Σ of degree 12.*

In the special case where S is the inverse image of the theta divisor in a principally polarized Abelian 3-fold, the canonical map is a degree two morphism onto a sextic surface Σ in $\mathbb{P}^3$. In this case, the singularities of Σ are in general: a plane cubic Γ which is a double curve of nodal type for Σ and, moreover, a strictly even set of 32 nodes for Σ. Also, in this special case, the normalization of Σ is in fact the quotient of S by an involution ι on A having only isolated fixed points (on A), of which exactly 32 lie on S.

Proof. Observe that the natural map $\mathrm{H}^0(\Omega_A^2) \to \mathrm{H}^0(\Omega_S^2)$ is injective because S is not a subabelian variety, moreover we get in this way a linear subsystem of $|K_S|$ which is base point free, since S embeds into A.

It is easy to observe that each translation, and also each involution ι with fixed points on A (multiplication by -1 for a suitable choice of an origin) acts trivially on the vector space $H^0(\Omega_A^2)$.

On the other hand, considering the exact sequence in Prop. 6.2,

$$0 \rightarrow \omega_A \rightarrow \omega_A(S) \rightarrow \omega_S \rightarrow 0$$

we see that the 3-dimensional system generated by $H^0(\Omega_A^2)$ maps isomorphically to $H^1(\omega_A)$, whereas $H^0(\mathcal{O}_A(S)) \cong H^0(\omega_A(S))$ maps to $H^0(\omega_S)$ under the following explicit map

$$f(z) \rightarrow f(z)(dz_1 \wedge dz_2 \wedge dz_3)/d\theta(z),$$

where $S = div(\theta(z))$, $f(z), \theta(z) \in H^0(\mathcal{O}_A(S))$

are expressed by even functions, and $(dz_1 \wedge dz_2 \wedge dz_3)/d\theta(z)$ stands for the Poincare' Residuum $\eta := \eta_i := (dz_1 \wedge dz_2 \wedge dz_3) \neg (\partial/\partial z_i)(\partial\theta(z)/\partial z_i)^{-1}$ ($\neg$ is the contraction operator).

Whence follows that the involution $z \rightarrow -z$ acts on the image of $H^0(\mathcal{O}_A(S))$ in $H^0(\mathcal{O}_S(K_S))$ as multiplication by -1.

Let us now choose in particular a surface S which is the inverse image of a theta divisor Θ on J: then the subspace V_{++} coming from $H^0(\Omega_A^2)$ is the pull back of $H^0(\Omega_\Theta^2)$, so it consists of the sections in $H^0(\Omega_S^2) = H^0(\mathcal{O}_S(K_S))$ which are invariant under the fixed point free covering involution $z \rightarrow z + \eta$ for the double cover $\pi : S \rightarrow \Theta$.

On our particular surface S acts the group $(\mathbb{Z}/2)^2$ generated by $z \rightarrow z + \eta$ and by $z \rightarrow -z$ for our choice of the origin (c.f. Prop. 6.1), and we see that, if we define V_{--} as the one dimensional space coming from $H^0(\omega_A(S))$, then V_{--} is an eigenspace with eigenvalue -1 for both the involutions above.

In particular, it follows that the involution ι defined by $\iota(z) = -z + \eta$ acts trivially on the space $H^0(\mathcal{O}_S(K_S))$. Therefore the canonical map of such a special S factors through the involution ι.

Geometry of the situation for special surfaces

Let $Z := S/\iota$.

Lemma 6.5. *The involution ι has exactly 32 isolated fixed points on S.*

Proof of the lemma. Let us find the fixed points of ι recalling that $\iota(z) = -z + \eta$. Then z yields a fixed point on A iff $2z \equiv \eta \ (mod\Lambda)$. The fixed points moreover lie on S if and only if they project in $J = \mathbb{C}^3/\Lambda(\Omega)$ to a (2-torsion) point which lies on Θ, i.e., to an odd thetacharacteristic.

Set $\Lambda' := \Lambda(\Omega)$, thus $\eta \in \Lambda'$, whence for such a fixed point $2z \in \Lambda'$ and its image in $\Lambda'/\Lambda \cong \mathbb{Z}/2$ is non trivial.

Therefore the number of the odd thetacharacteristics which are image of such a fixed points are in bijection with the set $N \subset ((\mathbb{Z}/2)^3)^2$ defined by the following equations:

$$N = \{(x, y) |^t xy = 1, x_1 = 1\}.$$

Whence, $card(N) = 16$ and there are exactly 32 fixed points on S.

□

Remark 6.6. *Since the double cover $S \to Z$ is ramified exactly on the 32 corresponding nodes of Z, these form an even set according to the definition of Catanese [1981].*

Note moreover that ι acts as multiplication by -1 on the space of global $1-$forms, therefore the quotient surface Z has $q(Z) = 0, p_g(Z) = 4, K_Z^2 = 6$.

Then the canonical map of S, for S special, factors through Z. In turn, since V_{++} is base point free, this means that there is a point $O \in \mathbb{P}^3 - \Sigma$ so that the projection with centre O to $\mathbb{P}^2$ yields the composition of the projection onto Θ with the canonical map of Θ.

On the other hand, as well known, Θ is the symmetric product $C^{(2)}$ of a curve C of genus 3, which, since Θ is smooth, is a smooth plane quartic curve $C = C_4 \subset \mathbb{P}^2$.

CLAIM: the canonical map of $C^{(2)}$ sends the divisor $P + Q$ to the line generated by P and Q.

Proof of the claim. If $\omega_1, \omega_2, \omega_3$ are a basis of $\mathrm{H}^0(\Omega_C^1)$, then a basis of the canonical system of $C^{(2)}$ is given, on the Cartesian product C^2, by $\omega_i(P) \wedge \omega_j(Q) + \omega_i(Q) \wedge \omega_j(P)$, but this vector is the wedge product of the two vectors $\omega_i(P)$ and $\omega_j(Q)$.

That this is a morphism follows e.g. since its base locus on C^2 is just the diagonal Δ, but $\epsilon^* |K_{C^{(2)}}| = |K_{C^2} - \Delta|$, whence $|K_{C^{(2)}}|$ is free from base points.

□

We let now Y be the quotient of Θ by the multiplication by -1, whence $Y = S/(\mathbb{Z}/2)^2$: Y has $K_Y^2 = 3$, $q(Y) = 0, p_g(Y) = 3$ and its canonical map is a triple cover of $\mathbb{P}^2$, branched on the dual curve $C^\vee$ of C. In fact, multiplication by -1 on Θ corresponds to residuation with respect to K_C on $C^{(2)}$.

Y has 28 nodes, corresponding to the odd thetacharacteristics of C. The covering $Z \to Y$ is etale, except over 12 of the nodes of Y: as we saw, Z has exactly 32 nodes lying above the remaining 16 nodes of Y, over these 12 nodes lie instead 12 smooth points of Z.

Remark 6.7. *1) The bicanonical system of $C^{(2)}$ (cf. Catanese, Ciliberto and Mendes-Lopes [1998]) factors through the bicanonical system of Y, which embeds Y in $\mathbb{P}^6$, since it is induced by the sections of $\mathrm{H}^0(J, \mathcal{O}_J(2\Theta))$.*

2) The monodromy of $\Theta \to \mathbb{P}^2$ is the full symmetric group $\mathcal{S}_4$. The monodromy of the canonical map of Z is instead the symmetric group $\mathcal{S}_3$.

Lemma 6.8. *The image Σ of Z is a surface of degree 6(hence, birational to Z).*

Proof of the lemma. Consider the morphism $f : Z \to \mathbb{P}^2$, obtained as the composition of the canonical map ϕ of Z with the projection p with centre O of Σ to $\mathbb{P}^2$.

It cannot be that $\deg(\Sigma) = 2$, otherwise p would be branched on a plane conic, whereas the branch curve of f is the irreducible curve $C^\vee$.

If instead $\deg(\Sigma) = 3$, then there would be a covering involution i for ϕ. Since there is already a covering involution j for f, gotten from the double cover $Z \to Y$, we let G be the group of covering involutions for f. Since the canonical map of S does not factor through the one of Θ, it follows that $i \neq j$.

Then G is a group of order $h \geq 4$ with h dividing 6, thus $h = 6$ and f should be Galois.

This is however a contradiction, since the inverse image of the branch curve $C^\vee$ has components of multiplicity both 1 and 2; this holds because $Z \to Y$ is etale in codimension 1, while $Y \to \mathbb{P}^2$ has simple branching on the curve $C^\vee$, and the general tangent to $C^\vee$ is not a bitangent.

<div align="right">□</div>

With the result of the previous lemma in our hands, we can finish the proof of the theorem. Assume that the canonical map of S were always not birational.

Since for special S the degree equals 2, we would have that the canonical map always factors through the involution ι. But, since S always admits the involution $z \to -z$, then S would be stable under the involution $z \to z + \eta$, i.e., would be a pull back of a theta divisor. Contradicting that the Kuranishi family has dimension 7 and not 6.

Finally, in the special case, the surface Z is a canonical model with $K^2 = 6, p_g = 4, q = 0$ and with birational canonical map. Therefore, the double curve of Σ is a plane cubic, cf. Catanese [1984b].

<div align="right">Q.E.D.</div>

In the special case, the equations of Σ can be written explicitly. In fact, giving an unramified double covering of a non hyperelliptic curve C of genus 3 is equivalent (cf. e.g. Catanese [1981]) to writing the equation of its canonical model as the determinant of a 2×2 symmetric matrix of quadratic forms.

We have, more precisely, coordinates x_0, x_1, x_2 in $\mathbb{P}^2$ and quadratic forms $Q_{33}(x), Q_{34}(x), Q_{44}(x)$ such that

$$C = \{(x_0, x_1, x_2) | Q_{33}(x)Q_{44}(x) - Q_{34}(x)^2 = 0\};$$

moreover, the double unramified covering C' of C is the genus 5 curve whose canonical model in $\mathbb{P}^5$ is defined as the following intersection of three quadrics:

$$C' = \{(x_0, x_1, x_2, y_3, y_4) | y_3^2 = Q_{33}(x), y_4^2 = Q_{44}(x), y_3 y_4 = Q_{34}(x)\}.$$

Now, there is a natural surjection of $(C')^2$ onto S. In fact, Θ is the symmetric square of C, and thus dominated by C^2, and S is the quotient of $(C')^2$ under the $(\mathbb{Z}/2)^2$ action permuting the the coordinates and acting with the diagonal action of the involution $\iota : C' \to C'$.

We can then read the canonical map of S as the map corresponding to the $(\mathbb{Z}/2)^2$-invariant sections of $K_{(C')^2}$.

Recall that on the first curve of the product $(C')^2$ a basis of $H^0(K_{C'})^+$ is given by x_0, x_1, x_2, and a basis for $H^0(K_{C'})^-$ is given by y_3, y_4. Similarly we have a basis w_0, w_1, w_2, z_3, z_4 for the second curve.

We find therefore that a basis for the $(\mathbb{Z}/2)^2$-invariant sections of $K_{(C')^2}$ is provided by $u_0 := x_1 w_2 - x_2 w_1, u_1 := x_0 w_2 - x_2 w_0, u_2 := x_0 w_1 - x_1 w_0, v :=$

$y_3z_4 - y_4z_3$(these are just ι-invariant Plücker coordinates of the line spanned by the two points of C').

Let a, b, c be the symmetric 3×3 matrices yielding the respective quadratic forms $Q_{33}(x), Q_{34}(x), Q_{44}(x)$: then the entries of the matrix α are polynomial functions in the respective entries of a, b, c and in the coordinates (u_0, u_1, u_2, v) on $\mathbb{P}^3$.

The shape of α^+ is

$$\begin{pmatrix} v^5 + Av^3 + Bv & C \\ C & v \end{pmatrix}$$

where for instance $A =^t u(-2\Lambda^2 b + \Lambda^2(a + c) - \Lambda^2 a - \Lambda^2 b)u$, and

$$C = \det \begin{pmatrix} {}^txax & {}^twax & {}^twaw \\ {}^txbx & {}^twbx & {}^twbw \\ {}^txcx & {}^twcx & {}^twcw \end{pmatrix}.$$

We have not yet found a compact expression for B, the one we have is too long to be reproduced anywhere.

7.   Irregular surfaces with $p_g = 4, q = 2$

This section will be devoted to the description of another interesting example, of surfaces with the following invariants: $p_g = 4, q = 2, K^2 = 18$ and birational canonical morphism onto its image.

The surfaces are obtained as $(\mathbb{Z}/2\mathbb{Z})^2$-Galois covers of a principally polarized Abelian surface A, with branch locus consisting of 3 divisors D_1, D_2, D_3 which are algebraic equivalent to the theta divisor Θ. We shall follow the notation of Catanese [1984a].

We choose then L_1, L_2, L_3 divisors which are also algebraically equivalent to Θ, and such that

$$2L_i \equiv D_j + D_k, \forall i \neq j \neq k \neq i.$$

We take the corresponding $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover $\pi : S \to A$ such that

$$\pi_* \mathcal{O}_S = \mathcal{O}_A \bigoplus (\oplus_{i=1,2,3} \mathcal{O}_A(-L_i)), \pi_* \omega_S = \mathcal{O}_A \bigoplus (\oplus_{i=1,2,3} \mathcal{O}_A(L_i)).$$

It follows immediately that the constructed surfaces have the numerical invariants as desired: for instance, since K_S is the ramification divisor R, and $2R \equiv \pi^*(D)$, where $D = D_1 + D_2 + D_3$, we have $K_S^2 = R^2 = D^2 = 9\Theta^2 = 18$.

We recall the standard notation, by which $D_i = div(x_i)$, $R_i = div(z_i)$ so that S is defined by the equations

$$w_i^2 = x_j x_k, \ w_i x_i = w_j w_k$$

in the rank 3 bundle $(\oplus_{i=1,2,3} \mathcal{O}_A(L_i))$, and we have $z_i^2 = x_i, w_i = z_j z_k$.

We also let ϕ_i be the unique section of $\mathcal{O}_A(L_i))$, and $C_i := div(\phi_i)$.

With this notation, there are 4 sections of the canonical sheaf ω_S, namely : $\omega := z_1 z_2 z_3$, and $\forall i = 1, 2, 3, \omega_i := \omega\phi_i/w_i = z_i\phi_i$.

We obtain immediately that the base locus of the canonical system projects down in A to the set $D \cap (\cap_{i=1,2,3}(D_i + C_i))$.

Remark 7.1. *The surface S has base point free canonical system provided the 6 curves D_i, C_i have no point common to three of them. Since any three of the six curves can be chosen as arbitrary translates of the theta divisor, it follows easily that for a general choice there are no base points of K_S.*

We assume henceforth the canonical system to be base-point free, so that we have the canonical morphism

$$\Phi : S \to \Sigma \subset \mathbb{P}^3$$

and we use the characters of the Galois group in order to study the geometry of the map Φ and more generally the canonical ring of S.

We have here $\mathcal{R}(S) = \bigoplus_{m=0}^{\infty} \mathrm{H}^0(\mathcal{O}_S(mK_S))$ where

$$\mathrm{H}^0(\mathcal{O}_S(2mK_S)) = \mathrm{H}^0(\mathcal{O}_A(mD)) \bigoplus \mathrm{H}^0(\oplus_i \mathcal{O}_A(-L_i + mD))$$

$$\mathrm{H}^0(\mathcal{O}_S((2m+1)K_S)) = \omega \, \mathrm{H}^0(\mathcal{O}_A(mD)) \bigoplus (\oplus_i z_i \, \mathrm{H}^0(\mathcal{O}_A(+L_i + mD))).$$

We have 4 generators for $\mathcal{R}(S)$ in degree 1, namely, $\omega, \omega_1, \omega_2, \omega_3$, moreover we observe the following dimensions for the four respective eigenspaces in degree 2: $\dim(\mathcal{R}(S)_0) = 9$, $\dim(\mathcal{R}(S)_i) = 4$.

Lemma 7.2. $\Phi(S)$ *is not a quadric (if the canonical system is base point free).*

Proof. It suffices to show the linear independence of the 10 monomials ω^2, ω_i^2 and $\omega\omega_i, \omega_j\omega_k$. Any linear relation is a sum of linear relations in each eigenspace, and clearly, if i, j, k are distinct indices, then $\omega\omega_i, \omega_j\omega_k$ are independent since their divisors are $2R_i + R_j + R_k + C_i'$, $R_j + R_k + C_j' + C_k'$ respectively, C_i' being the inverse image of the divisor C_i.

Moreover, a linear relation of the form $\Sigma_{i=0,1,2,3}c_i\omega_i^2 = 0$ would translate into a relation $c_0 x_1 x_2 x_3 + \Sigma_{i=1,2,3}c_i x_i \phi_i^2 = 0$ and since w.l.o.g. we may assume that $c_3 = 1$, we obtain that ω_3 vanishes at the points where $x_1 = x_2 = 0$, contradicting that the canonical system is base point free.

<div align="right">Q.E.D.</div>

Theorem 7.3. *For general choice of the three divisors D_i the canonical map Φ is birational onto its image.*

Proof. Consider the 8 points $P \in C_1' \cap C_2' = \{\phi_1 = \phi_2 = 0\}$. They map to $(z_1 z_2 z_3(P), 0, 0, z_3\phi_3(P))$. Moreover, the inverse image of these points is contained in $\omega_1 = \omega_2 = 0$ which consists of these 8 points, plus points in R_1 or in R_2 which therefore map to points where the first coordinate equals 0.

Thus, the inverse image of the punctured line $y_1 = y_2 = 0$, $y_0 \neq 0$ consists of these 8 points, which form two $(\mathbb{Z}/2)^2$ orbits. For each point $(a, 0, 0, b)$ in the image, since by generality we may assume $a \neq b \neq 0 \neq a$, the inverse image consists therefore of either 2 or 4 points. However, $\deg(\Phi)\deg(\Sigma) = 18$, whence the only possibility that Φ may not be birational is that $\deg(\Phi) = 2$.

Assume this to be the case: then, since $(\mathbb{Z}/2)^2$ acts Φ-equivariantly on S and on $\mathbb{P}^3$, then we would have an involution i on A which would lift to S, and actually in such a way to centralize the Galois action. This however implies that i leaves the three branching divisors D_i invariant.

Consider then the curve D_1, which has genus 2. It possesses then only the hyperelliptic involution, or at most a finite number of involutions whose quotient is an elliptic curve. Since however we may choose D_2 to cut D_1 in any assigned pair of points of D_1, we easily get the desired contradiction.

<div align="right">Q.E.D.</div>

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