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Deformation in the large of some complex manifolds, I

This article is dedicated to the memory of Fabio Bardelli

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1. Introduction

The main theme of the present note is the study of the deformations in the large of compact complex manifolds. Even when these manifolds are Kähler, we shall study their deformations without imposing the Kähler assumption.

Recall in fact that (cf. e.g. [K-M71]) a small deformation of a Kähler manifold is again Kähler. This is however (cf. [Hir62]) false for non-small deformations.

In general, Kodaira defined two complex manifolds X', X to be **directly deformation equivalent** if there is a proper holomorphic submersion $\pi : \Xi \to \Delta$ of a complex manifold Ξ to the unit disk in the complex plane, such that X, X' occur as fibres of π. If we take the equivalence relation generated by direct deformation equivalence, we obtain the relation of **deformation equivalence**, and we say that X is a deformation of X' in the large if X, X' are deformation equivalent.

These two notions extend the classical notions of irreducible, resp. connected components of moduli spaces. However, outside of the realm of projective manifolds, not so much is known about deformations in the large of compact complex manifolds, since the usual deformation theory considers only the problem of studying the small deformations.

Just to give an idea of how limited our knowledge is, consider that only in a quite recent paper ([Cat02]), of which this one is a continuation, we gave a positive answer to a basic question raised by Kodaira and Spencer (cf. [K-S58], Problem 8, Section 22, p. 907 of volume II of Kodaira's collected works), showing that any deformation in the large of a complex torus is again a complex torus (in [K-S58] only the case $n = 2$ was solved, cf. Theorem 20.2).

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The usual strategy to determine the deformation equivalence class of a compact complex manifold X is to construct as big a family $\Xi \to \mathcal{B}$ as possible, where $\mathcal{B}$ is a connected analytic space, and then try to prove:

- The family is versal, i.e., for each fibre X_0, we get a local surjection $(\mathcal{B}, 0) \to$ *Def*(X_0) onto the Kuranishi family of X_0.
- Given any 1-parameter family $\pi : \Xi' \to \Delta$ with the property that there exists a sequence $t_\nu \to 0$ such that X_{t_ν}, $\forall \nu$, occurs as a fibre in the family Ξ, then the same property is also enjoyed by X_0.

In this paper we shall begin to go further, considering manifolds which are torus fibrations, and since we will make extensive use of results and techniques from [Cat02], in the second section we will reproduce and extend some of those for the reader's and our benefit. In particular one can find a complete proof of:

Theorem 2.1. *Every deformation of a complex torus of dimension n is a complex torus of dimension n.*[1]

In Section 3 we shall consider the general situation of a holomorphic fibration $f : X \to Y$ with all the fibres complex tori (equivalently, by Theorem 2.1, with all the fibres smooth and with one fibre isomorphic to a complex torus).

We shall set up a standard notation, and we shall give criteria for f to be a holomorphic torus bundle, respectively a principal holomorphic torus bundle. The main result is Proposition 3.4, describing explicitly all principal holomorphic torus bundles over curves.

Proposition 3.4. *Any principal holomorphic torus bundle X over a curve Y is a quotient* $X = L/N$ *of a suitable holomorphic* $(\mathbb{C}^*)^d$*-bundle L over Y by a suitable discrete cocompact subgroup N of* $(\mathbb{C}^*)^d$.

Section 4 treats first the general problem of determining the small deformations of torus bundles, and then the main **Theorem 4.4** asserts that when the base is a curve of genus ≥ 2, then all the small deformations of a principal holomorphic torus bundle are again holomorphic torus bundles.

The short Section 5 is devoted to showing that the families of principal holomorphic torus bundles on curves constructed in Section 4 yield all the deformations in the large of such manifolds.

Theorem 5.1. *A deformation in the large of a holomorphic principal torus bundle over a curve C of genus* $g \geq 2$ *with fibre a complex torus T of dimension d is again a holomorphic principal torus bundle over a curve* C' *of genus g.*

Section 6 is instead quite long and develops a classification theory for principal holomorphic torus bundles over tori. This theory bears close similarities with the theory of line bundles over tori, as follows. Namely, assume that we have such a torus bundle over $Y = V/\Gamma$ and with fibre $T = U/\Lambda$.

[1] Marco Brunella pointed out that a weaker result in this direction had been obtained by A. Andreotti and W. Stoll in the paper "Extension of holomorphic maps", Ann. Math. **72**, 312–349 (1960).

Then the homomorphism of fundamental groups $\pi_1(X) \to \pi_1(Y) = \Gamma$ is a central extension with kernel Λ and is completely classified by a bilinear alternating form $A : \Gamma \times \Gamma \to \Lambda$.

We have a vector analogue of the Riemann bilinear relation, since viewing A as a real element of

$$\Lambda^2(\Gamma \otimes \mathbb{C})^\vee \otimes (\Lambda \otimes \mathbb{C}) = \Lambda^2(V \oplus \bar{V})^\vee \otimes (U \oplus \bar{U}),$$

its component in $\Lambda^2(\bar{V})^\vee \otimes (U)$ is zero.

The Riemann bilinear relation enables us to construct a universal family of such bundles, for each given choice of the extension class A. Namely, we take first the possible subspaces $U \subset (\Gamma \otimes \mathbb{C})$, $V \subset (\Lambda \otimes \mathbb{C})$ such that the Riemann bilinear relation holds, and in this way we obtain a family which we call the standard Appell–Humbert family.

From this family we obtain the so-called complete Appell–Humbert family, which contains all such principal holomorphic bundles with given form A, as stated by:

Theorem 6.8. *Any holomorphic principal torus bundle with extension class isomorphic to $\epsilon \in H^2(\Gamma, \Lambda)$ occurs in the complete Appell–Humbert family $\mathcal{T}'\mathcal{B}_A$.*

The similarity with the theory of line bundles on tori, namely with the theorem of Appell–Humbert, is that we are able to explicitly write the classifying group cocycle as a linear function easily determined by the extension class and by the complex structures V on the base, respectively U on the fibre.

Later we show the advantages of having such a realization of these bundles via explicit cocycles, namely how one can explicitly calculate several holomorphic invariants of such bundles using the bilinear algebra data of the extension class, we have for instance:

Theorem 6.10. *The cokernel of $0 \to H^0(\Omega^1_Y) \to H^0(\Omega^1_X)$ is the subspace of $U^\vee$ which annihilates the image of the Hermitian part of A, i.e., of the component B'' in $[(V \otimes \bar{V})^\vee \otimes (U)]$.*

It follows in particular that X is parallelizable if and only if the Hermitian part of A is zero.

Corollary 6.11. *The space $H^0(d\mathcal{O}_X)$ of closed holomorphic 1-forms on X contains the pull-back of $H^0(\Omega^1_Y)$ with cokernel the subspace U^* of $U^\vee$ which annihilates the image of A, i.e., $U^* = \{\beta | \beta \circ A(z, \gamma) = 0 \; \forall \gamma, \; \forall z\}$.*

Theorem 6.12. *The cokernel of $0 \to H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_X)$ is the subspace of $\bar{U}^\vee$ which annihilates the image of the anti-complex component B' of A, i.e., of the conjugate of the component in $[(\Lambda^2 V)^\vee \otimes (U)]$.*

As the reader might have noticed, we give necessary and sufficient conditions for the parallelizability of X, thus showing (a fact already pointed out by Nakamura many years ago, cf. [Nak75]) easily that this notion is not deformation invariant.

Via similar calculations, we determine (Proposition 6.16) the cohomology algebra $\oplus_i H^i(\mathcal{O}_X)$ of the structure sheaf, and (Corollary 6.15) the cohomology groups of the tangent sheaf of X.

With these calculations we are able to show (Theorem 6.17) that the complete Appell–Humbert family is versal for the manifolds X corresponding to smooth points of the base of the family, if suitable assumptions hold for A, and in the case where the fibre dimension is 1.

Due to lack of time we defer to the future the investigation of the small deformations in the general case, which is a necessary step for the investigation of the deformations in the large.

We finish the section by considering in detail the classical example of the Iwasawa 3-fold, and of its small deformations.

Finally, in Section 7, we recall the definition (cf. [Cat02]) of Blanchard–Calabi torus fibrations, and using the result of Section 5 we obtain as a consequence that the space of complex structures on the differentiable manifold underlying the product of a curve with a complex torus of dimension 2 has several distinct deformation types, namely we have the following:

Corollary 7.8. *The space of complex structures on the product of a curve C of genus $g \geq 2$ with a four-dimensional real torus contains manifolds which are not deformation equivalent to each other, namely, Blanchard–Calabi 3-folds which are not Kähler and holomorphic principal bundles in the family $\mathcal{F}_{g,0}$.*

2. Tori

It is well known that complex tori are parametrized by a connected family (with smooth base space), inducing all the small deformations (cf. [K-M71]).

In fact, we have a family parametrized by an open set $\mathcal{T}_n$ of the complex Grassmann manifold $Gr(n, 2n)$, image of the open set of matrices

$\{\Omega \in Mat(2n, n; \mathbb{C}) | det(\Omega\overline{\Omega}) > 0\}$.

We recall this well-known fact: if we consider $\mathbb{Z}^{2n}$ as a fixed lattice, to each matrix Ω as above we associate the subspace $U = (\Omega)(\mathbb{C}^n)$, so that $U \in Gr(n, 2n)$ and $\mathbb{Z}^{2n} \otimes \mathbb{C} \cong U \oplus \bar{U}$.

Finally, to Ω we associate the torus $U/p_U(\mathbb{Z}^{2n})$, $p_U : U \oplus \bar{U} \to U$ being the projection onto the first addendum.

The above family will be the main object we shall have in mind.

We also want to recall from [Cat02] the following first result:

Theorem 2.1. *Every deformation of a complex torus of dimension n is a complex torus of dimension n.*

In fact this proof uses several useful lemmata (some of them well known) which will be here slightly generalized, for further use in the following. The first two are due to Kodaira (for the first, cf. [Ko64], Theorem 2, p. 1392 of collected works, vol. III)

Lemma 2.2. *On a compact complex manifold X one has an injection*

$$H^0(d\mathcal{O}_X) \oplus \overline{H^0(d\mathcal{O}_X)} \to H^1_{DR}(X, \mathbb{C}).$$

Proof. It suffices to show that the map, $H^0(d\mathcal{O}_X) \to H^1_{DR}(X, \mathbb{R})$, sending

$$\omega \to \omega + \bar{\omega}$$

is an injection. Or else, there is a function f with $df = \omega + \bar{\omega}$, whence $\partial f = \omega$ and therefore $\bar{\partial}\partial f = d(\omega) = 0$. Thus f is pluriharmonic, hence constant by the maximum principle. It follows that $\omega = 0$. □

Recall (cf. [Bla56] and [Ue75]) that for a compact complex manifold X, the Albanese variety $Alb(X)$ is the quotient of the complex dual vector space of $H^0(d\mathcal{O}_X)$ by the minimal closed complex Lie subgroup containing the image of $H_1(X, \mathbb{Z})$.

The Albanese map $\alpha_X : X \to Alb(X)$ is given as usual by fixing a base point x_0, and defining $\alpha_X(x)$ as the class in the quotient of the functional given by integration on any path connecting x_0 with x.

One says the the Albanese Variety is **good** if the image $H_1(X)$ of $H_1(X, \mathbb{Z})$ is discrete in $H^0(d\mathcal{O}_X)$, and **very good** if it is a lattice (a discrete subgroup of maximal rank). Moreover, the **Albanese dimension of** X is defined as the dimension of the image of the Albanese map.

With this terminology, we can state an important consequence of an inequality due to Kodaira:

Lemma 2.3. *On a compact complex manifold X one has an injection*

$$\overline{H^0(d\mathcal{O}_X)} \to H^1(X, \mathcal{O}_X).$$

In particular, if $b_1(X)$ is the first Betti number of X, we have the inequalities

$$2h^1(X, \mathcal{O}_X) \geq h^1(X, \mathcal{O}_X) + h^0(d\mathcal{O}_X) \geq b_1(X).$$

If both equalities hold, then X has a very good Albanese variety, of dimension $h^1(X, \mathcal{O}_X) = h^0(d\mathcal{O}_X) = \frac{1}{2}\, b_1(X)$.

Proof. We claim that the map $\overline{H^0(d\mathcal{O}_X)} \to H^1(X, \mathcal{O}_X) = H^1_{\bar{\partial}}(X, \mathcal{O}_X)$, is injective. Or else, we have $\bar{\omega}$ with $\partial\bar{\omega} = 0$ and $\bar{\partial}\bar{\omega} = 0$ such that there is a function f with $\bar{\partial} f = \bar{\omega}$, whence $\bar{\partial}\partial f = -d(\omega) = 0$. Thus we conclude as in the preceding lemma: f is pluriharmonic, whence constant, thus $\omega = 0$.

The second assertion follows easily from the exact cohomology sequence

$$H^0(d\mathcal{O}_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$$

and from the first.

Finally, if equality holds it follows that the injection

$$H^0(d\mathcal{O}_X) \oplus \overline{H^0(d\mathcal{O}_X)} \to H^1_{DR}(X, \mathbb{C})$$

is an isomorphism, whence $H_1(X, \mathbb{Z})/(Torsion)$ maps isomorphically to a lattice $H_1(X)$ in the dual vector space of $H^0(d\mathcal{O}_X)$, thus $Alb(X) = H^0(d\mathcal{O}_X)^\vee/(H_1(X))$. □

Lemma 2.4. *Assume that $\{X_t\}_{t\in\Delta}$ is a 1-parameter family of compact complex manifolds over the 1-dimensional disk, such that there is a sequence $t_\nu \to 0$ with X_{t_ν} satisfying the weak 1-Hodge decomposition*

$$H^1_{DR}(X_0,\mathbb{C}) = H^0(d\mathcal{O}_{X_0}) \oplus \overline{H^0(d\mathcal{O}_{X_0})}.$$

Then the weak 1-Hodge decomposition also holds on the central fibre X_0.

Proof. We have $f : \Xi \to \Delta$ which is proper and smooth, and $f_*(\Omega^1_{\Xi|\Delta})$ is torsion free, whence (Δ is smooth of dimension 1) it is locally free of rank $h \geq q := (1/2)\, b_1(X_0)$.

In fact, the base-change theorem asserts that there is (cf. [Mum70], II 5, and [GR84]) a complex of vector bundles on Δ,

$$(*) \qquad E^0 \to E^1 \to E^2 \to \ldots E^n \quad \text{s.t.}$$

1) $R^i f_*(\Omega^1_{\Xi|\Delta})$ is the i-th cohomology group of $(*)$; whereas,
2) $H^i(X_t, \Omega^1_{X_t})$ is the i-th cohomology group of $(*) \otimes \mathbb{C}_t$.

We may shrink the disk Δ so that the rank of $E_i \to E_{i+1}$ is constant for $t \neq 0$, and thus, as a consequence, there is an isomorphism between $H^0(X_t, \Omega^1_{X_t})$ and the stalk $f_*(\Omega^1_{\Xi|\Delta}) \otimes \mathbb{C}_t$.

We first see what happens on the central fibre instead, where the space $H^0(X_0, \Omega^1_{X_0})$ can have a higher dimension

Claim 1. There are holomorphic 1-forms $\omega_1(t), \ldots\omega_h(t)$ defined in the inverse image $f^{-1}(U_0)$ of a neighbourhood (U_0) of 0, and such that their restriction to X_t are linearly independent $\forall t \in U_0$.

Proof of Claim 1. Assume that $\omega_1(t), \ldots\omega_h(t)$ generate the direct image sheaf $f_*(\Omega^1_{\Xi|\Delta})$, but $\omega_1(0), \ldots\omega_h(0)$ are linearly dependent. Then, w.l.o.g. we may assume $\omega_1(0) \equiv 0$, i.e. there is a maximal m such that $\hat{\omega}_1(t) := \omega_1(t)/t^m$ is holomorphic. Then, since $\hat{\omega}_1(t)$ is a section of $f_*(\Omega^1_{\Xi|\Delta})$, there are holomorphic functions α_i such that $\hat{\omega}_1(t) = \Sigma_{i=1,..h}\alpha_i(t)\omega_1(t)$, whence it follows that $\hat{\omega}_1(t)(1 - t^m\alpha_1(t)) = \Sigma_{i=2,..h}\alpha_i(t)\omega_1(t)$.

This, however, contradicts the fact that $f_*(\Omega^1_{\Xi|\Delta})$ is locally free of rank h.

Claim 2. $H^0(d\mathcal{O}_{X_0})$ has dimension $\geq q$.

Proof of Claim 2. Let d_v be the vertical part of exterior differentiation, i.e., the composition of $d : \Omega^1_\Xi \to \Omega^2_\Xi$ with the projection $(\Omega^2_\Xi \to \Omega^2_{\Xi|\Delta})$.

It clearly factors through an $f^*\mathcal{O}_\Delta$-linear map $d'_v : \Omega^1_{\Xi|\Delta} \to \Omega^2_{\Xi|\Delta}$.

Taking direct images, we get a homomorphism of coherent sheaves $f_*(d'_v) : f_*(\Omega^1_{\Xi|\Delta}) \to f_*(\Omega^2_{\Xi|\Delta})$ whose kernel will be denoted by $\mathcal{H}$.

By our assumption, for each t_ν we have a q-dimensional subspace $H_{t_\nu} := H^0(d\mathcal{O}_{X_{t_\nu}})$ of $H^0(X_{t_\nu}, \Omega^1_{X_{t_\nu}})$.

Therefore, as $t_\nu \to 0$, we have a limit in the Grassmann manifold $Grass(q,h)$ of the family $\{H_{t_\nu}\}$, whence a subspace H_0 of dimension q. Taking suitable bases, $\omega_i(0)$ of H_0, resp. $\omega_i(t_\nu)$ of H_{t_ν}, we see that, since $\omega_i(t_\nu)$ is d_v-closed, by continuity it follows that also $\omega_i(0) \in H^0(d\mathcal{O}_{X_0})$.

End of the proof. It follows from Lemma 2.2 that, being $b_1 = 2q$, $H^1_{DR}(X_0, \mathbb{C}) = H^0(d\mathcal{O}_{X_0}) \oplus H^0(d\bar{\mathcal{O}}_{X_0})$.

Actually, by the base change theorem it follows that $\mathcal{H}$ is free of rank q and it enjoys the base change property that its stalk $\mathcal{H} \otimes \mathbb{C}_t$ corresponds to the subspace $H^0(d\mathcal{O}_{X_t})$. □

Corollary 2.5. *Assume that* $\{X_t\}_{t\in\Delta}$ *is a 1-parameter family of compact complex manifolds over the 1-dimensional disk, such that there is a sequence* $t_\nu \to 0$ *with* X_{t_ν} *satisfying the weak 1-Hodge property, and moreover, such that* X_{t_ν} *has Albanese dimension* $= a$.

Then the central fibre X_0 *has a very good Albanese variety, and has also Albanese dimension* $= a$.

Proof. We use the fact (cf. [Cat91]) that, when the Albanese variety is good, then the Albanese dimension of X is equal to $max\{i|\ \Lambda^i H^0(d\mathcal{O}_X) \otimes \Lambda^i H^0(d\bar{\mathcal{O}}_X) \to H^{2i}_{DR}(X, \mathbb{C})$ has non-zero image$\}$.

If the weak 1-Hodge decomposition holds for X, then the Albanese dimension of X equals (1/2) max$\{j|\Lambda^j H^1(X, \mathbb{C})$ has non-zero image in $H^j(X, \mathbb{C})\}$. But this number is clearly invariant by homeomorphisms.

Finally, the Albanese variety for X_0 is very good since the weak 1-Hodge decomposition holds for X_{t_ν} whence also for X_0 by the previous lemma. □

Remark 2.6. As observed in ([Cat95], 1.9), if a complex manifold X has a generically finite map to a Kähler manifold, then X is bimeromorphic to a Kähler manifold. This applies in particular to the Albanese map.

Theorem 1.1 follows from the following statement:

Theorem 2.7. *Let* X_0 *be a compact complex manifold such that its Kuranishi family of deformations* $\pi : \Xi \to \mathcal{B}$ *enjoys the property that the set* $\mathcal{B}(torus) := \{b|X_b$ *is isomorphic to a complex torus*$\}$ *has* 0 *as a limit point.*

Then X_0 *is a complex torus.*

We will use the following "folklore":

Lemma 2.8. *Let* Y *be a connected complex analytic space, and* Z *an open set of* Y *such that* Z *is closed for holomorphic 1-parameter limits (i.e., given any holomorphic map of the 1-disk* $f : \Delta \to Y$*, if there is a sequence* $t_\nu \to 0$ *with* $f(t_\nu) \in Z$*, then also* $f(0) \in Z$*). Then* $Z = Y$.

Proof of the lemma. By choosing an appropriate stratification of Y by smooth manifolds, it suffices to show that the statement holds for Y, a connected manifold.

Since it suffices to show that Z is closed, let P be a point in the closure of Z, and let us take coordinates such that a neighbourhood of P corresponds to a compact polycylinder H in $\mathbb{C}^n$.

Given a point in Z, let H' be a maximal coordinate polycylinder contained in Z. We claim that H' must contain H, or else, by the holomorphic 1-parameter limit property, the boundary of H' is contained in Z, and since Z is open, by compactness we find a bigger polycylinder contained in Z, a contradiction which proves the claim. □ for the lemma.

Proof of Theorem 2.7. It suffices to consider a 1-parameter family ($\mathcal{B} = \Delta$) whence we may assume w.l.o.g. (cf. the proof of Lemma 2.4) that the weak 1-Hodge decomposition holds for each $t \in \Delta$.

By integration of the holomorphic 1-forms on the fibres (which are closed for t_ν and for 0), we get a family of Albanese maps $\alpha_t : X_t \to J_t$, which fit together in a relative map $a : \Xi \to J$ over Δ (J_t is the complex torus $(H^0(d\mathcal{O}_{X_t}))^\vee / H_1(X_t, \mathbb{Z})$).

Apply once more vertical exterior differentiation to the forms $\omega_i(t)$: $d_v(\omega_i(t))$ vanishes identically on X_0 and on X_{t_ν}, whence it vanishes identically in a neighbourhood of X_0, and therefore these forms $\omega_i(t)$ are closed for each t.

Therefore our map $a : \Xi \to J$ is defined everywhere and it is an isomorphism for $t = t_\nu$. Whence, for each t, α_t is surjective and has degree 1.

To show that α_t is an isomorphism for each t it suffices therefore to show that a is finite.

Assume the contrary: then there is a ramification divisor R of a, which is exceptional (i.e., if $B = a(R)$, then $dim B < dim R$).

By our hypothesis α_t is an isomorphism for $t = t_\nu$, thus R is contained in a union of fibres, and since it has the same dimension, it is a finite union of fibres. But if R is not empty, we reach a contradiction, since then there are some t's such that α_t is not surjective. □

We have also a more abstract result:

Proposition 2.9. *Assume that X has the same integral cohomology algebra of a complex torus and that $H^0(d\mathcal{O}_X)$ has dimension equal to $n = dim(X)$. Then X is a complex torus.*

Proof. Since $b_1(X) = 2n$, it follows from 1.2 that the weak 1-Hodge decomposition holds for X and that the Albanese variety of X is very good.

That is, we have the Albanese map $\alpha_X : X \to J$, where J is the complex torus $J = Alb(X)$. We want to show that the Albanese map is an isomorphism. It is a morphism of degree 1, since α_X induces an isomorphism between the respective fundamental classes of $H^{2n}(X, \mathbb{Z}) \cong H^{2n}(J, \mathbb{Z})$.

There remains to show that the bimeromorphic morphism α_X is finite.

To this purpose, let R be the ramification divisor of $\alpha_X : X \to J$, and B its branch locus, which has codimension at least 2. By means of a sequence of blowing ups of J with non-singular centres we can dominate X by a Kähler manifold $g : Z \to X$ (cf. [Cat95] 1.8, 1.9).

Let W be a fibre of α_X of positive dimension such that $g^{-1}W$ is isomorphic to W. Since Z is Kähler, $g^{-1}W$ is not homologically trivial, whence we find a differentiable submanifold Y of complementary dimension which has a positive intersection number with it. But then, by the projection formula, the image g_*Y has positive intersection with W, whence W is also not homologically trivial. However, the image of the class of W is 0 on J, contradicting that α_X induces an isomorphism of cohomology (whence also homology) groups. □

Remark 2.10. The second condition holds true as soon as the complex dimension n is at most 2. For $n = 1$ this is well known, for $n = 2$ this is also known, and due to Kodaira ([Ko64]): since for $n = 2$ the holomorphic 1-forms are closed, and moreover $h^0(d\mathcal{O}_X)$ is at least $[(1/2)b_1(X)]$.

For $n \geq 3$, the real dimension of X is greater than 5, whence, by the s-cobordism theorem ([Maz63]), the assumption that X is homeomorphic to a complex torus is equivalent to the assumption that X is diffeomorphic to a complex torus.

André Blanchard ([Bla53]) constructed in the early 50s an example of a non-Kähler complex structure on the product of a rational curve with a two-dimensional complex torus. In particular his construction (cf. [Somm75]) was rediscovered by Sommese, with a more clear and more general presentation, who pointed out that in this way one would produce exotic complex structures on complex tori.

However, the following remains open:

Question. Let X be a compact complex manifold of dimension $n \geq 3$ and with trivial canonical bundle such that X is diffeomorphic to a complex torus: is X then a complex torus?

The main problem here is to show the existence of holomorphic 1-forms, so it may well happen that this question also has a negative answer.

3. Generalities on holomorphic torus bundles

Throughout the rest of the paper, our set up will be the following: we have a holomorphic submersion between compact complex manifolds

$$f : X \to Y,$$

such that one fibre F (whence all the fibres, by Theorem 2.1) is a complex torus.

We shall denote this situation by saying that f is differentiably a torus bundle.

We let $n = dim X$, $m = dim Y$, $d = dim F = n - m$.

The case where $d = 1$ is very special because the moduli space for 1-dimensional complex tori exists and is isomorphic to $\mathbb{C}$. Whence it follows that in this situation (unlike the case $d \geq 2$) f is a holomorphic fibre bundle.

In any dimension, we have (e.g. by a much more general theorem of Grauert and Fischer (cf. [FG65])) that f is a holomorphic bundle if and only if all the smooth fibres are biholomorphic.

For any differentiable torus bundle we have a local system on Y,

(1) $$\mathbb{H} := \mathcal{R}^1 f_* \mathbb{Z}_X$$

and, by the exponential sequence $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$, if we define

(2) $$\mathcal{V}^\vee := \mathcal{R}^1 f_* \mathcal{O}_X,$$

then we get another exact sequence

(3) $$0 \to W^\vee \to \mathbb{H} \otimes \mathcal{O}_Y \to \mathcal{V}^\vee \to 0$$

of holomorphic vector bundles, where

(4) $$W^\vee := f_* \Omega^1_{X|Y}.$$

Here, $\mathbb{H}$ yields an $\mathbb{R}$-basis of $\mathcal{V}^\vee$ at each point, and the same holds for $\mathbb{H}^\vee := Hom_{\mathbb{Z}}(\mathbb{H}, \mathbb{Z})$ via the exact sequence

$$0 \to \mathcal{V} \to \mathbb{H}^\vee \otimes \mathcal{O}_Y \to W \to 0 \tag{5}$$

so that one obtains another differentiable bundle of complex tori, the so-called Jacobian of X, which is a bundle of Lie groups

$$Jac(X) := W/\mathbb{H}^\vee.$$

By Hodge symmetry the real flat bundle $\mathbb{H}^\vee \otimes \mathcal{O}_Y$ splits as a direct sum $\mathcal{V} \oplus \bar{\mathcal{V}}$, and $W \equiv \bar{\mathcal{V}}$ as complex vector bundles. We finally have the cotangent sheaves exact sequence

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to f^*W^\vee \to 0 \tag{6}$$

by which it follows

$$K_X \equiv f^*(K_Y + detW^\vee). \tag{7}$$

In general, the derived direct image cohomology sequence of (6)

$$0 \to \Omega^1_Y \to f_*\Omega^1_X \to f_*\Omega^1_{X|Y} \to \Omega^1_Y \otimes \mathcal{R}^1 f_*\mathcal{O}_X \to .. \tag{8}$$

is such that the coboundary map is given by the Kodaira–Spencer class in $H^0(\Omega^1_Y \otimes \mathcal{R}^1 f_*\mathcal{H}om(\Omega^1_{X|Y}, \mathcal{O}_X))$, which vanishes exactly when f is a holomorphic fibre bundle. In this case we have then an exact sequence

$$0 \to \Omega^1_Y \to f_*\Omega^1_X \to f_*\Omega^1_{X|Y} \to 0. \tag{8}$$

Remark 3.1. An immediate corollary of (6) is that X is complex parallelizable, i.e., Ω^1_X is trivial, only if Y is parallelisable and W is trivial on Y. If Ω^1_Y and W are trivial, then X is parallelizable if furthermore the extension class of (6), is trivial. This extension class lies in $H^1(f^*(\Omega^1_Y \otimes W))$. If, moreover, we have a holomorphic bundle, the class lies in $H^1(\Omega^1_Y \otimes W)$ and X is complex parallelizable if and only if Y is parallelizable, W is trivial on Y, and (8) splits.

Assume now that we have a holomorphic torus fibre bundle, thus we have the exact sequence

$$0 \to \Omega^1_Y \to f_*\Omega^1_X \to W^\vee \to 0. \tag{8'}$$

We have the well known

Proposition 3.2. *A differentiable bundle of d-dimensional tori is a principal holomorphic bundle if and only if we have an exact sequence*

$$0 \to \Omega^1_Y \to f_*\Omega^1_X \to \mathcal{O}^d_Y \to 0. \tag{8'}$$

Proof. f is a holomorphic bundle if and only if the Kodaira–Spencer class is identically zero, i.e., (8) is exact. In general (cf. [BPV84]) if T is a complex torus, we have an exact sequence of complex Lie groups

$$0 \to T \to Aut(T) \to \mathbb{M} \to 1,$$

where $\mathbb{M}$ is discrete. Taking sheaves of germs of holomorphic maps with source Y we get

$$0 \to \mathcal{H}(T)_Y \to \mathcal{H}(Aut(T))_Y \to \mathbb{M} \to 1$$

and we know that holomorphic bundles with base Y and fibre T are classified by the cohomology group $H^1(Y, \mathcal{H}(Aut(T))_Y)$. The exact sequence

$$0 \to H^1(Y, \mathcal{H}(T)_Y) \to H^1(Y, \mathcal{H}(Aut(T)))_Y \to H^1(Y, \mathbb{M})$$

determines when a holomorphic bundle is a principal holomorphic bundle. In this case the cocycles are in $H^1(Y, \mathcal{H}(T)_Y)$, i.e., they have values in the translation group, whence W is a trivial bundle.

Conversely, if W is trivial, we may first choose local coordinates (y) on a small neighbourhood $U \subset Y$ and local coordinates $(x) = (u', y)$ on X with $f(x) = y$, then we may choose a basis $w_1(y), ..w_d(y)$ of $W^\vee$, lift these to local holomorphic 1-forms on $f^{-1}(U)$, $\omega_1(y), ..\omega_d(y)$ and then take the linear coordinates $u_i := \int_{(0,y)}^x \omega_i(u', y)$.

On the universal cover of $f^{-1}(U)$, we get functions $(y_1, ..y_m), (u_1, ..u_d)$ whose differentials give a basis of Ω^1_X. Moreover, if we go to another open set $V \subset Y$, the new linear coordinates $(v_1, ..v_d)$ are such that $v_i - u_i$ is a function of $y \in U \cap V$. □

In the case of a principal holomorphic bundle it is useful to write $T = \mathbb{C}^d/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2d}$, thus we have the exact sequence

$$\begin{aligned} \to H^0(\mathcal{H}(T)_Y) \to H^1(Y, \Lambda) \to H^1\big(Y, \mathcal{O}_Y^d\big) \\ \to H^1(\mathcal{H}(T)_Y) \to^c \to H^2(Y, \Lambda). \end{aligned} \tag{9}$$

The significance of the homomorphism c is readily offered by topological considerations.

In fact, the homotopy exact sequence of a bundle f

$$\pi_2(Y) \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1 \tag{10}$$

determines an extension

$$1 \to \pi_1(F) \to \Pi \to \pi_1(Y) \to 1, \tag{10'}$$

where the action of $\pi_1(Y)$ on $\pi_1(F)$ by conjugation is precisely the monodromy automorphism. For a principal torus bundle the monodromy is trivial, indeed $\mathbb{H} \otimes \mathcal{O}_Y$ is a trivial differentiable bundle, thus the extension (10') is central, and is therefore classified by ($\Lambda = \pi_1(T)$, and $F = T$) a cohomology class $\epsilon \in H^2(Y, \Lambda)$.

The following situation is very interesting: let us consider a Stein manifold Y' whose cohomology group $H^2(Y', \mathbb{Z}) = 0$, and a holomorphic map $h : Y' \to Y$.

Then we may replace our principal bundle with compact base Y, with its pull-back via h. It follows from the previous considerations that the pull-back is a product $Y' \times T$ (since $\Lambda \cong \mathbb{Z}^d$, and in view of (9)). Actually, this holds more generally if the pull-back to Y' of ϵ is trivial.

Noteworthy special cases may be:

- Y' is the universal cover of Y, e.g. if Y is a curve or a complex torus.
- Y' is an open set of Y, for instance, if Y is a curve, any open set $\neq Y$ satisfies our hypotheses.
- Y' is $(\mathbb{C}^*)^m$ and the pull-back of the class ϵ is trivial.

Nevertheless, the case where Y is a curve can be more easily described via the following construction (cf. e.g. [BPV84, pp. 143–149]):

Example 3.3. Let Y be a complex manifold and let L be a principal holomorphic $(\mathbb{C}^*)^d$-bundle over Y, thus classified by a cohomology class $\xi \in H^1(Y, \mathcal{O}_Y^*)^d$. Let us consider any complex torus $T = \mathbb{C}^d/\Lambda$, and let us realize it as a quotient $(\mathbb{C}^*)^d/N$. Then we can form the quotient L/N, which is a holomorphic fibre bundle with fibre $\cong T$.

Proposition 3.4. *Any principal holomorphic torus bundle X over a curve Y is a quotient $X = L/N$ of a suitable holomorphic $(\mathbb{C}^*)^d$-bundle L over Y.*

Proof. It suffices (cf. loc. cit.) to show that there is a primitive embedding $i : \mathbb{Z}^d \to \Lambda$ and a class η in $H^2(Y, \mathbb{Z}^d)$ such that the class ϵ corresponding to X equals $i_*(\eta)$. Consider, however, that $H^2(Y, \Lambda) \cong H^2(Y, \mathbb{Z}) \otimes \Lambda$: thus there is a primitive embedding of $\mathbb{Z}$ in Λ with the property that the fundamental class of Y maps to ϵ. It suffices to extend the primitive embedding of $\mathbb{Z}$ to one of $\mathbb{Z}^d$ into Λ, and define η as the image of the fundamental class for the homomorphism induced by the embedding $\mathbb{Z} \subset \mathbb{Z}^d$. □

4. Small deformations of torus bundles

Let us briefly consider again the case of a general differentiable torus bundle.

Consider the exact sequence

$$0 \to f^*(W) \to \Theta_X \to f^*(\Theta_Y) \to 0 \tag{12}$$

and the derived direct image sequence

$$\begin{aligned} 0 \to (W) \to f_*\Theta_X \to (\Theta_Y) \to \\ \to (\mathcal{V}^\vee \otimes W) \to \mathcal{R}^1 f_*\Theta_X \to (\Theta_Y) \otimes \mathcal{V}^\vee \to \\ \to (\Lambda^2(\mathcal{V}^\vee) \otimes W) \to \mathcal{R}^2 f_*\Theta_X \to (\Theta_Y) \otimes \Lambda^2(\mathcal{V}^\vee) \to, \end{aligned} \tag{13}$$

where we used that $\mathcal{R}^2 f_*(\mathcal{O}_X) \cong \Lambda^2(\mathcal{V}^\vee)$.

Notice that the coboundaries are given by cup product with the Kodaira–Spencer class. Thus, in the case where we have a holomorphic bundle, all rows are exact.

Remark 4.1. In general, by a small variation of a theorem of E. Horikawa concerning the deformations of holomorphic maps, namely Theorem 4.9 of [Cat91], we obtain in particular that, under the assumption $H^0((\Theta_Y) \otimes V^\vee) = 0$, we have a smooth morphism $\mathit{Def}(f) \to \mathit{Def}(X)$.

Let's now assume that we have a holomorphic principal torus bundle, thus W and V are trivial holomorphic bundles on Y.

Proposition 4.2. *Let $f : X \to Y$ be a holomorphic principal torus bundle and assume that $H^0(\Theta_Y) = 0$. Then every small deformation of X is a holomorphic principal torus bundle over some small deformation of Y.*

Proof. Since $\mathcal{V}$ is trivial, by the previous remark, every small deformation of X is induced by a deformation of the map $f : X \to Y$.

That is, if we consider the respective Kuranishi families, we have holomorphic maps of $\mathit{Def}(f)$ to $\mathit{Def}(X)$, $\mathit{Def}(Y)$, such that the first is onto and smooth.

On the other hand, $H^1(f^*W)$ is the subspace $T^1(X|Y)$ in Flenner's notation (cf. [Flen79]), kernel of the tangent map $\mathit{Def}(f) \to \mathit{Def}(Y)$, thus we infer by the smoothness of $\mathit{Def}(f) \to \mathit{Def}(X)$, and since we have an exact sequence

$$0 \to H^1(f^*W) \to H^1(\Theta_X) \to H^1(\Theta_Y) \to \dots,$$

that indeed we have an isomorphism $\mathit{Def}(f) \cong \mathit{Def}(X)$.

Then we have a morphism $\mathit{Def}(f) \cong \mathit{Def}(X) \to \mathit{Def}(Y)$ and therefore, since any deformation of a torus is a torus, we get that any deformation of X is a torus bundle over a deformation of Y.

However, the local system $\mathbb{H}$ is trivial, and therefore it also remains trivial after deformation. Whence, it follows that also the bundles V, W remain trivial, since they are both generated by d global sections, by virtue of (3) and (5).

Therefore, the Jacobian bundles remain trivial, so all the fibres are isomorphic and every small deformation is a holomorphic bundle.

Since the classifying class in $H^1(Y, \mathbb{M})$ is locally constant, we finally infer that we get only principal holomorphic bundles. □

Remark 4.3. The cohomology group $H^1(f^*W)$, by the Leray spectral sequence, has a filtration whose graded quotient maps to a subspace of

$$H^1(W) \oplus H^0\big(Y, \mathcal{R}^1 f_* \mathcal{O}_X \otimes W\big) = H^1\big(\mathcal{O}_Y^d\big) \oplus H^0\big(Y, \mathcal{O}_Y^d \otimes \mathcal{O}_Y^d\big) = H^1\big(\mathcal{O}_Y^d\big) \oplus \mathbb{C}^{d \times d}.$$

The meaning of the above splitting is explained by the exact sequence (9): the left-hand side stands for the deformation of the given bundle with fixed fibre, the right-hand side stands for the local deformation of the torus T.

A typical case where the previous result applies is the one where Y is a curve of genus $g \geq 2$.

Theorem 4.4. *Let $f : X \to Y$ be a holomorphic principal torus bundle over a curve Y of genus $g \geq 2$, and with fibre a d-dimensional complex torus T. Then the Kuranishi family of deformations of X is smooth of dimension $3g - 3 + dg + d^2$,* *it has a smooth fibration onto* $Def(Y) \times Def(T)$*, its fibres over a point* (Y', T') *are given by the deformations parametrized by* $H^1(\mathcal{O}_{Y'}{}^d)$ *and inducing all the holomorphic principal torus bundles over* Y' *with fibre* T' *and given topological class in* $H^2(Y', \mathbb{Z}^{2d})$.

Proof. We can use Proposition 3.4 to construct, once a multidegree $m := (m_1, ..m_d)$ is fixed, a family $\mathcal{F}_{g,m}$ with smooth base, parametrizing quotients $X = L/N$, where L is a $(\mathbb{C}^*)^d$-bundle with multidegree m over a smooth curve Y.

Here, the parameter N varies in a smooth connected d^2-dimensional parameter space for complex tori $(\mathbb{C}^*)^d/N$, while the pair (Y, L) varies in the fibre product of the Picard bundles Pic^{m_i} over a universal family parametrizing all smooth curves of genus g.

By the proof of our previous proposition, our smooth family $\mathcal{F}_{g,m}$ has a bijective Kodaira–Spencer map, whence it is locally isomorphic to the Kuranishi family $Def(X)$.

The other assertions then follow easily. □

Remark 4.5. Looking more carefully, we see that the choice of a multidegree $m := (m_1, ..m_d)$ is not unique. Indeed, one has a well-defined element in $H^2(Y, \Lambda) = H^2(Y, \mathbb{Z}) \otimes \Lambda \cong \Lambda$, where the isomorphism is unique since a complex structure on Y fixes the orientation. Thus we have as invariant a vector in a lattice $\Lambda \cong \mathbb{Z}^{2d}$, whose only invariant is the divisibility index μ (we define $\mu = 0$ if the class is zero).

Whence, we can reduce ourselves to consider only families $\mathcal{F}_{g,\mu}$.

5. Deformations in the large of torus bundles over curves

As was already pointed out, the examples of Blanchard, Calabi and Sommese (cf. especially [Somm75]) show that a manifold diffeomorphic to a product $C \times T$, where C is a curve of genus $g \geq 2$, and T is a complex torus, need not be a holomorphic torus bundle over a curve. The deformation theory of Blanchard–Calabi varieties and, more generally, of differential torus bundles offers interesting questions (cf. [Cat02]).

But, for the more narrow class of varieties which are holomorphic torus bundles over a curve of genus ≥ 2 (this includes as a special case the products $C \times T$, where T is a complex torus, which were shown in [Cat02] to be closed for taking limits) we have a result quite similar to the one we have for tori.

Theorem 5.1. *A deformation in the large of a holomorphic principal torus bundle over a curve* C *of genus* $g \geq 2$ *with fibre a complex torus* T *of dimension* d *is again a holomorphic principal torus bundle over a curve* C' *of genus* g.

This clearly follows from the more precise statement. Fix an integer $g \geq 2$*, and a multidegree* $m := (m_1, ..m_d) \in \mathbb{Z}^d$*: then every compact complex manifold* X_0 *such that its Kuranishi family of deformations* $\pi : \Xi \to \mathcal{B}$ *enjoys the property that the set* $\mathcal{B}'' := \{b | X_b$ *is isomorphic to a manifold in the family* $\mathcal{F}_{g,m}$ *considered in* Theorem 4.4$\}$ *has* 0 *as a limit point. Then* X_0 *is also isomorphic to a manifold in the family* $\mathcal{F}_{g,m}$.

Proof. Observe first of all that $\mathcal{B}''$ is open in the Kuranishi family $\mathcal{B}$, by the property that the Kuranishi family induces a versal family in each neighbouring point, and by Theorem 4.4.

Whence, by Lemma 2.8 we can limit ourselves to consider the situation where $\mathcal{B}$ is a 1-dimensional disk, and there exists a sequence $t_\nu \to 0$ such that X_{t_ν}, $\forall \nu$, is a principal holomorphic torus bundle over a curve C_{t_ν}.

Claim 1. The first important property is that, for such a holomorphic principal bundle $X = X_{t_\nu}$ as above, the subspace $f_{t_\nu}^*(H^1(C_{t_\nu}, \mathbb{C}))$ is a subspace V of $H^1(X, \mathbb{C})$, of dimension $2g$, such that each isotropic subspace U of $H^1(X, \mathbb{C})$ (i.e., such that the image of $\Lambda^2(U) \to H^2(X, \mathbb{C})$ is zero) is contained in V.

Moreover, if the bundle is not topologically trivial, V itself is an isotropic subspace, whereas, if the bundle is topologically trivial, then any maximal isotropic subspace has dimension g.

In both cases, if an isotropic subspace U has dimension r and $U + \bar{U}$ has dimension $2r$, then $r \leq g$.

Proof of Claim 1. Assume that U is isotropic. Notice that we have an injection $U \subset V \oplus W$, where W injects into $H^1(F, \mathbb{Q})$ by virtue of the Leray spectral sequence yielding

$$H^1(X, \mathbb{Q})/H^1(Y, \mathbb{Q}) = ker(H^0(Y, \mathcal{R}^1 f_*(\mathbb{Q})) \to H^2(Y, \mathbb{Q})).$$

Observe that $H^0(Y, \mathcal{R}^1 f_*(\mathbb{Q})) = H^1(F, \mathbb{Q})$ since we have a principal bundle, and, moreover, that the above homomorphism in the spectral sequence is determined by the cohomology class $\epsilon \in H^2(Y, \Lambda)$.

In particular, the above map is trivial if and only if the bundle is topologically trivial.

In this case it follows by the Künneth formula that the wedge product yields an injection $(W \otimes V) \oplus (\Lambda^2 W) \to H^2(X, \mathbb{C})$.

Whence, if we have two non-proportional cohomology classes with trivial wedge product in cohomology, $(v_1 + w_1) \wedge (v_2 + w_2) = 0$ and we assume w.l.o.g. $w_2 \neq 0$, then first of all $\exists\, c \in \mathbb{C}$ $s.t.$ $w_1 = cw_2$.

Then also $c \neq 0$, $v_1 = cv_2$; which is a contradiction.

Assume now the bundle not to be topologically trivial: then $H^0(Y, \mathcal{R}^1 f_*(\mathbb{C})) \to H^2(Y, \mathbb{C}))$ is non trivial and $H^2(Y, \mathbb{C})$ maps to zero in the cohomology of X, thus V is an isotropic subspace.

However, then the Leray spectral sequence at least guarantees that $H^1(Y, H^0(Y, \mathcal{R}^1 f_*(\mathbb{C})) = V \otimes H^1(F, \mathbb{C})$ is a direct summand in $H^2(X, \mathbb{C})$.

Since moreover W is killed by the linear form $\epsilon : H^1(F, \mathbb{Q}) \to H^2(Y, \mathbb{C}) \cong \mathbb{C}$, $(\Lambda^2 W)$ embeds into the quotient $H^2(X, \mathbb{C})/V \otimes H^1(F, \mathbb{C})$.

Thus again we have an injection $(W \otimes V) \oplus (\Lambda^2 W) \to H^2(X, \mathbb{C})$, and the same argument as in the trivial case applies.

Step 2. *There is a morphism $F : \Xi \to \mathcal{C}$, where $\mathcal{C} \to \mathcal{B}$ is a smooth family of curves of genus g.*

Proof of Step 2. For this purpose we use ideas related to the classical Castelnuovo–de Franchis theorem and to the isotropic subspace theorem (cf. [Cat91]).

From the differentiable triviality of our family $\Xi \to \Delta$ and by Claim I follows that we have a uniquely determined subspace V of the cohomology of X_0, that we may freely identify to the previously considered subspace V, for each X_{t_ν}.

Now, for each t_ν, we have a decomposition $V = U_{t_\nu} \oplus \overline{U_{t_\nu}}$, where $U_{t_\nu} = f^*(H^0(\Omega^1_{C_{t_\nu}}))$.

By the compactness of the Grassmann variety and by the weak 1-Hodge decomposition in the limit, the above decomposition also holds for X_0, and U_0 is an isotropic subspace in $H^0(d\mathcal{O}_{X_0})$. The Castelnuovo–de Franchis theorem applies, and we get a holomorphic map f_0 to a curve Y_0 of genus $\geq g$. But this genus must equal g, since we noticed in the statement of Claim I that if U_0 isotropic, and $U_0 + \bar{U}_0 = U_0 \oplus \bar{U}_0$, then $dim(U_0) \leq g$.

Consider the vector bundle $f_* \Omega^1_{\Xi|\Delta}$. As in the proof of Lemma 2.4 we infer the existence of a vector (sub-)bundle $\mathcal{H}$ such that its stalk at t yields a subspace of $H^0(d\mathcal{O}_{X_t})$.

We have a map of the complex vector bundle $\mathcal{H} \bigoplus \bar{\mathcal{H}}$ to the trivial vector bundle $H^1(X_t, \mathbb{C}) = V \oplus W'$.

Since the construction of the Kuranishi family can be done by fixing the underlying real analytic structure, we may assume w.l.o.g. that this map of complex vector bundles has real analytic coefficients.

Now, $\forall t$, by Step I, $Ker(\mathcal{H}_t \to W'_t)$ has dimension $\leq g$, and dimension equal to g for $t = 0$.

Whence, this kernel provides a subspace U_t $\forall t$ in a neighbourhood of 0. The corresponding map to the Grassmann manifold is holomorphic in a non-empty open set and real analytic, and is therefore holomorphic $\forall t$ in a neighbourhood of 0.

We can now put the maps f_t together by choosing a basis of $f_*(U_t)$, and integrating these holomorphic 1-forms: we get the desired morphism $F : \Xi \to \mathcal{C}$ to the desired family of curves of genus g.

Final step. We have produced a morphism $F : \Xi \to \mathcal{Y}$, where $\mathcal{Y} \to \mathcal{B}$ is a family of curves.

Moreover, the fibres of f_0 are deformations in the large of complex tori, whence, by Theorem 2.1, these fibres are complex tori of dimension d. Consider, $\forall t_\nu$ the Kodaira–Spencer map of Y_{t_ν}: it is identically zero in a neighbourhood of t_ν, thus it is identically zero in a neighbourhood of 0.

Whence, f_0 is also a holomorphic torus bundle, and it is also principal because the cohomology group $H^1(Y_t, \mathbb{M})$ is a subspace of the cohomology group $H^1(Y_t, Aut(\mathbb{H}))$, and as such the cohomology classes whose triviality ensure that a bundle is principal are locally constant in t. □

Remark 5.2. Consider the family $\mathcal{F}_{g,m}$ of principal holomorphic torus bundles: it is an interesting question to decide which manifolds in the given family are Kähler manifolds, and which are not.

6. Holomorphic torus bundles over tori

In this section we shall consider the case where we have a holomorphic principal bundle $f : X \to Y$ with base a complex torus $Y = V/\Gamma$ of dimension m, and fibre a complex torus $T = U/\Lambda$ of dimension d.

In this case, our manifold X is a $K(\pi, 1)$ and, as we already saw, its fundamental group $\pi_1(X) := \Pi$ is a central extension

$$(10') \qquad 1 \to \Lambda \to \Pi \to \Gamma \to 1,$$

where the extension of Γ by Λ is classified by a cohomology class $\epsilon \in H^2(Y, \Lambda)$.

Tensoring the above exact sequence with $\mathbb{R}$, we obtain an exact sequence of Lie Groups

$$(10') \qquad 1 \to \Lambda \otimes \mathbb{R} \to \Pi \otimes \mathbb{R} \to \Gamma \otimes \mathbb{R} \to 1$$

such that as a differentiable manifold our X is the quotient

$$M := \Pi \otimes \mathbb{R}/\Pi.$$

We want to give a holomorphic family of complex structures on M such that all holomorphic principal bundles $f : X' \to Y'$ with extension class isomorphic to ϵ occur in this family.

This construction is completely analogous to the construction of the standard family of complex tori, parametrized by an open set $\mathcal{T}_n$ in the complex Grassmann manifold $Gr(n, 2n)$.

To this purpose, we must explain the analogue of the first Riemann bilinear relation in this context.

In abstract terms, it is just derived from the exact cohomology sequence

$$(9') \qquad \begin{aligned} H^1\big(Y, \mathcal{O}_Y^d\big) \cong H^1(Y, \mathcal{H}(U)_Y) \to H^1(\mathcal{H}(T)_Y) \\ \to^c\to H^2(Y, \Lambda) \to H^2(Y, \mathcal{H}(U)_Y) \end{aligned}$$

telling that the class ϵ maps to zero in $H^2(Y, \mathcal{H}(U)_Y)$. This condition can be, in more classical terms, interpreted as follows:

Remark 6.1 (first Riemann relation for prin. hol. torus bundles). There exists an alternating bilinear map $A : \Gamma \times \Gamma \to \Lambda$, (representing the cohomology class ϵ uniquely) such that, viewing A as a real element of

$$\Lambda^2(\Gamma \otimes \mathbb{C})^\vee \otimes (\Lambda \otimes \mathbb{C}) = \Lambda^2(V \oplus \bar{V})^\vee \otimes (U \oplus \bar{U}),$$

its component in $\Lambda^2(\bar{V})^\vee \otimes (U)$ is zero.

Proof. By Dolbeault's theorem, the second cohomology group $H^2(Y, \mathcal{H}(U)_Y)$ of the sheaf of holomorphic funtions with values in the complex vector space U, since Y is a torus, is isomorphic to the space of alternating complex antilinear functions on $V \times V$ with values in U. It is easy to verify that the coboundary operator corresponds to the projection of the cohomology group

$$H^2(Y, \Lambda \otimes \mathbb{C}) \to H^2(Y, \mathcal{H}(U)_Y) \cong \Lambda^2(\bar{V})^\vee \otimes (U). \qquad \square$$

Definition 6.2. *Given A as above, we define $\mathcal{TB}_A$ as the subset of the product of Grassmann manifolds $Gr(m, 2m) \times Gr(d, 2d)$ defined by $\mathcal{TB}_A := \{(V, U)|$* the component of A in $(\Lambda^2(\bar{V})^\vee \otimes (U))$ *is* $= 0\}$.

Remark 6.3. Observe that $\mathcal{TB}_A$ is a complex analytic variety, of codimension at most $d\ \frac{m(m-1)}{2}$.

Proof. We may in fact choose a basis for Γ, resp. for Λ so that A is then represented by a tensor $A^k_{i,j}$. Let as usual V correspond to a $2m \times m$ matrix V (thus the matrix $(V, \bar{V})$ yields the identity of $\Gamma \otimes \mathbb{C}$ with arrival basis the chosen one and with initial basis $v_1, \dots v_m, \bar{v_1}, \dots \bar{v_m}$.

Our desired condition is that

$$\forall h \neq \ell,\ w_{h,\ell} := \left[\Sigma_k\left(\Sigma_{i,j}\ v_{i,h} A^k_{i,j} v_{j,\ell}\right) e_k\right] \in U.$$

But this condition (observe moreover that $w_{h,\ell} = -w_{\ell,h}$) means that each vector $w_{h,\ell}$ is linearly dependent upon $u_1, \dots u_d$. Locally in the Grassmannian $Gr(d, 2d)$ the condition is given by d polynomial equations. □

Definition 6.4. *The standard (Appell–Humbert) family of torus bundles parametrized by $\mathcal{TB}_A$ is the family of principal holomorphic torus bundles $X_{V,U}$ on $Y := V/\Gamma$ and with fibre $T := U/\Lambda$ determined by the cocycle in $H^1(\Gamma, \mathcal{H}(T)_Y)$ which is obtained by taking $f_\gamma(z)$, which is the class* mod (Λ) *of $F_\gamma(z) := -A(z, \gamma)$, $\forall z \in V$.*

Remark 6.5. Observe that we may write A as $B + \bar{B}$, where

$$B \in [\Lambda^2(V)^\vee \otimes (U)] \oplus [(V \otimes \bar{V})^\vee \otimes (U)].$$

For later use, we write $B = B' + B''$, with $B' \in [\Lambda^2(V)^\vee \otimes (U)]$, $B'' \in [(V \otimes \bar{V})^\vee \otimes (U)]$, and will say that B'' is the Hermitian component of A, and B' is the complex component of A.

Clearly:

- $A(z, \gamma) = B(z, \gamma)$, $\forall z \in V$, thus $F_\gamma(z)$ is complex linear in z with values in U.
- $F_\gamma(z)$ is a cocycle with values in $T := U/\Lambda$ since

$$F_{\gamma_1+\gamma_2}(z) - F_{\gamma_1}(z + \gamma_2) - F_{\gamma_2}(z) = -F_{\gamma_1}(\gamma_2) = A(\gamma_1, \gamma_2) \in \Lambda.$$

Definition 6.6. *The complete Appell–Humbert family of torus bundles parametrized by $\mathcal{T}'\mathcal{B}_A$ is the family of principal holomorphic torus bundles $X_{V,U,\phi}$ on $Y := V/\Gamma$ and with fibre $T := U/\Lambda$ determined by the cocycle in $H^1(\Gamma, \mathcal{H}(T)_Y)$ which is obtained by taking the sum of $f_\gamma(z)$ with any cocycle $\phi \in H^1(\Gamma, \mathcal{O}^d_Y) = H^1(\Gamma, \mathcal{H}(U)_Y)$.*

Remark 6.7. Observe that any $\phi \in H^1(\Gamma, \mathcal{O}^d_Y) = H^1(\Gamma, \mathcal{H}(U)_Y)$ is represented by a cocycle with constant values in U. Therefore the cocycle $f_\gamma(z) + \phi$ is a linear cocycle (it is a polynomial of degree at most 1 in z).

Theorem 6.8. *Any holomorphic principal torus bundle with extension class isomorphic to $\epsilon \in H^2(\Gamma, \Lambda)$ occurs in the complete Appell–Humbert family $\mathcal{T}'\mathcal{B}_A$.*

Proof. Using sheaf cohomology, the proof is a standard consequence of the exactness of sequence $(9)'$. □

Remark 6.9. The previous theorem is an extension to the torus bundle case of the classical theorem of linearization of the system of exponents for holomorphic line bundles on complex tori (cf. [Sieg73, pp. 49–62], for an elementary proof, and [C-C91, p. 38], for a simpler elementary proof).

The above elementary proofs do not work verbatim here, since [Sieg73] uses integrals which cannot be taken with values in T, and [C-C91] uses the existence of a maximal isotropic sublattice of Γ (this no longer exists if A has values in a lattice Λ of arbitrary rank).

Let us now indicate the modifications required in order to obtain an elementary proof of Theorem 6.8.

Proof n. II. We have a cocycle $f_\gamma(z)$ with values in $T = U/\Lambda$, and we can lift it to a holomorphic function $F_\gamma(z)$ defined on the complex vector space V and with values in U.

The cocycle condition tells us that

$$F_{\gamma_1+\gamma_2}(z) - F_{\gamma_1}(z+\gamma_2) - F_{\gamma_2}(z) = -F_{\gamma_1}(\gamma_2) = A(\gamma_1, \gamma_2) \in \Lambda,$$

therefore $dF_\gamma(z)$ is a cocycle with values in $H^0(V, d(\mathcal{O}_V^d)) \subset H^0(V, \mathcal{O}_V^d)$.

By the classical linearization theorem there is a holomorphic 1-form g such that

$$(II-1)\; dF_\gamma(z) = g(z+\gamma) - g(z) + L_\gamma(z),$$

where $L_\gamma(z)$ is linear (polynomial coefficients of degree at most 1).

Define $\psi(z) := dg(z)$: by (II-1) follows that $\psi(z+\gamma) - \psi(z)$ has constant coefficients, thus the coefficients of ψ are linear (since their derivatives are Λ-periodic).

Moreover, since $d\psi = d^2(g) = 0$, it follows that there is a holomorphic 1-form Q with quadratic coefficients such that $dQ = \psi$.

Since $d(g-Q) = 0$, there is a holomorphic function $\Phi(z)$ such that $d\Phi = g-Q$.

Now, F_γ is cohomologous to $P_\gamma := F_\gamma - \Phi(z+\gamma) + \Phi(z)$. Now, $dP_\gamma = L_\gamma(z) + Q(z) - Q(z+\gamma)$, thereby proving:

Claim 1. The cocycle f_γ is cohomologous to a cocycle represented by P_γ, whose coefficients are polynomials of degree at most 3.

We end the proof via the following:

Claim 2. The cocycle f_γ is cohomologous to a cocycle represented by G_γ, whose coefficients are polynomials of degree at most 1.

Here, the proof runs as classically, since the relation (valid for every cocycle)

$$P_{\gamma_1}(z+\gamma_2) - P_{\gamma_1}(z) = P_{\gamma_2}(z+\gamma_1) - P_{\gamma_2}(z)$$

shows that, if we choose a basis of $\mathbb{C}^m$ where Γ has basis $e_1, \dots e_m, \tau_1, \dots \tau_m$, then if P_{e_i} is linear $\forall i = 1, \dots . m$, it follows then that any P_γ is linear.

To linearize $P_{e_i}, \forall i = 1, \dots m$, one can use the methods of solving linear difference equations as in [Sieg73, pp. 54–58]. □

The explicit standard normal form for cocycles is very useful to calculate several holomorphic cohomology groups of X:

Theorem 6.10. *The cokernel of* $0 \to H^0(\Omega^1_Y) \to H^0(\Omega^1_X)$ *is the subspace of* $U^\vee$ *which annihilates the image of the Hermitian part of A, i.e., of the component* B'' *in* $[(V \otimes \bar{V})^\vee \otimes (U)]$.

It follows in particular that X is parallelizable if and only if the Hermitian part of A is zero.

Proof. We have the exact sequence

$$0 \to H^0\big(\Omega^1_Y\big) \to H^0\big(\Omega^1_X\big) \to H^0\big(f_*\Omega^1_{X|Y}\big) \cong U^\vee,$$

thus we immediately get that our cokernel is a subspace of the complex vector space $U^\vee$.

To see which subspace we get, let us take a holomorphic 1-form ω and lift it back to the product $V \times T \cong \mathbb{C}^m \times T$, with coordinates (z, u).

Hence we write $\omega = \alpha + \beta = \Sigma_i \alpha_i(z,u) dz_i + \Sigma_j \beta_j(z,u) du_j$. Since the coefficients are holomorphic functions of u, and by the previous exact sequence we infer that β has constant coefficients, and that the coefficients of α depend only on z, we can write: $\omega = \Sigma_i \alpha_i(z) dz_i + \Sigma_j \beta_j du_j$. We also know that if $\beta = 0$ then the coefficients $\alpha_i(z)$ are constant.

The condition that ω is a pull-back from X is equivalent to $\gamma^*(\omega) = \omega \, \forall \gamma \in \Gamma$.

Since $\gamma(z,u) = (z + \gamma, u + f_\gamma(z))$, our condition reads out as

$$(**) \qquad \alpha(z+\gamma) - \alpha(z) = -\beta \circ df_\gamma(z) := -\Sigma_k \beta_k A^k(dz, \gamma).$$

$(**)$ implies that the coefficients of α may be assumed w.l.o.g. to be linear homogenous functions of z, thus there is a complex bilinear function B^* such that $\alpha = B^*(dz, z)$. Comparing again equation $(**)$, we infer $B^*(dz,\gamma) = -\beta \circ A(dz,\gamma)$.

Since, however, we have $\beta \circ A(dz,\gamma) = \beta \circ B(dz,\gamma)$ the above equation may hold iff $B^*(dz,\gamma) = -\beta \circ B'(dz,\gamma)$ and $\beta \circ B''(z,\gamma) = 0 \, \forall \gamma, \ \forall z$. Thus there exists an ω with vertical part $= \beta$ if and only if β annihilates the image of B'', as claimed. □

Corollary 6.11. *The space* $H^0(d\mathcal{O}_X)$ *of closed holomorphic 1-forms on X contains the pull-back of* $H^0(\Omega^1_Y)$ *with cokernel the subspace* U^* *of* $U^\vee$ *which annihilates the image of A, i.e.,* $U^* = \{\beta | \beta \circ A(z,\gamma) = 0 \, \forall \gamma, \ \forall z\}$.

Proof. We know that there exists a holomorphic 1-form ω with vertical part β iff β annihilates the Hermitian part B'' of A, and then we may assume $\omega = \beta \circ B'(dz, z) + \Sigma_j \beta_j du_j$.

Let us calculate the differential of ω: we get $d\omega = \beta \circ B'(dz, dz)$. Since, however, B' is antisymmetric, we get that $d\omega = 0$ if and only if $\beta \circ B'(z', z) = 0$, $\forall z, z' \in V$. Putting the two conditions together we obtain the desired conclusion that β should annihilate the total image of A. □

Theorem 6.12. *The cokernel of* $0 \to H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_X)$ *is the subspace of* $\bar{U}^\vee$ *which annihilates the image of the anti-complex component of A, i.e., of the conjugate of the component* B' *in* $[(\Lambda^2 V)^\vee \otimes (U)]$.

Proof. We calculate $H^1(\mathcal{O}_X)$ through the Leray spectral sequence for the map f, yielding the exact sequence

$$(*) \qquad 0 \to H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_X) \to H^0(\mathcal{R}^1 f_* \mathcal{O}_X) \to H^2(\mathcal{O}_Y).$$

We will interpret the above as Dolbeault cohomology groups, keeping in mind that $\mathcal{R}^1 f_* \mathcal{O}_X$ is a trivial holomorphic bundle of rank $d = dim(U)$.

Let η be a $\bar\partial$-closed (0, 1)-form, which we represent through its pull-back to $V \times T$, with coordinates (z, u).

Hence we write $\eta = \alpha + \beta = \Sigma_i \alpha_i(z, u) d\bar z_i + \Sigma_j \beta_j(z, u) d\bar u_j$.

The meaning of (*) is that is that η is $\bar\partial$-cohomologous to a form such that the functions β_j are constant. Let us then assume this to be the case, and let us impose the two conditions $\bar\partial(\eta) = 0$, and that η be a pull-back from X.

The first implies $\bar\partial_u(\alpha) = 0$, thus the functions $\alpha_i(z, u)$ are holomorphic in u, hence they depend only upon z.

Moreover, then $\bar\partial(\eta) = 0$ is equivalent to $\bar\partial_z(\alpha)(z) = 0$, and therefore is equivalent to the existence of a function $\phi(z)$ on $\mathbb{C}^n$ such that $\bar\partial_z(\phi(z)) = (\alpha)(z)$.

Then, the condition that η be a pull-back means that η is invariant under the action of Γ such that

$$z \to z + \gamma, \; u \to u + f_\gamma(z),$$

whence it writes out as $\alpha(z + \gamma) - \alpha(z) = -\Sigma_k \beta_k \overline{d f_\gamma(z)_k}$, or, equivalently,

$$(***) \qquad \bar\partial\phi(z + \gamma) - \bar\partial\phi(z) = -\Sigma_k \beta_k A^k(\gamma, d\bar z).$$

As before, we can write A as $A' + A''$ where, B'' being the Hermitian part of the tensor, $A'' = B'' + \bar B''$.

Assume first that $\beta \circ B'(\gamma, \bar z) := \Sigma_k \beta_k B'^k(\gamma, \bar z)$ is identically zero; then $\beta \circ B''(\gamma, \bar z) = \beta \circ A(\gamma, \bar z)$ and we simply choose $\phi := \beta \circ B''(z, \bar z)$ which clearly satisfies equation $(***)$.

Then $\eta := \bar\partial(\phi(z)) + \Sigma_j \beta_j d\bar u_j$ is a $\bar\partial$-closed 1-form on X with vertical part $\beta = \Sigma_j \beta_j d\bar u_j$.

Conversely, the meaning of the exact sequence $(*)$ is as follows:

given any vertical part $\beta = \Sigma_j \beta_j d\bar u_j$, we first find a differential (0, 1)-form on X

$\eta = \alpha + \beta = \Sigma_i \alpha_i(z) d\bar z_i + \Sigma_j \beta_j d\bar u_j$ with the given vertical part.

Then, the vertical part comes from $H^1(\mathcal{O}_X)$ only if its image in $H^2(\mathcal{O}_Y)$, provided by $\bar\partial(\eta)$, is zero.

For this purpose we have to solve the equation

$$\alpha_i(z + \gamma) - \alpha_i(z) = -\Sigma_k \beta_k A^k(\gamma, \bar e_i) = -\beta \circ A(\gamma, \bar e_i).$$

But an easy solution is provided by setting

$$\alpha_i(z) = -\Sigma_k \beta_k B'^k(\bar z, \bar e_i) - \Sigma_k \beta_k B''^k(z, \bar e_i) = -\beta \circ B'(\bar z, \bar e_i) - \beta \circ B''(z, \bar e_i).$$

But now, $\bar\partial(\eta) = \bar\partial\alpha = -\beta \circ B'(d\bar z, d\bar z)$ which is a (0-2)-form with constant coefficients, and is therefore equal to zero if and only if all of its coefficients are zero, thus iff $-\beta \circ B'(\bar v, d\bar z w) = 0 \; \forall v, w \in V$, which proves our assertion. □

Remark 6.13. The previous theorem allows us to write explicitly the basis of the complete Appel–Humbert family. Locally on $Gr(d, 2d)$ we may assume after a permutation of coordinates that our subspace U is the space of vectors $\{(u', u'')|u'' = U^*(u')\}$, for some $(d \times d)$-matrix U^*.

Then our equations are that we consider the matrices (V, U^*) and the vectors $v \in \mathbb{C}^m$, $\beta \in \mathbb{C}^d$ such that, in the notation of Remark 6.3, $\forall h \neq \ell$, $w_{h,\ell} := [(\Sigma_{i,j}\ v_{i,h} A_{i,j} v_{j,\ell})]$ satisfies

$$w''_{h,\ell} = U^*\big(w'_{h,\ell}\big), \quad \beta\big(w'_{h,\ell}\big) = 0.$$

Let us consider the easiest possible case, where $m = 2, d = 1$, then our parameter variety is a pull-back of the variety in $\mathbb{C}^4$ with equations

$$w'' = u^* w', \quad \beta w' = 0,$$

a product of $\mathbb{C}$ with $\{(\beta, w') \in \mathbb{C}^2 |\ \beta w' = 0\}$, a reducible variety.

In order to analyse the problem of describing the small deformations of principal holomorphic torus bundles over tori we need to calculate the cohomology groups of the tangent sheaf Θ_X. This can be accomplished by the above multilinear algebra methods.

For ease of calculations recall that the canonical bundle Ω^n_X is trivial, therefore, by Serre duality, it is sufficient to determine the cohomology groups

$$H^{n-i}\big(\Omega^1_X\big) \cong H^i(\Theta_X)^\vee.$$

Theorem 6.14. *$H^{n-i}(\Omega^1_{X|Y})$ fits into a short exact sequence*

$$0 \to coker\beta_{n-i-1} \to H^{n-i}\big(\Omega^1_{X|Y}\big) \to ker\beta_{n-i} \to 0,$$

where $\beta_i : U^\vee \otimes H^i(\mathcal{O}_X) \to V^\vee \otimes H^{i+1}(\mathcal{O}_X)$ is given by cup product and contraction with $B'' \in [(\bar{V}) \otimes V)^\vee \otimes (U)]$.

Proof. Recall the exact sequence (6)

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to f^*W^\vee \to 0 \tag{6}$$

where W is a trivial bundle, and the extension class is a pull-back of the extension class of the sequence

$$0 \to \Omega^1_Y \to f_*\Omega^1_X \to W^\vee = f_*\Omega^1_{X|Y} \to 0, \tag{8}$$

which coincides with the Hermitian component B'' of A, as it is easy to verify (observe that $B'' \in [(\bar{V}) \otimes V)^\vee \otimes (U)] \cong H^1_{\bar\partial}(V^\vee \otimes U \otimes \mathcal{O}_Y) \cong H^1(W \otimes \Omega^1_Y) \subset H^1(f^*W \otimes f^*\Omega^1_Y))$.

Therefore, in the exact cohomology sequence of (6), the coboundary operator is given by cup product with B'', whence we can write this exact sequence as

$$\begin{aligned} \ldots H^{n-i}\big(\Omega^1_X\big) \to H^{n-i}\big(\Omega^1_{X|Y}\big) &= H^{n-i}(U^\vee \otimes \mathcal{O}_X) \\ \to H^{n-i+1}\big(f^*\Omega^1_Y\big) &= H^{n-i+1}(V^\vee \otimes \mathcal{O}_X) \end{aligned}$$

whence the desired assertion follows. □

Corollary 6.15. *$H^i(\Theta_X)$ fits into a short exact sequence*

$$0 \to coker\, b_{i-1} \to H^i(\Theta_X) \to ker\, b_i \to 0,$$

where $b_i : V \otimes H^i(\mathcal{O}_X) \to U \otimes H^{i+1}(\mathcal{O}_X)$ is given by cup product and contraction with $B'' \in [(\bar{V}) \otimes V)^\vee \otimes (U)]$.

Proof. Serre duality. □

For completeness we state without proof a more general result than the one that we need:

Proposition 6.16. *The Leray spectral sequence for the sheaf $\mathcal{O}_X$ and for the map f yields a spectral sequence which degenerates at the E_3 level and with E_2 term $= (H^i(\mathcal{R}^j f_* \mathcal{O}_X), d_2)$, where $d_2 : (H^i(\mathcal{R}^j f_* \mathcal{O}_X) = H^i(\Lambda^j(V)) = \Lambda^i(\bar{V}^\vee) \otimes \Lambda^j(\bar{U}^\vee) \to \Lambda^{i+2}(\bar{V}^\vee) \otimes \Lambda^{j-1}(\bar{U}^\vee)$ is provided by cup product and contraction with $\bar{B}' \in \Lambda^2(\bar{V}^\vee) \otimes (\bar{U})$.*

We observe now some deformation theoretic consequences:

Theorem 6.17. *Let $f : X \to Y$ be a holomorphic principal bundle with base a complex torus $Y = V/\Gamma$ of dimension m, and fibre an elliptic curve T ($T = U/\Lambda$ has dimension 1).*

Assume, moreover, $\pi_1(X) := \Pi$ to be a non-trivial central extension

$$1 \to \Lambda \to \Pi \to \Gamma \to 1 \qquad (10')$$

classified by a cohomology class $\epsilon \neq 0 \in H^2(Y, \Lambda)$ whose associated bilinear form A has an image of dimension $= 2$.

1) *Then every limit of manifolds in the complete Appell–Humbert family is again a holomorphic principal bundle $f' : X' \to Y'$ with fibre an elliptic curve T', and thus occurs in the complete Appell–Humbert family (6.6 and 6.8).*
2) *Every small deformation of such a manifold X is induced by the complete Appell–Humbert in the case where $h^1(\mathcal{O}_X) = m$.*

Proof. By Corollary 6.11 we get that $H^0(d\mathcal{O}_X)$, the space of closed holomorphic 1-forms on X equals the pull-back of $H^0(\Omega^1_Y) = H^0(d\mathcal{O}_Y)$ since the antisymmetric form A representing ϵ is non zero.

Likewise, again the Leray spectral sequence for cohomology shows that $H^1(X, \mathbb{C})$ contains the pull-back of $H^1(Y, \mathbb{C})$ with cokernel equal to the subspace of $\Lambda^\vee$ annihilated by the image of A. This image is the whole Λ, whence $b_1(X) = b_1(Y)$ and it follows immediately that $f : X \to Y$ is the Albanese map of X.

Proof of Assertion 1. By Lemma 2.4 and Corollary 2.5 any limit X' in a 1-parameter family X_t of such holomorphic bundles has a surjective Albanese map $f' : X' \to Y'$ onto a complex torus $Y' = V'/\Gamma$ of dimension m.

Since any deformation in the large of an elliptic curve is an elliptic curve, the general fibre of f' is an elliptic curve.

We have finished if we show that f' is a submersion, since an elliptic fibration without singular fibres is a holomorphic bundle. Moreover, since we have that the small deformations of $f' : X' \to Y'$ contain principal holomorphic bundles, then the same thing holds for $f' : X' \to Y'$.

That f' is a submersion follows by purity of branch locus (cf. [Mum78]): we would then have a ramification divisor R_0 for the central fibre, but $R_{t_\nu} = \emptyset$ for the special fibres, yielding the same contradiction as in the proof of Theorem 2.7.

Proof of Assertion 2. It suffices to show that every small deformation X' of X has an Albanese map $f' : X' \to Y'$ onto a complex torus $Y' = V'/\Gamma$ of dimension m.

However, by semicontinuity, $h^1(\mathcal{O}_{X'}) \leq m$, on the other hand we may apply Lemma 2.3 to conclude that X' has a very good Albanese map, and again by purity of branch locus we conclude that we have a submersion. □

Remark 6.18. We are interested more generally to see whether the complete Appell–Humbert family, which does not have a smooth base but is pure of dimension $m^2 + m$, has a surjective Kodaira–Spencer map (this is a first step towards the proof of its versality). It is necessary for this purpose to calculate the cohomology group $H^1(\Theta_X)$ in an explicit way.

We write, according to Theorem 6.12, $H^1(\Theta_X) = (\bar{V}^\vee) \oplus ker(B')$, where we see B' as a linear map $(B') : (\bar{U}^\vee) \to \Lambda^2(\bar{V}^\vee)$.

We have then, according to Corollary 6.14, an exact sequence

$$V \to U \otimes (\bar{V}^\vee \oplus kerB') \to H^1(\Theta_X) \to V \otimes (\bar{V}^\vee \oplus kerB')$$
$$\to U \otimes (\Lambda^2(\bar{V}^\vee) \oplus (\bar{V}^\vee \otimes (kerB')),$$

where the first and last map are given by contraction with B''.

We first analyse when holds the desired dimensional estimate $dim H^1(\Theta_X) = m^2 + m$.

Case 1. B'' is zero, whence X is parallelizable, thus $dim H^1(\Theta_X) = (m+1)h^1(\mathcal{O}_X)$. But in this case B' is non zero, whence $h^1(\mathcal{O}_X) = m$ and we are done.

Case 2. $B'' \neq 0$ and $B' \neq 0$, then $h^1(\mathcal{O}_X) = m$ and thus follows $dim H^1(\Theta_X) \leq (m+1)h^1(\mathcal{O}_X) = m(m+1)$.

We omit the not difficult verification that in both cases the Kodaira–Spencer map is surjective.

Case 3. $B' = 0$, and $B'' \neq 0$. In this case $A = B'' + \bar{B}''$, whence we observe that the image of B'' has dimension $= 1$.

We leave aside this case, where the rank of the antisymmetric matrix B'' also plays a role.

We finally describe the most well-known example of such elliptic bundles over 2-dimensional tori, namely, the so-called **Iwasawa manifold** (cf. [K-M71]).

One lets N (biholomorphic to $\mathbb{C}^3$) be the unipotent group of 3×3 upper trangular matrices with all eigenvalues equal to 1.

N contains the discrete cocompact subgroup Π which is the subgroup of the matrices with entries in the subring $\Lambda \subset \mathbb{C}$, $\Lambda := \mathbb{Z}[i]$.

Obviously, $X := N/\Pi$ is a fibre bundle $f : X \to \mathbb{C}^2/\Lambda^2$, where f is induced by the homomorphism $F : N \to \mathbb{C}^2$ given by the coordinates z_{12}, z_{23}.

X is parallelizable being a quotient of a complex Lie group by a discrete subgroup.

Moreover, in our case, the alternating form A is obtained as the antisymmetrization of the product map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, which induces $\Lambda \times \Lambda \to \Lambda$.

Thus in this case there is no Hermitian part, confirming the parallelizability, and then $h^1(\mathcal{O}_X) = 2$, and $dim H^1(\Theta_X) = 6$.

That's why the Kuranishi family of X is smooth of dimension 6, and gets small deformations which are not parallelizable (cf. [Nak75]).

7. Blanchard–Calabi Manifolds

The Sommese–Blanchard examples ([Bla53], [Bla56], [Somm75]) provide non-Kähler complex structures X on manifolds diffeomorphic to a product $C \times T$, where C is a compact complex curve and T is a 2-dimensional complex torus.

In fact, in these examples, the projection $X \to C$ is holomorphic and all the fibres are 2-dimensional complex tori.

Remark 7.1. Start with a Sommese–Blanchard 3-fold with trivial canonical bundle and deform the curve C and the line bundle L in such a way that the canonical bundle becomes the pull-back of a non-torsion element of $Pic^0(C)$: then this is the famous example showing that the Kodaira dimension is not deformation invariant for non-Kähler manifolds (cf. [Ue80]).

Also Calabi ([Cal58]) showed that there are non-Kählerian complex structures on a product $C \times T$.

We briefly sketch (cf. [Cat02] for precise results in the case where the base manifold Y is a curve and X has dimension 3) how to vastly generalize these constructions.

Definition 7.2 (Blanchard–Calabi Jacobian manifolds). *Let Y be a compact complex manifold, and let W be a rank d holomorphic vector bundle admitting $2d$ holomorphic sections $\sigma_1, \sigma_2, .., \sigma_{2d}$, which are everywhere $\mathbb{R}$-linearly independent.*

Then the quotient X of W by the $\mathbb{Z}^{2d}$-action acting fibrewise by translations : $w \to w + \Sigma_{i=1,..2d}\, n_i \sigma_i$ is a complex manifold diffeomorphic to the differentiable manifold $Y \times T$, where T is a d- dimensional complex torus. X will be called a **Jacobian Blanchard–Calabi manifold**.

Remark 7.3. The canonical divisor of X equals $K_X = \pi^*(K_Y - detW)$, where $\pi : X \to Y$ is the canonical projection. Moreover, if Y is Kähler, X is Kähler only if the bundle W is trivial (i.e., iff X is a holomorphic product $Y \times T$).

Indeed, if $d = 2$, one has $h^0(\Omega^1_X) = h^0(\Omega^1_Y)$ unless W is trivial, as follows from the dual of the sequence (5)

$$0 \to V \to (\mathcal{O}_Y)^{2d} \to W \to 0.$$

Proof. If X is Kähler, then, since the first Betti number of X equals $2d + 2h^0(\Omega^1_Y)$, then it must hold that $h^0(\Omega^1_X) = h^0(\Omega^1_Y) + d$, in particular $h^0(W^\vee) \geq d$.

But $(W^\vee)$ is a sub-bundle of a trivial bundle of rank $2d$, whence d linearly independent sections of $(W^\vee)$ yield a composition $(\mathcal{O}_Y)^d \to (W^\vee) \to (\mathcal{O}_Y)^{2d}$ whose image is a trivial bundle of rank $\leq d$.

Indeed the rank must be d, else the d sections would not be $\mathbb{C}$-linearly independent.

Assume now that $h^0(W^\vee) = r$: then the corresponding image of the composition $(\mathcal{O}_Y)^r \to (W^\vee) \to (\mathcal{O}_Y)^{2d}$ would be a trivial bundle of rank r, thus $(W^\vee)$ has a trivial summand I_r of rank r . Then we have a direct sum $(W^\vee) = I_r \oplus Q$ and dually W is a direct sum $I_r \oplus Q^\vee$. We use now the hypothesis that there are $2d - 2r$ holomorphic sections of W which are $\mathbb{R}$-independent at any point: it follows then that if $d = 2$ and $0 < r < d = 2$, then $Q^\vee$ has rank 1 and admits holomorphic sections which are everywhere $\mathbb{R}$-independent, thus it is trivial. □

Definition 7.4. *Given a Jacobian Blanchard–Calabi manifold $\pi : X \to Y$, any X-principal homogeneous space $\pi' : Z \to Y$ will be called a Blanchard–Calabi manifold.*

Remark 7.5. Given a Jacobian Blanchard-Calabi manifold $X \to Y$, the associated X-principal homogeneous spaces are classified by $H^1(Y, W)$.

Thus, the main existence problem is the one concerning the Jacobian Blanchard–Calabi manifolds.

Proposition 7.6. *Let G be the d^2-dimensional projective Grassmann variety $G\mathbb{P}(d-1, 2d-1)$ of $d-1$-dimensional projective subspaces in $\mathbb{P}^{2d-1}$. The datum of Jacobian Blanchard–Calabi manifold $f : X \to Y$ is equivalent to the datum of a "totally non-real" holomorphic map $h : Y \to G$, i.e., such that the corresponding ruled variety R_h given by the union of the $d-1$-dimensional projective subspaces $h(y)$ has no real points.*

Proof. Let G be the Grassmann variety $G\mathbb{P}(d-1, 2d-1) = Gr(d, 2d)$, so G parametrizes the d-dimensional vector subspaces in $\mathbb{C}^{2d}$: observe that the datum of an exact sequence

$$0 \to V \to (\mathcal{O}_C)^{2d} \to W \to 0$$

is equivalent to the datum of a holomorphic map $h : Y \to G$, since for any such f we let V, W be the respective pull-backs of the universal sub-bundle U and of the quotient bundle Q.

The condition that the standard $2d$ sections are $\mathbb{R}$-linearly independent means that there is no $y \in Y$ and no real vector $v \in \mathbb{R}^{2d}$ such that v belongs to the subspace corresponding to $h(y)$. □

If h is constant, W is trivial and we have a product.

Else, we assume for simplicity that h is generically finite onto its image, and we consider the deformation of the holomorphic map h.

We have the following exact sequence:

$$0 \to \Theta_Y \to (f)^* \Theta_G \to N_h \to 0,$$

where N_h, the normal sheaf of the morphism h, governs the deformation theory of the morphism h, in the sense that the tangent space to Def(h) is the space $H^0(N_h)$, while the obstructions lie in $H^1(N_h)$.

By virtue of the fact that $\Theta_G = Hom(U, Q)$, and of the cohomology sequence associated to the above exact sequence, we get $0 \to H^0(\Theta_Y) \to H^0(V^\vee \otimes W) \to H^0(N_h) \to H^1(\Theta_Y) \to H^1(V^\vee \otimes W) \to H^1(N_f) \to 0$, and we conclude that the deformations of the map are unobstructed provided $H^1(V^\vee \otimes W) = 0$.

In [Cat02] we saw that this holds, in the special case where $W = L \oplus L$, if the degree d of L satisfies $d \geq g$. We showed in this way that if $d \geq g$ the dimension of the space of deformations of the map f is given by $3g - 3 + 4h^0(2L) = 4d + 1 - g$, and this number clearly tends to infinity together with $d = deg(L)$.

Whence we got (loc. cit.) the following:

Corollary 7.7. *The space of complex structures on the product of a curve C with a four-dimensional real torus has unbounded dimension.*

As a corollary of our Theorem 4.1 we also obtain:

Corollary 7.8. *The space of complex structures on the product of a curve C of genus $g \geq 2$ with a four-dimensional real torus contains manifolds which are not deformation equivalent to each other, namely, Blanchard–Calabi 3-folds which are not Kähler and holomorphic principal bundles in the family $\mathcal{F}_{g,0}$.*

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