

FROM ABEL'S HERITAGE: TRANSCENDENTAL OBJECTS IN ALGEBRAIC GEOMETRY AND THEIR ALGEBRIZATION.

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**Welcome the Abel Prize and long live the memory of Abel,
long live mathematics!** ¹

1. INTRODUCTION

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The Encyclopedic Dictionary of Mathematics (edited in Japan) is not a book devoted to the history of mathematics, but tries instead to briefly introduce the reader to the current topics of mathematical research. By non only lexicographical coincidence it starts with "Abels, Niels Henrik" as topic 1.

It contains a succinct biography of Abel

" Niels Henrik Abel (august 5, 1802 - april 6 , 1829) ... In 1822, he entered the University of Christiania [today's Oslo] .. died at twenty-six of tuberculosis. His best known works are: the result that algebraic equations of order five or above cannot in general be solved algebraically; the result that ³*Abelian equations [i.e., with Abelian Galois group] can be solved algebraically; the theory of *binomial series and of *elliptic functions; and the introduction of *Abelian functions. His work in both algebra and analysis, written in a style conducive to easy comprehension, reached the highest level of attainment of his time."

Talking about Abel's heritage entails thus talking about a great part of modern mathematics, as it is shown by the ubiquity of concepts such as Abelian

¹The present research, an attempt to treat history and sociology of mathematics and mathematics all at the same time, took place in the framework of the Schwerpunkt "Globale Methode in der komplexen Geometrie", and of the EC Project EAGER.

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¹Was the greatest enemy of mathematics Alexander the Great, who cut the Gordian knot instead of peacefully writing a book about it?

²This topic has in fact been treated quite extensively in the contribution by Ciliberto.

³A star next to a theme denotes that a section of the dictionary is devoted to the discussion of the topic

groups, Abelian integrals and functions, Abelian Varieties, and of relatives of theirs as anabelian geometry, nonabelian Hodge theory...

Writing is certainly a more difficult task than talking, when the time limits force us to plan our way on secure and direct tracks: for this reason we decided that the present text, with the exception of a couple of protracted mathematical discussions, should essentially be the text of our oral exposition at the Abel Bicentennial Conference. Thus its aim is just to lead the reader along quite personal views on history and development of mathematics, and on certain topics in the still very alive subject of transcendental algebraic geometry.

One could declare its Leitmotiv to be G.B. Vico's theory of cycles in the history of mankind, adapted to the analysis of mathematical evolutions and revolutions:

Geometry in ancient Greece, Algebra by the Arabs and in early Renaissance, Geometry again by B. Cavalieri and his indivisibles, Analysis and Physics by the Bernoulli's,.. and later on an intricate succession of points of view and methods, often alternative to each other, or striving in directions opposite one to the other, which all together enriched our knowledge and understanding of the mathematical reality.

Therefore, if we conceive algebra, geometry .. more as methodologies than as domains of knowledge, comes out naturally the difficult question: which way of doing mathematics is the one we are considering ?

This question, probably a sterile question when considering the history of mathematics, is however a very important one when we are making choices for future directions of mathematical research: to purport this assertion it will suffice only to cite the (for me, even exaggerated) enthusiasm of nowadays algebraic geometers for the new insights coming to their field by physical theories, concepts and problems.

In any case, in our formerly bourgeois world, idle questions with provocative answers to be defended at tea time at home or in a Cafe', have often motivated interesting discussions, and it is just my hope to be able to do the same thing here.

2. ABEL, THE ALGEBRAIST?

So I will start, as due, by citing Hermann Weyl's point of view (cf. [Weyl31],[Weyl] and also [Yag], pp. 26 and 151, for comments), expressed in an address directed towards mathematics teachers, and later published in the Journal : Unterrichtsblätter für Mathematik und Naturwissenschaften, Band 38 (1933), s. 177-188.

There are two Classes of mathematicians:

- ALGEBRAISTS: as Leibniz, Weierstrass
- GEOMETERS-PHYSICISTS: as Newton, Riemann, Klein

and people belonging to different classes may tend to be in conflict with each other.

The tools of the algebraists are : logical argumentation, formulae and their clever manipulation, algorithms.

The other class relies more on intuition, and graphical and visual impressions. For them it is more important to find a new truth than an elegant new proof.

The concept of rigour is the battlefield where the opposite parties confront themselves, and the conflicts which hence derived were sometimes harsh and longlasting.

The first well known example is the priority conflict between Leibniz and Newton concerning the invention of the Calculus (which however was invented independently by the two scientists, as it is currently agreed upon).

The inputs which the two scientists provided did indeed integrate themselves perfectly. On one hand the pure algebraic differential quotient dy/dx would be a very dry concept (algebraists nevertheless are still nowadays very keen on inflicting on us the abstract theory of derivations!) without the intuition of velocities and curve tangents; on the other hand, in the analysis of several phenomena, a physical interpretation of Leibniz's rule can turn out to be amazingly complicated .

More closely related with Abelian integrals and their periods was Weierstrass' constructive criticism of the "Principles" by Riemann and Dirichlet.

As also pointed out in the contribution⁴ by Schappacher, this conflict soon became the Berlin-Göttingen conflict, and almost deflagrated between Weierstrass and Felix Klein (who continued on the way started as a student of Clebsch)⁵ .

Klein's antipathy for Weierstrass was more intellectual than personal: Klein put a special enphasys on geometrical and physical intuition, which he managed to develop in the students by letting them construct solid (plaster, or metal) models of curves and surfaces in ordinary 3-space, or letting them draw very broad (1 x 2 meters) paper tables of cubic plane curves with an explicit plotting of their \mathbb{Q} -rational points.

A concrete witness to this tradition is the exhibition of plaster models of surfaces which are to be found still nowadays in the Halls of the mathematical Institute in Göttingen. These models were then produced by the publishing company L. Brill in Darmstadt, later by the Schilling publishing company, and sold around the world: I have personally seen many of those in most of the older Deparments I have visited (cf. the 2 Volumes edited by G.Fischer on "Mathematical Models" [Fisch86]⁶).

A similar trick, with computer experiments replacing the construction of models, is still very much applied nowadays in the case where professors have to supervise too many more students' theses than they can really handle.

⁴Here and after, I will refer to the oral contributions given at the Abel Bicentennial Conference, and not to the articles published in this Volume.

⁵We have noticed that another article devoted to this topic has appeared after we gave the talk, namely [Bott02] by U. Bottazzini.

⁶Writes G. Fischer : " There were certainly other reasons than economic for the waning interest in models. More and more general and abstract viewpoints came to the forefront of mathematics..... Finally Nicolas Bourbaki totally banned pictures from his books."

Weierstrass had a victory, in the sense that not only the theory of calculus, but also the theory of elliptic functions is still nowadays taught almost in the same way as it was done in his Berlin lectures.

But Klein's "defeat" (made harder by the long term competition with Poincaré about the proof of the uniformization theorem, see later) was however a very fertile humus for the later big growth of the Göttingen influence, and certainly Weyl's meditations which we mentioned above were reflecting this historically important controversy.

Where does then Abel stay in this classification? I already took position, with my choice of the title of this section: Abel is for me an algebraist and I was glad ⁷ to hear Christian Houzel stressing in his contribution the role of Abel's high sense of rigour. Abel's articles on the binomial coefficients, on the summation of series, and on the solution of algebraic equations testify his deep concern for the need of clear and satisfactory proofs ⁸.

Of course, like many colours are really a mixture of pure colours, the same occurs for mathematicians, and by saying that he was deep down an "algebraist" I do not mean to deny that he possessed a solid geometrical intuition, as we shall later point out.

In fact, Abel himself was proud to introduce himself during his travels as 'Professor of Geometry' ⁹.

The best illustration of his synthetical point of view is shown by the words (here translated from French) with which he begins his Memory (XII-2 "Mémoire sur une propriété générale d'une classe très étendue de fonctions transcendentes."): ¹⁰

"The transcendental functions considered until nowadays by the geometers are a very small number. Almost all the theory of transcendental functions is reduced to the one of logarithmic, exponential and circular functions, which are essentially one only kind. Only in recent times one has begun to consider some other functions. Among those, the elliptic transcendentals, about which M. Legendre developed so many elegant and remarkable properties, stay in the first rank."

The statement "are essentially one only kind" is the one we want now to comment upon.

⁷For many years on, until after the first world war, there used to be a course in Italian Universities entitled "Lezioni di analisi algebrica ed infinitesimale". This shows that the birth of Analysis as a new separated branch, trying to appear more on the side of applied mathematics, is a relative novelty which ends the reconciliation made for the dualism Leibniz-Newton.

⁸cf. the article by: J J O'Connor and E F Robertson in <http://www-history.mcs.st-andrews.ac.uk/References/Abel.html>, citing his letter to Holmboe from Berlin. Here we can read: "In other words, the most important parts of mathematics stand without foundation".

⁹To be perfectly honest, all we know is that in 1826 he signed himself in at the "Goldenes Schiff" in Predazzo as 'Abel, professore della geometria.'

¹⁰This important article was lying at the centre of the contribution by Griffiths, and is also amply commented upon in Kleiman's contribution.

Algebraists like indeed short formulae, and these are in this case available. It suffices to consider the single formula : $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and then, by considering $\exp(ix)$ and the inverse functions of the ones we can construct by easy algebraic manipulations, we get easily ahold of the wild proliferation of functions which for instance occupy the stage of the U.S.A. Calculus courses (sin, cos, sec, cosec, tg, cotg, and their hyperbolic analogues).

No doubt, synthesis is a peculiarity of pure mathematics, and applied or taught mathematics may perhaps need so many different names and functions: but, pretty sure, Abel stood by the side of synthesis and conciseness.

His well known saying, that he was able to learn so rapidly because he had been: " studying the Masters and not their pupils", is quite valid nowadays. Today there is certainly an inflation of books and divulgations, many are second, third hand or even further. Abel's point of view should be seriously considered by some pedagogists who want to strictly regulate children's learning, forcing them to study n-th hand knowledge. Perhaps this is a strictly democratic principle, by which one wants to prevent some children from becoming precociously wise (as Abel did), and possibly try to stop their intellectual growth (this might be part of a more general ambitious program, sponsored by Television Networks owners).

In any case, Abel had read the masters, and he knew many functions: he still belongs to the mathematical era where functions are just concretely defined objects and not subsets of a Cartesian product satisfying a geometrical condition. One of Abel's main contributions was to consider not only ample classes of functions defined by integrals of algebraic functions, but to study their inverse functions and their periodicity (the so called elliptic functions being among the latter).

At his time did not yet exist the concept of "Willkürliche Funktionen" (= "arbitrary functions"), quite central e.g. for Weierstrass, Dini, Peano and Hilbert (cf. [Dini78], [Weier78], [G-P84]) and which motivated much of the developments in the theory of sets leading to the construction of several pathological situations (as Lebesgue's non constant function with derivative almost everywhere zero, [Leb02]).

Most of the functions he considered were in fact written as $\int_{x_0}^x y(t)dt$, where $y(x)$ is an **algebraic function** of x , which simply means that the function is defined on some interval in \mathbb{R} and that there exists a polynomial $P(x, y) \in \mathbb{C}[x, y]$ ¹¹ such that $P(x, y(x)) \equiv 0$. The functions given by these integrals, or by sums of several of these, are nowadays called **Abelian functions**.

The above statement is by and large true, with however a single important exception, concerning Abel's treatment of functional equations: there he considers quite generally the functions which occur as solution of certain **functional equations**.

¹¹Observe that we wrote $\mathbb{C}[x, y]$ instead of $\mathbb{R}[x, y]$. It is commonly agreed that, if we want to summarize in two words which the greatest contribution of Abel and Jacobi was, then it was to consider the elliptic integrals not just as functions of a real variable, but also as functions of a complex variable. So, we owe to them the birth of the theory of holomorphic functions.

As example, we take the content of an article also considered in Houzel's talk, VI-Crelle I (1826). This article is a real gem: it anticipates S.Lie's treatment of Lie group germs, and yields actually a stronger result (although, under the assumption of commutativity):

Theorem 2.1. *Let $f : U \rightarrow \mathbb{R}$ be a germ of function defined on a neighbourhood of the origin, $\mathbb{R}^2 \supset U \ni 0$, such that*

$$f(z, f(x, y))$$

is symmetric in x, z, y (i.e., in today's terminology, we have an Abelian Lie group germ in 1-variable). Then there exists a germ of change of variable $\psi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that our Lie group becomes $(\mathbb{R}, +)$, or, more concretely, such that

$$\psi(f(x, y)) = \psi(x) + \psi(y).$$

Among the Masters' work which Abel studied was certainly, as already mentioned, M. Legendre and his theory of elliptic integrals. For these, already considered by Euler and Lagrange, Legendre devised a **normal form** (here $R(x)$ is a rational function of x):

$$\int_{x_0}^t \frac{R(x)}{\sqrt{(1-x^2)(1-k^2x^2)}} dx.$$

In the remarkable paper XVI-1 (published on Crelle, Bd. 2,3 (1827, 1828)), entitled "Recherches sur les fonctions elliptiques", Abel, as we already mentioned, writes clearly, after observing that the study of these elliptic integrals can be reduced to the study of integrals of the first, second and third kind ¹²

$$\int \frac{d\theta}{\sqrt{1-c^2\sin^2\theta}}; \int d\theta \sqrt{1-c^2\sin^2\theta}, \int \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-c^2\sin^2\theta}},$$

" These three functions are the ones that M. Legendre has considered, especially the first, which enjoys the most remarkable and the simplest properties. I am proposing myself, in this Memoir, to consider the inverse function, i.e., the function $\phi(a)$, determined by the equations

$$a = \int \frac{d\theta}{\sqrt{1-c^2\sin^2\theta}}$$

$$\sin\theta = \phi(a) = x."$$

In the following pages, where he manages to give a simple proof of the double periodicity of the given function ϕ , Abel shows the clarity of his geometric intuition.

He simply observes that in Legendre's normal form one should win the natural resistance to consider non real roots, and actually it is much better to

¹²Since an elliptic integral is the one where we consider a square root $\sqrt{P(x)}$ where P is a polynomial of degree 3 or 4, one can reduce it, after applying a projective transformation of the line \mathbb{P}^1 , to a square root $\sqrt{Q(y^2)}$, where Q is quadratic, and then we can view it as an integral on the unit circle, whose projection to the x -line \mathbb{P}^1 yields a double cover.

consider the case where $k^2 < 0$, and thus he considers (his notation) the integral

$$\int_{x_0}^t \frac{1}{\sqrt{(1-c^2x^2)(1+e^2x^2)}} dx,$$

where c, e are strictly positive real numbers.

The roots of the radical are the points $\pm 1/c, \pm(1/e)\sqrt{-1}$, we have a rectangular symmetry around the origin and we have two periods $\omega, \tilde{\omega}$ obtained by integrating on the two closed paths lying over the segments joining pairs of opposite roots:

$$\omega = 4 \int_0^{1/c} \frac{1}{\sqrt{(1-c^2x^2)(1+e^2x^2)}} dx, \quad \tilde{\omega} = 4 \int_0^{i/e} \frac{1}{\sqrt{(1-c^2x^2)(1+e^2x^2)}} dx.$$

It is straightforward to observe that the two periods (of the **real curve**) are such that $\omega \in \mathbb{R}$, respectively $\tilde{\omega} \in i\mathbb{R}$, thus they are linearly independent over the real numbers.

The conclusion is that the inverse function is **doubly periodic** with one real and one imaginary period, so that its fundamental domain is a rectangle with sides parallel to the real, resp. imaginary axis. Since every elliptic integral of the first kind can be reduced to this form, their inverse functions are all doubly periodic (unlike the circular functions, which possess only one period).

Abel does not bother to highlight the geometry underlining his argument, it is clear that he has developed a good geometrical intuition, but his style is extremely terse and concise. This conciseness becomes almost abrupt in the other article XIII-Vol. 2¹³, "Theorie des transcendentes elliptiques".

This long memoir starts *nulla interposita more* (it was probably unfinished): "For more simplicity I denote the radical by \sqrt{R} , whence we have to consider the integral

$$\int \frac{P dx}{\sqrt{R}},$$

P denoting a rational function of x ."

It is divided into three chapters, the first devoted to the reduction of elliptic integrals by means of algebraic functions, the second to the reduction of elliptic integrals by means of logarithmic functions, and finally the third is entitled "A remarkable relation which exists among several integrals of the form

$$\int \frac{dx}{\sqrt{R}}, \int \frac{x dx}{\sqrt{R}}, \int \frac{x^2 dx}{\sqrt{R}}, \int \frac{dx}{(x-a)\sqrt{R}}.$$

¹³The second volume of the edition [Abel81] by Sylow and Lie, Christiania 1881, contains the unpublished papers of Abel, with a few exceptions. As the editors remark, this edition, posterior by more than 30 years to the edition of 1839 edited by the friend and colleague of Abel, Holmboe, was financed by the Norwegian Parliament after the great demand for Abel's works (Holmboe's edition went rapidly out of print) of which especially the French Mathematical Society made itself interpreter. The two editors decided to omit in the second volume three articles which were partly based on an erroneous memoir of his youth, written in Norwegian, where Abel thought he could prove that the general equation of degree n can be solved by radicals. This is the only published article which is not appearing in the Holmboe edition, nor in Volume I of the edition by Sylow and Lie.

The memoir contains the explicit discussions of several concrete problems concerning such reductions, and contains for many of those problems explicit references to Legendre. It looks to me a rather early work, because we directly see the influence of the study of Legendre, but rather important for two reasons.

The first reason, already clear from the title of chapter III, is that Abel here for the first time considers the question of the relations holding among sums of elliptic integrals. This problem will be considered more generally for all algebraic integrals of arbitrary genus g in his fundamental Memoir XII-1, entitled "Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendentes", presented on october 10 1826 to the Académie des Sciences de Paris, and published only in 1841.

The main theorem of the latter was so formulated: " If we have several functions whose derivatives can be roots of the same algebraic equation [if $y(x)$ is an algebraic function of x , i.e., there is a polynomial P such that $P(x, y(x)) \equiv 0$, then for each rational function $f(x, y)$ there is a polynomial $F(x, y)$ such that $F(x, f(x, y(x))) \equiv 0$]¹⁴, with coefficients rational functions of one variable $[x]$, one can always express the sum of an arbitrary number of such functions by means of an algebraic and a logarithmic function, provided that one can establish among the variables of these functions a certain number of algebraic relations".

The first theorem is then given through formula (12):

$$\int f(x_1, y_1)dx_1 + \int f(x_2, y_2)dx_2 + \dots \dots + \int f(x_\mu, y_\mu)dx_\mu = v[(t_1, ..t_k)] :$$

here $f(x, y)$ is a rational function, we take the μ points which form the complete intersection of $[P(x, y) = 0]$ and $[G_t(x, y) = 0]$ where G_t depends rationally upon the parameter $t = (t_1, ..t_k)$, and the conclusion is, as we said, that v is the sum of a rational and of a logarithmic function.

Abel also explains clearly in the latter memoir that the number of these relations is a number, which later on was called the genus of the curve C birational to the plane curve of equation $P(x, y) = 0$. The way we understand the hypothesis of the theorem nowadays is through the geometric condition: if the Abel sum of these points is constant in the Jacobian variety of C . I will come back to the geometric interpretations in the next section, let me now return to the second reason of importance of the cited Memoir XIII, 2.

For instance, in Chapter I, Abel gives very explicit formulae, e.g. for the reduction of integrals of the form

$$\int \frac{x^m dx}{\sqrt{R}},$$

where $R(x)$ is a polynomial of degree 3, 4, to the integrals

$$\int \frac{dx}{\sqrt{R}}, \int \frac{xdx}{\sqrt{R}}, \int \frac{x^2 dx}{\sqrt{R}}.$$

¹⁴Here and elsewhere, [\dots] stands for an addition of the present author

The above integral is calculated by recursions, starting from the equation

$$d(Q\sqrt{R}) = S \frac{dx}{\sqrt{R}},$$

and writing explicitly

$$S = \phi(0) + \phi(1)x + \cdots + \phi(m)x^m$$

$$Q = f(0) + f(1)x + \cdots + f(m-3)x^{m-3}.$$

This is an example of Abel's mastery in the field of **Differential Algebra**. Although the modern reader, as well as Sylow and Lie, may underscore the impact of these very direct calculations, it seems to me that there has been a resurgence of this area of mathematics, especially in connection with the development of computer algorithms and programs which either provide an explicit integration of a given function by elementary functions¹⁵, or decide that the given function does not admit an integration by elementary functions (this problem was solved by Risch, and later concrete decision procedures and algorithms were given by J Davenport and Trager [Risch], [Dav79-1], [Dav79-2],[Dav81], [Trag79]).

Differential algebra is also the main tool in the article (XVII-2) "Mémoire sur les fonctions transcendentes de la forme $\int y dx$ ou y est une fonction algébrique de x ."

This paper looks very interesting and somehow gave me the impression (or at least I liked to see it in this way) of being a forerunner of the applications of Abelian integrals to questions of transcendence theory.

This time I will state the main theorem by slightly altering Abel's original notation

Theorem 2.2. *Assume that ϕ is a [non trivial] polynomial*

$$\phi(r_1, \dots, r_\mu, w_1, \dots, w_p)$$

and that $\phi \equiv 0$ if we set

$$r_i = \int y_i(x) dx, \quad w_j = u_j(x),$$

where y_i, u_j are algebraic functions. Then there is a [non trivial] linear relation $\sum_i c_i \int y_i(x) dx = P(x)$, with constant coefficients c_i and with P an algebraic function.

Corollary 2.3. *Let $y_i(x) dx$ be linearly independent Differentials of the I Kind on a Riemann surface. Then the respective integrals are algebraically independent.*

Nowadays, we would use the periodicity of these functions on the universal cover of the algebraic curve, or the order of growth of the volume of a periodic hypersurface (this argument was the one used later by Cousin, cf. [Cous02]) to infer that the polynomial must be linear (vanishing on the complex linear span of the periods). Abel's argument is instead completely algebraic in nature, he

¹⁵I.e., by rational functions or by logarithms (more generally, one can consider algebraic functions and logarithms of these).

chooses in fact the polynomial ϕ to be of minimal degree with respect to r_μ , and applies d/dx to the relation in order to obtain (Abel's notation)¹⁶

$$\sum_j \phi'(r_j) y_j + \phi'(x) = 0.$$

Abel shows then that ϕ has degree 1 in r_μ , since writing

$$R = r_\mu^k + P_0 r_\mu^{k-1} + P_1 r_\mu^{k-2} + \dots = 0$$

he gets:

$$0 = \frac{d}{dx}(R) = r_\mu^{k-1}(k y_\mu + P'_0) + \{\dots\} r_\mu^{k-2} + \dots = 0$$

whence, by the minimality of the degree k , this polynomial in r_μ has all coefficients identically zero, in particular $(k y_\mu + P'_0) \equiv 0$, therefore

$$r_\mu = \int y_\mu dx = -(1/k)P_0$$

is the desired degree one relation. By induction Abel derives the full statement that ϕ is linear in r_1, \dots, r_μ .

3. THE GEOMETRIZATION OF ABEL'S METHODS.

The process of geometrization and of a deeper understanding of Abel's discoveries went a long way, with alternate phases, for over 150 years. We believe that ¹⁷ a fundamental role for the geometrization was played by the italian school of algebraic geometry, which then paved the way for some of the more abstract developments in algebraic geometry.

Although it was very depressing for Abel that his fundamental Memoir XII-1 was not read by A. Cauchy (this is the reason why it took more than 15 years before it was published), still in Berlin Abel found the enthusiastic support of L. Crelle, who launched his new Journal by publishing the articles of Abel and Jacobi. Recall that finally fame and recognition were reaching Abel through the offer of a professorship in Berlin, which, crowning the joint efforts of Crelle and Jacobi, arrived however a few days after Abel had died.

Especially inspired by the papers of Abel was Jacobi, in Berlin, who also wrote a revolutionary article on elliptic function theory, entitled "Fundamenta nova theoriae functionum ellipticarum".

It was Jacobi who introduced the words 'Abelian integrals', 'Abelian functions': Jacobi's competing point of view (few competitions however were so positive and constructive in the history of mathematics) started soon to prevail.

Jacobi introduced the so called **elliptic theta functions**, denoted $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$, (cf. e.g. [Tric] or [Mum3]) and expressed the elliptic functions, like Abel's ϕ , inverse of the elliptic integral of the first kind, as a ratio of theta functions.

¹⁶Why was not Abel using the notation $\partial/\partial r_j$?

¹⁷As amply illustrated by Ciliberto in his contribution.

A much more general definition of **theta-series** (the expression was later coined by Rosenhain and Göpel, followers of Riemann) was given by B. Riemann, who defined his **Riemann Theta function** as the following series of exponentials

$$\theta(z, \tau) := \sum_{n \in \mathbb{Z}^g} \exp(2\pi i [\frac{1}{2} {}^t n \tau n + {}^t n z])$$

where $z \in \mathbb{C}^g$, $\tau \in \mathcal{H}_g = \{\tau \in \text{Mat}(g, g, \mathbb{C}) \mid \tau = {}^t \tau, \text{Im}(\tau) \text{ is positive definite}\}$.

The theta function converges because of the condition that $\text{Im}(\tau)$ is positive definite, it admits \mathbb{Z}^g as group of periods, being a Fourier series, and it has moreover a $\tau\mathbb{Z}^g$ -quasi-periodicity which turns out to be the clue for constructing $2g$ -periodic meromorphic functions as quotients of theta series.

With a small variation (cf. [Mum3]) one defines the theta-functions with characteristics

$$\theta[a, b](z, \tau) := \sum_{n \in \mathbb{Z}^g} \exp(2\pi i [\frac{1}{2} {}^t (n+a) \tau (n+a) + {}^t (n+a)(z+b)]),$$

and the Jacobi functions $\theta_{a,b}(z, \tau)$ are essentially the functions $\theta[a/2, b/2](2z, \tau)$.

Beyond the very explicit and beautiful formulae, what lies beyond this apparently very analytic approach is the **pioneeristic principle** that any **meromorphic function** f on a complex manifold X can be written as

$$f = \frac{\sigma_1}{\sigma_2}$$

of two relatively prime sections of a unique **Line Bundle** L on X .

This formulation came quite long after Jacobi, but Jacobi's work had soon a very profound impact. For instance, one of the main contributions of Jacobi was the solution of the **inversion problem** explicitly for genus $g = 2$.

Concretely, Jacobi considered a polynomial $R(x)$ of degree 6, and then, given the two Abelian integrals

$$u_1(x_1, x_2) := \int_{x_0}^{x_1} \frac{dx}{\sqrt{R}} + \int_{x_0}^{x_2} \frac{dx}{\sqrt{R}},$$

$$u_2(x_1, x_2) := \int_{x_0}^{x_1} \frac{x dx}{\sqrt{R}} + \int_{x_0}^{x_2} \frac{x dx}{\sqrt{R}},$$

he found that the two symmetric functions

$$s_1 := x_1 + x_2, s_2 := x_1 x_2,$$

are 4-tuple periodic functions of u_1, u_2 .

Under the name **Jacobi inversion problem** went the generalization of this result for all genera g , and the solution to the Jacobi inversion problem was one of the celebrated successes of Riemann.

Nowadays the result is formulated as follows: given a compact Riemann surface C of genus g , let $\omega_1, \dots, \omega_g$ be a basis of the space $H^0(\Omega_C^1)$ of holomorphic differentials on C **adapted** to a **symplectic** basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ for the Abelian group of closed paths $H_1(C, \mathbb{Z})$: this means that

$$\left(\int_{\alpha_i} \omega_j \right) = (\delta_{i,j}) = I_g, \left(\int_{\beta_i} \omega_j \right) = (\tau_{i,j}),$$

where I_g is the Identity ($g \times g$) Matrix and $\tau \in \mathcal{H}_g$, and that the intersection matrices satisfy $(\alpha_i, \alpha_j) = 0$, $(\beta_i, \beta_j) = 0$, $(\alpha_i, \beta_j) = I_g$.

Then the **Abel-Jacobi** map $C^g \rightarrow \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g) := Jac(C)$, associating to the g -tuple P_1, \dots, P_g of points of C the sum of integrals (taken modulo $(\mathbb{Z}^g + \tau\mathbb{Z}^g)$)

$$a_g(P_1, \dots, P_g) := \int_{P_0}^{P_1} (\omega) + \dots + \int_{P_0}^{P_g} (\omega),$$

(ω being the vector with i -th component ω_i),

is **surjective** and yields a **birational map** of the **symmetric product**

$$C^{(g)} := Sym^g(C) := C^g / \mathcal{S}_g,$$

\mathcal{S}_g being the symmetric group of permutation of g elements, onto the **Jacobian Variety of C**

$$Jac(C) := \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

We see the occurrence of a matrix τ in the so called Siegel upper half space \mathcal{H}_g of symmetric matrices with positive definite imaginary part: this positive definiteness ensures the convergence of Riemann's theta series, and indeed Riemann used explicitly his theta function to express explicitly the symmetric functions of the coordinates of a g -tuple P_1, \dots, P_g of points of C as rational functions of theta factors.

Today, we tend to forget about these explicit formulae, and we focus our attention to the geometric description of the Abel-Jacobi maps for any n -tuple of points of C in order to grasp the power of the discoveries of Abel, Jacobi and Riemann.

The **Abel-Jacobi** maps $C^n \rightarrow \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g) := Jac(C)$, given by

$$a_n(P_1, \dots, P_n) := \int_{P_0}^{P_1} (\omega) + \dots + \int_{P_0}^{P_n} (\omega),$$

and which naturally factor through the symmetric product $C^{(n)}$, enjoy the following properties (cf. the nice and concise lecture notes by D. Mumford [Mum75] for more on this topic, and also [corn76] for a clear modern presentation)

- (1) **Abel's Theorem:** The fibres are projective spaces corresponding to **linearly equivalent** divisors, i.e. $a_n(P_1, \dots, P_n) = a_n(Q_1, \dots, Q_n)$ if and only if there is a rational function f on C with polar divisor $P_1 + \dots + P_n$ and divisor of zeros $Q_1 + \dots + Q_n$.
- (2) For $n = g - 1$ we have that the image of a_{g-1} equals, up to translation, the hypersurface $\Theta \subset Jac(C)$ whose inverse image in \mathbb{C}^g is given by the vanishing of the Riemann theta function $\Theta = \{z | \theta(z, \tau) = 0\}$.
- (3) For $n = g$ we have that a_g is onto and birational.
- (4) For $n \geq 2g - 1$ we have that a_n is a fibre bundle with fibres projective spaces \mathbb{P}^{n-g} .
- (5) For $n=1$ we have an embedding of the curve C inside $Jac(C)$, and in such a way that C meets a general translate of the theta divisor Θ in exactly g points. And in this way one gets an explicit geometrical

description of the inverse to the Abel Jacobi map a_g . Indeed, to a point $y \in \text{Jac}(C)$ we associate the g -tuple of points of C given by the intersection $C \cap (\Theta + y)$ (here we think of $C \subset \text{Jac}(C)$ under the embedding a_1).

Many of the above properties are very special and lead often to a characterization of curves among algebraic varieties.

The above formulations are moreover the fruit of a very long process of maturation whose evolution is not easy to trace. For instance, when did the concept of the **Jacobian variety** of a curve (Riemann surface) C make its first appearance?

This notion certainly appears in the title of the papers by R. Torelli in 1913 ([Tor13]) but apparently ¹⁸ the name was first used by F. Klein and became very soon extremely popular. Observe however that in the classical treatise by Appell and Goursat ([App-Gours95]), dedicated to analytical functions on Riemann surfaces, and appeared first in 1895, although Jacobi's inversion theorem is amply discussed, no Jacobian or whatsoever variety is mentioned.

The French school of Humbert, Picard, Appell and Poincaré was very interested about the study of the so called "hyperelliptic varieties", generalization of the elliptic curves in the sense that they were defined as algebraic varieties X of dimension n admitting a surjective entire holomorphic map ¹⁹ $u : \mathbb{C}^n \rightarrow X$.

Among those are the so called **Abelian Varieties** ²⁰, which are the projective varieties which have a structure of an algebraic group.

In particular, Picard proved a very nice result in dimension $d = 2$, which was observed by Ciliberto (cf. his article in the present volume, also for related historical references) to hold quite generally. We want to give here a simple proof of this result, which we found during the Conference, and which makes clear one basic aspect in which the higher dimensional geometry has a different flavour than the theory of curves ²¹: namely there is no ramification in passing from Cartesian to symmetric products.

Theorem 3.1. *Let X be an algebraic variety of dimension $d \geq 2$ and assume that there is a natural number n such that the n -th symmetric product $X^{(n)}$ is birational to an Abelian variety A . Then $n = 1$ (whence, X is birational to an Abelian variety).*

¹⁸We heard this claim from S.J. Patterson in Göttingen, soon after he had written the article [Pat99].

¹⁹The letter u clearly stands for "uniformization map".

²⁰Which however, at the time of Torelli's paper, 1913, and also afterwards, were called Picard Varieties . With the Prize winning Memoir by Lefschetz, [Lef21], the terminology Abelian varieties became the only one in use.

²¹One could argue whether the beauty of 1-dimensional geometry bears similarities to the surprising isomorphisms of classical groups of small order. These also, by the way, are related to Abel's heritage. For instance, the isomorphism $\mathcal{S}_4 \cong A(2, (\mathbb{Z}/2))$, whence $\mathcal{S}_3 \cong A(2, (\mathbb{Z}/2))/(\mathbb{Z}/2)^2 \cong PGL(2, (\mathbb{Z}/2)) \cong \text{Aut}(\mathbb{H})$. Here, \mathbb{H} is the group of order 8 of unit integral quaternions, and the last isomorphism is related to some later development in the theory of algebraic curves, namely to Recilla's tetragonal construction, see [Rec74], [Don81]

Moreover, this result illustrates another main difference between the geometry of curves and the one of higher dimensional varieties X : the latter has a quite different flavour, because only seldom one can resort to the help of subsidiary Abelian varieties for the investigation of a higher dimensional variety X .

Proof. W.l.o.g. we may assume that X be smooth. Let us consider the projection π of the Cartesian product onto the symmetric product, and observe that $\pi : X^n \rightarrow X^{(n)}$ is unramified in codimension 1.

We get a rational map $f : X^n \rightarrow A$ by composing π with the given birational isomorphism, and then we observe that every rational map to an Abelian variety is a morphism.

It is moreover clear from the construction that f is not branched in codimension 1: in particular, it follows that X^n is birational to an unramified covering of A , whence X^n is birational to an Abelian variety.

Let us introduce now the following notation: for a smooth projective variety Y we consider the algebra of global holomorphic forms $H^0(\Omega_Y^*) := \bigoplus_{i=0, \dots, \dim(Y)} H^0(\Omega_Y^i)$ (**holomorphic algebra**, for short).

This graded algebra is a birational invariant, and for an Abelian variety A it is the free exterior algebra over $H^0(\Omega_A^1)$.

Now, the holomorphic algebra $H^0(\Omega_{X^n}^*)$ of a Cartesian product X^n is the tensor product of n copies of the holomorphic algebra $H^0(\Omega_X^*)$ of X .

Denote $H^0(\Omega_X^*)$ by B : then we reached the conclusion that $B^{\otimes n}$ is a free exterior algebra over its part of degree 1, $B_1^1 \oplus B_2^1 \cdots \oplus B_n^1$.

It follows that also B is a free exterior algebra (i.e., X enjoys the property that its holomorphic algebra $H^0(\Omega_X^*)$ is a free exterior algebra over $H^0(\Omega_X^1)$).

Moreover, the holomorphic algebra of the symmetric product is the invariant part of this tensor product (for the natural action of the symmetric group in n -letters \mathcal{S}_n), and by our assumption $C := (B^{\otimes n})^{\mathcal{S}_n}$ is also a free exterior algebra. However, $C^1 = B^1$, and $r := \dim(B^1)$ is also the highest degree i such that $B^i \neq 0$. But then $C^{nr} \neq 0$, contradicting the property that C is a free exterior algebra with $\dim(C^1) = r$, if $n \neq 1$.

Thus $n = 1$, and X is birational to an Abelian variety. □

Remark 3.2. *In the case of dimension $d = 1$, the same algebraic arguments easily yield that the n -th symmetric product of a curve C is not birational to an Abelian variety if $n \neq g$, $g := \dim(H^0(\Omega_C^1))$.*

Once more, the algebra of differential forms, as in Abel's work, has played the pivotal role.

The importance of this algebra was observed also by Mumford ([Mum68]) who used it to show that on an algebraic surface X with $H^0(\Omega_X^2) \neq 0$, the group of 0-cycles (Sums $\sum_i m_i P_i$ of points $P_i \in X$ with integer multiplicities $m_i \in \mathbb{Z}$) modulo rational equivalence is not finite dimensional, contrary to the hope of Severi, (D. Mumford sarcastically wrote: "One must admit that in this

case the *technique* of the italians was superior to their vaunted intuition”²²) who unfortunately was basing his proposed theory on a wrong article ([Sev32], where not by chance the error was an error of ramification).

It must be again said that the italian school, and especially Castelnuovo, gave a remarkable impetus to the geometrization of the theory of Abelian varieties.

This approach, especially through the work of Severi, influenced Andre’ Weil who understood the fundamental role of Abelian varieties for many questions of algebraic number theory. Weil used these ideas to construct ([Weil-VA]) the Jacobian variety of a curve as a quotient of the symmetric product $C^{(g)}$, and then, for a d -dimensional variety with $d \geq 2$, the Albanese variety $Alb(X)$ as a quotient (in the category of Abelian varieties) of the Jacobian $J(C)$ of a sufficiently general linear section $C = X \cap H_1 \cap H_2 \cdots \cap H_{d-1}$.

It must be however said that also the later geometric constructions were deeply influenced by the bilinear relations which Riemann, through a convenient dissection of his Riemann surface C , showed to hold for the periods of the Abelian integrals of the first kind of C .

Nowadays, the usual formulation is (according to Auslander and Tolimieri, [A-T] pages 267 and 274, the first formulation is essentially due to Gaetano Scorza in [Scor16], while the second is essentially due to Hermann Weyl in [Weyl34], [Weyl36], with refinements from A. Weil’s book [Weil-VK])

Definition 3.3. *Let Γ be a discrete subgroup of a complex vector space V , such that the quotient V/Γ is compact (equivalently, $\Gamma \otimes \mathbb{R} \cong V$): then we say that the complex torus V/Γ satisfies the two Riemann bilinear relations if*

- I) *There exists an alternating form $A : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ such that A is the imaginary part of an Hermitian form H on V*
- II) *H is positive definite.*

Remark 3.4. *Or, alternatively, a complex structure on $\Gamma \otimes \mathbb{C}$, i.e., a decomposition $\Gamma \otimes \mathbb{C} = V \oplus \bar{V}$ and an element $A \in \Lambda^2(\Gamma)^\vee$ yield a polarized Abelian variety if the component of A in $\Lambda^2(V)^\vee \subset \Lambda^2(\Gamma \otimes \mathbb{C})^\vee$ is zero and then its component in $(V)^\vee \otimes (\bar{V})^\vee$ is a positive definite Hermitian form.*²³

The basic theorem characterizing complex Abelian varieties is however due to Henri Poincaré ([Poi84], [Poi02]) who proved the linearization of the system of exponents, i.e., the more difficult necessary condition in the theorem, by an averaging procedure (integrating the ambient Hermitian metric of $X \rightarrow \mathbb{P}^N$ with respect to the translation invariant measure of $X = V/\Gamma$, he obtained a translation invariant Hermitian metric).

Theorem 3.5. *A complex torus $X = V/\Gamma$ is an algebraic variety if and only if the two Riemann bilinear relations hold true for V/Γ .*

²²However, as well known, there are Italians with techniques and ideas, and others who are not perfect. In particular, while it is not difficult to find errors or wrong assertions in Enriques and Severi, it is rather hard to do this with Castelnuovo.

²³The second characterization is very useful for the study of fibre bundles of Abelian varieties, as we had opportunity to experience ourselves, cf. [Cat02-2]

Both conditions are equivalent to the existence of a meromorphic function f on the complex vector space V whose group of periods is exactly Γ (i.e., $\Gamma = \{v \in V | f(z+v) \equiv f(z)\}$).

Poincaré had an extensive letter exchange with Klein (cf. Klein's Collected Works, where pages 587 to 621 of Vol. III are devoted to the "Briefwechsel" between the two, concerning the problem of uniformization, and their early attempts, which were based on a 'principle of continuity' which was not so easy to justify²⁴), especially related to the study of discontinuous groups, acting not only on \mathbb{C}^n as in the case of tori, but also on the hyperbolic upper half plane \mathbb{H} . The main result, whose complete proof was obtained in 1907 by a student of Klein, Koebe, and by Poincaré independently, was the famous uniformization theorem that again we state in its modern formulation for the sake of brevity.

Theorem 3.6. *If a Riemann surface is not the projective line $\mathbb{P}_{\mathbb{C}}^1$, the complex plane \mathbb{C} , nor \mathbb{C}^* or an elliptic curve, then its universal covering is the (Poincaré) upper half plane \mathbb{H} .*

The reason why the upper half plane "belongs" to Poincaré is that Klein preferred to work with the biholomorphically equivalent model given by the unit disk $\mathbb{D} := \{z \in \mathbb{C} | |z| < 1\}$. In this way Klein was capable of making us the gift of beautiful symmetries given by tessellations of the disk by fundamental domains for the action of very explicit Fuchsian groups (discrete subgroups Γ of $PSU(1, 1, \mathbb{C})$) with compact quotient \mathbb{D}/Γ .

To summarize the highlights of the turn of the century, when geometry was a very central topic, one should say that several new geometries came to birth at that time: but the new developments were based on new powerful analytic tools, which were the bricks of the new building.

However, although the birth of differential geometry lead to new geometrical theories based on infinite processes where metric notions played a fundamental role, algebraic geometry went on with alternating balance between geometrical versus algebraic methods.

4. ALGEBRAIZATION OF THE GEOMETRY

At this moment, a witty reader, tired of the distinction "algebra"- "non algebra", might also remind us that the popular expression "This is algebra for me" simply means: 'I do not understand a single word of this'.

There is a serious point to it: the concept algebra is slightly ambiguous, and a very short formula could be not very inspiring without a thoroughful explanation of its meaning(s), and of all the possible consequences and applications.

One of the best ways to understand a formula is for instance to relate it to a picture, to see it thus related to a geometrical or dynamical process.

Needless to say, the best example of such an association is the Weierstrass equation of a plane cubic curve

$$C_3^1 = \{(x, y, z) \in \mathbb{P}_{\mathbb{C}}^2 | y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3\}.$$

²⁴In his unpublished Fermi Lectures held in Pisa in 1976, D. Mumford explained how this approach was working, using clarifications due to Chabauty, [Chab50]

To this equation we immediately associate the picture yielding the group law of $C = C_3^1$, i.e., a line L intersecting C in the three points P, Q and $T = -(P+Q)$.

Where have we seen this picture first?²⁵ Well, in my case, I (almost) saw it first in the book by Walker on Algebraic curves, exactly in the last paragraph, in the section 9.1 entitled "Additions of points on a cubic". The book was written in 1949, and if we look at books on algebraic curves written long before, the group law is not mentioned there. For instance, Coolidge's book "A treatise on algebraic plane curves" has a paragraph entitled "elliptic curves", pages 302-304, and the main theorems are first that an elliptic curve is birational to a plane cubic, and then the **Cross-Ratio Theorem** asserting that if $P \in C_3^1 \subset \mathbb{P}^2$ is any point, through P pass exactly 4 tangent lines, and their cross ratio is independent of the choice of $P \in C$. The Weierstrass equation, and the explanation that the cubic curve is uniformized through the triple $(1, \mathcal{P}, \mathcal{P}')$, where \mathcal{P} is the Weierstrass function, comes later, as due after the Riemann-Roch theorem²⁶, on pages 363-367 in the paragraph "Curves of genus 1".

Going to important textbooks of the Italian tradition, like Enriques and Chisini's 4 Volumes on the "Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche"²⁷ we see that Volume IV contains the Book 6, devoted to "Funzioni ellittiche ed Abeliane". Here Abel's theorem is fully explained, and on page 77 we see Abel's theorem for elliptic curves, on page 81 the addition theorem for the \mathcal{P} function of Weierstrass: the geometry of the situation is fully explained, i.e., that three collinear points sum to zero in the group law given by the sum of Abelian integrals of the first kind

$$u_1 + u_2 + u_3 = \int_{x_0}^{x_1} \frac{dx}{y} + \int_{x_0}^{x_2} \frac{dx}{y} + \int_{x_0}^{x_3} \frac{dx}{y} = 0.$$

It is also observed that the inverse of the point $P = (1, x, y)$ is the point $P = (1, x, -y)$, as a consequence of the fact that \mathcal{P} is an even function.

This Book 6 is clearly influenced by Bianchi's Lecture Notes "Lezioni sulla teoria delle Funzioni di variabile complessa"²⁸ whose Part 2 is entirely devoted to "Teoria delle funzioni ellittiche", and in the pages 315-322 the addition theorem for \mathcal{P} is clearly explained, moreover "Alcune applicazioni geometriche" are given in the later pages 415-418.

So, the picture is there, is however missing the wording: a plane cubic is an Abelian group through the sum obtained via linear equivalence of divisors, namely, the sum of three points P, Q, T is zero if and only if the divisor $P + Q + T$ is linearly equivalent to a fixed divisor D of degree 3 (in the Weierstrass model, D is the divisor $3O$, O being the flex point at infinity).

²⁵Not always mathematical concepts have a birthday. But sometimes it happens, as for another Legacy by Abel: the concept of an abstract group was born in 1878, with a "Desiderata and suggestions" by A. Cayley [Cay78-1]. In this note appears first the multiplication table of a group. Immediately afterwards, however, in [Cay78-2], Cayley realized that it is much better to work with what is now called the "Cayley graph" of a group endowed with a set of generators.

²⁶"With proof communicated to the author by prof. Osgood verbally in Nov. 1927".

²⁷Edited by Zanichelli in the respective years 1915, 1918, 1924, 1934.

²⁸Spoerri, Pisa, 1916

As it was explained to me by Norbert Schappacher (cf. [Schap90]), the works of Mordell and Weil in the 1920's are responsible of this new wording and perspective. In fact, these authors considered a cubic curve C whose equations has coefficients in a field K , and noticed that the set $C(K)$ of K -rational points, i.e., the points whose coordinates are in K , do indeed form a subgroup. For this they did not need that one flex point should be K -rational, since essentially, once we have a K -rational point O , we can reembed the elliptic curve C by the linear system $|3O|$, and then obtain a new cubic C' whose K rational points are exactly those of C .

Through these works started an exciting new development, namely the geometrization of arithmetic, which was one of the central developments in the 20-th century mathematics.

For instance, the theory of elliptic curves over fields of finite characteristic was (cf.[Weil-29]) built by Weil and then Tate who (cf. the quite late appearing in print [Tate74]), starting from the Weierstrass equation, slightly modified into

$$y^2z = x^3 - p_2xz^2 - p_3z^3$$

started to construct analogues of the theory of periods.²⁹

Going back to Bianchi, it is Klein's and Bianchi's merit to have popularized the geometric picture of elliptic curves, and actually Bianchi went all the way through in some of his papers to describe the beautiful geometry related to the embeddings of elliptic curves as non degenerate curves of degree n in \mathbb{P}^{n-1} ($\forall n \geq 3$).

It took however quite long till a purely algebraic interpretation of Weierstrass' equation made its way through.

Nowadays we would associate to an elliptic curve C and to a point $O \in C$ the \mathbb{N} -graded ring

$$\bigoplus_{m=0}^{\infty} H^0(C, \mathcal{O}_C(mO)) := \mathcal{R}(C, \mathcal{L})$$

where C is defined over a (non algebraically closed) field K of characteristic $\neq 2$, O is a K -rational point and $\mathcal{L} := \mathcal{O}_C(O)$.

As a consequence of the Riemann Roch theorem we obtain the following

Theorem 4.1.

$$\mathcal{R}(C, \mathcal{L}) \cong K[u, \xi, \eta]/(\eta^2 - \xi^3 + p_2\xi u^4 + p_3u^6),$$

where $\deg(u) = 1$, $\deg(\xi) = 2$, $\deg(\eta) = 3$, $\text{div}(u) = O$.

To go back to the original Weierstrass equations it suffices to observe that

$$\mathcal{P} = \xi/u^2, \mathcal{P}' = \eta/u^3,$$

and that $x := \xi u$, $y := \eta$, $z := u^3$ are a basis of the vector space $H^0(C, \mathcal{O}_C(3O))$.

One sees also clearly how the Laurent expansion at O of \mathcal{P} is determined by and determines p_2, p_3 .

²⁹De hoc satis, because the talks by Faltings and Wiles were exactly dealing with these aspects of Abel's legacy.

Surprisingly, the following general problem is still almost completely open, in spite of a lot of research in this or similar directions

Problem 4.2. *Describe the graded ring $\mathcal{R}(A, \mathcal{L})$ for \mathcal{L} an ample divisor on an Abelian variety, for instance in the case where \mathcal{L} yields a principal polarization.*

Before explaining the status of the question, I would first like to explain its importance.

Take for instance the case of an elliptic curve C whose ring is completely described (the ring does not depend upon the choice of O because we have a transitive group of automorphisms provided by translations for the group law of C).

We want for instance to describe the geometry of the embedding of C as a curve of degree 4 in \mathbb{P}^3 . We observe that, at least in the case where K is algebraically closed, any such embedding is given by the linear system $|4O|$, for a suitable choice of O .

The coordinates of the map are given by a basis of the vector space $H^0(C, \mathcal{O}_C(4O))$, i.e., by 4 independent homogeneous elements of degree 4 in our graded ring $\mathcal{R}(C, \mathcal{L})$. These are easily found to be equal to $s_0 := u^4, s_1 := u^2\xi, s_2 := \xi^2, s_3 := u\eta$. Then we obviously have the two equations

$$s_0s_2 = s_1^2, \quad s_3^2 = s_1s_2 + p_2s_0s_1 + p_3s_0^2,$$

holding for the image of C (the second is obtained by the "Weierstrass" equation once we multiply by u^2).

These are all the equations, essentially by Bezout's theorem, since C maps to a curve of degree 4 and $4 = 2 \times 2$.

From an algebraic point of view, what we have shown is the process of determining a subring of a given ring, and the nowadays computer algebra programs like "Macaulay" have standard commands for this operation (even if sometimes the computational complexity of the process may become too large if one does not use appropriate tricks).

Classically, a lot of attention was devoted to the geometric study of the maps associated to the linear systems $|m\Theta|$, where Θ is the divisor yielding a principal polarization of the given Abelian variety.

For instance, the 4-ic Kummer surface is the image in \mathbb{P}^3 of a principally polarized Abelian surface under the linear system $|2\Theta|$, yielding a $2 : 1$ morphism which identifies a point v to $-v$, and blows up the 16 2-torsion points to the 16 nodal singularities of the image surface.

There is a wealth of similar results, which can be found for instance in the books of Krazer³⁰, Krazer-Wirtinger, and Coble ([Kraz], [K-W], [Coble]), written in the period 1890-1926.

The first books are directly influenced by the Riemann quadratic relations, i.e., linear relations between degree two monomials in theta functions with characteristics (and their coefficients being also products of "Thetanullwerte", i.e., values in 0 of such thetas with characteristics), and show an attempt to use geometrical methods starting from analytic identities. Coble's book is entitled

³⁰"Theorie der Thetafunktionen, Teubner, 1894.

”Algebraic geometry and theta functions”, and is already influenced by the breakthrough made by Lefschetz in his important Memoir ([Lef21]).

Lefschetz used systematically the group law to show that if $s_1(v), \dots, s_r(v) \in H^0(m\Theta)$ and we choose points a_1, \dots, a_r such that $\sum_i a_i = 0$, then the product $s_1(v + a_1) \dots s_r(v + a_r) \in H^0(rm\Theta)$. Then he chooses sufficiently many and sufficiently general points a_1, \dots, a_r so that these sections separate points and tangent vectors. Thus Lefschetz proves in particular

Theorem 4.3. $|m\Theta|$ yields a morphism for $m \geq 2$ and an embedding for $m \geq 3$.

The direction started by Lefschetz was continued by many authors, notably Igusa, Mumford, Koizumi, Kempf, who proved several results concerning the equations of the image of an Abelian variety (e.g., that the image of $|m\Theta|$ is an intersection of quadrics for $m \geq 5$).

I will later return on some new ideas related to these developments, I would like now to focus on the status of the problem I mentioned.

The case where A has dimension $g = 1$ being essentially solved by Weierstrass, the next question is whether the answer is known for $g = 2$. This is the case, since the description of the graded ring was obtained by A. Canonaco in 2001 ([Can02]); an abridged version of his result is as follows

Theorem 4.4. *Let A be an Abelian surface and Θ be an effective divisor yielding a principal polarization: then the graded ring $\mathcal{R}(A, \mathcal{O}_A(\Theta))$ has a presentation with 11 generators, in degrees $(1, 2^3, 3^5, 4^2)$, and 37 relations, in degrees $(4, 5^6, 6^{17}, 7^{10}, 8^3)$.*

Remark 4.5. *1) Canonaco gives indeed explicitly the 37 relations, whose shape however is not always the same. One obtains in this way an interesting stratification of the Moduli space of p.p. Abelian surfaces. Does this stratification have a simple geometrical meaning in terms of invariant theory?*

The proof uses at a certain point some computer algebra aid, since the equations are rather complicated.

Nevertheless, we would like to sketch the simple geometric ideas underlying the algebraic calculations, since, as one can easily surmise, they are related to the aforementioned Riemann’s developments of Abel’s investigations.

Proof. The key point is thus that, in the case where Θ is an irreducible divisor, Θ is isomorphic to a smooth curve C of genus 2 (the other case where Θ is reducible is easier, since then A is a product of elliptic curves, and Θ is the union of a vertical and of a horizontal curve).

One uses first of all the exact sequence (for $n \geq 2$, since for $n = 1$ the right arrow is no longer surjective)

$$0 \rightarrow H^0(A, \mathcal{O}_A((n-1)\Theta)) \rightarrow H^0(A, \mathcal{O}_A(n\Theta)) \rightarrow H^0(\Theta, \mathcal{O}_\Theta(n\Theta)) \rightarrow 0$$

and of the isomorphism

$$H^0(\Theta, \mathcal{O}_\Theta(n\Theta)) \cong H^0(C, \mathcal{O}_C(nK_C)).$$

One relates thus our graded ring to the canonical ring of the curve C , which is well known, the canonical map of C yielding a double covering of \mathbb{P}^1 branched on 6 points.

In more algebraic terms, there is a homogeneous polynomial $R(y_0, y_1)$ of degree 6 such that

$$\mathcal{R}(C, \mathcal{O}_C(K_C)) \cong K[y_0, y_1, z]/(z^2 - R(y_0, y_1)).$$

One can summarize the situation by observing that, if $\mathcal{R}_A := \mathcal{R}(A, \mathcal{O}_A(\Theta))$, $\mathcal{R}_C := \mathcal{R}(C, \mathcal{O}_C(K_C))$, then \mathcal{R}_A surjects onto the subring \mathcal{R}' of \mathcal{R}_C defined by $\mathcal{R}' := \bigoplus_{n \geq 2} H^0(C, \mathcal{O}_C(nK_C))$.

To lift the ring structure of \mathcal{R}' , which is not difficult to obtain, to the ring structure of \mathcal{R}_A we use again Abel's theorem, i.e. the sequence of maps

$$C^2 \rightarrow C^{(2)} \rightarrow A$$

where the last is a birational morphism contracting to a point the divisor E consisting of the set of pairs $\{(P, i(P)) | P \in C\}$ in the canonical system of C (here i is the canonical involution of the curve C).

Letting D_i be the pull back of a fixed Weierstrass point of C under the i -th projection of $C \times C$ onto C , we obtain that

$$H^0(A, \mathcal{O}_A(n\Theta)) \cong H^0(C^2, \mathcal{O}_{C^2}(n(D_1 + D_2 + E)))^+,$$

where the superscript $+$ denotes the $+1$ -eigenspace for the involution of C^2 given by the permutation exchange of coordinates. It also helps to consider that the Θ divisor of A is the image of a vertical divisor $\{P\} \times C$. Instead, the diagonal Δ_C of C^2 enters also in the picture because the pull back (under $h = \phi_K \times \phi_K : C \times C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, ϕ_K being the canonical map of C) of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ is exactly the divisor $\Delta_C + E$, whence $\Delta_C + E$ is linearly equivalent to $2D_1 + 2D_2$.

We omit the more delicate parts of the proof, which are however based on the above linear equivalences and on the action of the dihedral group $D_{4 \times 2}$ on $C \times C$ (this is a lift, via the $(\mathbb{Z}/2)^2$ Galois cover $h : C \times C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of the permutation exchange of coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$).

□

Remark 4.6. 1) *This approach should work in principle for the more general case of the Jacobian variety of a hyperelliptic curve. In fact, for each Jacobian variety we have the sequence of maps*

$$C^g \rightarrow C^{(g)} \rightarrow A$$

and the Θ divisor is the image of a big vertical divisor $\{P\} \times C^{g-1}$. Again we have a $(\mathbb{Z}/2)^g$ Galois cover $h : C^g \rightarrow (\mathbb{P}^1)^g$ and a semidirect product of the Galois group with the symmetric group in g letters \mathcal{S}_g (the group of the g -dimensional cube).

2) *Another question is whether there does exist a more elegant, or just shorter presentation for the ring.*

A more conceptual understanding of the several identities of general theta functions came through the work of Mumford ([Mum66-7] , cf. also [Igu] and [Weil-64]).

In Mumford's articles and in Igusa's treatise one finds a clear path set by choosing representation theory as a guide line, especially as developed by Weyl, Heisenberg and von Neumann.

The basic idea is shortly said: let G be a compact topological group, endowed henceforth with the (translation invariant) Haar measure $d\mu_G$.

Consider then the vector space $V = L^2(G, \mathbb{C})$: then we have an action τ of the group G on V by translations, $\tau_\gamma(f)(g) := f(g\gamma^{-1})$.

Defining the group of characters G^* as $G^* := Hom(G, \mathbb{C}^*)$ we have an action of G^* on V given by multiplication $\chi f(g) := \chi(g)f(g)$.

The two actions fail to commute, but by very little, since

$$\chi[\tau_\gamma(f)](g) = \chi(g)f(g\gamma^{-1})$$

$$\tau_\gamma(\chi f)(g) = \chi(g\gamma^{-1})f(g\gamma^{-1}) = \chi(\gamma)^{-1}\chi(g)f(g\gamma^{-1})$$

thus commutation fails just up to multiplication with the constant function $\chi(\gamma)^{-1}$.

Together, the action of G and of G^* generate a subgroup of the Heisenberg group, a central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow Heis(G) \rightarrow G \times G^* \rightarrow 1.$$

It turns out that the algebra of theta functions is deeply related to the representation theory of the Heisenberg group of the Abelian variety A (we see A as the given group G).

But, as Mumford pointed out, we have a more precise relation which takes into account a given line bundle \mathcal{L} .

In the case of an Abelian variety A , the group of characters is endowed with a complex structure viewing it as the **Picard variety** $Pic^0(A)$, the connected component of 0 in $H^1(A, \mathcal{O}_A^*)$. $Pic^0(A)$ is also called the dual Abelian variety, and a non degenerate line bundle \mathcal{L} is one for which the homomorphism $\phi_{\mathcal{L}} : A \rightarrow Pic^0(A)$, defined by $\phi_{\mathcal{L}}(x) = T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ (T_x denoting translation by x), is surjective (hence with finite kernel $K(\mathcal{L})$).

Mumford introduces the finite Heisenberg group associated to \mathcal{L} via the so called Thetagroup of \mathcal{L} , defined as $\Theta(\mathcal{L}) := \{(x, \psi) | \psi : \mathcal{L} \cong T_x^*(\mathcal{L})\}$.

$\Theta(\mathcal{L})$ is a central extension of $K(\mathcal{L})$ by \mathbb{C}^* , but since $K(\mathcal{L})$ is finite, if n is the exponent of $K(\mathcal{L})$, the central extension is induced (through extension of scalars) by another central extension

$$1 \rightarrow \mu_n \rightarrow \Theta(\mathcal{L}) \rightarrow K(\mathcal{L}) \rightarrow 1,$$

where μ_n is the group of n -th roots of unity.

Moreover, the alternating form $\alpha : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ given by the Chern class of \mathcal{L} gives a non degenerate symplectic form on $K(\mathcal{L})$ with values in μ_n , thus allowing to easily obtain from $\Theta(\mathcal{L})$ the Heisenberg group of a finite group G .

The geometry of the situation is that the group $K(\mathcal{L})$ acts on the projective space associated to the vector space $H^0(A, \mathcal{L})$: but if we want a linear representation on the vector space $H^0(A, \mathcal{L})$ we must see this vector space as a representation of the finite Heisenberg group $\Theta(\mathcal{L})$ (thus we have a link with Schur's theory of multipliers of a projective representation).

In the case of $\mathcal{L} = \mathcal{O}_A(n\Theta)$, $K(\mathcal{L})$ consists of the subgroup A_n of n -torsion points, and another central idea, when we have to deal with a field of positive characteristic, is to replace the vector space $\Gamma \otimes \mathbb{R}$ with the inverse limit of the subgroups A_n .

The story is too long and too recent to be further told here: Mumford used this idea in order to study the Moduli space of Abelian varieties over fields of positive characteristics, and in turn this was used to take the reduction modulo primes of Abelian varieties defined over number fields.

These results were crucial for arithmetic applications, especially Faltings' solution ([Falt83]) of the

Mordell conjecture. Let a curve C of genus $g \geq 2$ be defined over a number field K : then the set $C(K)$ of its rational points is finite.

It must be furthermore said that the algebraic calculations allowed by the study of the characters of representations of the finite Heisenberg groups has lead also to a better concrete understanding of equations and geometry of Abelian varieties.

Surprisingly enough, even in the case of elliptic curves this has led, together with Atiyah's study of vector bundles on elliptic curves (cf.[Ati57]) to a deeper understanding of the geometry of symmetric products of elliptic curves and their maps to projective spaces (cf. [Ca-Ci93]).

The recent literature is so vast that we have chosen to mention just a single but quite beautiful example, due to Manolache and Schreyer (cf. [Man-Schr]).

The authors give several equivalent descriptions of the moduli space $X(1, 7)$ of Abelian surfaces S with a polarization L whose elementary divisors are $(1, 7)$.

Their main result is that this moduli space is birational to the Fano 3-fold V_{22} of polar hexagons to the Klein plane quartic curve C (of equation $x^3y + y^3z + z^3x = 0$) which is a compactification of the moduli space $X(7)$ of elliptic curves E with a level 7 structure, i.e., elliptic curves given with an additional isomorphism of the group E_7 of torsion points with $(\mathbb{Z}/7)^2$.

The Klein quartic is rightly famous because it admits then as group of automorphisms the group $\mathbb{P}SL(2, \mathbb{Z}/7) = SL(2, \mathbb{Z}/7)/\{\pm I\}$, a group of cardinality 168, and as it is well known this makes the Klein quartic the curve of genus 3 with the maximal number of automorphisms (cf. [Acc94]).

Now, it is easy to suspect some connection between the pairs (S, L) and the pairs $(E, \mathcal{L} := \mathcal{O}_E(7O))$ once we have learnt of the finite Heisenberg group: in fact, the respective groups for L and for \mathcal{L} are isomorphic, and the respective complete linear systems $|L|$ and $|\mathcal{L}|$ yield embeddings for E , respectively for the general S , into \mathbb{P}^6 .

That is, we view both E and S as Heisenberg invariant subvarieties of the same \mathbb{P}^6 with an action of $(\mathbb{Z}/7)^2$ provided by the projectivization of the standard representation of the Heisenberg group on $\mathbb{C}^{\mathbb{Z}/7}$.

The geometry of the situation tells us that E is an intersection of $28 - 14 = 14$ independent quadrics, while we expect S to be contained in $28 - 28 = 0$ quadrics, so we seem to be stuck without a new idea.

The central idea of the authors is to think completely in algebraic terms, looking at a self dual locally free Hilbert resolution of the ideal of S , which has length 5 instead of 4 (because of $H^1(S, \mathcal{O}_S) \neq 0$).

It turns out that the middle matrix, because of Heisenberg symmetry, boils down to a 3×2 matrix of linear forms on a certain \mathbb{P}^3 . Then, the 3 determinants of the 2×2 -minors yield three quadric surfaces whose intersection is a twisted cubic curve Γ_S which is shown to completely determine S . In this way one realizes the moduli space as a certain subvariety of the Grassmann variety of 3-dimensional vector subspaces of the vector space $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ of quadrics in the given \mathbb{P}^3 (one takes the locus of subspaces where a certain antisymmetric bilinear map of vector bundles restricts to zero).

This subvariety is the Fano 3-fold V_{22} mentioned above, which is a rational variety.

At this point we don't want to deprive the reader of the pleasure of learning the intricate details from the original sources ([Man-Schr], [Schr]): but we need at least to explain what is a polar hexagon of a plane quartic curve C with equation $f(x) = 0$.

Observe that the polynomial $f(x)$ depends upon exactly 15 coefficients, while, if we take 6 linear forms $l_i(x)$, they depend on 18 affine parameters, and we expect therefore to have a 3-dimensional variety $Hex(C)$ parametrizing the 6-tuples of such linear forms such that $f(x) = \sum_i l_i(x)^4$.

$Hex(C)$ is called the variety of polar hexagons, and it is indeed a 3-fold in the case of the Klein quartic. Finally, these constructions allow to find the Klein quartic as the discriminant of the net of quadrics in \mathbb{P}^3 associated to the Hilbert resolution of S .

Also the identification of the Fano 3-fold with $Hex(C)$ is based on the study of higher syzygies, but beyond this many other beautiful classical results are used, which are due to Klein, Scorza, and Mukai (cf. [Klein78], [Scor99], [Muk92]).

Especially nice is the old theorem of Scorza ([Scor99]), proved in 1899, that the variety of plane quartic curves is birational to the variety of pairs of a plane quartic curve D given together with an even theta-characteristic (this amounts to writing the equation of D as the discriminant of a net of quadrics in \mathbb{P}^3). This theorem is a clear example of the geometrization of ideas coming from the theory of theta functions (which, as we saw, are certain Fourier series, and therefore, seemingly, purely analytical objects).

ABELIAN VARIETIES AND MULTILINEAR ALGEBRA

The roots of these developments, which historically go under the name "The problem of Riemann matrices", and occupied an important role for the birth

of the theory of rings, modules and algebras, are readily explained by the following basic

Remark 4.7. *Given two tori $T = V/\Gamma, T' = V'/\Gamma'$, any holomorphic map $f : T \rightarrow T'$ between them is induced by a complex linear map $F : V \rightarrow V'$ such that $F(\Gamma) \subset \Gamma'$.*

Whence, for a g -dimensional torus, the ring

$$\text{End}(T) := \{f : T \rightarrow T \mid f \text{ is holomorphic, } f(0) = 0\}$$

is the subring of the ring of matrices $\text{Mat}(2g, 2g, \mathbb{Z})$ given by

$$\text{End}(T) : \{B \in \text{End}(\Gamma) \mid B \in (V^\vee \otimes V) \oplus (\bar{V}^\vee \otimes \bar{V}) \subset (\Gamma \otimes \mathbb{C})^\vee \otimes (\Gamma \otimes \mathbb{C})\}$$

since then the restriction of B to $V \subset (\Gamma \otimes \mathbb{C})$ is complex linear.

In general, the study of endomorphism rings of complex tori is not completely achieved.

The main tool which makes the case of Abelian Varieties easier is the famous **Poincaré's complete reducibility theorem** (cf. [Poi84])

Theorem 4.8. *Let A' be a subabelian variety of an Abelian variety A : then there exists another Abelian variety A'' and an **isogeny** (a surjective homomorphism with finite kernel) $A' \times A'' \rightarrow A$.*

Proof. The datum of A' amounts to the datum of a sublattice $\Gamma' \subset \Gamma$ which is saturated (Γ/Γ' is torsion free) and complex (i.e., there is a complex subspace $W \subset V$ with $W \oplus \bar{W} = \Gamma' \otimes \mathbb{R} \subset \Gamma \otimes \mathbb{R} = V \oplus \bar{V}$).

Now, given the alternating form A , its orthogonal in Γ yields a sub-lattice Γ'' spanning the complex subspace U orthogonal to W for the Hermitian bilinear product associated to H : since H is positive definite, we obtain an orthogonal direct sum $V = W \oplus U$, and we define $A'' := U/\Gamma''$.

□

We sketched the above proof just with the purpose of showing how the language of modern multilinear algebra is indeed very appropriate for these types of questions.

The meaning of the reducibility theorem is that, while for general tori a subtorus $T' \subset T$ only yields a quotient torus T/T' , here we get a direct sum if we consider an equivalence relation which identifies two isogenous Abelian varieties.

Algebraically, the winning trick was thus to classify first Endomorphism Rings tensored with the rational integers, because

Remark 4.9. *If $T = V/\Gamma, T' = V/\Gamma'$ are isogenous tori, then $\text{End}(T) \otimes \mathbb{Q} \cong \text{End}(T') \otimes \mathbb{Q}$.*

And then the study is restricted to the one of **Simple Abelian Varieties**, i.e., of the ones which do not admit any Abelian subvariety whatsoever (naturally, this concept was very much inspired by the analogous concept of curves which do not admit a surjective and not bijective mapping onto a curve of positive genus).

The classification of Endomorphism rings of Abelian varieties was achieved through a long series of works by Scorza, Rosati, Lefschetz and Albert (cf. e.g. [Scor16], [Lef21], [Albe], [Ros29]) and today one can find an exposition in Chapters 5 and 9 of the book by Lange and Birkenhake [L-B], cf. also, for an historical account, the article by Auslander and Tolimieri [A-T].

Although the methods of Scorza and Rosati were more geometrical, certainly more than the later ones by Albert, who essentially worked in the new direction set up by Emmy Noether, i.e., of the abstract algebra, a central role is played by a notion due to Rosati, the so-called **Rosati involution**.

Given an endomorphism with integral matrix B , the Rosati involution associates to it (A being a Riemann integral matrix as in (3.3), (3.4)) the matrix $B' : A^{-1} {}^t B A$. The Rosati involution is positive in the sense that the symmetric bilinear form $(B_1, B_2) := \text{Tr}(B'_1 B_2 + B_1 B'_2)$ yields a positive definite scalar product.

It turns out that the classification of Riemann matrices is very close to the study of rational Algebras with a positive involution, and abstract arguments imply that these simple algebras are skew fields \mathbb{F} of finite dimension over \mathbb{Q} of two types

- (I) The centre K of \mathbb{F} is a totally real number field and, if $K \neq \mathbb{F}$, then \mathbb{F} is a quaternion algebra over K . Moreover, for every embedding $\sigma : K \rightarrow \mathbb{R}$, $\mathbb{F} \otimes \mathbb{R}$ is always definite ($\mathbb{F} \otimes \mathbb{R} \cong \mathbb{H}$), or always indefinite ($\mathbb{F} \otimes \mathbb{R} \cong M(2, \mathbb{R})$).
- (II) The centre K of \mathbb{F} is a totally complex quadratic extension of a totally real number field K_0 , and then $\mathbb{F} \otimes \mathbb{C}$ is a matrix algebra $M(r, \mathbb{C})$ such that the positive involution extends to the standard involution $C \rightarrow {}^t \overline{C}$.

The analytic moduli theory of Abelian varieties owes much to the work of Siegel and to his 'Symplectic geometry' ([Sieg43]): today the space of matrices $\mathcal{H}_g := \{\tau \in \text{Mat}(g, g, \mathbb{C}) \mid \tau = {}^t \tau, \text{Im}(\tau) > 0\}$ is called the Siegel upper half space, and it is a natural parameter space for Abelian varieties, since, depending on the polarization, there is a subgroup Γ of $Sp(2g, \mathbb{Q})$ such that the moduli space is, analytically, the quotient \mathcal{H}_g/Γ .

The moduli theory of Abelian varieties with a certain polarization and endomorphism structure was pursued relatively recently by Shimura (cf. [Shim63]), and it is a currently very active field of research for the arithmetic applications of the theory of such Shimura varieties.

I do not need to cite for instance the (recently proved) so called Shimura-Taniyama-Weil conjecture about the modularity of elliptic curves defined over \mathbb{Q} : I can simply refer to the talk by Wiles.

In this direction, however, the current tendency is to develop also much the geometry, since one has to look at the reduction of these modular varieties modulo primes. The hope is that this study will play a primary role for the pursuing of the so called Langlands program, which is a vast generalization of the previously cited conjecture, proposing to relate modular forms arising in different contexts (cf. [Langl70], and [Langl76], [Langl79], [Del79] for early accounts of the story).

I hope that some more competent author than me will report about this development in the present volume.

I want instead to end this section by pointing out (cf. [Zap91]) how important the role of Scorza was for the development of the field of abstract algebra in Italy: his path started with correspondences between curves, but, as we contended here, his researches centered about Riemann matrices made him realize about the relevance of the powerful new algebraic concepts.

5. FURTHER LINKS TO THE ITALIAN SCHOOL

We mentioned in the previous section how the research of Rosati and Scorza was very much influenced by the new geometric methods of the Italian school of algebraic geometry.

As we said, a crucial role was played by Castelnuovo: concerning Abel's theorem, in the article [Cast93], entitled "The 1-1 correspondences between groups of p points on a curve of genus p ", he explained how one could e.g. formulate the fundamental theorem about the inversion of Abelian integrals as a consequence of the theory of linear series on a curve (a development starting with the geometric interpretation of the Riemann-Roch theorem).

It is interesting to observe that, when he wrote a final note in the edition appearing in his collected works, he points out that the results can be formulated in a simpler way if one introduces the concept of the Jacobian variety of the curve.

These notes added around 1935 are rather interesting: for instance, in the note to the paper entitled "On simple integrals belonging to an irregular surface" ([Cast05]) he pointed out that exactly in this Memoir he introduced the concept of the so called **Picard variety**, applying this concept to the study of algebraic surfaces. In fact, the theorem of Picard to which Castelnuovo refers, proved by Picard in [Pic95], and with precisions by Painlevé in [Pain03], is indeed the characterization of the Abelian varieties, (we add to it a slight rewording in modern language)

Theorem 5.1. *Let V^p be a p -dimensional algebraic variety admitting a transitive p -dimensional abelian group of birational transformations: then the points $\xi \in V$ are uniformized by entire $2p$ -periodic functions on \mathbb{C}^p ,*

$$\xi_k = \phi_k(u_1, \dots, u_p)$$

(i.e., V is birational to a complex torus of dimension p).

The first main result of Castelnuovo in [Cast05] is the construction of the so called Albanese variety and Albanese map of an algebraic surface X . Recall that, in modern language, the Albanese variety of a projective variety X is the Abelian variety $(H^0(\Omega_X^1))^\vee / H_1(X)$ where $H_1(X)$ is the lattice, in the dual vector space of $H^0(\Omega_X^1)$, given by integration along closed paths.

The Albanese map, defined up to translation, as a result of the choice of a base point x_0 , associates to a point x the linear functional $\alpha(x) := \int_{x_0}^x (\text{mod } H_1(X))$.³¹

The second result, obtained independently by F. Severi in [Sev05], concerns the equality of the irregularity of an algebraic surface and the dimension of the space of holomorphic one forms. Both proofs were relying on a shaky proof given by Enriques one year before, in [Enr05], claiming the existence a continuous system Σ of dimension $q := p_g - p_a$ of "inequivalent curves" (i.e., such that for a generic curve $C \in \Sigma$, the set of curves in Σ which are linearly equivalent to C has dimension zero).

Fortunately, a correct analytical proof was later found by Poincaré in [Poi10]. Enriques and Severi tried for a long time to repair the flaw in Enriques' geometrical arguments, although in the end it started to become clear the need for higher order differential elements (i.e., higher order terms in the Taylor expansion of the curve variation). The fruit of the researches carried on much later in the 50's was to show that indeed, for varieties defined over algebraically closed fields of positive characteristic, the **arithmetic irregularity** $q := p_g - p_a$ was in general larger than the **geometric irregularity** defined as the maximal dimension of a continuous system of "inequivalent curves". We refer the reader to Mumford's classical book [MumLC], relating this question to the non reducedness of the **Picard scheme** $H^1(\mathcal{O}_X^*)$.

Castelnuovo was instead more interested in the applications of the previous theorem, the most important one being the theorem [Cast05-2] that an algebraic surface with arithmetic genus p_a smaller than -1 is birationally ruled.

This theorem is indeed one of the key theorems of the classification of algebraic surfaces, since it also implies the well known

Castelnuovo rationality criterion: a surface is rational if and only if the bigenus P_2 and the irregularity q vanish.

Without opening a new story, I would like to observe that the so called "Enriques classification" of algebraic surfaces, done by Castelnuovo and Enriques, was one of the most interesting cooperations in the history of mathematics, which took place in the years from 1892 to 1914 (and especially intense in the period 1892 to 1906).

Besides the published papers, one can consult today the book entitled "Riposte armonie" [RA] ("Hidden harmonies", as are the ones governing algebraic

³¹Giacomo Albanese emigrated from Italy in 1936 to Sao Paulo, Brasil, where, soon after the war, he became closely acquainted with André Weil, who taught there, as well as Zariski. Weil is responsible for the name "Albanese variety", but Ciliberto and Sernesi in [Alb-CP] write: "Whilst the attribution of this concept to Albanese is dubious, ..." . Indeed the basis for this is an article, [Alb34], where Albanese studies correspondences between algebraic surfaces through the consideration of the induced action on the space of holomorphic 1-forms. The coupled names "Albanese" and "Picard" appear in the title of the article by W.L. Chow " Abstract theory of the Picard and Albanese varieties", [Chow59].

Indeed, in the 50's, one main purpose was to distinguish among the two dual varieties, which are only isogenous, and not in general isomorphic. According to the historical note of Lang on page 52 of [Lang-AV], Matsusaka was the first to give a construction of the Albanese variety using the generic curve.

surfaces), which contains around 670 letters (or postcards) written from Federico Enriques to Guido Castelnuovo.

Naturally, also Castelnuovo wrote quite many letters, but apparently Enriques did not bother to keep them. This already shows where Castelnuovo and Enriques respectively belong, in the rough distinction made by H. Weyl which we quoted above, although both of them were obviously geometers.

In fact, Enriques used to discuss mathematics with his assistants during long walks in gardens or parks, and would only sometime stop to write something with his stock on the gravel. Moreover, as Guido Castelnuovo wrote of him³² with affection, he was a "mediocre reader, who saw in a page not what was written, but what he wanted to see"; certainly his brain was always active like a vulcano. After the first world war the collaboration of the two broke up, more on the side of Castelnuovo. As his daughter Emma Castelnuovo writes, Enriques would regularly visit his sister (Castelnuovo's wife) at their house, and after dinner the two mathematicians went to a separate sitting room, where Enriques wanted to discuss his many new ideas, while Castelnuovo had prudently instructed his wife to come after some time and interrupt their conversation with some excuse.

Castelnuovo was 6 years older than Enriques, was always calm and mature, and, after the appearance of his ground breaking two notes over algebraic surfaces ([Cast91]) he was a natural reference for the brilliant student Enriques, who graduated in Pisa in 1891 (just at the time when Guido became a full professor). Enriques wanted first to perfection his studies under the guidance of Segre in Torino, but instead got a fellowship in Rome by Cremona, and there, in 1892, started the intense mathematical interchange with Castelnuovo.

In his first Memoir ([Enr93]) Enriques, after an interesting historical introduction, sketches the main tools to be used for the birational study of algebraic surfaces, namely: the theory of linear systems of curves, the canonical divisor and the operation of adjunction. Some results, as the claim that two birational quartic surfaces are necessarily projectively equivalent, are today known not to hold. It took a long time to make things work properly, and it is commonly agreed that the joint paper [C-E14] marks the achievement of the classification theory.

For the later steps, (main ones, as well known to anyone who understands the structure of the classification theorem) a very important role played the

(IP) De Franchis' theorem on irrational pencils ([D-F05], [Cast05-2]) and the

(HS) Classification of hyperelliptic surfaces .

(IP) De Franchis' theorem, obtained independently also by Castelnuovo and Enriques, asserts that if on a surface S there are several linearly independent one forms $\omega_i, i = 1, \dots, r$ which are pointwise proportional, then there is a mapping $f : S \rightarrow C$ to an algebraic curve C such that these forms are pull backs of holomorphic one forms on C . It was used by Castelnuovo to show

³²In the commemoration opening vol. I of his selected papers in geometry, [EnrMS], IX - XXII.

that under the inequality $p_g \leq 2q - 4$ there is a mapping to a curve of positive genus.

This theorem leads to a typical example of the algebraization of the geometry: although the Hodge theory of Kähler varieties was established with the use of hard analytic tools which underlie Hodge's theorem on harmonic integrals ([Hod52], [Kod52]), what turned out to be very fertile was the simple algebraic formulation in terms of the cohomology algebra of a projective variety.

Using this, Z. Ran, M. Green and myself ([Ran81], [Cat91]) were independently able to extend the result of de Franchis to the case of higher dimensional varieties and higher dimensional targets. In this way the ideas of the Italian school came back to intense life, and became an important tool for the investigation of the fundamental groups of algebraic varieties (for instance N. Mok [Mok97] tried to extend this result to infinite dimensional representations, with the hope of using such a result for the solution of the so called Shafarevich conjecture about the universal coverings of algebraic varieties).

(HS) The second main work of de Franchis ³³, together with Bagnera, was the classification of hyperelliptic surfaces, i.e. of surfaces whose universal covering space is biholomorphic to \mathbb{C}^2 (cf. [Ba-DF07], [Ba-DF07-P], [Ba-DF07], [E-S07], [E-S08], [E-S09], [E-S10]). This classification was also obtained by Enriques and Severi, and the Bordin Prize was awarded to Enriques and Severi in 1907, and to the Sicilian couple in 1909 ³⁴. Strange as it may seem that two couples get two prizes for the same theorem, instead of sharing one, this story is even more complicated, since the first version of the paper by Enriques and Severi was withdrawn after a conversation of Severi with de Franchis, and soon corrected. Bagnera and de Franchis were only a little later, since they had to admit a restriction (a posteriori useless, since no curve on an Abelian surface is contractible, cf. [D-F36]): their proof was however simpler, and further simplified by de Franchis much later ([D-F36-2]).

Another beautiful result, and this one even more related to Abelian integrals, is the famous

Torelli's Theorem: Let C, C' be two algebraic curves whose Jacobian varieties are isomorphic as polarized Abelian varieties (equivalently, admitting the same matrix of periods for Abelian integrals of the I kind): then C and C' are birationally isomorphic, [Tor13].

Torelli was born in 1884 and was a student of Bertini, in Pisa, where he attended also Bianchi's lectures: he was for short time assistant of Severi, and died prematurely in the first world war, in 1915.

These years at the beginning of the 20-th century in Italy were thus quite exciting. In the book by J. Dieudonné, [Dieu] vol.1, Chapter VI, entitled "Développement et chaos", contains a paragraph dedicated to "L' école italienne et la théorie des systèmes linéaires", namely, devoted to the second generation of the Italian school. As third generation of the Italian school, we became very

³³Michele de Franchis, born in Palermo, 1875, was very much influenced by the teaching of Scorza. He was also the Director of the Rendiconti Circolo Matematico di Palermo in the years 1914-1946, as the successor of the founder G.B. Guccia

³⁴Lefschetz was instead recipient of the Bordin Prize in 1919 for the cited Memoir [Lef21].

interested, also because of this criticism, to become fully acquainted with the results of these precursors.

What I found as a very interesting peculiarity reading the book by Enriques on "Algebraic surfaces" ([Enr49]), was the mixture of theorems, proofs, speculations, and history of the genesis of the mathematical ideas.

For instance, chapter IX, entitled "Irregular surfaces and continuous systems of disequivalent curves", has a section 6 on "History of the theory of continuous systems" (pages 339-347). One can read there that in 1902 Francesco Severi, who had just graduated in Torino, following the advice of his master C. Segre, accepted a position in Bologna as assistant of Severi. Under the influence of Enriques, Severi started the investigation of the particular surfaces which occur as the symmetric square of a curve (cf. [Sev03]). According to Enriques, this research lead him to consider the **Problem of the base** for the class of divisors modulo numerical equivalence.

We may also observe that these surfaces have a very special geometry, and, although a general characterization has not yet been described in general, we have some quite recent (cf. [C-C-ML98], [H-P01], [Pir01]) results

Theorem 5.2. *The symmetric squares $C^{(2)}$ of a curve of genus 3 are the only irregular surfaces of general type with $p_g \geq 3$ presenting the **non-standard case** for the bicanonical map, i.e., such that*

- *Their bicanonical map ϕ_{2K} is not birational onto its image*
- *S does not contain any continuous system of curves of genus 2.*

Moreover, any algebraic surface with $p_g = q = 3$ is either such a symmetric product (iff $K^2 = 6$ for its minimal model), or has $K^2 = 8$ and is the quotient of a product $C_1 \times C_2$ of two curves of respective genera 2 and 3 by an involution $i = i_1 \times i_2$ where C_1/i_1 has genus 1, while i_2 operates freely.

The symmetric square $C^{(2)}$ of a curve of genus 2 occurs in another characterization of the **non-standard case** given by Ciliberto and Mendes-Lopes ([Ci-ML00], [Ci-ML02]).

Theorem 5.3. *The double covers S of a principal polarized Abelian surface A , branched on a divisor algebraically equivalent to 2Θ , are the only irregular surfaces of general type with $p_g = 2$ presenting the **non-standard case** for the bicanonical map.*

In these theorems plays an essential role the continuous system of **paracanonical curves**, i.e., of those curves which are algebraically equivalent to a canonical divisor. To this system is devoted section 8 of the cited Chapter IX of [Enr49], and there Enriques, after mentioning false attempts by Severi and himself to determine the dimension of the paracanonical system $\{K\}$, analyses the concrete case of a surface $C^{(2)}$, with C of genus 3, in order to conjecture that $\dim\{K\} = p_g$. The assumptions conjectured by Enriques were not yet the correct ones, but, under the assumptions that the surface does not contain any irrational pencil of genus ≥ 2 , the conjecture of Enriques was proved by Green and Lazarsfeld, via the so called "Generic vanishing theorems" (cf. [G-L87], [G-L91]).

I will not dwell further on this very interesting topic, referring the reader to the survey, resp. historical, articles [Cat91-B] [Cil91-B].

I should however point out that further developments are taking place in this direction, following a seminal paper by Mukai ([Muk81]) who extended the concept of Fourier transforms (don't forget that theta functions are particular Fourier series!) to obtain an isomorphism between the derived category of coherent sheaves on an Abelian variety X and the one of its dual Abelian variety $\hat{X} := Pic^0(X)$.

One specimen is the combination of Mukai's technique with the theory of generic vanishing theorems by Green and Lazarsfeld to obtain limitations on the singularities of divisors on an Abelian variety (cf. [Hac00], where one can also find references to previous work by Kollar, Ein and Lazarsfeld).

Speaking about links with the Italian school I should not forget the beautiful lectures I heard in Pisa from Aldo Andreotti on complex manifolds and on complex tori. Through his work with F. Gherardelli ([A-G76]), I got in touch with a problem of transcendental nature which occupies a central place in Severi's treatise on Quasi Abelian Varieties.

Quasi Abelian varieties, in Severi's terminology, are the Abelian complex Lie groups which sit as Zariski open sets in a projective variety.

Whence, they are quotients \mathbb{C}^n/Γ where Γ is a discrete subgroup of \mathbb{C}^n , thus of rank $r \leq 2n$, and the above algebraicity property leads again to the two Riemann bilinear relations:

- I) There exists an alternating form $A : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ such that A is the imaginary part of an Hermitian form H on V
- II) there exists such an H which is positive definite (in this case H is not uniquely determined by A).

Andreotti and Gherardelli conjectured that **The Riemann bilinear relations hold if and only if there is a meromorphic function of \mathbb{C}^n with group of periods equal to Γ** . This conjecture was the first Ph.D. problem I gave, and after some joint efforts, Capocasa and I were able to prove it in [C-C91].

HOMOLOGICAL ALGEBRA for Abelian and irregular Varieties

As already mentioned, David Mumford's ground breaking articles ([Mum66-7]) set up the scope of laying out a completely algebraic theory of theta functions. His attempt was not the only one, for instance Barsotti (cf. [Bars64-6],[Bars70],[Bars81],[Bars83-5], and [Lang-AV] for references about his early work) had another approach to Abelian varieties, based on power series, Witt vectors and generalizations of them (Witt covectors cf. [Bars81]), and the so called "Prostapheresis formula" (cf. [Bars83-5]).

Discussing here the respective merits of both approaches would be difficult, but at least I can say that, while Barsotti's work is mainly devoted to Abelian varieties in positive characteristic, the theory of theta groups, as already mentioned, is also a very useful tool in characteristic zero.

The title of Mumford's series of articles is "On the equations defining abelian varieties", which has a different meaning than "The equations defining abelian varieties"³⁵. Thus, he set up a program which has been successfully carried out in the case of several types of Abelian varieties. More generally, one can set as a general target the one of studying the equations of irregular varieties, i.e., of those which admit a non trivial Albanese map.

Since otherwise the problem is set in too high a generality, let me give a concrete example (for many topics I will treat now, consult also the survey paper [Cat97], which covers developments until 1996).

Chapter VIII of the book by F. Enriques (finished by his assistants Pompilj and Franchetta after the death of Enriques) is devoted to the attempt to find explicitly the canonical surfaces S of low canonical degree in \mathbb{P}^3 : i.e., one considers surfaces with $p_g = 4$ and with birational canonical map $\phi := \phi_K : S \rightarrow \mathbb{P}^3$. If the canonical system has no base points, then we will have a surface of degree $d = K^2 \geq 5$. The cases $K^2 = 5, 6$ are easy to describe, and for $K^2 \geq 7$ Enriques made some proposals to construct some regular surfaces (with a different method, Ciliberto [Cil81] was able to construct these for $7 \leq K^2 \leq 10$, and to sketch a classification program, later developed in [Cat84], based essentially on Hilbert's syzygy theorem).

It was possible to treat the irregular case (cf. [C-S02]) using a new approach based on Beilinson's theorem ([Bei78]) for coherent sheaves \mathcal{F} on \mathbb{P}^n , which allows to write every such sheaf as the cohomology of a **monad** (a complex with cohomology concentrated at only one point) functorially associated with \mathcal{F} .

The natural environment for rings not necessarily generated in degree 1 is however the **weighted projective space**, which is the projective spectrum $Proj(\mathcal{A})$ of a polynomial ring $\mathcal{A} := \mathbb{C}[x_0, \dots, x_n]$ graded in a non standard way, so that the indeterminates x_i have respective degrees m_i which are positive integers, possibly distinct.

Canonaco ([Can00]) was first able to extend Beilinson's theorem to the weighted case under some restriction on the characteristic of the base field, and later ([Can02]) not only removed this restriction, but succeeded to construct a unique functorial Beilinson type complex, making use of a new theory, of so called **graded schemes**. A concrete application given was to determine the canonical ring of surfaces with $K^2 = 4, p_g = q = 2$ (for these surfaces there is no good canonical map to the ordinary 3-space \mathbb{P}^3).

Although a general theory appears to be very complicated, it thus turns out that Abelian varieties (and for instance related symmetric products of curves) offer crucial examples (admitting sometime a geometrical characterization) for the study of irregular varieties. Some of them were already discussed before, we want to give a new one which is particularly interesting, and yields (cf. [C-S02] for more details) an easy counterexample to an old "conjecture" by Babbage (cf. [Cat81] and [Bea79] for references and the first counterexample).

³⁵Computers however have not only helped us to do some explicit calculations, but, according to F.O. Schreyer, they have also made us wiser: by showing us explicit equations which need several pages to be written down, they make us wonder whether having explicit equations means any better understanding.

Example 5.4. *Let $J = J(C)$ be the Jacobian of a curve C of genus 3, and let $A \rightarrow J$ be an isogeny of degree 2. The inverse image S of the theta divisor $\Theta \cong_{\text{bir}} C^{(2)}$ provides A with a polarization of type $(1, 1, 2)$.*

The canonical map of S factors through an involution ι with 32 isolated fixed points, and the canonical map of the quotient surface $\Sigma := S/\iota$, whose image is a surface Y of degree 6 in \mathbb{P}^3 having 32 nodal isolated singularities, and a plane cubic as double curve. Moreover, Σ is the normalization of Y .

It is now difficult for me to explain and to foresee exactly what principles should these examples illustrate, let me however try.

In Theorem 3.1 I tried to give an explicit example of the "algebraization" of the geometry, showing how the question of the birationality to Abelian varieties of symmetric products of varieties can be reduced to pure exterior algebra arguments. Abelian varieties are just, so to say, the complex incarnation of exterior algebras.

On the other hand, a companion article by Bernstein-Gelfand-Gelfand ([BGG78]) appeared next to the cited article [Bei78] by Beilinson. In abstract setting, it shows the equivalence of the derived category of coherent sheaves on \mathbb{P}^n and the category of finite modules over the exterior algebras. While it is not yet completely clear how to extend this result to the weighted case, quite recently in [E-F-S] it was shown how the BGG method allows to write functorially not only the sheaves, but also the homomorphisms in the Beilinson monad.

Note that the exterior algebra of BGG is the exterior algebra over the indeterminates of a polynomial ring, and is therefore apparently geometrically unrelated for the moment to the exterior algebra of an Abelian variety. Given however a morphism $f : X \rightarrow A$, we attach to it the induced homomorphism between the respective holomorphic algebras of A and X , $f^* : H^0(\Omega_A^*) \rightarrow H^0(\Omega_X^*)$. Whence, we obtain a module over the exterior algebra, and we associate to it a Beilinson monad. This procedure shows that to f we associate some geometric objects related to the Gauss maps corresponding to f .

It is just the converse which looks more problematic, is it possible to associate, to a map $f : X \rightarrow \mathbb{P}^n$ to a projective space, a geometric map to an Abelian variety giving a realization of the module $f_*\mathcal{O}_X$?

In general, progress on the question of canonical rings or equations of irregular varieties might require at least further combinations of the several existing techniques which we have mentioned.

6. MORE NEW RESULTS AND OPEN PROBLEMS

6.1. The Torelli problem

The Torelli theorem, mentioned in the previous section, was again at the centre of attention in the 50's, when several new proofs were found, by Weil, Matsusaka, Andreotti (and many others afterwards).

Particularly geometrical was the proof given by A. Andreotti ([Andr58]), who showed that, given a Jacobian variety $J(C)$ with a theta divisor $\Theta \cong_{\text{bir}} C^{(g-1)}$, the Gauss map of Θ is ramified on the dual variety $C^\vee \subset \mathbb{P}^{g-1}$ of the canonical image of the curve C . Using projective duality, one sees therefore that $(J(C), \Theta)$ determines C (Andreotti had then to treat the hyperelliptic case separately).

Andreotti and Mayer ([A-M67]) pushed the study of the geometry of canonical curves, especially of the quadrics of rank 4 containing them, to obtain some equations valid for the period matrices of curves inside the Siegel upper half plane. This paper, written at a time when the fashion was oriented in quite different directions, had a great impact on the revival of classical researches about algebraic curves, Abelian varieties (cf. [ACGH84] and [L-B] for references).³⁶

In the same years Philip Griffiths (cf. [Grif68], [Grif70], [Grif84] and references therein) greatly extended the theory of the periods of Abelian integrals, proposing to use the Hodge structures of varieties, i.e. the isomorphism class of the datum

$$H^*(X, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p,q} H^{p,q}(X)$$

to study their moduli (note that a modern formulation of Torelli's theorem is that the birational isomorphism class of an algebraic curve is determined by the Hodge structure on its cohomology algebra).

A prominent role played in his program the attempt to find a reasonable generalization of Torelli's theorem, and indeed (cf. [Grif84]) a lot of Torelli type theorems were proved for very many classes of special varieties.

Since then, a basic question has been the one of finding sufficient conditions for the validity of an infinitesimal Torelli theorem for the period map of holomorphic n -forms (i.e., for the Hodge structure on $H^n(X, \mathbb{Z}) \otimes \mathbb{C}$) for a variety X of dimension n . The question is then, roughly speaking, whether the period map is a local embedding of the local moduli space. Observe moreover that the n -forms are the only forms which surely exist on simple cyclic ramified coverings Y_D of a variety X of general type, branched on pluricanonical divisors D (divisors $D \in |mK_X|$), and in this context the Torelli problem is a quantitative question about how large m has to be in order that the variation of Hodge structure distinguishes the Y_D 's (cf. the article [Mig95] which gave a very interesting application of these ideas to families of higher dimensional varieties, opening a new direction of research).

The validity of such an infinitesimal Torelli theorem can be formulated in purely algebraic terms as follows: is the cup product

$$H^1(X, T_X) \times H^0(X, \Omega_X^n) \rightarrow H^1(X, \Omega_X^{n-1})$$

non degenerate in the first factor?

³⁶I came to read this beautiful paper under the instigation of Francesco Gherardelli. He explained to me that the citation from "Il teatro alla moda" by Benedetto Marcello: "As a first duty shall the modern poet ignore all about the ancient Roman and Greek poets, because these last too ignored everything about modern poetry" was motivated by the extreme difficulties that Aldo had when he wanted to lecture in Paris about this result without first explaining the excellence of the rings of coefficients he was using, or use the notation g_r^1 without explaining the representability of such functors.

Classical examples by Godeaux and Campedelli and modern ones (cf.[BPV84] for references) produce surfaces of general type with $p_g = q = 0$ (thus with a trivial Hodge structure), yet with moduli.

In view of Andreotti's interpretation of Torelli's theorem, one suspects then that a good condition might be the very ampleness of the canonical divisor, i.e., the condition that the canonical map be an embedding.

This might unfortunately not be the case, as shown in a joint paper with I. Bauer [B-C02]:

Theorem 6.1. *There are surfaces of general type with onjective canonical morphism and such that the infinitesimal Torelli theorem for holomorphic 2-forms does not hold for each surface in the moduli space. Examples of such behaviour are quotients $(C_1 \times C_2)/(\mathbb{Z}/3)$, where $C_1 \rightarrow C_1/(\mathbb{Z}/3) \cong \mathbb{P}^1$ is branched on $3k + 2$ points, and $(\mathbb{Z}/3)$ acts freely on C_2 with a genus 3 quotient.*

It would be interesting to establish stronger geometrical properties of the canonical map which guarantee the validity of the infinitesimal Torelli theorem for the holomorphic n -forms.

6.2. The Q.E.D. problem

If higher dimensional varieties were products of curves, life would be much simpler. It obviously cannot be so, since there are plenty of varieties which are simply connected (e.g., smooth hypersurfaces in \mathbb{P}^n with $n \geq 3$), without being rational.

Can life be simpler ?

It is a general fact of life that, in order to make the study of algebraic varieties possible, one must introduce some equivalence relation.

The most classical one is the so called birational equivalence, which allows in particular not to distinguish between the different projective embeddings of the same variety.

Definition 6.2. *Let X and Y be projective varieties defined over the field K : then they are said to be birational if their fields of rational functions are isomorphic : $K(X) \cong K(Y)$.*

Moreover, one must allow algebraic varieties to depend on parameters, for instance the complex hypersurfaces of degree d in \mathbb{P}^n depend on the coefficients of their equations: but if these are complex numbers, we can have uncountably many birational classes of algebraic varieties.

To overcome this difficulty, Kodaira and Spencer introduced the notion of **deformation equivalence** for complex manifolds: they ([K-S58]) defined two complex manifolds X', X to be **directly deformation equivalent** if there is a proper holomorphic submersion $\pi : \Xi \rightarrow \Delta$ of a complex manifold Ξ to the unit disk in the complex plane, such that X, X' occur as fibres of π . If we take the equivalence relation generated by direct deformation equivalence, we obtain the relation of deformation equivalence, and we say that X is a deformation of X' in the large if X, X' are deformation equivalent.

These two notions extend to the case of compact complex manifolds the classical notions of irreducible, resp. connected, components of moduli spaces.

It was recently shown however([Man01], [K-K01],[Cat01],) that it is not possible to give effective conditions in order to guarantee the deformation equivalence of algebraic varieties, as soon as the complex dimension becomes ≥ 2 .

Thus in [CatQED] I introduce the following relation

Definition 6.3. *Let n be a positive integer, and consider, for complex algebraic varieties X, Y of dimension n , the equivalence relation generated by*

- (1) *Birational equivalence*
- (2) *Flat deformations with connected base and with fibres having only at most canonical singularities*
- (3) *Quasi étale maps, i.e., morphisms which are unramified in codimension 1.*

This equivalence will be denoted by $X \cong_{QED} Y$ (QED standing for: quasi-étale-deformation).

Remark 6.4. • *Singularities play here an essential role. Note first of all that, without the restriction on these given in (2), we obtain the trivial equivalence relation (since every variety is birational to a hypersurface).*

- *Assume that a variety X is rigid, smooth with trivial algebraic fundamental group: then X has no deformations, and there is no non trivial quasi-étale map $Y \rightarrow X$.*

In this case the only possibility, to avoid that X be isolated in its QED-equivalence class, is that there exists a quasi-étale map $f : X \rightarrow Y$.

If f is not birational, however, the Galois closure of f yields another quasi-étale map $\phi : Z \rightarrow X$, thus it follows that f is Galois and we have a contradiction if $\text{Aut}(X) = \{1\}$.

It does not look so easy to construct such a variety X .

Are there invariants for this equivalence?

A recent theorem of Siu ([Siu02]) shows that the Kodaira dimension is invariant by QED equivalence.

It is an interesting question to determine the QED equivalence classes inside the class of varieties with fixed dimension n , and with Kodaira dimension k . For curves and special surfaces, there turns out to be only one class ([CatQED]):

Theorem 6.5. *In the case $n \leq 2, k \leq 1$ the following conditions are equivalent*

- (i) $X \cong_{QED} Y$
- (ii) $\dim X = \dim Y = n, \text{Kod}(X) = \text{Kod}(Y) = k$

The previous result uses heavily the Enriques classification of algebraic surfaces. We can paraphrase the problem, for the open case of surfaces of general type, using Enriques' words ([Enr49]): "We used to say in the beginning that, while curves have been created by God, surfaces are the work of the devil. It

appears instead that God wanted to create for surfaces a finer order of hidden harmonies..”

Are here then new hidden harmonies to be found?

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Bibliographical remark.

We are not in the position to even mention the most important references. However, some of the references we cite here contain a vast bibliography, for instance [Sieg73], pages 193-240, and [Zar35], pages 248-268.

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