Some new surfaces with $p_g = q = 0$.

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0   Introduction

It is well known that an algebraic curve of genus zero is isomorphic to the projective line. The search for an analogous statement in the case of algebraic surfaces led Max Noether to conjecture that a smooth regular (i.e., $q(S) = 0$) algebraic surface with vanishing geometric genus ($p_g(S) = 0$) should be a rational surface. The first counterexample to this conjecture was provided by F. Enriques ([EnrMS], I), who introduced the so called Enriques surfaces by considering the normalization of sextic surfaces in 3-space double along the edges of a tetrahedron. Nowadays a large number of surfaces of general type with $p_g = q = 0$ is known, but the first ones were constructed in the thirties by L. Campedelli and L. Godeaux (cf. [Cam], [God]: in their honour minimal surfaces of general type with $K^2 = 1$ are called numerical Godeaux surfaces, and those with $K^2 = 2$ are called numerical Campedelli surfaces).
In the seventies, after rediscoveries of these old examples, many new ones were found through the efforts of several authors (cf. [BPV], pages 234-237 and references therein). In particular, in the spirit of Godeaux' method to produce interesting surfaces as quotients $S = Z/G$ of simpler surfaces by the free action of a finite group G, A. Beauville proposed a very simple construction by taking as Z the product $X = C_1 \times C_2$ of two curves of respective genera $g_1, g_2 \geq 2$, together with an action of a group G of order $(g_1 - 1)(g_2 - 1)$ (this method produces surfaces with $K^2 = 8$). He also gave

an explicit example as quotient of two Fermat curves (in section 2, we shall indeed show that his example leads to exactly two non isomorphic surfaces). In this paper we will discuss Beauville's construction starting from the bottom, i.e., as the datum of two appropriate coverings of $\mathbb{P}^1$ and address the problem of classification of these surfaces, which we are unable for time limits to achieve in this note.

The interest on this issue stems from the open problem that David Mumford set forth at the Montreal Conference in 1980 : "Can a computer classify all surfaces of general type with $p_g = 0$? Our purpose is to show how complex this question is (and probably computers are needed even if one asks a more restricted question).

First of all, all known surfaces of general type with $p_g = q = 0, K^2 = 8$ are quotients $\mathbb{H} \times \mathbb{H}/\Gamma$ of the product of two upper half planes by a discrete cocompact group. Besides the cited examples, there are also quotients which are not related to products of curves, and were constructed long ago by Kuga and Shavel using quaternion algebras (cf. [Ku], [Sha]).

It is still a difficult open question whether one can have a fake quadric, i.e., a surface of general type which is homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Studying the special case where S is the quotient of a product of two curves, we want to show how huge is the number of components of the corresponding moduli space, and how detailed and subtle the classification is.

An important feature is also the question of rigidity: for some of these surfaces the moduli space consists of one or two points (cf. [Cat00], [Cat03]), for others it has strictly positive dimension, and in any case the construction yields connected components of the moduli space.

Surfaces with $p_g = q = 0$ were also investigated from other points of view. We would like to mention several articles by M. Mendes Lopes and R. Pardini ([Pa], [MLP1], [MLP2]) where the authors study the problem of describing and classifying the failure of birationality of the bicanonical map.

We will here classify all smooth algebraic surfaces $S = C_1 \times C_2/G$, where C_1, C_2 are as above curves of genus at least two and G is a finite abelian group acting freely on $C_1 \times C_2$ by a product action, and yielding a quotient surface with $p_g = q = 0$. In this case K^2 has to be equal to 8, and we will see that there are already several cases. Our first main result is

Theorem 0.1 *Let S be a surface with $p_g = q = 0$ isogenous to a higher product $C_1 \times C_2/G$ of unmixed type. If G is abelian, then G is one of the*

following groups: $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/3\mathbb{Z})^2$, $(\mathbb{Z}/5\mathbb{Z})^2$.
Each of these groups really occur.

We will then give a complete description of the connected components of the moduli space that arise from these surfaces. We remark again that for $G = (\mathbb{Z}/5\mathbb{Z})^2$, we get two isolated points, i.e., these surfaces are rigid, but there are two different ones. For the other cases the group determines a positive dimensional irreducible connected component of the moduli space.

In section 4 we calculate explicitly (theorem 4.3) the torsion group $T(S) = H_1(S, \mathbb{Z})$ of our surfaces with G abelian: it turns out that in some cases $T(S)$, which has a natural surjection onto G, is strictly bigger than G, but it is exactly $G = (\mathbb{Z}/5\mathbb{Z})^2$ for the two Beauville surfaces. Whence, these are only distinguished by their fundamental group, and not by the first homology group.
A classification of surfaces with $p_g = q = 0$ isogenous to a product (i.e., releasing the hypothesis that the group be abelian) is possible, but it is quite complicated and there are many more cases as we will show in the last section, where we give a list of examples of surfaces with $p_g = q = 0$ isogenous to a higher product $C_1 \times C_2/G$ with non abelian G, some of them already known, others new. We will postpone the complete classification to a forthcoming article (in the non abelian case a non trivial problem is also the one of determining the Hurwitz equivalence or inequivalence of certain systems of generators of a finite group).
Quite similar is the case of surfaces isogenous to a product of curves and with $q = p_g = 1$. The classification of these surfaces is also related to the determination of the so called non standard case for the non birationality of the bicanonical map. We refer the reader for this topic to the forthcoming Ph.D. Thesis of F. Polizzi.

1 Basic invariants of surfaces isogenous to a product.

Let S be a smooth connected algebraic surface over the complex numbers. First we will recall the notion of surfaces isogenous to a higher product of curves. By prop. 3.11 of [Cat00] the following two properties 1) and 2) of a surface are equivalent.

Definition 1.1 *A surface S is said to be* isogenous to a higher product *if and only if, equivalently, either*
1) S admits a finite unramified covering which is isomorphic to a product of curves of genera at least two, or
2) S is a quotient $S := (C_1 \times C_2)/G$, where the C_i's are curves of genus at least two, and G is a finite group acting freely on $Z := (C_1 \times C_2)$.
We have two cases: the mixed case *where the action of G exchanges the two factors (and then C_1, C_2 are isomorphic), and the* unmixed case *where G acts via a product action.*

We recall briefly the following results from [Cat00] (cf. also [Cat03]):

- Let S be isogenous to a product, and let S' be another surface with the same fundamental group as S and such that $K_S^2 = K_{S'}^2$ (equivalently, $\chi(S) = \chi(S')$ or $e(S) = e(S')$): then S' is orientedly diffeomorphic to S and either S' or its complex conjugate surface $\bar{S}'$ belongs to an irreducible smooth family, yielding a connected component of the moduli space of surfaces of general type.

- There is a unique minimal realization $S := (C_1 \times C_2)/G$ (i.e., the genera g_1, g_2 of the two curves C_1, C_2 are minimal). It follows that G, g_1, g_2 are invariants of the fundamental group of S.

- The minimal realization provides an explicit realization of the above family as the datum of two branched coverings $C_i \to C_i' = C_i/G$ whose topological type is completely determined by the two orbifold exact group sequences $1 \longrightarrow \pi_1(C_i) \longrightarrow \Pi(i) \longrightarrow G \longrightarrow 1$, obtained from the fundamental group exact sequence of the quotient map $C_1 \times C_2 \to S$

$$(**) 1 \longrightarrow \Pi_{g_1} \times \Pi_{g_2} \longrightarrow \pi_1(S) \longrightarrow G \longrightarrow 1$$

  by moding out the normal subgroup $\Pi_{g_{i+1}}$.

We obtain an easier picture in the case where $q(S) = 0$, or, equivalently, $C_1' \cong C_2' \cong \mathbb{P}^1$.

Definition 1.2 *1) Let G be a group. Then a* **spherical system of generators** *of G (* **S.G.S. of G** *) is an ordered sequence $\mathcal{A} = (a_1, \ldots a_n)$ of generators of G with the property that their product $a_1 \ldots a_n = 1$.*

2) If we choose n points $P_1, \ldots P_n \in \mathbb{P}^1$, and a geometric basis $\gamma_1, \ldots \gamma_n$ of $\pi_1(\mathbb{P}^1 - \{P_1, \ldots P_n\})$ (γ_i is a simple counterclockwise loop around P_i, and they follow each other by counterclockwise ordering around the base point), then a S.G.S. of G determines a surjective homo0morphism $\psi : \pi_1(\mathbb{P}^1 - \{P_1, \ldots P_n\}) \to G$.

Now, the braid group of the sphere $\pi_0(Diff(\mathbb{P}^1 - \{P_1, \ldots P_n\}))$ operates on such homomorphisms, and their orbits are called Hurwitz equivalence classes of spherical systems of generators.

With the above notation we obtain

Theorem 1.3 *Let S be a surface isogenous to a product, of unmixed type and with $q(S) = 0$. Then to S we attach its finite group G (up to isomorphism) and the equivalence classes of an unordered pair of two S.G.S.'s $\mathcal{A}, \mathcal{A}'$ of G, under the equivalence relation generated by*

1) Hurwitz equivalence for $\mathcal{A}$,
1') Hurwitz equivalence for $\mathcal{A}'$,
2) simultaneous conjugation for $\mathcal{A}, \mathcal{A}'$, i.e., for $\phi \in Aut(G)$, we let $(\mathcal{A} = (a_1, \ldots a_n), \mathcal{A}' = (a'_1, \ldots a'_n))$ be equivalent to $(\phi(\mathcal{A}) = (\phi(a_1), \ldots \phi(a_n)), \phi(\mathcal{A}') = (\phi(a'_1), \ldots \phi(a'_n)))$.

Then two surfaces S, S" are deformation equivalent if and only if the corresponding equivalence classes of pairs of S.G.S.'s of G are the same.

Proof. If S, S' are deformation equivalent, then they have an isomorphic fundamental group exact sequence $(**)$. Whence, we get pairs of isomorphic orbifold exact sequences, compatible with an identification of G with a fixed group. Now, the orbifold exact sequences determine homomorphisms $\psi_1 : \pi_1(\mathbf{C}'_1 - \{P_1, \ldots P_n\}) \to G$, $\psi_2 : \pi_1(\mathbf{C}'_2 - \{P'_1, \ldots P'_m\}) \to G$. One sees immediately that these pairs are defined up to equivalence (for instance, 2) follows by the fact that a G-covering space is determined by the kernel of the surjection of the fundamental group onto G, and not by the specific homomorphism).

Conversely, we see easily that if the equivalence classes are the same, then the surfaces are deformation equivalent.

Q.E.D.

Remark 1.4 Observe that, if the group G is abelian, then Hurwitz equivalence of $\mathcal{A} = (a_1, \ldots a_n)$ is simply permutation equivalence of the sequence $(a_1, \ldots a_n)$.

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We shall assume throughout that we have a surface S isogenous to a higher product and that we are in the unmixed case, thus we have a finite group G acting on two curves C_1, C_2 with genera $g_1, g_2 \geq 2$, and acting freely by the product action on $Z := C_1 \times C_2$.
Since

$$K_S^2 = 8\chi(\mathcal{O}_S) = \frac{8(g_1 - 1)(g_2 - 1)}{|G|}$$

the assumption $p_g(S) = q(S) = 0$ implies that $K_S^2 = 8$.

Remark 1.5 We have the following elementary but crucial formulae:
1) $8(g_1 - 1)(g_2 - 1) = K_{C_1 \times C_2}^2 = |G| \cdot K_S^2 = 8 \cdot |G|$, whence

$$|G| = (g_1 - 1)(g_2 - 1).$$

2) Since $q(S) = 0$ we have $C_i/G \cong \mathbb{P}^1$ for $i = 1, 2$, so by the Hurwitz formula we get:

$$|G| = \frac{2}{-2 + \sum_j (1 - \frac{1}{m_j})}(g_i - 1),$$

where $i = 1, 2$ and m_j is the branching index of a branch point P_j of $C_i \longrightarrow \mathbb{P}^1$. In particular, in view of 1) it must hold:

$$\frac{2}{-2 + \sum_j (1 - \frac{1}{m_j})} \in \mathbb{N}.$$

It is easy to see that the number of branch points of the two coverings $C_i \longrightarrow \mathbb{P}^1$ cannot be too high.

Lemma 1.6 *Let $S = C_1 \times C_2/G$ be as above. Then the number of branch points of each covering $C_i \longrightarrow \mathbb{P}^1$ is at most eight.*

Proof. Assume e.g. that $C_1 \longrightarrow C_1/G = \mathbb{P}^1$ has at least 9 branch points. Then

$$|G| \leq \frac{2}{-2 + \sum_{j=1}^{9}(1 - \frac{1}{2})}(g_1 - 1) = \frac{4}{5}(g_1 - 1),$$

contradicting $g_2 \geq 2$. Therefore we can have at most 8 branch points. Q.E.D.

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2   The case: G abelian

We will assume from now on that G is a finite abelian group. In this section we will show that the only abelian groups which give rise to a surface isogenous to a product $S = C_1 \times C_2/G$, of unmixed type and with $p_g = q = 0$, are $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/3\mathbb{Z})^2$ and $(\mathbb{Z}/5\mathbb{Z})^2$.

Our first step is to limit the order of the group G.

Proposition 2.1 *Let G be a finite abelian group and let C be a smooth algebraic curve of genus $g \geq 2$ admitting an action of G such that $C/G = \mathbb{P}^1$. We denote by r the number of branch points of the morphism $C \longrightarrow C/G$. If $r \geq 4$ then*

$$|G| \leq 4(g-1)$$

except for the case $r = 4$ and where the multiplicities of the branch points are $(2, 2, 3, 3)$ (then $G = \mathbb{Z}/6$).

Proof. Recall that, by the Riemann existence theorem, giving a Galois covering $C \to C/G = \mathbb{P}^1$, with branch points $P_1, \ldots P_r$ and branching indices $m_1, \ldots m_r$ is equivalent, in the case where G is abelian, to giving

- Elements $a_1, \ldots a_r$ of G of respective orders $m_1, \ldots m_r$ (here a_i is the image in G of a geometric loop around P_i) such that

- $a_1 + \ldots a_r = 0$

- $a_1, \ldots a_r$ generate G.

Note that the elements $a_1, \ldots a_r$ are unique up to ordering.
If $r \geq 5$, then $\sum(1 - \frac{1}{m_j}) \geq \frac{5}{2}$, whence

$$|G| \leq \frac{2}{-2 + \frac{5}{2}}(g-1) = 4(g-1).$$

Therefore it remains to analyse the case $r = 4$. We assume that the multiplicities are $m_1 \leq m_2 \leq m_3 \leq m_4$. $(2, 2, 2, 2)$ is obviously not possible, since it contradicts $g \geq 2$. $(2, 2, 2, n)$, for $n \geq 3$, is not possible, since $a_1 + a_2 + a_3 = -a_4$ has order 2 contradicting the fact that a_4 has order n. Suppose now that $(m_1, \ldots, m_4) = (2, 2, 3, n)$. Then

$$\frac{2}{-2 + \sum_j (1 - \frac{1}{m_j})} = \frac{6n}{2n - 3} \leq 4,$$

for $n \geq 6$. We remark that $n = 4$ or 5 is not possible, since $-a_4 = a_1 + a_2 + a_3$ has order 3 or 6. Therefore the only possible case is $(2, 2, 3, 3)$. Here we have $a_1 + a_3 = -(a_2 + a_4)$ has order 6, whence $G = \mathbb{Z}/6$ and $g = 2$. For the remaining cases $(2, 2, \geq 4, \geq 4)$, $(2, \geq 3, \geq 3, \geq 3)$ and $(\geq 3, \geq 3, \geq 3, \geq 3)$ it is immediate that

$$\frac{2}{-2 + \sum_j (1 - \frac{1}{m_j})} \leq 4.$$

<div align="right">Q.E.D.</div>

Our second step is to show that the group G cannot be cyclic:

Proposition 2.2 *Let S be a surface isogenous to a higher product $C_1 \times C_2 / G$ such that $q = 0$. Then G cannot be cyclic.*

Proof of prop. 2.2. Both maps $C_i \longrightarrow C_i/G \cong \mathbb{P}^1$ determine the following situation: $G \cong \mathbb{Z}/d$ is generated by elements $a_1, \ldots a_r$ of respective orders $m_1, \ldots m_r$, respectively by elements $b_1, \ldots b_s$ of respective orders $n_1, \ldots n_s$. We claim that G cannot act freely on $C_1 \times C_2$. In fact, the stabilizers of some point in the first curve C_1 are exactly the subgroups generated by some element a_i. Since G is cyclic, the union $\mathcal{S}$ of the stabilizers is the set of elements whose order divides some m_i. If $\mathcal{S}'$ is the union of the stabilizers for the action on the second curve C_2, we want $\mathcal{S} \cap \mathcal{S}' = \{0\}$. This amounts to requiring that $\forall i = 1, \ldots r, j = 1, \ldots s$, the integers m_i and n_j are relatively prime. The condition that the a_i's generate is however equivalent to d being the least common multiple of the m_i's. Since d is also the least common multiple of the n_j's, we obtain a contradiction.

<div align="right">Q.E.D.</div>

We proceed discussing the case $r = 3$, and we assume again that the multiplicities are (m_1, m_2, m_3) with $m_1 \leq m_2 \leq m_3$.

Remark 2.3 We observe that $g.c.d.(m_1, m_2) = 1$ implies that $m_3 = m_1 \cdot m_2$ and G is cyclic of order m_3, a posibility which was already excluded.

We are now ready to prove the following:

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Proposition 2.4 *Let S be a surface isogenous to a higher product $C_1 \times C_2/G$ such that $p_g = q = 0$. Then either*
1) $g_1, g_2 \leq 5$, i.e. $|G| = (g_1 - 1)(g_2 - 1) \leq 16$,
or
2) $G = (\mathbb{Z}/5\mathbb{Z})^2$.

Before proving the above proposition we will prove the following weaker form.

Proposition 2.5 *Let S be a surface isogenous to a higher product $C_1 \times C_2/G$ such that $p_g = q = 0$. Then either*
1) $g_1, g_2 \leq 5$, i.e. $|G| = (g_1 - 1)(g_2 - 1) \leq 16$,
or
2) for one of the two curves the datum $(m_1, m_2, m_3; G)$ of branching orders plus occuring group yields a priori only one of the following possibilities:
a) $(2, 6, 6; \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z})$,
b) $(2, 8, 8; \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$,
c) $(2, 12, 12; \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z})$,
d) $(2, 20, 20; \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/20\mathbb{Z})$,
e) $(3, 6, 6; \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z})$,
f) $(4, 4, 4; \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})$,
g) $(5, 5, 5; (\mathbb{Z}/5\mathbb{Z})^2)$.

Proof. We have already seen that if $C \longrightarrow C/G = \mathbb{P}^1$ has ≥ 4 branch points, then

$$|G| \leq 4(g - 1)$$

except for the case $r = 4$ and the multiplicities of the branch points are $(2, 2, 3, 3)$ (then $G = \mathbb{Z}/6$). But by prop. 2.2 we know that this case cannot occur. Therefore we can assume that $C_1 \longrightarrow C_1/G$ has $r = 3$ branch points. We write again the multiplicities (m_1, m_2, m_3) with $m_1 \leq m_2 \leq m_3$. They correspond again to elements $a_i \in G$ of order m_i, generating G such that $a_1 + a_2 + a_3 = 0$.
$(2, 2, n)$ is not possible since then $-2 + \sum_j (1 - \frac{1}{m_j}) < 0$.

1) $\mathbf{m_1 = 2}$:
Then $m_2 \geq 4$, since for $m_2 = 3$, we must have $m_3 = 6$, whence $\alpha := -2 + \sum_j (1 - \frac{1}{m_j}) = 0$.
If $m_2 = 4$, then $m_3 = 4$ and $\alpha = 0$, which is not possible.
m_2 odd implies that G is cyclic, so we can exclude all these cases by prop.

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2.2.
If $m_2 = 6$, then $m_3 = 6$ and $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.
If $m_2 = 8$, then $m_3 = 8$ and $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.
For $m_2 = 10, 14, 16, 18$ we see that m_3 has to be equal to m_2, but then $\frac{2}{\alpha} \notin \mathbb{N}$.
If $m_2 = 12, 20$, then we are in the cases c) resp. d) of the claim.
We assume now that $m_3 \geq m_2 \geq 22$. Then we have

$$|G| \leq 2(-2 + \frac{1}{2} + \frac{21}{22} + \frac{21}{22})^{-1}(g_1 - 1) = \frac{44}{9}(g_1 - 1).$$

Therefore if $\frac{2}{\alpha} \in \mathbb{N}$, then $\frac{2}{\alpha} \leq 4$.

2) $\mathbf{m_1 = 3}$:
$m_2 = 3$ implies $\alpha = 0$, whereas $m_2 = 4, 5$ imply that G is cyclic. Therefore
we can assume $m_2 \geq 6$.
$m_2 = 6$ implies $m_3 = 6$ and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.
$m_2 > 6$ implies that either G is cyclic or $m_2 \geq 9$. If $m_3 \geq m_2 \geq 9$, then

$$|G| \leq 2(-2 + \frac{2}{3} + \frac{8}{9} + \frac{8}{9})^{-1}(g_1 - 1) = \frac{9}{2}(g_1 - 1).$$

Therefore if $\frac{2}{\alpha} \in \mathbb{N}$, then $\frac{2}{\alpha} \leq 4$.

3) $\mathbf{m_1 = 4}$:
$m_2 = 4$ implies that $m_3 = 4$ and $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.
$m_2 = 5$ implies again that G is cyclic, which is not possible; therefore we can
assume that $m_3 \geq m_2 \geq 6$. Then

$$|G| \leq 2(-2 + \frac{3}{4} + \frac{5}{6} + \frac{5}{6})^{-1}(g_1 - 1) = \frac{24}{5}(g_1 - 1).$$

Therefore, if $\frac{2}{\alpha} \in \mathbb{N}$, then $\frac{2}{\alpha} \leq 4$.

4) $\mathbf{m_1 = 5}$:
$m_2 = 5$ implies that $m_3 = 5$ and $G = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.
Therefore we have $m_3 \geq m_2 \geq 6$. Then

$$|G| \leq 2(-2 + \frac{4}{5} + \frac{5}{6} + \frac{5}{6})^{-1}(g_1 - 1) = \frac{60}{14}(g_1 - 1).$$

Whence, if $\frac{2}{\alpha} \in \mathbb{N}$, then $\frac{2}{\alpha} \leq 4$.

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5) **$m_1 \geq 6$** :
In this case we have

$$|G| \leq 2(-2 + \frac{15}{6})^{-1}(g_1 - 1) = 4(g_1 - 1).$$

Therefore we have proven our claim.

<div align="right">Q.E.D.</div>

In order to prove proposition 2.4 we have now to exclude the cases $2a) - 2f)$ of the previous result. This will be done in the following lemma.

Lemma 2.6 *Let S be a surface isogenous to a higher product $C_1 \times C_2/G$ such that $p_g = q = 0$. Then G cannot be one of the following groups:*
a) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$,
b) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$,
c) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$,
d) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/20\mathbb{Z}$,
e) $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$,
f) $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Proof.
a) In this case the multiplicities of the branch points for C_1 have to be $(2, 6, 6)$. Then the union of the stabilizers is equal to $\{(1, 0), (0, x), (1, 5), (1, 3), (1, 1)\}$. Therefore there are only 2 elements of order 3 left, and they cannot generate G. So there is no possibility that $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ acts freely on $C_1 \times C_2$. The cases $b) - e)$ are excluded exactly in the same way.
f) Let a_1, a_2, a_3 generate $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Then we can assume w.l.o.g. that a_1, a_2 is a $\mathbb{Z}/4\mathbb{Z}$ basis. But then, if Σ resp Σ' denotes the set of stabilizers of C_1 resp. C_2, we have

$$\sharp(\Sigma \cap ((\mathbb{Z}/2)^2 - \{0\})) \geq 2,$$

and the same for Σ'. In particular $\Sigma \cap \Sigma' \neq \emptyset$. Q.E.D.

This proves theorem 2.4.
We are now ready to formulate the main result of this section.

Theorem 2.7 *Let S be a surface with $p_g = q = 0$ isogenous to a higher product $C_1 \times C_2/G$. If G is abelian, then G is one of the following groups: $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/3\mathbb{Z})^2$, $(\mathbb{Z}/5\mathbb{Z})^2$.*

Proof. We know by our previous considerations that $G = (\mathbb{Z}/5\mathbb{Z})^2$ or $|G| \leq 16$.

Moreover, G cannot be cyclic, whence $|G| \in \{4, 8, 12, 16\}$.

Obviously, $G = (\mathbb{Z}/2\mathbb{Z})^2$ is not possible and this excludes the case $|G| = 4$.

If $|G| = 8$, then either $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ or $G = (\mathbb{Z}/2\mathbb{Z})^3$. Assume that $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Since G is not generated by elements of order 2, there must be at least one generator of order 4 for each of the curves C_1 and C_2. But there is exactly one non trivial element, namely $(0, 2)$, which is the double of any element of order 4. Hence the stabilizers of the two curves cannot intersect trivially and therefore G cannot act freely on $C_1 \times C_2$.

If $|G| = 12$, then G can only be $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and this case was excluded before.

If $|G| = 16$, then G is one of the following groups: $(\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}$, $(\mathbb{Z}/4)^2$, $(\mathbb{Z}/2)^4$. $(\mathbb{Z}/4)^2$ was already excluded and $(\mathbb{Z}/2)^2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is excluded in the same way as $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/4\mathbb{Z}$, since there is also only one element which can be the double of an element of order 4. Q.E.D.

3 The moduli of surfaces with $p_g = q = 0$ isogenous to a higher product (with abelian group).

In this section we will show that the groups in theorem 2.7 really occur. More precisely, we will describe exactly the corresponding moduli spaces.

3.1 $G = (\mathbb{Z}/2\mathbb{Z})^3$

Since every element of G has order 2, we clearly need $r \geq 5$ branch points for each covering $C_i \longrightarrow \mathbb{P}^1$. It is now easy to see that $r_1 = 5$ and $r_2 = 6$. We denote by $\mathcal{S}_i$ the union of the stabilizers of the covering $C_i \longrightarrow \mathbb{P}^1$. Then, since $\mathcal{S}_i$ contains a basis, it has cardinality at least 3. Since however $\mathcal{S}_1$, $\mathcal{S}_2$ are disjoint, and their union has cardinality at most 7, we see that $\mathcal{S}_1$ must contain exactly 4 elements (since the sum of the five elements is zero). We may then assume that

$$(a_1, a_2, a_3, a_4, a_5) = (e_1, e_2, e_3, e_1, e_2 + e_3),$$

where e_1, e_2, e_3 is a suitable $\mathbb{Z}/2\mathbb{Z}$ - basis of $(\mathbb{Z}/2\mathbb{Z})^3$. Then there is only one possibility (up to permutation) left for $(b_1, b_2, b_3 b_4, b_5, b_6)$, namely

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (e_1 + e_2, e_1 + e_3, e_1 + e_2 + e_3, e_1 + e_2, e_1 + e_3, e_1 + e_2 + e_3).$$

Therefore we have shown the following

Theorem 3.1 *The surfaces with $p_g = 0$ isogenous to a product with group $G = (\mathbb{Z}/2\mathbb{Z})^3$ form an irreducible connected component of dimension 5 in their moduli space.*

Remark 3.2 This result was already shown by R. Pardini in [Pa], where she classifies surfaces with $p_g = 0$, $K^2 = 8$, which are double planes. In fact the above surfaces are the only ones in our list having non birational bicanonical map.

3.2 $\quad G = (\mathbb{Z}/2\mathbb{Z})^4$

Again, since there are only elements of order 2 in G we see that the number of branch points for each covering $C_i \longrightarrow \mathbb{P}^1$ has to be at least 5. But since $|G| \leq 4(g_i - 1)$ for both curves, we see that $r_1 = r_2 = 5$. For the first curve C_1 we can assume

$$(a_1, a_2, a_3, a_4, a_5) = (e_1, e_2, e_3, e_4, e := e_1 + \ldots + e_4),$$

where e_1, e_2, e_3, e_4 is a $\mathbb{Z}/2\mathbb{Z}$ - basis of $(\mathbb{Z}/2\mathbb{Z})^4$. Then the problem reduces to finding $v_1, \ldots, v_5 \in G$ such that
1) $\sum_{i=1}^{5} v_i = 0$;
2) $rank\ (v_1, \ldots, v_5) = 4$;
3) v_i is of weight $w = 2$ or 3 (since e is the only vector in G of weight 4 and the e_i's are the only vectors of weight $w(e_i)$ equal to 1 in G).

Remark 3.3 $\sum_{i=1}^{5} v_i = 0$ implies that $\sum_{i=1}^{5} w(v_i) \equiv 0(2)$. Therefore the number n_3 of vectors of weight 3 in $\{v_1, \ldots, v_5\}$ has to be even.

Lemma 3.4 *Only the case $n_3 = 2$ is possible.*

Proof. Since there are 4 elements of weight 3 in $(\mathbb{Z}/2\mathbb{Z})^4$ we have to exclude the cases $n_3 = 4$, $n_3 = 0$. Assume $n_3 = 4$. Then w.l.o.g. $v_i = e + e_i$ for

$i = 1, \ldots, 4$. But then $v_5 = \sum_{i=1}^{4} v_i = e$, which contradicts $w(v_5) \in \{2, 3\}$. Assume that $n_3 = 0$. But since

$$\sum_{v:w(v)=2} v = e,$$

five of these 6 vectors of weight 2 can never have sum zero. $\qquad$ Q.E.D.

Therefore without loss of generality we can assume that $v_1 = e + e_1$, $v_2 = e + e_2$. Then v_3, v_4, v_5 have all weight two and their sum is equal to $e_1 + e_2$. We observe that we cannot have:

$$\left| \bigcup_{i=3}^{5} supp(v_i) \right| = 3,$$

because this would imply $\sum_{i=3}^{5} v_i = 0$. Therefore we can assume that $v_3 + v_4 = e$ and then $v_5 = e_3 + e_4$. Then we have two possibilities for v_3 and v_4, namely

$$v_3 = e_1 + e_3, \quad v_4 = e_2 + e_4;$$

or

$$v_3 = e_1 + e_4, \quad v_4 = e_2 + e_3.$$

But these two possibilities give rise to isomorphic surfaces, since they are equivalent by the permutation of v_1 and v_2.

Therefore we have shown the following:

Theorem 3.5 *The surfaces with $p_g = 0$ isogenous to a product with group $G = (\mathbb{Z}/2\mathbb{Z})^4$ form an irreducible connected component of dimension 4 in their moduli space.*

3.3 $G = (\mathbb{Z}/3\mathbb{Z})^2$

Examples of this type have already been given by Dolgachev in [Dolg98]. In this case $G - \{0\} = (\mathbb{Z}/3\mathbb{Z})^2 - \{0\}$ has 8 elements, whence the union of the stabilizers of each covering has to consist of exactly four elements, i.e. $|\mathcal{S}_1| = |\mathcal{S}_2| = 4$. Moreover we know that the number of branch points of each covering $C_i \longrightarrow \mathbb{P}^1$ is 4. Thus we have up to permutation:

$$(a_1, a_2, a_3, a_4) = (a, b, -a, -b),$$

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and
$$(b_1, b_2, b_3, b_4) = (a', b', -a', -b'),$$

where a, b (resp. a', b') is a basis of $(\mathbb{Z}/3\mathbb{Z})^2$. Therefore we have shown the following

Theorem 3.6 *The surfaces with $p_g = 0$ isogenous to a product with group $G = (\mathbb{Z}/3\mathbb{Z})^2$ form an irreducible connected component of dimension 2 in their moduli space.*

3.4 $G = (\mathbb{Z}/5\mathbb{Z})^2$

These surfaces are a particular case of examples that were introduced by A. Beauville (cf. [Bea]).

We see that we have for both coverings $C_i \longrightarrow \mathbb{P}^1$ 3 branch points and the multiplicity is always 5. In particular, these surfaces are rigid. We shall see that here we have two components of the moduli space, i.e. there are two non isomorphic Beauville surfaces.

In order to give a Beauville surface it is equivalent to give the following data:

- a_1, a_2, $a_3 \in G$ of order 5 such that they generate G and their sum is zero;

- b_1, b_2, $b_3 \in G$ of order 5 such that they generate G and their sum is zero.

Moreover they have to fulfill the following condition:

$$(< a_1 > \cup < a_2 > \cup < a_3 >) \cap (< b_1 > \cup < b_2 > \cup < b_3 >) = \{0\}.$$

We denote the set of sixtuples $(a_1, a_2, a_3, b_1, b_2, b_3)$ satifying the above conditions by $\widetilde{\mathfrak{M}}$. On $\widetilde{\mathfrak{M}}$ the group $\mathcal{G} := Gl(2, \mathbb{Z}/5\mathbb{Z}) \times \mathfrak{S}_3 \times \mathfrak{S}_3$ acts in the natural way. We remark that $|\mathcal{G}| = 24 \cdot 20 \cdot 6 \cdot 6$.

Up to a permutation of the b_i's we can write every element of $\widetilde{\mathfrak{M}}$ as $(e_1, e_2, -(e_1 + e_2), \lambda(e_1 + 2e_2), \mu(3e_1 + 4e_2), \rho(e_1 + 4e_2))$. This is possible since $(\mathbb{Z}/5\mathbb{Z})^2 - \{< e_1 >, < e_2 >, < e_1 + e_2 >\}$ determines exactly three stabilizer groups. Since $\lambda(e_1 + 2e_2) + \mu(3e_1 + 4e_2) + \rho(e_1 + 4e_2) = 0$ we see immediately that $\lambda = \mu = \rho$, which implies that there are at most 4 different Beauville surfaces.

Theorem 3.7 *There are exactly two non isomorphic surfaces with $p_g = 0$ isogenous to a product with group $G = (\mathbb{Z}/5\mathbb{Z})^2$.*

Proof. Since the cardinality of $\tilde{\mathfrak{M}}$ is $24 \cdot 20 \cdot 12 \cdot 2$, there are at leasts two orbits of $\mathcal{G}$. But obviously the two elements $(e_1, e_2, -(e_1 + e_2), \lambda(e_1 + 2e_2), \lambda(3e_1 + 4e_2), \lambda(e_1 + 4e_2))$ and $(e_1, e_2, -(e_1 + e_2), -\lambda(e_1 + 2e_2), -\lambda(3e_1 + 4e_2), -\lambda(e_1 + 4e_2)) \in \tilde{\mathfrak{M}}$ are equivalent under $(\varphi, (1\ 2), (1\ 2)) \in \mathcal{G}$, where $\varphi \in Gl(2, \mathbb{Z}/5\mathbb{Z})$ is given by $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$. This proves the claim.

<div align="right">Q.E.D.</div>

4 $H_1(S, \mathbb{Z})$ for surfaces isogenous to a product with G abelian

In [BPV], p. 237, there is a list of examples of minimal surfaces of general type with $p_g = q = 0$. While for $1 \leq K^2 \leq 6$ for each example the first homology group is given, in the case $K^2 = 8, 9$ there is a question mark.
This motivated us to calculate $H_1(S, \mathbb{Z})$ for surfaces $S = C_1 \times C_2/G$ isogenous to a higher product of unmixed type with G abelian.
Let's recall again some facts from [Cat00]. Let g_i be the genus of the curve C_i and denote by Π_g the fundamental group of a compact Riemann surface of genus g. Then we have the following exact sequence

$$(**)\ 1 \longrightarrow \Pi_{g_1} \times \Pi_{g_2} \longrightarrow \pi_1(S) \longrightarrow G \longrightarrow 1.$$

Since we have assumed that S is of unmixed type, Π_{g_1} and Π_{g_2} are both normal subgroups of $\pi_1(S)$. We define $\Pi(i+1) := \pi_1(S)/\Pi_{g_i}$, where $i+1$ is considered as element in $\mathbb{Z}/2\mathbb{Z}$. Then we get two exact sequences

$$1 \longrightarrow \pi_1(C_i) \longrightarrow \Pi(i) \longrightarrow G \longrightarrow 1,$$

which are exactly the orbifold fundamental group exact sequences of the coverings $C_i \longrightarrow C_i/G =: C'_i$, in particular $\Pi(i) = \pi_1^{orb}(C'_i - B|m)$ is the orbifold fundamental group of $C_i \longrightarrow C_i/G$ (for the definition and properties of the orbifold fundamental group we refer again to [Cat00]).
We henceforth have an exact sequence

$$1 \longrightarrow \Pi_{g_1} \times \Pi_{g_2} \longrightarrow \Pi(1) \times \Pi(2) \longrightarrow G \times G \longrightarrow 1,$$

where $\pi_1(S)$ is the inverse image of $\varphi : \Pi(1) \times \Pi(2) \longrightarrow G \times G$ of G diagonally embedded in $G \times G$. In particular, we have the following exact sequence

$$(*) \quad 1 \longrightarrow \pi_1(S) \longrightarrow \Pi(1) \times \Pi(2) \longrightarrow G \longrightarrow 1,$$

where $\Pi(1) \times \Pi(2) \longrightarrow G$ is the composition of φ with the map $G \times G \longrightarrow G$, $(a, b) \mapsto a - b$.

We observe the following.

Remark 4.1 Let

$$1 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 1$$

be an exact sequence of groups and assume G to be abelian. Then the following sequence is exact:

$$1 \longrightarrow A^{ab} \longrightarrow \overline{B} := B/[A, A] \longrightarrow G \longrightarrow 1.$$

and moreover the abelianization of B equals the abelianization of $\overline{B}$.

We apply this remark repeatedly: first to the exact sequence $(**)$, obtaining

$$1 \to H_1 \times H_2 \to \overline{\pi_1(S)} \to G \to 1$$

which embeds into the exact sequence

$$1 \to H_1 \times H_2 \to \overline{\Pi(1)} \times \overline{\Pi(2)} \to G \times G \to 1.$$

It follows that an element $(h_1, h_2) \in H_1 \times H_2$ is a commutator in $\overline{\pi_1(S)}$ if and only if h_1 is a commutator in $\overline{\Pi(1)}$ and h_2 is a commutator in $\overline{\Pi(2)}$. Therefore, if we define G_i as the abelianization of $\Pi(i)$, or equivalently of $\overline{\Pi(i)}$, the sequence

$$0 \longrightarrow \pi_1(S)^{ab} \longrightarrow G_1 \times G_2 \longrightarrow G \longrightarrow 0.$$

is exact and we have proven the following

Proposition 4.2 *Let $S = C_1 \times C_2/G$ be a surface isogenous to a higher product of unmixed type with G abelian. Furthermore, denote by G_i the abelianization of the orbifold fundamental group of $C_i \longrightarrow C_i/G$. Then*

$$H_1(S, \mathbb{Z}) = ker(G_1 \times G_2 \longrightarrow G \times G \longrightarrow G),$$

where the last map is obviously given by $(a, b) \mapsto a - b$.

In the rest of the paragraph we will use our classification result of the previous section in order to calculate the torsion groups of all surfaces isogenous to a higher product with G abelian and $p_g = 0$.

Theorem 4.3 *Let S be a surface with $p_g = q = 0$ isogenous to a higher product $C_1 \times C_2/G$ (not of mixed type) and assume G to be abelian. Then we get the following values of $H_1(S, \mathbb{Z})$:*

1) $H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^6$ for $G = (\mathbb{Z}/2\mathbb{Z})^3$,

2) $H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^4$ for $G = (\mathbb{Z}/2\mathbb{Z})^4$,

3) $H_1(S, \mathbb{Z}) = (\mathbb{Z}/3\mathbb{Z})^4$ for $G = (\mathbb{Z}/3\mathbb{Z})^2$,

4) $H_1(S, \mathbb{Z}) = (\mathbb{Z}/5\mathbb{Z})^2$ for $G = (\mathbb{Z}/5\mathbb{Z})^2$.

Proof. 1) In this case $C_1 \longrightarrow C_1/G = \mathbb{P}^1$ has 5 branch points $p_1, \ldots p_5$ of multiplicities $(2, 2, 2, 2, 2)$, whence the orbifold fundamental group $\Pi(1) = \pi_1^{orb}(\mathbb{P}^1 - \{p_1, \ldots, p_5\}; (2, 2, 2, 2, 2))$ equals $< a_1, \ldots a_5 |\ a_1^2 = \ldots = a_5^2 = a_1 \cdot \ldots \cdot a_5 = 1 >=< a_1, \ldots a_4 | a_1^2 = \ldots a_4^2 = (a_1 \cdot \ldots \cdot a_4)^2 = 1 >$. Therefore $G_1 = \Pi(1)^{ab} = (\mathbb{Z}/2\mathbb{Z})^4$. Since $C_2 \longrightarrow C_2/G = \mathbb{P}^1$ has 6 branch points, again each of multiplicitiy 2, we see that $G_2 = (\mathbb{Z}/2\mathbb{Z})^5$. Therefore

$$H_1(S, \mathbb{Z}) = ker((\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/2\mathbb{Z})^5 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^3) = (\mathbb{Z}/2\mathbb{Z})^6.$$

2) Here $C_i \longrightarrow C_i/G = \mathbb{P}^1$ has 5 branch points $p_1, \ldots p_5$ resp. $q_1, \ldots q_5$ of multiplicities $(2, 2, 2, 2, 2)$, whence the orbifold fundamental group $\Pi(i) = \pi_1^{orb}(\mathbb{P}^1 - \{p_1, \ldots, p_5\}; (2, 2, 2, 2, 2))$ (resp. $\pi_1^{orb}(\mathbb{P}^1 - \{q_1, \ldots, q_5\}; (2, 2, 2, 2, 2))$) equals $< a_1, \ldots a_5 : a_1^2 = \ldots a_5^2 = a_1 \cdot \ldots a_5 = 1 >=< a_1, \ldots a_4 : a_1^2 = \ldots a_4^2 = (a_1 \cdot \ldots \cdot a_4)^2 = 1 >$. Therefore $G_i = \Pi(1)^{ab} = (\mathbb{Z}/2\mathbb{Z})^4$. Therefore

$$H_1(S, \mathbb{Z}) = ker((\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/2\mathbb{Z})^4 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^4) = (\mathbb{Z}/2\mathbb{Z})^4.$$

3) Here $C_i \longrightarrow C_i/G = \mathbb{P}^1$ has 4 branch points, all of multiplicity 3, and as above we see that $G_i = (\mathbb{Z}/3\mathbb{Z})^3)$, whence

$$H_1(S, \mathbb{Z}) = ker((\mathbb{Z}/3\mathbb{Z})^3 \oplus (\mathbb{Z}/3\mathbb{Z})^3 \longrightarrow (\mathbb{Z}/3\mathbb{Z})^2) = (\mathbb{Z}/3\mathbb{Z})^4.$$

4) In this case $C_i \longrightarrow C_i/G = \mathbb{P}^1$ has 3 branch points, all of multiplicity 5, and as before we see that $G_i = (\mathbb{Z}/5\mathbb{Z})^2)$, whence

$$H_1(S, \mathbb{Z}) = ker((\mathbb{Z}/5\mathbb{Z})^2 \oplus (\mathbb{Z}/5\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/5\mathbb{Z})^2) = (\mathbb{Z}/5\mathbb{Z})^2.$$

Q.E.D.

5 Some new examples with G non abelian

We postpone the classification of surfaces S with $p_g = 0$ isogenous to a higher product with G non abelian to a forthcoming article.

In the rest of the paper we will give however some new examples of surfaces isogenous to a product with non abelian group. We remark that several examples were already given by Mendes Lopes and Pardini (cf. [Pa], [MLP1]). We observe that in the non abelian case we cannot find such a low upper bound for the cardinality of the group G (as in (2.4)), in fact we will exhibit examples of surfaces S with $p_g = 0$ isogenous to a higher product with $G = \mathfrak{A}_5$ and $G = \mathfrak{S}_4$. The reason is that, in the case $r = 3$, the branching indices here do not need to satisfy the condition that m_3 be a divisor of the least common multiple of m_1, m_2.

In the non abelian case, however, more restrictions come from the condition that the two stabilizer sets $\mathcal{S}_1, \mathcal{S}_2$ have an empty intersection. In fact, here $\mathcal{S}_1$ is the union of the conjugacy classes of the cyclic subgroups generated by $a_1, \ldots a_n$. Therefore, knowledge of the conjugacy classes of G will help to find examples, while knowledge of the branching indices plus Sylow's theorems help to show that some cases do not occur.

5.1 $G = \mathfrak{A}_5$

Observe that in this case the group contains exactly three non trivial conjugacy classes, completely determined by the order of the elements in the class ($m = 2$ gives the class of the 15 double transpositions which form five Klein subgroups $\mathcal{K}_i \cong (\mathbb{Z}/2\mathbb{Z})^2$, $m = 3$ gives the conjugacy class of the 20 three cycles, $m = 5$ yields the conjugacy class of the 24 five cycles).

It follows that for one of the two curves only one branching index can occur. In this case the formulae of section 1 read:

$$|G| = 60 = (g_1 - 1)(g_2 - 1),$$

$$|G| = 60 = \frac{2}{-2 + \sum_j (1 - \frac{1}{m_j})}(g_i - 1).$$

Denoting $\frac{2}{-2 + \sum_j (1 - \frac{1}{m_j})}$ by α_1 resp. α_2 we remark that branching of pure type give the following values for α_i:

$$(2, \ldots, 2) =: 2^r \quad \Longrightarrow \quad \alpha_i = \frac{4}{r-4} \leq 4;$$

$$(3, \ldots, 3) =: 3^h \quad \Longrightarrow \quad \alpha_i = \frac{3}{h-3} \leq 3;$$

$$(5, \ldots, 5) =: 5^n \quad \Longrightarrow \quad \alpha_i = \frac{5}{2n-5} \leq 5.$$

Therefore we need "mixed branching" for at least one of the two curves. Observe moreover that the integrality of α_i implies $r \in \{5, 6, 8\}$, $h \in \{4, 6\}$, $n \in \{3, 5\}$.

Example 1.

For C_1 we take pure branching of type 3^4, i.e. $g_1 = 4$, $\alpha_1 = 3$, and for C_2 we take branching of type $(2, 5, 5)$, i.e. $g_2 = 21$, $\alpha_1 = 20$.

Since here obviously the union of the stabilizer subgroups for each curve have trivial intersection (remark that conjugating elements of order 2, 3, 5, you get again elements of order 2, 3, 5), the problem is reduced to finding elements a_1, a_2, a_3, a_4 of order three such that their product is 1, generating $\mathfrak{A}_5$ and elements b_1, b_2, b_3 of orders $(2, 5, 5)$, such that their product is 1 generating $\mathfrak{A}_5$.

1) We set $a_1 = (123)$, $a_2 = (345)$, $a_3 = (432)$, $a_4 = (215)$. It is obvious that these are elements of order 3 of $\mathfrak{A}_5$ and that their product is 1. Therefore it remains to verify that $\mathfrak{A}_5$ is generated by these elements. But we observe that $a_1 \cdot a_2 = (12345)$, which is an element of order 5, and $a_1 \cdot a_3 \cdot a_1 = (14)(23)$, which has order 2. Therefore the subgroup generated by a_1, a_2, a_3, a_4 has order at least 30. Since $\mathfrak{A}_5$ is simple it cannot have a subgroup of order 30, whence a_1, a_2, a_3, a_4 generate $\mathfrak{A}_5$.

2) We set $b_1 = (24)(35)$, $b_2 = (21345)$, $b_3 = (12345)$. Obviously $b_1 \cdot b_2 \cdot b_3 = 1$. In order to show that b_1, b_2, b_3 generate $\mathfrak{A}_5$ it suffices to find an element of order 3 in $< b_1, b_2, b_3 >$. E.g. $b_3 \cdot b_1 \cdot b_3 = (152)$.

Therefore we have constructed a surface $S = C_1 \times C_2/\mathfrak{A}_5$, where $g(C_1) = 4$, $g(C_2) = 21$.

In [Pa] the author gives another surface isogenous to a product with group $\mathfrak{A}_5$. This surfaces is obviously different to ours since in her case $g(C_1) = 5$, $g(C_2) = 16$. We will return to these examples later.

Example 2.

For C_1 we take pure branching of type 5^3, i.e. $g_1 = 6$, $\alpha_1 = 5$, and for C_2 we

take branching of type $(2, 2, 2, 3)$, i.e. $g_2 = 13$, $\alpha_1 = 12$.

Again the union of the stabilizer subgroups for each curve have trivial intersection, hence we have to find elements a_1, a_2, a_3 of order five such that their product is 1, generating $\mathfrak{A}_5$ and elements b_1, b_2, b_3, b_4 of orders $(2, 2, 2, 3)$, such that their product is 1 generating $\mathfrak{A}_5$.

We set $a_1 = (12534)$, $a_2 = (12453)$, $a_3 = (12345)$; $b_1 = (12)(34)$, $b_2 = (24)(35)$, $b_3 = (14)(35)$ and $b_4 = (234)$. It is now easy to see that these choices satisfy the required conditions and we obtain a new surface $S = C_1 \times C_2/\mathfrak{A}_5$ with $g(C_1) = 6$, $g(C_2) = 13$.

Example 3.

For C_1 we take pure branching of type 2^5, i.e. $g_1 = 5$, $\alpha_1 = 4$, and for C_2 we take branching of type $(3, 3, 5)$, i.e. $g_2 = 16$, $\alpha_1 = 15$.

Again the union of the stabilizer subgroups for each curve have trivial intersection, hence we have to find elements a_1, a_2, a_3, a_4, a_5 of order two such that their product is 1, generating $\mathfrak{A}_5$ and elements b_1, b_2, b_3 of orders $(3, 3, 5)$, such that their product is 1 generating $\mathfrak{A}_5$.

We set $a_1 = (12)(34))$, $a_2 = (13)(24)$, $a_3 = (14)(23)$, $a_4 = (14)(25)$, $a_3 = (14)(25)$; $b_1 = (123)$, $b_2 = (345)$, $b_3 = (54321)$. It is now easy to see that these choices satisfy the required conditions and we obtain a new surface $S = C_1 \times C_2/\mathfrak{A}_5$ with $g(C_1) = 5$, $g(C_2) = 16$. These surfaces were already constructed by R. Pardini in [Pa].

5.2 $G = \mathfrak{D}_4 \times \mathbb{Z}/2\mathbb{Z}$

In order to avoid misunderstanding we note that for us $\mathfrak{D}_4$ is the group generated by x, y with the relations $x^4 = y^2 = e$ and $yxy = x^{-1}$.

Observe that in $\mathfrak{D}_4$ the centre consists of $\{e, x^2\}$, and there are three more conjugacy classes, namely, $\{x, x^{-1}\}$, $\{y, yx^2\}$ and $\{xy, yx\}$.

Example 4.

We will now rewrite an example which was already constructed by R. Pardini (cf. [Pa]) in our algebraic setting. For the curve C_1 we take pure branching of type $(2, 2, 2, 2, 2, 2)$, whereas for C_2 we take branching of type $(2, 2, 2, 4)$. Whence, $g_1 = 9$, $g_2 = 3$.

We set $a_1 = (y, 0)$, $a_2 = (yx, 1)$, $a_3 = (yx^2, 0)$, $a_4 = (yx, 1)$, $a_5 = (x^2, 1)$, $a_6 = (x^2, 1)$. Then obviosly $a_1, \ldots a_6$ generate $\mathfrak{D}_4 \times \mathbb{Z}/2\mathbb{Z}$ and their product

is $(e, 0)$. Furthermore for C_2 we set $b_1 = (e, 1)$, $b_2 = (y, 1)$, $b_3 = (xy, 0)$, $b_4 = (x, 0)$. Again these elements generate G and have trivial product. We obtain thus a surface $S = C_1 \times C_2/\mathfrak{D}_4 \times \mathbb{Z}/2\mathbb{Z}$ with $g(C_1) = 9$, $g(C_2) = 3$.

5.3 $\mathfrak{S}_4$

Here there is only the following algebraic possibility of a surface $S = C_1 \times C_2/\mathfrak{S}_4$ with $p_g = 0$.

Example 5.
For the curve C_1 we take branching of type $(2, 2, 2, 2, 2, 2)$, whereas for C_2 we take branching of type $(3, 4, 4)$. Whence, $g_1 = 13$, $g_2 = 3$.
We set $a_1 = a_2 = (12)$, $a_3 = a_4 = (23)$, $a_5 = a_6 = (34)$. Obviously their product is 1 and they generate.
For C_2, we set $b_1 = (123)$, $b_2 = (1234)$, $b_3 = (1243)$. One immediately verifies that $b_1 b_2 b_3 = 1$. Moreover, if $H' := < b_1, b_2, b_3 >$, then H' contains the transposition (34) and acts double transitively, since it contains $b_1 = (123)$. Whence $H' = \mathfrak{S}_4$.
We obtain thus a surface $S = C_1 \times C_2/\mathfrak{S}_4$ with $g(C_1) = 13$, $g(C_2) = 3$. Again this example was already constructed by R. Pardini (cf. [Pa]).

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