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THE MAXIMUM LIKELIHOOD DEGREE

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AMIT KHETAN, and BERND STURMFELS

Abstract. Maximum likelihood estimation in statistics leads to the problem of maximizing a product of powers of polynomials. We study the algebraic degree of the critical equations of this optimization problem. This degree is related to the number of bounded regions in the corresponding arrangement of hypersurfaces, and to the Euler characteristic of the complexified complement. Under suitable hypotheses, the maximum likelihood degree equals the top Chern class of a sheaf of logarithmic differential forms. Exact formulae in terms of degrees and Newton polytopes are given for polynomials with generic coefficients.

1. Introduction. In algebraic statistics [14, 23, 24], a *model for discrete data* is a map $\mathbf{f}: \mathbf{R}^d \rightarrow \mathbf{R}^n$ whose coordinates f_1, \dots, f_n are polynomial functions in the parameters $(\theta_1, \dots, \theta_d) =: \theta$. The parameter vector θ ranges over an open subset \mathcal{U} of \mathbf{R}^d such that $f(\theta)$ lies in the positive orthant $\mathbf{R}_{>0}^n$. The image $f(\mathcal{U})$ represents a family of probability distributions on an n -element state space, provided we make the extra assumption that $f_1 + \dots + f_n - 1$ is the zero polynomial.

A given data set is a vector $u = (u_1, \dots, u_n)$ of positive integers. The problem of *maximum likelihood estimation* is to find parameters θ which best explain the data u . This leads to the following optimization problem:

$$(1) \quad \text{Maximize } f_1(\theta)^{u_1} f_2(\theta)^{u_2} \cdots f_n(\theta)^{u_n} \quad \text{subject to } \theta \in \mathcal{U}.$$

Under suitable assumptions we have an optimal solution $\hat{\theta}$ to the problem (1), which is an algebraic function of the data u . Our goal is to compute the degree of that algebraic function. We call this number the *maximum likelihood degree* of the model \mathbf{f} . Equivalently, the *ML degree* is the number of complex solutions to the critical equations of (1), for a general data vector u . In this paper we prove results of the following form:

THEOREM 1. *Let f_1, \dots, f_n be polynomials of degrees b_1, \dots, b_n in d unknowns. If the maximum likelihood degree of the model $\mathbf{f} = (f_1, \dots, f_n)$ is finite then it is less*

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than or equal to the coefficient of z^d in the generating function

$$(2) \quad \frac{(1 - z)^d}{(1 - zb_1)(1 - zb_2) \cdots (1 - zb_n)}.$$

Equality holds if the coefficients of the polynomials f_i are sufficiently generic.

We wish to point out that in the above theorem and in the rest of the article the word *generic* refers technically to belonging to a Zariski open subset of some complex affine or projective space. In applications, a random choice of coefficients of f_i from \mathbf{R} is sufficient to meet this genericity condition. As an example, consider a model given by $n = 4$ quadratic polynomials in $d = 2$ parameters. The solution to (1) satisfies the two critical equations

$$\frac{u_1}{f_1} \frac{\partial f_1}{\partial \theta_1} + \frac{u_2}{f_2} \frac{\partial f_2}{\partial \theta_1} + \frac{u_3}{f_3} \frac{\partial f_3}{\partial \theta_1} + \frac{u_4}{f_4} \frac{\partial f_4}{\partial \theta_1} = \frac{u_1}{f_1} \frac{\partial f_1}{\partial \theta_2} + \frac{u_2}{f_2} \frac{\partial f_2}{\partial \theta_2} + \frac{u_3}{f_3} \frac{\partial f_3}{\partial \theta_2} + \frac{u_4}{f_4} \frac{\partial f_4}{\partial \theta_2} = 0.$$

If the f_i are general quadrics then these equations have 25 complex solutions. Indeed, the formula for the maximum likelihood degree in Theorem 1 gives

$$\frac{(1 - z)^2}{(1 - 2z)^4} = 1 + 6z + \underline{25}z^2 + 88z^3 + 280z^4 + \cdots.$$

For special quadrics f_i , the ML degree can be much lower than 25. A familiar example is the *independence model for two binary random variables*:

$$(3) \quad f_1 = \theta_1\theta_2, \quad f_2 = (1 - \theta_1)\theta_2, \quad f_3 = \theta_1(1 - \theta_2), \quad f_4 = (1 - \theta_1)(1 - \theta_2).$$

Here the ML degree is only one because the maximum likelihood estimate $\hat{\theta}$ is a rational function (= algebraic function of degree one) of the data u :

$$\hat{\theta}_1 = \frac{u_1 + u_3}{u_1 + u_2 + u_3 + u_4} \quad \text{and} \quad \hat{\theta}_2 = \frac{u_1 + u_2}{u_1 + u_2 + u_3 + u_4}.$$

This paper is organized as follows. In Section 2 we present the algebraic geometry for studying critical points of a rational function $f = f_1^{u_1} \cdots f_n^{u_n}$ on an irreducible projective variety X . The critical equations $d \log(f) = 0$ are interpreted as sections of the sheaf $\Omega^1(\log D)$ of 1-forms with logarithmic singularities along the divisor D defined by f . In Theorem 5, we show that if D is a global normal crossing divisor then the ML degree equals the degree of the top Chern class of $\Omega^1(\log D)$. If X is projective d -space then this leads to Theorem 1. In Section 3 we study the case when X is a smooth toric variety, and we derive a formula for the ML degree when the f_i are Laurent polynomials which are generic relative to their Newton polytopes. For instance, Example 9 shows that the ML degree is

13 if we replace (3) by

$$f_i = \alpha_i + \beta_i\theta_1 + \gamma_i\theta_2 + \delta_i\theta_1\theta_2 \quad (i = 1, 2, 3, 4).$$

Section 4 is concerned with the relationship of the ML degree to the bounded regions of the complement of $\{f_i = 0\}$ in \mathbf{R}^d . The number of these regions is a lower bound to the number of real solutions of the critical equations, and therefore a lower bound to the ML degree. We show that for plane quadrics all three numbers can be equal. However, for other combinations of plane curves the ML degree and the number of bounded regions diverge, and we prove a tight upper bound on the latter in Theorem 13. An important instance is the case when the f_i are linear forms, and we will mention a result of Varchenko [28] (Theorem 14) that the ML degree coincides with the number of bounded regions of the arrangement of hyperplanes $\{f_i = 0\}$ when the f_i are (not necessarily generic) linear forms.

Section 5 revisits the ML degree for toric varieties, replacing the smoothness assumption by a much milder condition. Theorem 16 gives a purely combinatorial formula for the ML degree in terms of the Newton polytopes of the polynomials f_i . This section also discusses how resolution of singularities can be used to compute the ML degree for nongeneric polynomials.

Section 6 deals with topological methods for determining the ML degree. Theorem 20 shows that, under certain restrictive hypotheses, it coincides with the Euler characteristic of the complex manifold $X \setminus D$, and Theorem 23 offers a general version of the semi-continuity principle which underlies the inequality in Theorem 1. In Section 7 we relate the ML degree to the sheaf of logarithmic vector fields along D , which is the sheaf dual to $\Omega^1(\log D)$.

This paper was motivated by recent appearances of the concept of ML degree in statistics and computational biology. Chor, Khetan and Snir [7] showed that the ML degree of a phylogenetic model equals 9, and Geiger, Meek and Sturmfels [15] proved that an undirected graphical model has ML degree one if and only if it is decomposable. The notion of ML degree also makes sense for certain parametrized models for continuous data: Drton and Richardson [11] showed that the ML degree of a Gaussian graphical model equals 5, and Bout and Richards [5] studied the ML degree of certain mixture models. The ML degree always provides an upper bound on the number of local maxima of the likelihood function. Our ultimate hope is that a better understanding of the ML degree will lead to the development of custom-tailored algorithms for solving the critical equations $d \log(f) = 0$. There is a need for such new algorithms, given that methods currently used in statistics (notably the *EM-algorithm*) often produce only local maxima of (1).

Remark 2. After we submitted our manuscript two pieces of work were brought to our attention. The first is by Silvotti [26] where he proves our Theorem 20 with slightly different assumptions. Silvotti's Theorem 1.2 and Theorem

3.1 in [26] are the corresponding results. The second is by Damon [9] where he also proves our Theorem 20 with slightly different assumptions in his Corollary 6. The techniques he uses in proving this result are very similar to our arguments where we also employ Milnor numbers. Furthermore, Damon obtains the formula of our Theorem 1 in his Corollary 7.

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2. Critical points of rational functions. In this section we work in the following general set-up of algebraic geometry. Let X be a complete factorial algebraic variety over the complex numbers \mathbf{C} . We also assume that X is irreducible of dimension $d \geq 1$. In applications to statistics, the variety X will often be a smooth projective toric variety.

Suppose that $f \in \mathbf{C}(X)$ is a rational function on X . Since X is factorial, the local rings $\mathcal{O}_{X,x}$ are unique factorization domains. This means that the function f has a global factorization which is unique up to constants (see Proposition 6.11 in [17]):

$$(4) \quad f = F_1^{u_1} F_2^{u_2} \cdots F_r^{u_r}.$$

Here F_i is a prime section of an invertible sheaf $\mathcal{O}_X(D_i)$ where D_i is the divisor on X defined by F_i . In our applications we usually assume that $r \geq n$ where n is the number considered in the Introduction. For instance, if f_1, \dots, f_n are polynomials and $X = \mathbf{P}^d$ then $r = n + 1$; namely, F_1, \dots, F_n are the homogenizations of f_1, \dots, f_n using θ_0 , $F_{n+1} = \theta_0$ and $u_{n+1} = -\sum_{i=1}^n b_i u_i$ (see the proof of Theorem 1 for details).

By (4), we can write the divisor of the rational function f uniquely as

$$\operatorname{div}(f) = \sum_{i=1}^r u_i D_i,$$

where the u_i are (possibly negative) integers. Let D denote the reduced union of the codimension one subvarieties $D_i \subset X$, or, as a divisor, $D := \sum_{i=1}^r D_i$.

We are interested in computing the critical points of the rational function f on the open set $V := X \setminus D$ complementary to the divisor D . Especially, we wish to know the number of critical points, counted with multiplicities.

A critical point is by definition a point $x \in X$ where the differential 1-form df vanishes. If x is a smooth point on X , and x_1, \dots, x_d are local coordinates, then $df = \sum_{j=1}^d (\partial f / \partial x_j) dx_j$. Hence x is a critical point of f if and only if

$$(5) \quad \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \dots = \frac{\partial f}{\partial x_d} = 0.$$

We next rewrite the critical equations (5) using the factorization (4). Around each point $x \in X$, we may choose a local trivialization for the sheaf $\mathcal{O}_X(D_i)$ and express F_i locally by a regular function. By slight abuse of notation, we denote that regular function also by F_i . For instance, if $X = \mathbf{P}^d$ then this means replacing the homogeneous polynomial F_i by a dehomogenization.

Since f has neither zeros nor poles on the open set V , the vanishing of df is equivalent to the vanishing of the logarithmic derivative

$$(6) \quad d\log(f) = \frac{df}{f} = \sum_{i=1}^r u_i d\log(F_i) = \sum_{i=1}^r u_i \frac{dF_i}{F_i}.$$

We now recall some classical definitions and results concerning the sheaf of differential 1-forms with logarithmic singularities along D . The standard references on this subject are Déglise’s book [10] and Saito’s paper [25]. We define $\Omega_X^1(\log D)$ as a subsheaf of the sheaf $\Omega_X^1(D)$ of 1-forms with poles at most on D and of order one. This sheaf is the image of the natural map

$$\Omega_X^1 \oplus \mathcal{O}_X^r \longrightarrow \Omega_X^1(D)$$

which is given by the inclusion $\Omega_X^1 \rightarrow \Omega_X^1(D)$ and the homomorphisms sending $1 \in \mathcal{O}_X \rightarrow d\log(F_i)$. For experts we note that our definition differs from the one in [25] when D is not normal crossing. Saito’s sheaf is the double dual of our $\Omega_X^1(\log D)$, which explains why his is always locally free when X is a surface [25, Corollary 1.7]. Ours need not be locally free even for surfaces. However, our definition gives a natural exact sequence.

LEMMA 3. *If X is factorial and complete then we have an exact sequence*

$$(7) \quad 0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^1(\log D) \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{D_i} \rightarrow 0.$$

Proof. The local sections of the sheaf $\Omega_X^1(\log D)$ are rational 1-forms which can be written as $\omega = \sum_{i=1}^r \psi_i \cdot d\log(F_i) + \eta$, where η is a regular 1-form.

Since the D_i ’s are distinct prime divisors and X is factorial, the local rings \mathcal{O}_{X, D_i} are discrete valuation rings with parameter F_i . Thus F_j is invertible in this local ring for $j \neq i$, and ω is regular if and only if F_i divides ψ_i . This implies

that the homomorphism which sends ω to the vector $(\psi_i \pmod{F_i})_{i=1, \dots, r}$ is well defined, and it induces an isomorphism from the quotient $\Omega_X^1(\log D)/\Omega_X^1$ onto $\bigoplus_{i=1}^r \mathcal{O}_{D_i}$. □

Assume now that X is smooth. Then both sheaves $\Omega_X^1(D)$ and Ω_X^1 are locally free of rank $d = \dim(X)$. Hence the intermediate sheaf $\Omega_X^1(\log D)$ is torsion free of the same rank. Our next result shows that $\Omega_X^1(\log D)$ is locally free if and only if the divisors D_i are smooth and intersect transversally.

PROPOSITION 4. *Let $x \in X$ be a smooth point, x_1, \dots, x_d local coordinates at x and D_1, \dots, D_h the divisors which contain x . Then the sheaf $\Omega_X^1(\log D)$ is locally free at x if and only if the $h \times d$ -matrix $(\partial F_i / \partial x_j)$ has rank h at x .*

Proof. Any local section of $\Omega_X^1(\log D)$ can be written in the form

$$(8) \quad \omega = \sum_{i=1}^r \psi_i \cdot d\log(F_i) + \eta = \sum_{i=1}^h \psi_i \cdot d\log(F_i) + \sum_{j=1}^d \eta_j \cdot dx_j.$$

This observation gives rise to a local exact sequence

$$(9) \quad 0 \rightarrow \mathcal{O}_{X,x}^h \rightarrow \mathcal{O}_{X,x}^h \oplus \mathcal{O}_{X,x}^d \rightarrow \Omega_{X,x}^1(\log D) \rightarrow 0.$$

The surjective map on the right takes $((\psi_i), (\eta_j))$ to the sum on the right-hand side of (8). The injective map on the left takes the h -tuple (A_1, \dots, A_h) to

$$((\psi_i), (\eta_j)) \quad \text{with} \quad \psi_i = F_i A_i \quad \text{and} \quad \eta_j = - \sum_{l=1}^h A_l \frac{\partial F_l}{\partial x_j}.$$

The exactness of the sequence (9) follows from the proof of Lemma 3. If the section ω in (8) is identically zero in $\Omega_{X,x}^1(\log D)$ then ω is in particular regular, and so F_i divides each ψ_i .

Now, since X is reduced, a coherent sheaf \mathcal{F} is locally free of rank d if and only if $\dim_{\mathbb{C}} \mathcal{F} \otimes \mathbb{C}_x = d$ for each point x . Since tensor product is right exact, it follows that this condition is verified for $\Omega_X^1(\log D)$ if and only if the matrix of $\mathcal{O}_X^h \rightarrow \mathcal{O}_X^h \oplus \mathcal{O}_X^d$, evaluated at x , has rank precisely h . Since the functions F_1, \dots, F_h vanish at x , this is exactly the asserted condition that the Jacobian matrix $(\partial F_i / \partial x_j)_{i=1, \dots, h, j=1, \dots, d}$ has rank h at x . □

In the above situation where X is smooth and $\Omega_X^1(\log D)$ is locally free we shall say that the divisor D has *global normal crossings* (or GNC).

THEOREM 5. *Let X be smooth and assume that D is a GNC divisor. Then:*

- (1) *the section $d\log(f)$ of $\Omega_X^1(\log D)$ does not vanish at any point of D ,*

(2) if the divisor D intersects every curve in X (in particular, if D is ample) then $d\log(f)$ vanishes only on a finite subset of $V = X \setminus D$,

(3) if the above conclusions hold, then the number of critical points of f on V , counted with multiplicities, equals the degree of the top Chern class $c_d(\Omega_X^1(\log D))$.

Proof. We abbreviate $\sigma := d\log(f) = \sum_{i=1}^r u_i d\log(F_i)$. By the proof of Proposition 4 it follows that if $(\partial F_i / \partial x_j)_{i=1, \dots, h, j=1, \dots, d}$ has rank h at x , then $\Omega_X^1(\log D)$ is locally free of rank d with generators $d\log(F_i)$ and some choice of $d - h$ of the dx_j . If we write σ in this basis, the coefficients of $d\log(F_i)$ are the constants u_i while the coefficients of the dx_j are some regular functions. The first assertion follows immediately since the exponents u_i are all nonzero.

The second assertion follows from the first: let Z_σ be the zero set of the section σ . Since Z_σ does not intersect D , it follows that $\dim(Z_\sigma) = 0$.

Thirdly, if \mathcal{F} is a locally free sheaf of rank d on a smooth variety X of dimension d , and σ is a section of $H^0(\mathcal{F})$ with a zero scheme Z_σ of dimension 0, then the length of Z_σ equals the degree of the top Chern class $c_d(\mathcal{F})$. \square

The total Chern class of a sheaf \mathcal{F} is the sum $c_{tot}(\mathcal{F}) = \sum_{i=0}^d c_i(\mathcal{F})z^i$. This is a polynomial in z whose coefficients are elements in the Chow ring $A^*(X)$. Recall that every element in $A^*(X)$ has a well-defined degree which is the image of its degree d part under the degree homomorphism $A^d(X) \rightarrow \mathbf{Z}$.

COROLLARY 6. *Suppose that X is smooth and D is a GNC divisor on X which intersects every curve. Then the number of critical points of f , counted with multiplicities, is the degree of the coefficient of z^d in the following polynomial:*

$$(10) \quad c_{tot}(\Omega_X^1) \cdot \prod_{i=1}^r (1 - zD_i)^{-1} \in A^*(X)[z].$$

Proof. The total Chern class $c_{tot}(\mathcal{F})$ is multiplicative with respect to exact sequences, i.e., if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of sheaves, then $c_{tot}(B) = c_{tot}(A) \cdot c_{tot}(C)$. Hence the sequence (7) implies the result. \square

In the next section, we apply the formula (10) in the case when X is a smooth projective toric variety. The Chow group $A^d(X)$ has rank one and is generated by the class of any point. This canonically identifies $A^d(X)$ with \mathbf{Z} and so any top Chern class can be considered to be a number.

COROLLARY 7. *Suppose X is a smooth toric variety with boundary divisors $\Delta_1, \dots, \Delta_s$ and D is GNC and meets every curve. The number of critical points of f , counted with multiplicity, equals the coefficient of z^d in*

$$(11) \quad \frac{\prod_{j=1}^s (1 - z\Delta_j)}{\prod_{i=1}^r (1 - zD_i)} \in A^*(X)[z].$$

Proof. By virtue of equation (10) we need only compute the total Chern class $c_{tot}(\Omega_X^1)$. For this we use the exact sequence in [13, page 87],

$$0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^1(\log \Delta) \rightarrow \bigoplus_{j=1}^s \mathcal{O}_{\Delta_j} \rightarrow 0,$$

where $\Delta = \sum_{j=1}^s \Delta_j$, and the fact that $\Omega_X^1(\log \Delta)$ is trivial. □

3. Models defined by generic polynomials. We now apply the results of the previous section to algebraic statistics models $\mathbf{f}: \mathbf{R}^d \rightarrow \mathbf{R}^n$. To illustrate how this works, we first prove Theorem 1 for generic polynomials. The proof of the statement that the ML degree of generic polynomials is an upper bound on the ML degree of special polynomials (when this number is finite) is deferred to Theorem 8 which is a generalization of Theorem 1. See also Theorem 23 where this semi-continuity principle is stated in general.

Proof of Theorem 1 (generic case). The polynomials f_1, \dots, f_n are assumed to be generic among all (nonhomogeneous) polynomials of degrees b_1, \dots, b_n in $\theta_1, \dots, \theta_d$, and u_1, \dots, u_n are positive integers. We take X to be projective space \mathbf{P}^d with coordinates $(\theta_0 : \theta_1 : \dots : \theta_d)$. Our object of interest is the following rational function on $X = \mathbf{P}^d$:

$$F = (f_1^{u_1} f_2^{u_2} \dots f_n^{u_n}) \left(\frac{\theta_1}{\theta_0}, \frac{\theta_2}{\theta_0}, \dots, \frac{\theta_d}{\theta_0} \right).$$

The global factorization (4) of this F has $r = n + 1$ prime factors, namely,

$$F_i = \theta_0^{b_i} \cdot f_i \left(\frac{\theta_1}{\theta_0}, \dots, \frac{\theta_d}{\theta_0} \right) \quad \text{for } i = 1, \dots, n,$$

and $F_{n+1} = \theta_0$ with $u_{n+1} = -b_1 u_1 - b_2 u_2 - \dots - b_n u_n$. The Chow ring of $X = \mathbf{P}^d$ is $\mathbf{Z}[H]/\langle H^{d+1} \rangle$, where H represents the hyperplane class. By our genericity hypothesis, the $r = n + 1$ prime factors of F yield smooth divisors whose union is global normal crossing. They correspond to the following divisor classes:

$$D_1 = b_1 H, D_2 = b_2 H, \dots, D_n = b_n H \text{ and } D_{n+1} = H.$$

Projective space \mathbf{P}^d is a smooth toric variety with $d + 1$ torus-invariant divisors Δ_j , each having the same class H . Hence the formula in (11) specializes to

$$\frac{(1 - zH)^{d+1}}{(1 - zb_1 H) \cdots (1 - zb_n H)(1 - zH)} = \frac{(1 - zH)^d}{(1 - zb_1 H) \cdots (1 - zb_n H)}.$$

Since we work in the Chow ring of projective space \mathbf{P}^d , the coefficient of $(zH)^d$ is the same as the coefficient of z^d in the generating function in (2). \square

We now generalize our results from polynomials of fixed degrees to Laurent polynomials with fixed Newton polytopes. Recall that the *Newton polytope* of a Laurent polynomial $f(\theta_1, \dots, \theta_d)$ is the convex hull of the set of exponent vectors of the monomials appearing in f with nonzero coefficient. Given a convex polytope $P \subset \mathbf{R}^d$ with vertices in \mathbf{Z}^d , by a *generic Laurent polynomial with Newton polytope P* we will mean a sufficiently general \mathbf{C} -linear combination of monomials with exponent vectors in $P \cap \mathbf{Z}^d$.

In the next theorem we consider n Laurent polynomials f_1, f_2, \dots, f_n having respective Newton polytopes P_1, P_2, \dots, P_n . Because the f_i are Laurent polynomials, i.e., their monomials may have negative exponents, we only consider those critical points of the likelihood function $f = f_1^{u_1} f_2^{u_2} \dots f_n^{u_n}$ which lie in the algebraic torus $(\mathbf{C}^*)^d$. The number of such critical points (counted with multiplicity) will be called the *toric ML degree* of the rational function f .

Let $P = P_1 + P_2 + \dots + P_n$ denote the *Minkowski sum* of the given Newton polytopes, and let X be the projective toric variety defined by P . Let $\eta_1, \dots, \eta_s \in \mathbf{Z}^d$ be the primitive inner normal vectors of the facets of P . They span the rays of the fan of X . Let $\Delta_1, \dots, \Delta_s$ denote the corresponding torus-invariant divisors on X . Each of the Newton polytopes P_i is the solution set of a system of linear inequalities of the specific form

$$P_i = \{x \in \mathbf{R}^d \mid \langle x, \eta_j \rangle \geq -a_{ij} \text{ for } j = 1, \dots, s\}.$$

The divisor on X defined by the Laurent polynomial f_i is linearly equivalent to $D_i = \sum_{j=1}^s a_{ij} \Delta_j$. The a_{ij} 's are integers which can be positive or negative. The divisor on X defined by $f = f_1^{u_1} f_2^{u_2} \dots f_n^{u_n}$ is linearly equivalent to

$$(12) \quad \sum_{i=1}^n u_i D_i = \sum_{j=1}^s \left(\sum_{i=1}^n u_i a_{ij} \right) \cdot \Delta_j.$$

We abbreviate the *support* of this divisor by

$$(13) \quad I = \{j \in \{1, \dots, s\} \mid \sum_{i=1}^n u_i a_{ij} \neq 0\}.$$

THEOREM 8. *If the toric variety X is smooth and the toric ML degree of the rational function f is finite then it is bounded above by the coefficient of z^d in the following generating function with coefficients in the Chow ring of X :*

$$(14) \quad \frac{\prod_{j \notin I} (1 - z\Delta_j)}{\prod_{i=1}^n (1 - zD_i)}.$$

Equality holds if each f_i is generic with respect to its Newton polytope P_i .

Note that Theorem 1 is the special case of Theorem 8 when P_i is the standard d -dimensional simplex $\text{conv}\{0, e_1, \dots, e_d\}$ scaled by a factor of b_i .

Proof. Let us first assume that f_i is a generic Laurent polynomial with Newton polytope P_i . Let $\mathbf{C}[x_1, \dots, x_s]$ be the homogeneous coordinate ring (cf. [8]) of X with one variable for each torus-invariant divisor Δ_j . Given a Laurent polynomial $f_i(\theta)$ with Newton polytope P_i , the corresponding rational function on X is $F_i(x)/x^{D_i}$ where D_i is as defined above and F_i is homogeneous of degree D_i . Therefore the rational function on X we are interested in is

$$F = x^{-\sum u_i D_i} \prod F_i(x)^{u_i}.$$

We next show that the divisor of F is GNC. Note that F_i is a generic section of a line bundle on X that is generated by its sections. This implies that the divisors $\{F_i = 0\}$ meet transversally in the dense torus of X . If any of the variables x_i are zero we simply restrict to the smaller dimensional toric variety determined by the corresponding facet where the restricted F_i 's remain generic sections of the restricted bundles.

The reduced divisor of poles and zeros of F is $D = \sum D_i + \sum_{j \in I} \Delta_j$ where I is defined as in (13). Since $\sum D_i$ is the divisor corresponding to P it is ample on the toric variety X defined by P . So $\sum D_i$ meets every curve on X and therefore so does D and we can apply Corollary 7. A variable x_j appears as a factor in F if and only if $j \in I$, in which case $1 - z\Delta_j$ appears in both the numerator and denominator of (11), and we get the expression (14).

Consider now arbitrary Laurent polynomials f_1, \dots, f_n in $\theta_1, \dots, \theta_d$ such that $f = \prod f_i^{u_i}$ has only finitely many critical points in $(\mathbf{C}^*)^d$. Let ν be the coefficient of z^d in (14). Let \mathbf{C}^m be the space of all n -tuples of Laurent polynomials with the given Newton polytopes. Consider the critical equations of $f = \prod f_i^{u_i}$ and clear denominators. The resulting collection of d Laurent polynomials defines an algebraic subset \tilde{W} in the product space $\mathbf{C}^m \times (\mathbf{C}^*)^d$. Saturate \tilde{W} to remove any components along the hypersurfaces $\{f_i = 0\}$ and get a new algebraic subset W . The map from W onto \mathbf{C}^m is dominant and generically finite, and the generic fiber of this map consists of ν points.

Our given Laurent polynomials f_1, \dots, f_n represent a point ϕ in \mathbf{C}^m . Let $\theta^{(1)}, \dots, \theta^{(\kappa)}$ be the isolated critical points of f . For each i , consider any irreducible component $W^{(i)}$ of W containing the point $(\phi, \theta^{(i)})$ in $W \subset \mathbf{C}^m \times (\mathbf{C}^*)^d$. By Krull's Principal Ideal Theorem, the component $W^{(i)}$ of W has codimension $\leq d$ and hence it has dimension $\geq m$. As the generic fiber is finite, the dimension of $W^{(i)}$ is exactly m and the projection to \mathbf{C}^m is dominant. Since $\theta^{(i)}$ is an isolated solution of the critical equations, the projection map to \mathbf{C}^m is open, so the intersection of $W^{(i)}$ with an open neighborhood of $(\phi, \theta^{(i)})$ maps onto an open neighborhood of ϕ . Hence every generic point $\tilde{\phi}$ near ϕ has a preimage $(\tilde{\phi}, \tilde{\theta}^{(i)})$ near $(\phi, \theta^{(i)})$, and these preimages are distinct for $i = 1, \dots, \kappa$. Here we are using the "specialization

principle” stated in Mumford’s book [21, (3.10)]. See also Theorem 23 below. This argument also works when the $\theta^{(i)}$ have multiplicities. We conclude that $\kappa \leq \nu$. \square

We illustrate Theorem 8 with two examples which we revisit in Section 5.

Example 9. Consider n generic polynomials $f_1(\theta_1, \theta_2), \dots, f_n(\theta_1, \theta_2)$ where the support of f_i consists of monomials $\theta_1^p \theta_2^q$ with $0 \leq p \leq s_i$ and $0 \leq q \leq t_i$, and suppose the u_i ’s are generic. The Newton polytope of f_i is the rectangle

$$P_i = \text{conv}\{(0, 0), (s_i, 0), (0, t_i), (s_i, t_i)\}.$$

The Minkowski sum of these rectangles is another rectangle, and $X = \mathbf{P}^1 \times \mathbf{P}^1$. In the numerator of (14), the contribution of the two torus-invariant divisors D and E corresponding to the left and the bottom edge of this rectangle survives. The denominator comes from the product of the divisors of f_1, \dots, f_n :

$$\frac{(1 - zD)(1 - zE)}{(1 - (s_1D + t_1E)z)(1 - (s_2D + t_2E)z) \cdots (1 - (s_nD + t_nE)z)}.$$

Now, the coefficient of the term z^2 modulo the Chow ring relations

$$D^2 = 0, \quad E^2 = 0, \quad D \cdot E = 1$$

gives the toric ML degree

$$(15) \quad \binom{n}{i=1} s_i \binom{n}{j=1} t_j + \sum_{k=1}^n s_k t_k - \sum_{i=1}^n (s_i + t_i) + 1.$$

Example 10. Let f_1, f_2, f_3 be generic polynomials in θ_1 and θ_2 with supports

$$\begin{aligned} A_1 &= \{1, \theta_1, \theta_1\theta_2, \theta_1^2\}, \\ A_2 &= \{1, \theta_1, \theta_2, \theta_1\theta_2, \theta_1^2\}, \\ A_3 &= \{1, \theta_1\theta_2, \theta_1\theta_2^2\}. \end{aligned}$$

The corresponding Newton polygons P_1, P_2, P_3 are shown in Figure 1. The normal fan of the Minkowski sum has eight rays and is shown in Figure 2.

Theorem 8 applies because the toric surface X is smooth. We label the eight rays by x_1, \dots, x_8 in counterclockwise order, starting with $(1, 0)$. The Chow ring $A^*(X)$ is the polynomial ring $\mathbf{Z}[x_1, \dots, x_8]$ modulo the ideal

$$\langle x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_2x_8, \\ x_3x_5, x_3x_6, x_3x_7, x_3x_8, x_4x_6, x_4x_7, x_4x_8, x_5x_7, x_5x_8, x_6x_8, \\ x_1 - x_3 - x_4 - x_5 + x_7 + 2x_8, x_2 + x_3 - x_5 - x_6 - x_7 - x_8 \rangle.$$

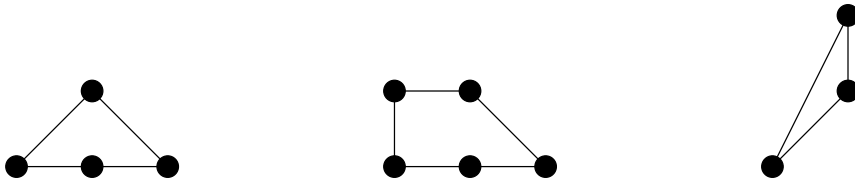


Figure 1. Three Newton polygons.

The three divisors corresponding to the polygons P_1, P_2, P_3 in Figure 1 are

$$D_1 = 2x_3 + 2x_4 + 2x_5 + x_6$$

$$D_2 = 2x_3 + 2x_4 + 2x_5 + x_6 + x_7 + x_8$$

$$D_3 = x_4 + 3x_5 + 2x_6 + x_7.$$

If all u_i 's are positive, then the support of the divisor $u_1D_1 + u_2D_2 + u_3D_3$ is $I = \{3, \dots, 8\}$. It follows that the toric ML degree is the coefficient of z^2 in

$$(1 - zx_1)(1 - zx_2)(1 - zD_1)^{-1}(1 - zD_2)^{-1}(1 - zD_3)^{-1}.$$

This coefficient is $14x_1x_2$, which means that the toric ML degree is 14.

The *toric ML degree of the model \mathbf{f}* is the toric ML degree defined above for generic u . In this case, there is no cancellation among the coefficients in (13), and I is the set of all indices j such that for some P_i the supporting hyperplane normal to η_j does not pass through the origin. The toric ML degree of \mathbf{f} is a numerical invariant of the polytopes P_1, \dots, P_n . A combinatorial formula for this invariant will be presented in Theorem 16 of Section 5.

4. Bounded regions in arrangements. As in the Introduction, we consider n polynomials f_1, \dots, f_n in d unknowns $\theta_1, \dots, \theta_d$. We now assume that all coefficients of the f_i 's are real numbers, and we also assume that u_1, \dots, u_n are positive

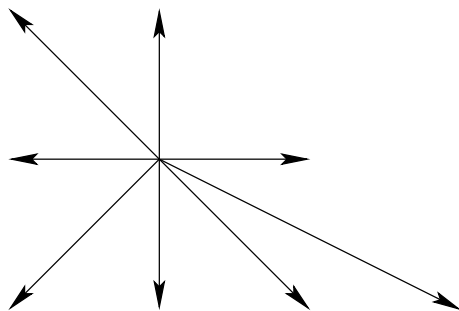


Figure 2. The fan of a smooth projective toric surface.

integers. However, we do not assume that the union of the divisors of the f_i 's has global normal crossings. This is the case of interest in statistics. Consider the arrangement of hypersurfaces defined by the f_i 's and let $V_{\mathbf{R}} = \mathbf{R}^d \setminus \bigcup_{i=1}^n \{f_i = 0\}$ be the complement of this arrangement. A connected component of $V_{\mathbf{R}}$ is a *bounded region* if it is bounded as a subset of \mathbf{R}^d . Then the following basic observation holds.

PROPOSITION 11. *For any polynomial map $\mathbf{f}: \mathbf{R}^d \rightarrow \mathbf{R}^n$ and any $u \in \mathbf{N}_{>0}^n$,*

$$\begin{aligned} \#\{\text{bounded regions of } V_{\mathbf{R}}\} &\leq \#\{\text{critical points of } f_1^{u_1} \cdots f_n^{u_n} \text{ in } \mathbf{R}^d\} \\ &\leq \text{ML degree of } \mathbf{f}. \end{aligned}$$

Proof. The function $f = f_1^{u_1} \cdots f_n^{u_n}$ is continuous, and on the boundary of the closure of each bounded region its value is zero. Hence it has to have at least one (real) critical point in the interior of each region. The second inequality holds trivially, since the ML degree was defined as the number of critical points of $f_1^{u_1} \cdots f_n^{u_n}$ in \mathbf{C}^d , counted with multiplicities. \square

This observation raises the question whether the inequalities above could be realized as equalities. We next show that this is the case when f_1, \dots, f_n are quadrics in the plane. Here the ML degree is $2n^2 - 2n + 1$ by Theorem 1.

PROPOSITION 12. *For each n , there are n quadrics f_1, \dots, f_n in \mathbf{R}^2 such that*

$$\#\{\text{bounded regions of } V_{\mathbf{R}}\} = \text{ML degree of } \mathbf{f} = 2n^2 - 2n + 1.$$

Hence all critical points are real.

Proof. We will take n quadrics that define “nested” ellipses with center at the origin, as suggested by Figure 3. The proof follows by induction: assume we have $2(n - 1)^2 - 2(n - 1) + 1$ bounded regions with $n - 1$ ellipses. Observe that the $(n - 1)$ st ellipse contains $2n - 3$ bounded regions. Then we add a new long and skinny ellipse which replaces the $2n - 3$ regions with $3(2n - 3) + 2$ regions. The total count comes out to be $2n^2 - 2n + 1$. \square

We will see such an equality holding for n linear hyperplanes in \mathbf{R}^d below. However, even in the plane \mathbf{R}^2 , the number of critical points and the number of bounded regions of $V_{\mathbf{R}}$ diverge for curves of degree ≥ 3 . Theorem 1 implies that for n generic plane curves of degrees b_1, \dots, b_n the ML degree is

$$\sum_{i=1}^n b_i(b_i - 2) + \sum_{i < j} b_i b_j + 1.$$

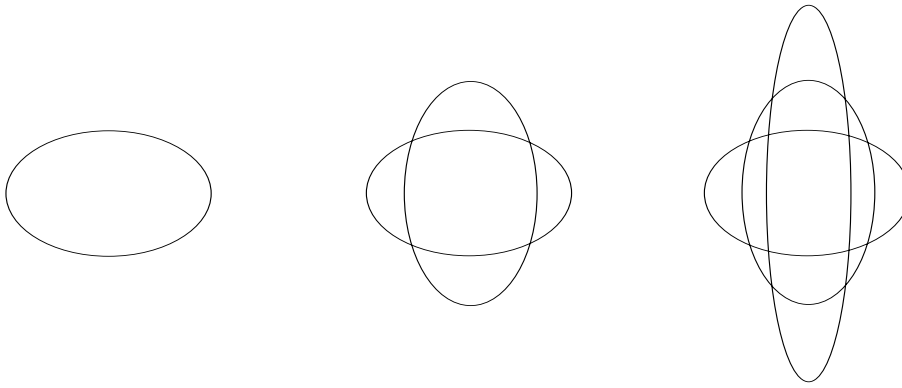


Figure 3. The “nested” ellipse construction.

The optimal upper bound for the number of bounded regions of $V_{\mathbf{R}}$ is smaller than the ML degree, by the following unpublished result due to Oleg Viro.

THEOREM 13. (Viro) *Let f_1, \dots, f_n be real plane curves of degrees b_1, \dots, b_n , and let K be the number of odd degree curves among them. Then*

$$\#\{\text{bounded regions of } V_{\mathbf{R}}\} \leq \sum_{i=1}^n \frac{(b_i - 1)(b_i - 2)}{2} + \sum_{i < j} b_i b_j + 1 - K,$$

and this bound is optimal.

Sketch of proof. The proof is by induction. For $n = 1$ the bound above is Harnack’s inequality [16], and it is optimal. Suppose the statement is proved for $n - 1$ curves, and f_n defines a curve of degree b_n . We will take this new curve so that it has the maximum number of bounded regions allowed by Harnack’s inequality, i.e. $B_n := (b_n - 1)(b_n - 2)/2 + 1$ if b_n is even, and one less than that if b_n is odd. One can achieve B_n by taking a curve with $B_n - 1$ unnested ovals and one more distinguished piece (that gives an extra bounded region when b_n is even) such that some line intersects this distinguished piece in exactly b_n points. We can arrange this last curve in such a way that when we superimpose the distinguished piece on the arrangement given by $n - 1$ curves, the last curve will intersect the curve given by f_i in $b_i b_n$ points. Now if we trace this last distinguished piece, every time we encounter an intersection point an extra bounded region is created, except the last point in case b_n is odd. Together with the remaining $B_n - 1$ ovals we get the bound. \square

In order to get any meaningful lower bound on the number of bounded regions of $V_{\mathbf{R}}$ one needs to make some assumptions. Without any assumptions the lower bound is zero: for f_i of even degree we take an empty (real) curve, and for f_i of odd degree we take the union of an empty curve with a line. If we let all the

lines intersect in a single point there will not be any bounded region. If we insist on at least having a GNC configuration, then by the same construction the lower bound that we get is the number of bounded regions in a generic arrangement of K lines where K is the number of odd degrees b_i . This idea leads us to studying the ML degree of a hyperplane arrangement.

THEOREM 14. *Let \mathbf{f} be given by n linear polynomials f_1, \dots, f_n with real coefficients. Then the ML degree of \mathbf{f} is equal to the number of the bounded regions of $V_{\mathbf{R}}$, and all critical points of the optimization problem (1) are real.*

This theorem does not assume any hypothesis such as global normal crossing. Under the GNC hypothesis, the hyperplanes would be in general position and the number of bounded regions equals $\binom{n-1}{d}$, as predicted by Theorem 1. Theorem 14 is Theorem 1.2.1 in [28], and we refer the reader to this reference.

Example 15. A family of important statistical models where Theorem 14 applies is the *linear polynomial model* of [24]. Such a model is given by a polynomial in r unknowns $x = (x_1, \dots, x_r)$ with indeterminate coefficients,

$$p(x) = \sum_{j=1}^d \theta_j x^{a_j} \quad (\text{with } a_j \in \mathbf{N}^r),$$

together with n data points $v_1, \dots, v_n \in \mathbf{R}^r$. The model is parametrized by

$$f_1(\theta) = \sum_{j=1}^d \theta_j v_1^{a_j}, \dots, f_n(\theta) = \sum_{j=1}^d \theta_j v_n^{a_j}.$$

The ML degree is the number of bounded regions of this arrangement.

5. Polytopes and resolution of singularities. We now return to the setting of Section 3, with the aim of relaxing the restrictive smoothness hypothesis in Theorem 8. Our aim is to derive a combinatorial formula for the toric ML degree of any model \mathbf{f} defined by generic Laurent polynomials satisfying a mild hypothesis. The derivation of Theorem 16 involves resolution of singularities in the toric category. At the end of the section we shall comment on using resolution of singularities for bounding the ML degree in general.

Given a polytope P in \mathbf{R}^d and a linear functional v on \mathbf{R}^d , we write

$$P^v = \{p \in P \mid \forall p' \in P: \langle v, p \rangle \leq \langle v, p' \rangle \}$$

for the face of P at which v attains its minimum. Two linear functionals v and v' are *equivalent* if $P^v = P^{v'}$. The equivalence classes are the relative interiors of cones of the *inner normal fan* Σ_P . If σ is a cone in Σ_P , or σ is a cone in any fan which refines Σ_P , then we write $P^\sigma = P^v$ for v in the relative interior of σ .

If f is a polynomial with Newton polytope P then f^σ denotes the *leading form* consisting of all terms of f which are supported on P^σ .

As in Section 3, let f_1, \dots, f_n be Laurent polynomials with Newton polytopes $P_1, \dots, P_n \subset \mathbf{R}^d$. Consider any fan Σ which is a common refinement of the inner normal fans $\Sigma_{P_1}, \dots, \Sigma_{P_n}$. Suppose τ is a cone in Σ and let k be the dimension of $(P_1 + \dots + P_n)^\tau$. There exists a k -dimensional linear subspace L of \mathbf{R}^d and vectors $q_1, \dots, q_n \in \mathbf{R}^d$ such that $q_i + P_i^\tau$ lies in L for all $i = 1, \dots, n$. The subspace L is unique and satisfies $L \cap \mathbf{Z}^d \simeq \mathbf{Z}^k$. Let $V(\cdot, \dots, \cdot)$ denote the *normalized mixed volume* on the subspace L . Here “normalized” refers to the lattice $L \cap \mathbf{Z}^d$, as is customary in toric geometry [13]. For any k -element subset $\{i_1, \dots, i_k\}$ of $\{1, 2, \dots, n\}$ we abbreviate

$$(16) \quad V(P_{i_1}, \dots, P_{i_k}; \tau) = V(q_{i_1} + P_{i_1}^\tau, \dots, q_{i_k} + P_{i_k}^\tau) \quad \text{if } \text{codim}(\tau) = k,$$

and $V(P_{i_1}, \dots, P_{i_k}; \tau) = 0$ if $\text{codim}(\tau) > k$. If $k = d$ and $\tau = \{0\}$ then we simply write $V(P_{i_1}, \dots, P_{i_d})$ for the mixed volume in (16). If $k = 0$ and τ is full-dimensional then (16) equals 1; this happens in the last sum of (17).

We are now ready to state our more general toric ML degree formula. As in Section 3, let X be the toric variety corresponding to the Minkowski sum $P = P_1 + \dots + P_n$ and Σ_X the normal fan with rays η_1, \dots, η_s . We consider the function $f = f_1^{u_1} \dots f_n^{u_n}$. Each polytope P_i corresponds to a divisor D_i so the divisor of f is $D = \sum u_i D_i$. Let I be the support of D as in (13). Label the rays of Σ_X so that $\{1, \dots, r\}$ are the indices not in I .

For each subset J of $\{1, \dots, r\}$ let τ_J denote the smallest cone of Σ which contains the vectors η_j for $j \in J$. If no such cone exists then τ_J is just a formal symbol and the expression (16) is declared to be zero for $\tau = \tau_J$. The mild smoothness hypothesis we need is that every singular cone of Σ contains at least one ray from I . Equivalently all cones τ_J are smooth.

THEOREM 16. *Suppose every singular cone of Σ_X contains some ray in the support of the divisor D . Then, the toric ML degree of the rational function f is bounded above by the following alternating sum of mixed volumes:*

$$(17) \quad \sum_{1 \leq i_1 \leq \dots \leq i_d \leq n} V(P_{i_1}, \dots, P_{i_d}) - \sum_{\substack{j \in \{1, \dots, r\} \\ 1 \leq i_1 \leq \dots \leq i_{d-1} \leq n}} V(P_{i_1}, \dots, P_{i_{d-1}}; \tau_{\{j\}}) \\ + \sum_{\substack{\{j_1, j_2\} \subset \{1, \dots, r\} \\ 1 \leq i_1 \leq \dots \leq i_{d-2} \leq n}} V(P_{i_1}, \dots, P_{i_{d-2}}; \tau_{\{j_1, j_2\}}) + \dots \\ + (-1)^d \sum_{\{j_1, \dots, j_d\} \subset \{1, \dots, r\}} V(\emptyset; \tau_{\{j_1, \dots, j_d\}}).$$

Equality holds if each f_i is generic relative to its Newton polytope P_i .

Proof. In order to apply Corollary 7 we must resolve the singularities of X . For toric varieties this is done in two steps. First we get a simplicial toric variety without adding any new rays to the fan. Second we resolve the remaining singular (but simplicial) cones by adding new rays. This procedure is described in detail in [13]. Typically the first step involves taking the pulling (reverse lexicographic) subdivision at each ray in the fan (see Chapter 8 in [27] for definitions). However, under the given hypothesis it is enough to perform pulling subdivisions only at the rays in the support of D to obtain a simplicial fan $\Sigma_{\bar{X}}$. This fine detail will be important below. Our hypothesis holds for this intermediate fan as well, and subsequently we take a smooth refinement $\Sigma_{X'}$ of $\Sigma_{\bar{X}}$ by adding new rays in the relative interiors of each of the singular cones. Let $\pi: X' \rightarrow X$ be the induced map.

We will show that we get no new critical points under the resolution. Hence the number of critical points can be computed on X' . We finally claim that the Chern class formula expands into the given combinatorial formula.

We investigate critical points of the pullback of our rational function:

$$F' = \pi^*(F) = \left(x^{-\sum u_i \pi^*(D_i)}\right) \prod \pi^*(F_i(x)).$$

For generic f_i , the same argument as in the proof of Theorem 8 shows that the reduced divisor of poles and zeros D' of F' is GNC. What we must show is that all critical points of F' on X' are off the exceptional locus hence actually critical points of F on X .

There are two types of new cones in $\Sigma_{X'}$. The first come from the triangulation step. By our construction, any such cone must contain a ray η_j in the support of D . This ray corresponding to the strict transform under π is in the support of $\sum u_i \pi^*(D_i)$, and its variable appears as a factor in F' . By part 1 of Theorem 5, F' has no critical points along the torus-invariant divisor Δ'_j , hence no critical points on any torus orbit contained in Δ'_j .

The second type of new cone comes from the desingularization step. These cones all contain at least one new ray η_E corresponding to an exceptional divisor Δ_E in X' . We will show there are no critical points on Δ_E . Equivalently we show that there are no critical points on each torus orbit contained in Δ_E .

Given a torus orbit let τ_E be the corresponding cone of Σ' containing η_E . There is some minimal cone τ of Σ_X containing τ_E . Let τ' be any cone of $\Sigma_{X'}$ containing τ_E that is maximal with respect to being contained in τ . Since τ is refined in $\Sigma_{X'}$ it must be a singular cone, and so by the hypothesis it has some generating ray in the support of $\sum u_i D_i$, or equivalently the linear function of this Cartier divisor is not identically zero on τ . The pullback keeps the same linear functional which cannot be zero on the subset τ' of τ . As a consequence, τ' contains a ray η_j in the support of $\sum u_i \pi^*(D_i)$. This means x_j^c appears as a factor in F' for some nonzero integer c .

If η_j is a generator of τ_E , then as above there are no critical points on Δ'_j and thus no critical points on the orbit corresponding to τ_E . Suppose on the other hand

that η_j is not a generator of τ_E . Let x_{E_1}, \dots, x_{E_k} be the variables corresponding to the generators of τ_E in an affine chart of X' that contains τ' . Note that the variable x_j corresponding to η_j is not among these variables. Because τ is the minimal cone of Σ_X containing τ_E , the face $P_i^{\tau_E}$ is contained in the face P_i^τ , and hence it is contained in the face $P_i^{\tau'}$. So $(F'_i)^{\tau_E}$, obtained by setting x_{E_1}, \dots, x_{E_k} to zero, does not contain any of the variables corresponding to τ' ; in particular it doesn't contain x_j . On the other hand, $F'_i = (F'_i)^{\tau_E} + G'_i$ where G'_i is in the ideal generated by x_{E_1}, \dots, x_{E_k} . Since x_j is not among the x_{E_i} we have

$$\left(\frac{\partial F'_i}{\partial x_j} \right) \Big|_{x_E=0} = \frac{\partial((F'_i)^{\tau_E})}{\partial x_j} = 0,$$

where $x_E = 0$ means $x_{E_1} = \dots = x_{E_k} = 0$. Hence we have

$$\left(\frac{\partial(\log F')}{\partial x_j} \right) \Big|_{x_E=0} = \frac{c}{x_j} + \sum_i \left(\frac{\partial F'_i}{\partial x_j} \right) \Big|_{x_E=0} \neq 0.$$

We conclude that F' has no critical points on the torus orbit corresponding to τ_E as desired.

Thus the toric ML degree of f on X' is the same as that on f . Since F' has no critical points on the exceptional locus and D is ample on X , $\pi^*(D)$ meets any curve off the exceptional locus and therefore the ML degree must be finite. It is computed in the Chow ring of X' as the coefficient of z^d in

$$\frac{\prod_{j \notin I} (1 - z\Delta'_j) \prod_k (1 - z\Delta_{E_k})}{\prod_{i=1}^n (1 - zD'_i)}.$$

Here Δ'_j are the strict transforms of the Δ_j not in the support of $\sum u_i D_i$. The Δ_{E_k} are those exceptional divisors which are not in the support of $\sum u_i \pi_i^*(D_i)$, and D'_i are the proper transforms of the divisor classes of the F_i .

We can now expand our Chern class product replacing $(1 - zD'_i)$ in the denominator by $1 + zD'_i + z^2(D'_i)^2 + \dots + z^d(D'_i)^d$ in the numerator. The intersection product of any collection of prime torus-invariant divisors is the cycle of the cone they span or 0 if there is no such cone. Hence the coefficient of z^d is the sum of all terms of the form

$$(-1)^k D'_{i_1} D'_{i_2} \cdots D'_{i_{d-c}} \cdot \tau^c.$$

Here $1 \leq i_1 \leq \dots \leq i_{d-c} \leq n$ and τ^c ranges over all dimension c cones of $\Sigma_{X'}$ spanned by rays not in the support of $\sum u_i D'_i$. This product is exactly the mixed volume $V(P_{i_1}, \dots, P_{i_{d-c}}; \tau^c)$.

To finish we note that if τ^c contains an exceptional divisor Δ_{E_k} , the minimal cone τ of Σ_X containing τ^c must have *strictly larger* dimension. This is because

τ^c does not have any rays in the support of $\sum u_i D'_i$ hence is not maximal in τ . As a consequence all of the faces $P_{ij}^{\tau^c}$ have a translate that lies in a subspace of dimension strictly less than c and the corresponding mixed volume is 0. In conclusion, the exceptional divisors do not contribute to the top Chern class product and the formula reduces to the stated one. \square

In two variables we recover a particularly simple formula:

COROLLARY 17. *Let f_1, \dots, f_n be generic Laurent polynomials in two variables (θ_1, θ_2) with Newton polygons P_1, \dots, P_n , and let $P = P_1 + \dots + P_n$ be their Minkowski sum. If the origin lies on none of the lines spanned by edges of P then the ML degree is equal to the area of P plus the areas of each of the P_i .*

Proof. This is a special case of Theorem 16 when no facets pass through the origin. Therefore the only term is

$$\sum_{i=1}^n \sum_{j=i}^n V(P_i, P_j).$$

The Euclidean area of each P_i is $V(P_i, P_i)/2$. The Euclidean area of the Minkowski sum $P = \sum P_i$ equals $\frac{1}{2} \sum_i V(P_i, P_i) + \sum_{i < j} V(P_i, P_j)$. The stated formula is the sum of these expressions. \square

Now that we are equipped with the volume formulas in Theorem 16 and Corollary 17, let us revisit the two-dimensional examples from Section 3.

Example 18. The Newton polygons P_1, \dots, P_n in Example 9 are axis-parallel parallelograms. The first term in the formula (15) is the area of their Minkowski sum $P_1 + \dots + P_n$, and the second term is the sum of the areas of the P_i , as in Corollary 17. The third and fourth term are the two correction terms stemming from the fact that the origin is a vertex of each Newton polytope. These terms disappear if we replace one f_i by $\theta_1 \theta_2 f_i$.

The number 14 in Example 10 can also be derived using Theorem 16. The three polygons in Figure 1 have areas $1, \frac{3}{2},$ and $\frac{1}{2}$ respectively. The area of their sum is 15. The two divisors corresponding to x_1 and x_2 pass through the origin and yield correction terms of 1 and 4 respectively. Finally we add back 1 for the vertex at the origin. Altogether $1 + \frac{3}{2} + \frac{1}{2} + 15 - 1 - 4 + 1 = 14$.

Our discussion so far indicates that we get the sharpest results when X is smooth and D is GNC. In the toric case these two hypotheses could essentially be removed as in Theorem 16. The GNC condition can also be relaxed for certain other cases as we saw in the previous section. In general, if the pair (X, D) does not satisfy the smoothness and GNC hypotheses then we must appeal to Hironaka’s theorem on resolution of singularities (see e.g. [18]). This furnishes a proper projective morphism $\pi: X' \rightarrow X$ such that X' is smooth and $\pi^{-1}(D)$ has GNC.

We need to compute the divisor $\pi^*(div(f))$ of the pullback of the function f . If D'_i is the proper transform of the divisor D_i and E_1, \dots, E_k are the exceptional divisors of π then

$$\pi^*(div(f)) = \sum_{i=1}^r u_i D'_i + \sum_{j=1}^k \mu_j E_j,$$

where μ_j are certain (possibly negative) integers. These integers are $\mu_j = \sum_{i=1}^r u_i m_{ij}$ where m_{ij} is the multiplicity of the full transform of D_i along E_j . The underlying reduced divisor is $D' := \sum_{i=1}^r D'_i + \sum_{j: \mu_j \neq 0} E_j$. The number of critical points is now gotten by applying Theorem 5 to (X', D') instead of (X, D) . This procedure can be very complicated in practice. We illustrate it with a simple example.

Example 19. Let $d = 2, n = 4, f_1 = x, f_2 = y, f_3 = (x - 1)^2 + (y - 1)^2 - 2,$ and $f_4 = (x + 1)^2 + 2(y - 2)^2 - 9$. The divisor D is not a GNC divisor since at the origin all the four curves defined by f_1, \dots, f_4 meet. In order to resolve this singularity we blow up $X = \mathbf{P}^2$ at $(0 : 0 : 1)$ to obtain X' which is smooth. We note that $c_{tot}(\Omega_{X'}^1)$ is $(1 - zE)(1 - zH)(1 - zH')^2$ where H is the proper transform of the generic hyperplane section, E is the exceptional divisor, and H' is the proper transform of a line through the origin.

We have four cases: we consider first the general case where $u_1 + u_2 + u_3 + u_4 \neq 0$ and $u_1 + u_2 + 2u_3 + 2u_4 \neq 0$. In this case D' consists of the proper transforms of the four original curves, the exceptional divisor, and the pullback of the line at infinity. After cancellations, we just need to compute the coefficient of z^2 in $\frac{1}{(1 - C_1z)(1 - C_2z)}$ where C_1 and C_2 are the irreducible divisors corresponding respectively to the circle and the ellipse. This coefficient is $C_1^2 + C_2^2 + C_1 \cdot C_2$. In X' the two curves intersect in three points, and their self-intersection also yields three points. Hence the ML degree is nine.

In the special case where $u_1 + u_2 + u_3 + u_4 = 0$, we need to compute the coefficient of z^2 in $\frac{1 - zE}{(1 - C_1z)(1 - C_2z)}$, which is $C_1^2 + C_2^2 + C_1 \cdot C_2 - E \cdot C_1 - E \cdot C_2$. Since $E \cdot C_i = 1$ the ML degree drops down to seven. If $u_1 + u_2 + 2u_3 + 2u_4 = 0$ then the coefficient of z^2 in $\frac{1 - zH}{(1 - C_1z)(1 - C_2z)}$ is $C_1^2 + C_2^2 + C_1 \cdot C_2 - H \cdot C_1 - H \cdot C_2$. Since $H \cdot C_i = 2$, the ML degree is five. Finally, if both $u_1 + u_2 + u_3 + u_4$ and $u_1 + u_2 + 2u_3 + 2u_4$ are zero, then the coefficient of z^2 in $\frac{(1 - zH)(1 - zE)}{(1 - C_1z)(1 - C_2z)}$ is $C_1^2 + C_2^2 + C_1 \cdot C_2 - H \cdot C_1 - H \cdot C_2 - E \cdot C_1 - E \cdot C_2 + H \cdot E$: since $H \cdot E = 0$, the ML degree further drops down to three.

The number of bounded regions of the complement of the four curves in \mathbf{R}^2 is seven. By Proposition 11, this is a lower bound on the ML degree when all u_i are positive. This example shows that, for specific *negative* values of the u_i 's, the number of critical points may be smaller than this lower bound.

6. ML degree and Euler characteristic. A well-known result in the theory of hyperplane arrangements [22] states that the number of bounded regions of a real arrangement equals the Euler characteristic of the complement of its com-

plexification. The Euler characteristic is $(-1)^{d+1}\mu(0)$ where $\mu(0)$ is the Möbius number of the intersection lattice, and, by Theorem 14, this is precisely the ML degree of the associated linear model. Here we extend this relationship between topology and the ML degree to statistical models which are given by nonlinear polynomials f_i . Working in the general setting of Section 2, we shall prove the following:

THEOREM 20. *Let X be a smooth complete algebraic variety over \mathbf{C} of dimension d , and let D be the reduced divisor associated to $f = f_1^{u_1} \cdots f_n^{u_n}$. Assume that the hypotheses (a), (b) and (c) below hold. Then the ML degree equals $(-1)^d e_{top}(X \setminus D)$ where e_{top} is the topological Euler characteristic.*

Invoking Hironaka’s theorem on resolution of singularities, we fix a blow up $\pi: \tilde{X} \rightarrow X$ such that \tilde{X} is smooth, and the rational function f pulls back to a proper morphism \tilde{f} onto $\mathbf{P}_{\mathbf{C}}^1$. The three hypotheses are as follows:

(a) The inverse image $D' := \pi^{-1}(D)$ of the divisor D can be written as $D' = \tilde{D} + D_H$ where \tilde{D} is the support of the divisor $\text{div}(\tilde{f})$ of the pullback \tilde{f} of the rational function f , while D_H is the horizontal divisor consisting of the sum of all components of D' which map onto $\mathbf{P}_{\mathbf{C}}^1$.

(b) The restriction of \tilde{f} to $D_H \setminus \tilde{D}$ is a topological fiber bundle over $\mathbf{C}^* = \mathbf{P}_{\mathbf{C}}^1 \setminus \{0, \infty\}$.

(c) The number of critical points of \tilde{f} on $\tilde{X} \setminus \tilde{D}$ is finite.

Remark 21. Hypothesis (a) is crucial and depends on the exponents u_i . For instance, consider the rational function $f = (y - x^2)(y + x^2)^{-1}$ on $X = \mathbf{P}_{\mathbf{C}}^2$. We get \tilde{X} by blowing up the origin twice. The exceptional curve of the first blow-up belongs to the fiber $\{\tilde{f} = 1\}$ and hence is not supported on $\text{div}(\tilde{f})$. Hypothesis (a) is not satisfied for this example. If we take instead $f = (y - x^2)(y + x^2)^{-2}$ then hypothesis (a) is satisfied because the exceptional curve belongs to the fiber $\{f = \infty\}$ and is hence supported on $\text{div}(\tilde{f})$,

Hypothesis (b) implies that the cohomology ranks of $D_H \setminus \tilde{D}$ can be computed from that of \mathbf{C}^* and the fibers using Künneth’s formula. The alternating sum of the ranks is zero for the fibers, and we get $e_{top}(D_H \setminus \tilde{D}) = 0$. In fact, the hypothesis (b) could be replaced by the more general condition $e_{top}(D_H \setminus \tilde{D}) = 0$.

Any proper map $f: X \rightarrow \mathbf{C}^*$ for X smooth is a topological fiber bundle if it is a submersion, i.e. $df \neq 0$ for all points in X . Therefore to check hypothesis (b), we need only find a controlled stratification (in the sense of Thom-Mather theory [19]) of D_H into locally closed smooth sets such that for all points on each strata S , $d(\tilde{f}|_S) \neq 0$. In particular, this last condition will imply that the critical points of \tilde{f} on $\tilde{X} \setminus \tilde{D}$ are the same as the critical points of f on $X \setminus D$.

Proof of Theorem 20. Our method follows closely the proof of the Lefschetz hyperplane section theorem (cf. [1, 2]). Moreover, since the complement is not necessarily compact we shall use Borel-Moore homology [4] (see also [6]). We note that for compact spaces X the ordinary homology groups coincide with the

Borel-Moore homology groups. In the Borel-Moore homology theory we have the following useful exact sequence to be used below: if X is locally compact, F is closed in X , and $U := X \setminus F$, we have

$$(18) \quad \dots \rightarrow H_i(F) \rightarrow H_i(X) \rightarrow H_i(U) \rightarrow \dots$$

Thus in this situation the Borel-Moore Euler characteristic is additive:

$$(19) \quad e_{BM}(X) = e_{BM}(F) + e_{BM}(U).$$

Finally, if U is an even-dimensional orientable manifold then Poincaré duality holds between Borel Moore homology and ordinary cohomology, and $e_{BM}(U)$ coincides with the topological Euler number $e_{top}(U)$. In our situation we get

$$e_{top}(X \setminus D) = e_{top}(\tilde{X} \setminus D') = e_{BM}(\tilde{X} \setminus D') = e_{BM}(\tilde{X} \setminus \tilde{D}) - e_{BM}(D' \setminus \tilde{D}).$$

The last equation follows from (19). Hypothesis (b) implies $e_{BM}(D' \setminus \tilde{D}) = 0$ (see Remark 21), and hence it suffices to show that the ML degree equals

$$e_{top}(X \setminus D) = e_{BM}(\tilde{X} \setminus \tilde{D}) = e_{top}(\tilde{X} \setminus \tilde{D}).$$

In other words, we may now simply erase the tilde and consider the case when X is smooth and f defines a proper morphism $X \setminus D \rightarrow \mathbf{C}^*$.

Let \mathcal{C} denote the set of critical points of f on X . By hypothesis (c), this set is finite and the ML degree equals its cardinality counting multiplicities:

$$\mu = \sum_{p \in \mathcal{C}} \mu_p.$$

The multiplicity μ_p of a critical point p of f is known as the *Milnor number* at p of the hypersurface $F_p = \{x \in X: f(x) = f(p)\}$. Milnor [20] showed that this algebraic invariant of a singularity has the following topological interpretation. Consider a coordinate chart around the point p and intersect the fiber $F_\epsilon := \{x \mid f(x) = f(p) + \epsilon\}$ with a ball of radius δ around p . For $\epsilon \ll \delta$ this intersection is the *Milnor fiber*. Milnor [20] showed that the Milnor fiber is homotopy equivalent to a bouquet of μ_p spheres of dimension $d - 1$.

Each singular fiber is obtained (cf. [3, 20]) from a smooth fiber by replacing the Milnor fiber by a contractible set. The Borel-Moore exact sequence (18) implies that the Euler number of a singular fiber F' is obtained from the Euler number of a smooth fiber F by adding $-(-1)^{d-1} \sum_{p \in F'} \mu_p$.

Then the Euler number of the union of the singular fibers equals $|\mathcal{C}|$ times the Euler number of a smooth fiber F plus the correction $-(-1)^{d-1} \mu$. Applying Künneth's formula to the fiber bundle defined by f on $X \setminus D$ minus the union of

the singular fibers, and then applying the additivity formula (19), we conclude that $e_{BM}(X \setminus D) = e_{top}(X \setminus D) = (-1)^d \mu$ as desired. \square

Example 22. For another illustration consider Example 19 with $X = \mathbf{P}_{\mathbf{C}}^2$. The generic ML degree was 9 but it decreased by 4 when $u = (u_1, u_2, u_3, u_4)$ is a general solution of $u_1 + u_2 + 2u_3 + 2u_4 = 0$. This is consistent with Theorem 20, because for such u the divisor D loses the component at infinity. The difference is a projective line minus 6 points, which has Euler number -4 . Consider our hypotheses when \tilde{X} is the blow-up of X at the origin. If $u_1 + u_2 + u_3 + u_4 \neq 0$ then the exceptional curve is part of \tilde{D} and Theorem 20 is valid. On other hand, if $u_1 + u_2 + u_3 + u_4 = 0$ then it maps to \mathbf{P}^1 under a rational map of degree ≥ 2 , so hypothesis (a) does not hold. The philosophy of this example is that, even if the divisor D is locally biholomorphic to an arrangement of hyperplanes, genericity of the exponents u_i may be necessary for the topological formula of Theorem 20 to hold.

Theorems 1, 8 and 16 offer combinatorial formulae for the ML degree and hence (using Theorem 20) for the Euler number of the complement of an arrangement of generic hypersurfaces $\{f_i = 0\}$ in $X = \mathbf{P}_{\mathbf{C}}^d$ or in a toric manifold. In each of these theorems, the combinatorial number becomes an upper bound for the ML degree when the coefficients of the f_i are special. This semi-continuity principle will be explained by the following general topological result, in which also the underlying manifold is allowed to vary.

THEOREM 23. *Assume we are given a one-parameter smooth proper family X_t of complex manifolds over the unit disk $B := \{t \in \mathbf{C} : |t| < 1\}$, and a family of rational functions f_t on X_t , such that:*

- (1) *for $t \neq 0$ the divisor $D(t)$ defined by f_t has GNC, and*
- (2) *for $t = 0$ the divisor $D(0)$ defined by f_0 has the same homology class as $D(t)$ for the natural differentiable trivialization of the family X_t .*

Then the ML degree of f_0 is less than or equal to the ML degree of f_t .

In order to understand the second hypothesis, let us recall Ehresmann’s Theorem [12]: any proper submersion $\phi: \mathcal{X} \rightarrow B$ of differentiable manifolds is a differentiable fiber bundle, i.e., if U is a sufficiently small open set in B , there is a local diffeomorphism between $\phi^{-1}(U)$ and $U \times F$, for a fiber F , and this diffeomorphism is compatible with the two projections to U .

Sketch of proof. Let $\mathcal{X} \rightarrow B$ be the total space of the family $\{X_t\}_{t \in B}$. We consider the function $\phi: \mathcal{X} \rightarrow B \times \mathbf{P}^1$, given by $\phi(x) = (t(x), f(x))$.

Consider the locus $\Xi \subset \mathcal{X}$ given by the vanishing of the vertical differential of ϕ : this is the local complete intersection defined by the d partials $\partial f / \partial x_i = 0$, where x_1, \dots, x_d are local coordinates on the fibers provided by the Implicit Function Theorem. At each critical point p of f_0 , the locus Ξ has dimension 1,

and thus, in a neighborhood of p , the morphism $\Xi \rightarrow B$ is finite, whence its degree is locally constant. This establishes the desired semi-continuity. \square

In general, it might be difficult to show that a given rational function f_0 has a perturbation as above, or we might want to calculate the ML degree with more algebraic precision. The results in the next section may help.

7. Logarithmic vector fields. In this section we will show that the formula for the ML degree given by the logarithmic Chern number (Theorem 5) holds in greater generality. Returning to the setting of Section 2, we consider logarithmic vector fields along D . Again, our definition differs slightly from the one given by Saito [25].

Definition 24. If D is a reduced divisor on a factorial variety X , the *sheaf of logarithmic vector fields* $\Theta_X(-\log D)$ is the dual $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1(\log D), \mathcal{O}_X)$.

Recall that the *tangent sheaf* Θ_X is $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$, the dual of the 1-forms on X . The inclusion of Ω_X^1 into $\Omega_X^1(\log D)$, studied in Lemma 3, dualizes to an inclusion of the logarithmic vector fields into the tangent sheaf.

PROPOSITION 25. *We have the following exact sequence of sheaves on X :*

$$0 \rightarrow \Theta_X(-\log D) \rightarrow \Theta_X \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{D_i}(D_i) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1(\log D), \mathcal{O}_X).$$

If X is smooth, then the rightmost homomorphism is onto, and the total Chern class of the sheaf of logarithmic vector fields equals

$$(20) \quad c_{tot}(\Theta_X(-\log D)) = \frac{c_{tot}(\Theta_X) \cdot c_{tot}(\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1(\log D), \mathcal{O}_X))}{\prod_{i=1}^r (1 + zD_i)}.$$

Proof. Dualizing the sequence $0 \rightarrow \mathcal{O}_X(-D_i) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{D_i} \rightarrow 0$, we get

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{D_i}, \mathcal{O}_X) = 0 \quad \text{and} \quad \mathcal{E}xt_{\mathcal{O}_X}^1(\mathcal{O}_{D_i}, \mathcal{O}_X) \cong \mathcal{O}_{D_i}(D_i).$$

Hence the $\mathcal{E}xt_{\mathcal{O}_X}$ -sequence gotten by dualizing the sequence (7) has the form

$$0 \rightarrow 0 \rightarrow \Theta_X(-\log D) \rightarrow \Theta_X \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{D_i}(D_i) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1(\log D), \mathcal{O}_X) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \rightarrow \dots$$

This is the first statement of Proposition 25. If X is smooth then the cotangent sheaf Ω_X^1 is free, and we have $\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) = 0$. The formula (20) follows from the multiplicativity of the total Chern class. \square

Remark 26. The two leftmost maps in the exact sequence of Proposition 25 characterize the logarithmic vector fields on X along D as those vector fields $\xi \in \Theta_X$ which satisfy $\xi(F_i) \equiv 0 \pmod{F_i}$ for all i . In other words, for each $i = 1, \dots, r$, the vector field $\xi = \sum_{j=1}^d \xi_j \partial/\partial x_j$ has the property that there exist functions ψ_i such that $\xi(F_i) := \sum_{j=1}^d \xi_j \partial F_i/\partial x_j = \psi_i F_i$.

Remark 27. An interesting case will be the one where X is smooth and the sheaf $\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1(\log D), \mathcal{O}_X)$ has zero-dimensional support and length p . Here the top Chern class of $\Theta_X(-\log D)$ and the top Chern class of $\Omega_X^1(\log D)$ differ by $(-1)^{d-1}p$, and the count in part 3 of Theorem 5 changes accordingly.

As in the proof of Theorem 5, set $\sigma := d\log(f)$ and let Z_σ be the subscheme of X defined by the vanishing of σ . The restriction of Z_σ to the open set $V = X \setminus D$ is the critical locus of f . The reason why the sheaf $\Theta_X(-\log D)$ is important is that it enters directly into the algebraic description of Z_σ .

Indeed, the section σ of $\Omega_X^1(\log D)$ corresponds to an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1(\log D) \rightarrow \mathcal{E} \rightarrow 0,$$

where \mathcal{O}_{Z_σ} is the kernel of $\mathcal{E}xt_{\mathcal{O}_X}^1(\mathcal{E}, \mathcal{O}_X) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_X^1(\log D), \mathcal{O}_X)$. Hence the $\mathcal{E}xt_{\mathcal{O}_X}$ -sequence gotten by dualizing the previous sequence has the form

$$(21) \quad 0 \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \rightarrow \Theta_X(-\log D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{Z_\sigma} \rightarrow 0.$$

This leads to the following more refined formula for the ML degree.

THEOREM 28. *Assume that X is smooth, D meets every curve, the sheaf $\Theta_X(-\log D)$ is locally free, and Z_σ does not intersect the divisor D . Then the number of critical points of f on $V = X \setminus D$ equals $(-1)^d c_d(\Theta_X(-\log D))$.*

Proof. Using (21), we can view Z_σ as the locus of zeros of the locally free sheaf dual to $\Theta_X(-\log D)$, whose top Chern class is $(-1)^d c_d(\Theta_X(-\log D))$. \square

Remark 29. The sequence (21) gives the following description of the equations defining the critical points. The ideal sheaf of the subscheme Z_σ is generated by the functions $\sum_{i=1}^r u_i \psi_i$, where ξ varies among the logarithmic vector fields and (ψ_1, \dots, ψ_r) is derived from ξ as in Remark 26. This holds because the section $\sigma = d\log(f)$ factors through the homomorphism $\mathcal{O}_X^r \rightarrow \Omega_X^1(\log D)$, thus the homomorphism dual to the section σ also factors through the dual homomorphism $\Theta_X(-\log D) \rightarrow \mathcal{O}_X^r, \xi \mapsto (\psi_1, \dots, \psi_r)$.

We next present two examples which illustrate the hypotheses of Theorem 28. Here X is a smooth surface (i.e. $d = 2$) with local coordinates x and y .

Example 30. Let $r = 2$ with $F_1 = x$ and $F_2 = x^v - y^u$. A vector field $\xi = a(x, y) \cdot \partial/\partial x + b(x, y) \cdot \partial/\partial y$ is logarithmic if and only if $a = x\psi_1$ and $b = (v/u)y\psi_1 + \lambda(x^v - y^u)$, for some function λ . Observe that $\psi_2 = v\psi_1 - y^{u-1}\lambda$. We conclude that the sheaf $\Theta_X(-\log D)$ is locally free of rank 2.

The origin does not belong to the subscheme Z_σ if there is a function $u_1\psi_1 + u_2\psi_2 = (u_1 + v \cdot u_2)\psi_1 - u_2y^{u-1}\lambda$, for some ψ_1 and λ , which does not vanish at the origin. This holds if and only if $u_1 + v \cdot u_2 \neq 0$.

Example 31. Let $r = 3$ with $F_1 = x, F_2 = y$ and $F_3 = x - y$. Logarithmic vector fields ξ have the form $x\psi_1 \cdot \partial/\partial x + y\psi_2 \cdot \partial/\partial y$ with $x\psi_1 - y\psi_2$ divisible by $x - y$. This implies $\psi_1 = \lambda + y \cdot \mu$, $\psi_2 = \lambda + x \cdot \mu$, and $\psi_3 = \lambda$, for any functions λ, μ . Thus $\Theta_X(-\log D)$ is locally free of rank two. The origin does not belong to the subscheme Z_σ if and only if $u_1 + u_2 + u_3 \neq 0$.

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