This article is dedicated to the memory of Boris Moisezon

CANONICAL SYMPLECTIC STRUCTURES AND DEFORMATIONS OF ALGEBRAIC SURFACES.

FABRIZIO CATANESE
UNIVERSITÄT BAYREUTH

Abstract. We show that a minimal surface of general type has a canonical symplectic structure (unique up to symplectomorphism) which is invariant for smooth deformation. Our main theorem is that the symplectomorphism type is also invariant for deformations which allow certain normal singularities, called Single Smoothing Singularities (and abbreviated as SSS), and moreover for deformations yielding $\mathbb{Q}$-Gorenstein smoothings of quotient singularities.

We use then the counterexamples of M.Manetti to the DEF = DIFF question (whether deformation type and diffeomorphism type coincide for algebraic surfaces) to show that these yield surfaces of general type which are not deformation equivalent but are canonically symplectomorphic.

1. Introduction

The present note takes up again and clarifies part of a previous preprint ([Cat02]), which contained some theorems and some conjecture.

The first Leitfaden was the sketch of a proposal: how to show that some examples, called 'abc' examples, yield, when the constants b and $a+c$ are fixed, diffeomorphic surfaces of general type.

This conjecture was later shown to be true in our joint paper with Wajnryb [CW04], thus exhibiting simply connected (minimal) surfaces of general type which are pairwise diffeomorphic but not deformation equivalent (in other words, they belong to different connected components of the moduli space).

The proposal was motivated by the fact that the first examples of diffeomorphic but not deformation equivalent surfaces, due to Marco Manetti ([Man01], were not simply connected, and one could suspect that their universal abelian covers (which are compact) could be deformation equivalent.

On the other hand, in [Cat02] it was shown that a minimal surface of general type S has a canonical symplectic structure (unique up to symplectomorphism) which is invariant for smooth deformation, and which is called 'canonical' because its cohomology class is exactly the class of the canonical divisor K_S.

The natural question was then whether two canonically symplectomorphic algebraic surfaces are deformation equivalent.

We take up here again the negative results of [Cat02] in this direction, which we had left aside for a while in the hope to be able to prove the stronger result that the 'abc' examples are canonically symplectomorphic.

Date: March 3, 2006.

We change however some proofs. First of all we give direct arguments, avoiding to resort to the theory developed by Auroux and Katzarkov ([A-K00]). Secondly, we prove a more general result (than in [Cat02]), which is needed in order to show that the Manetti examples are canonically symplectomorphic.

The examples of [Man01] are in fact pairs of surfaces of general type S, S' which are not deformation equivalent, but which admit respective degenerations to normal surfaces X, X' with singularities, which in turn are deformation equivalent to each other via an equisingular deformation.

Moreover, each degeneration yields a $\mathbb{Q}$-Gorenstein smoothing of the singularities.

Manetti used a result of Bonahon on the group of diffeomorphisms of lens spaces in order to show that the surfaces S, S' are diffeomorphic to each other.

Here, with a direct proof, more general results are shown to hold true: we admit first of all deformations with singular fibres which are normal and in the following class SSS

Definition 1.1. *A Single Smoothing Surface Singularity* (X_0, x_0) *is a normal surface singularity such that the smoothing locus*

$$\Sigma := \{t \in Def(X_0, x_0) | X_t \text{ is smooth}\}$$

is irreducible (cf. [L-W86], [K-SB88]*, and references therein for examples of S.S.S. singularities, such as complete intersections, some cusp and triangle singularities..).*

Our main technical result is the following

Theorem 1.2. *Let* $\mathcal{X} \subset \mathbb{P}^N \times \Delta$ *and* $\mathcal{X}' \subset \mathbb{P}^N \times \Delta'$ *be two flat families of normal surfaces over the disc of radius 2 in* $\mathbb{C}$.

Denote by $\pi : \mathcal{X} \to \Delta$ *and by* $\pi' : \mathcal{X}' \to \Delta$ *the respective projections and make the following assumptions on the respective fibres of* π, π'*:*

1) the central fibres X_0 *and* X'_0 *are surfaces with single smoothing singularities*

2) the other fibres X_t, X'_t, *for* $t, t' \neq 0$ *are smooth.*

Assume moreover that

3) the central fibres X_0 *and* X'_0 *are projectively equivalent to respective fibres (*$X_0 \cong Y_0$ *and* $X'_0 \cong Y_1$*) of an equisingular projective family* $\mathcal{Y} \subset \mathbb{P}^N \times \Delta$ *of surfaces.*

Set $X := X_1$, $X' := X'_1$*: then*

a) X *and* X' *are diffeomorphic*

b) if FS *denotes the symplectic form inherited from the Fubini-Study Kähler metric on* $\mathbb{P}^N$*, then the symplectic manifolds* (X, FS) *and* (X', FS) *are symplectomorphic.*

The same conclusion holds if hypothesis 1) is replaced by

1') The singularities of X_0*, resp.* X'_0*, are cyclic quotient singularities and the two flat families yield* $\mathbb{Q}$*-Gorenstein smoothings of them.*

An important step in the proof is furnished by the following

Theorem 1.3. *Let $X_0 \subset \mathbb{P}^N$ be a projective variety with isolated SS singularities. Then X can be approximated by symplectic submanifolds W_t of $\mathbb{P}^N$, which are diffeomorphic to the glueing of the 'exterior' of X_0 (the complement to the union B of suitable (Milnor) balls around the singular points) with the Milnor fibres, glued along the Milnor links.*

A corollary of the above theorems is the following

Theorem 1.4. *A minimal surface of general type S has a canonical symplectic structure, unique up to symplectomorphism, and stable by deformation, such that the class of the symplectic form is the class of the canonical sheaf $\omega_S = \Omega^2_S = \mathcal{O}_S(K_S)$. The same result holds for any projective smooth variety with ample canonical bundle.*

Remark 1.5. *Theorem 1.2 holds more generally for varieties of higher dimension with single smoothing isolated singularities.*

It also holds under the assumptions

i) $X_0 = X'_0$

ii) for each singular point x_0 of X_0, the two smoothings $\mathcal{X}, \mathcal{X}'$, correspond to paths in the same irreducible component of $Def(X_0, x_0)$.

As already mentioned, for the main application we need to apply Theorem 1.2 under hypothesis 1') above.

To this purpose we recall some known facts on the class of singularities given by the (cyclic) quotient singularities admitting a $\mathbb{Q}$-Gorenstein smoothing (cf. [Man01], Section 1, pages 34-35, or the original sources [K-SB88],[Man90],[L-W86]).

The simplest way to describe the singularities

$$\textbf{Cyclic quotient singularity } \frac{1}{dn^2}(1, dna-1) = A_{dn-1}/\mu_n$$

is to view them on the one side as quotients of $\mathbb{C}^2$ by a cyclic group of order dn^2 acting with the indicated characters $(1, dna-1)$, or on the other side as quotients of the rational double point A_{dn-1} of equation $uv - z^{dn} = 0$ by the action of the group μ_n of n-roots of unity acting in the following way:

$$\xi \in \mu_n \textbf{ acts by } : (u, v, z) \to (\xi u, \xi^{-1} v, \xi^a z).$$

This quotient action gives rise to a quotient family $\mathcal{X} \to \mathbb{C}^d$, where

$\mathcal{X} = \mathcal{Y}/\mu_n$, $\mathcal{Y}$ is the hypersurface in $\mathbb{C}^3 \times \mathbb{C}^d$ of equation $uv - z^{dn} = \Sigma_{k=0}^{d-1} t_k z^{kn}$ and the action of μ_n is extended trivially on the factor $\mathbb{C}^d$.

The heart of the construction is that $\mathcal{Y}$, being a hypersurface, is Gorenstein (this means that the canonical sheaf $\omega_{\mathcal{Y}}$ is invertible), whence such a quotient $\mathcal{X} = \mathcal{Y}/\mu_n$, by an action which is unramified in codimension 1, is (by definition) $\mathbb{Q}$-Gorenstein.

These smoothings were considered by Kollár and Shepherd Barron ([K-SB88], 3.7-3.8-3.9, cf. also [Man90]), who pointed out their relevance in the theory of compactifications of moduli spaces of surfaces, and showed that,conversely, any $\mathbb{Q}$-Gorenstein smoothing of a quotient singularity is induced by the above family (which has a smooth base, $\mathbb{C}^d$).

Riemennschneider ([Riem74]) had earlier shown that, for the cyclic quotient singularity $\frac{1}{4}(1,1)$, the basis of the semiuniversal deformation consists of two smooth components intersecting transversally, each one yielding a smoothing, but only one admitting a simultaneous resolution, and only the other yielding smoothings with $\mathbb{Q}$-Gorenstein total space.

Remark 1.6. *To conclude the above discussion, assume follows that hypothesis 1) in Theorem 1.2 is replaced by the weaker hypothesis 1'). Then we consider singularities having more than one smoothing component, but we consider only smoothings in a fixed irreducible smoothing component..*

As already mentioned, Manetti ([Man01]) produced the first examples of (an arbitrarily large number of, as was also achieved in [CW04]) surfaces of general type pairwise not deformation equivalent, but pairwise diffeomorphic.

His examples are based on a complicated but very ingenious construction of Abelian coverings of rational surfaces with group $(\mathbb{Z}/2)^m$, leading to families $\mathcal{X}$, $\mathcal{X}'$ as in Theorem 1.2, since the two families induce smoothings of cyclic quotient singularities which are $\mathbb{Q}$-Gorenstein. Whence follows right away

Theorem 1.7. *Manetti's surfaces (Section 6 in* [Man01]*) provide examples of surfaces of general type which are not deformation equivalent, but, endowed with their canonical symplectic structures, are symplectomorphic.*

2. Proof of the Theorems

Proof. **(of Theorem 1.2)**

Let us recall the well known Theorems of Ehresmann ([Ehr43]) and Moser ([Mos65])

Theorem 2.1. *(Ehresmann + Moser) Let $\pi : \mathcal{X} \to T$ be a proper submersion of differentiable manifolds with T connected, and assume that we have a differentiable 2-form ω on $\mathcal{X}$ with the property that*

(*) $\forall t \in T$ $\omega_t := \omega|_{X_t}$ yields a symplectic structure on X_t whose class in $H^2(X_t,\mathbb{R})$ is locally constant on T (e.g., if it lies on $H^2(X_t,\mathbb{Z})$).

Then the symplectic manifolds (X_t, ω_t) are all symplectomorphic.

Henceforth, applying the lemma to $T := \Delta - \{0\}$, and to the restrictions of the two given families $\mathcal{X}$, $\mathcal{X}'$, we can for both statements replace X by any X_t with $t \neq 0$ sufficiently small, and similarly replace X' by any $X'_{t'}$ with $t' \neq 0$.

In other words, assuming $X_0, X'_0 \subset \mathbb{P}^N$, we may assume that X and X' are very near to X_0, respectively X'_0.

Since the family $\mathcal{Y}$ is equisingular, to each singular point $x_0 \in Sing(X_0)$ corresponds a unique singular point $x'_0 \in Sing(X'_0)$(indeed X_0, X'_0 are homeomorphic by a homeomorphism carrying x_0 to x'_0).

For each $x_0 \in Sing(X_0)$, π induces a germ of holomorphic mapping F_{x_0} : $\Delta \to \mathcal{D}_{x_0} \subset Def(X_0, x_0)$, where $\mathcal{D}_{x_0}$ is the smoothing component; respectively, for $x'_0 \in Sing(X_0)$ the corresponding singular point, π' induces a germ $F'_{x'_0}$.

Let $\mathcal{Z}_{x_0} \subset \mathcal{D}_{x_0} \times \mathbb{P}^N$ be given by the restriction of the semiuniversal deformation of the germ (X_0, x_0), and, for each $t \in \Delta$, consider the corresponding singular point $y_0(t) \in Y_t$ (thus $y_0(0) = x_0, y_0(1) = x'_0$), and the semiuniversal deformation of the corresponding germ $(Y_t, y_0(t))$.

We obtain in this way a family of pairs of germs,

$$\mathcal{Y}'_0 \subset \mathcal{Z}_0 \subset \mathcal{D}_0 \times \mathbb{P}^N \to \mathcal{D}_0 \subset \Delta \times \mathbb{C}^m,$$

where $\mathcal{Y}'_0$ is the family of germs $(Y_t, y_0(t))$, induced by $\mathcal{Y}$.

Our assumption, that each $\mathcal{D}_{y_0(t)}$ is irreducible, implies immediately the irreducibility of $\mathcal{D}_0$.

For each $0 < \epsilon << 1, 0 < \eta << 1$ we consider the family of Milnor links

$$\mathcal{K}_{\epsilon,\eta} := \cup_t \mathcal{K}_{\epsilon,\eta}(t) := \cup_t [\mathcal{Z}_0 \cap (\{t\} \times B(0,\epsilon) \times S(y_0(t), \eta))]$$

where $B(0,\epsilon)$ is the ball of radius ϵ and centre the point $0 \in \mathcal{D}_{y_0(t)}$ corresponding to $(Y_t, y_0(t))$, while $S(y_0(t), \eta)$ is the sphere in $\mathbb{P}^N$ with centre $y_0(t)$ and radius η in the Fubini Study metric.

It is well known that, for $\eta << 1$ and $\epsilon << \eta$, the family $\mathcal{K}_{\epsilon,\eta} \to ((\Delta \times B(0,\epsilon)) \cap \mathcal{D}_0)$ is differentially trivial (either in the sense of stratified sets, cf. [Math70], or, as suffices to us, in the weaker sense that when we pull it back through a differentiable map $\Delta \to (B(0,\epsilon) \cap \mathcal{D}_{x_0})$ we get a differentiable product).

We use now, to prove statement a), a variant with boundary of Ehresmann's theorem

Lemma 2.2. *Let $\pi : \mathcal{M} \to T$ be a proper submersion of differentiable manifolds with boundary, such that T is a ball in $\mathbb{R}^n$, and assume that we are given a fixed trivialization ψ of a closed family $\mathcal{N} \to T$ of submanifolds with boundary. Then we can find a trivialization of $\pi : \mathcal{M} \to T$ which induces the given trivialization ψ.*

Proof. It suffices to take on $\mathcal{M}$ a Riemannian metric where the sections $\psi(p, T)$, for $p \in \mathcal{N}$, are orthogonal to the fibres of π. Then we use the customary proof of Ehresmann's theorem, integrating liftings orthogonal to the fibres of standard vector fields on T. □

Proof of a): we apply lemma 2.2 several times:

- i) first we apply it in order to thicken the trivialization of the Milnor links to a closed tubular neighbourhood in the family $\mathcal{Z}_0$,
- ii-a) then we apply it in order to get a compatible trivialization of the family of exteriors in Y_t of the balls $B(y_0(t), \eta/2)$
- ii-b) then we apply it to the restriction of the families $\mathcal{X} \to \Delta$, $\mathcal{X}' \to \Delta$, to a ball of radius δ where δ is so chosen that $F_{x_0}(\{t| \, |t| < \delta\}) \subset B(0, \epsilon/2)$ (resp. for F'_{x_0}), and to the exterior of the balls $B(x_0, \eta/2)$,resp. $B(x'_0, \eta/2)$, so that we get compatible trivializations of the exterior in $\mathcal{X}$ to the balls $B(x_0, \eta/2)$, resp. in $\mathcal{X}'$ to the balls $B(x'_0, \eta/2)$

- iii) we finally use our assumptions that the images of F'_{x_0}, resp. F_{x_0} land in $\mathcal{D}_0$ which is irreducible: it follows that there is a holomorphic mapping $G : \Delta \to \mathcal{D}_0$ whose image contains the two points $F_{x_0}(t_0)$, $F'_{x'_0}(t'_0)$ and is contained in $(\Delta \times B(0, \epsilon/2)) \cap \Sigma$ (Σ being as before the smoothing locus).

  We consider then the pull back to Δ under G of the family of closed Milnor fibres

  $$\mathcal{M}_{\epsilon,\eta} := \cup_t \mathcal{M}_{\epsilon,\eta}(t) := \cup_t [\mathcal{Z}_0 \cap (\{t\} \times B(0, \epsilon) \times \overline{B(y_0(t), \eta)})].$$

  To this family we apply again 2.2, in order to obtain a trivialization of the family of Milnor fibres which extends the given trivialization on the family of (closed) tubular neighbourhoods of the Milnor links.

We are now done, since we obtain the desired diffeomorphism between X and X' by glueing together (in the intersection with $B(x_0, \eta) - \overline{B(x_0, \eta/2)}$, resp. with $B(x'_0, \eta) - \overline{B(x'_0, \eta/2)}$) the two diffeomorphisms provided by the restrictions of the respective trivializations ii) (to the intersection of the complement to $\overline{B(x_0, \eta/2)}$, resp. $\overline{B(x'_0, \eta/2)}$) and iii) (to the intersection with $B(x_0, \eta)$, resp.$B(x'_0, \eta)$): they glue because they both extend the trivialization i).

Step I in the proof of b. We want to use the previous construction and considerations in order to construct a family of differentiable embeddings of the same differentiable manifold $V \cong X \cong X'$ into $\mathbb{P}^N$, which includes the embeddings X and X'. We shall later show in step II that we can manage that every fibre inherits a symplectic structure from the Fubini-Study form and we can then finally apply Moser's theorem.

First of all, observe that V is obtained from X by writing X as the glueing of two manifolds with boundary, namely the union M of the Milnor fibres, and the 'exterior' of X, i.e., the closure $\mathcal{E}$ of the complement $X \setminus B$ ($B =$ union of the Milnor balls $B(x_0, \eta)$), which both have as boundary the union $\mathcal{K}$ of the Milnor links.

We consider now the product $V \times [-1, 1]$ and we first map it to $\mathbb{P}^N \times [-1, 1]$ via the product of a piecewise differentiable map ψ and the second projection. Later on we shall approximate ψ by a differentiable map ϕ which is an embedding when restricted to each fibre.

The key idea to construct ψ is to make a little bit longer the neck around the Milnor link, and to use the trivialization of the family of Milnor links and of Milnor fibres.

We define ψ_s, for $-1 \leq s \leq -\frac{1}{2}$ on $\mathcal{E}$ by using a path $\tau(s)$ in $\mathcal{D}_{x_0}$ from the point t corresponding to X, and reaching the origin (X_0) for $s = -\frac{1}{2}$: thus $t := \tau(-1)$, and $0 := \tau(-\frac{1}{2})$. We map then the exterior $\mathcal{E}$ to the exterior of $X_{F_{x_0}(\tau(s))}$, we do a completely similar operation for $\frac{1}{2} \leq s \leq 1$ (thus, for $s = 1$ $\mathcal{E}$ maps to the exterior of X').

Instead, for $-\frac{1}{2} \leq s \leq \frac{1}{2}$, we use a path $\nu(s)$ in the parameter space for the family $\mathcal{Y}$, and map the exterior $\mathcal{E}$ to the exterior of $Y_{\nu(s)}$.

For the interiors, i.e., the Milnor fibres minus a collar $\mathcal{C}$, we send them to the Milnor fibres corresponding to a path $\sigma(s)$ in $\mathcal{D}_0$ connecting the points

$F_{x_0}(t), F'_{x'_0}(t')$ corresponding to X, respectively to X'. We require of course that the first projection of $\sigma(s)$ onto Δ equals $\nu(s)$ for $-\frac{1}{2} \leq s \leq \frac{1}{2}$, while for $-1 \leq s \leq -\frac{1}{2}$ the first coordinate of $\sigma(s)$ equals 0, similarly for $\frac{1}{2} \leq s \leq 1$ it equals 1.

Finally, we use the collar and the trivialization of the Milnor links to join the boundary of the Milnor fibres with the boundary of the exterior.

We have now, for each s, a map ψ_s which is a differentiable embedding in the exterior $\forall s$, while it is an embedding for $s = -1, +1$.

Without loss of generality we may assume that $N \geq 4$, else there is nothing to prove, and we indeed may assume $N \geq 5$ by choosing if $N = 4$ the standard embedding $\mathbb{P}^4 \to \mathbb{P}^5$.

We obtain the desired family of embeddings ϕ_t by applying the following variation of Whitney's embedding theorem

Lemma 2.3. *Let W be a differentiable manifold with boundary of dimension k, and let $\psi : W \to \mathbb{R}^n$ be a continuos map which is a differentiable embedding around the boundary ∂W.*

If $n \geq 2k+1$, then ψ can be approximated by an embedding ϕ which coincides with ψ in a neighbourhood of ∂W.

Proof of the Lemma First of all, by a standard convolution process, we can approximate ψ by a differentiable map $\varphi : W \to \mathbb{R}^n$ which coincides with ψ in a neighbourhood of ∂W.

After that, we construct a differentiable map $f : W \to \mathbb{R}^m$ which equals zero in a neighbourhood of ∂W, and is such that $\varphi \oplus f : W \to \mathbb{R}^n \oplus \mathbb{R}^m$ is an embedding.

After that, we construct ϕ composing $\varphi \oplus f$ via a sufficiently general projection $\mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n$ close to the first projection of the direct sum.

□

We are not yet finished with the proof of b), because we have that the pull back ω of the Fubini Study form is non degenerate on the exterior of the Milnor balls, and in the interior of the Milnor fibres, but it could be degenerate in a neighbourhood of the collar of the Milnor link $\mathcal{K}$.

The procedure to obtain nondegeneracy everywhere will be now given while proving Theorem 1.3.

Proof. **(of Theorem 1.3)** Without loss of generality, and to simplify our notation, assume that X_0 has only one singular point x_0, and let $B := B(x_0, \eta)$ be a Milnor ball around the singularity, let moreover $\mathcal{D} := \mathcal{D}_{x_0} \subset Def(X_0, x_0)$, and, for $t \in \mathcal{D} \cap B(0, \epsilon)$ we consider the Milnor fibre $\mathcal{M}_{\epsilon,\eta}(t)$, whereas we have the two Milnor links

$$\mathcal{K}_0 := X_0 \cap S(x_0, \eta) \text{ and } \mathcal{K}_t := \mathcal{Z}_t \cap S(x_0, \eta - \delta)$$

.

We can consider the Milnor collars $\mathcal{C}_0(\delta) := X_0 \cap (\overline{B(x_0, \eta)} \setminus B(x_0, \eta - \delta))$, and $\mathcal{C}_t(\delta) := \mathcal{Z}_t \cap (\overline{B(x_0, \eta)} \setminus B(x_0, \eta - \delta))$.

We restrict now t to vary in a holomorphic disc Δ mapping to a path through the origin and intersecting the complement of the smoothing locus Σ only at the origin.

Now, with this restriction, the Milnor collars fill up a complex submanifold of dimension $dim X_0 + 1 := n+1$, and we 'join the two Milnor links' by a differentiable embedding of the abstract Milnor collar (i.e., $\mathcal{C}_0(\delta)$ as a differentiable manifold maps in such a way that its boundary maps onto $\mathcal{K}_0 \cup \mathcal{K}_t$).

For $\epsilon << \delta$ the tangent spaces to the image of the abstract Milnor collar can be made very close to the tangent spaces of the Milnor collars $\mathcal{M}_{\epsilon,\eta}(t)$, and we can conclude the proof via the following well known lemma which lies at the heart of Donaldson's work ([Don96-2])

Lemma 2.4. *Let $W \subset \mathbb{P}^N$ be a differentiable submanifold of even dimension $2n$, and assume that the tangent space of W is close to be complex in the sense that for each tangent vector v to W there is another tangent vector v' such that*

$$Jv = v' + u, |u| < |v|.$$

Then the restriction to W of the Fubini Study Form ω_{FS} makes W a symplectic submanifold of $\mathbb{P}^N$.

Proof. Let A be the symplectic form on projective space, so that for each tangent vector v we have:

$|v|^2 = A(v, Jv) = A(v, v') + A(v, u)$.

Since $|A(v, u)| < |v|^2$, $A(v, v') \neq 0$ and A restricts to a nondegenerate form. □

$Q.E.D.$for Theorem 1.3

Remark 2.5. *By Moser's theorem the symplectic manifolds W_t are symplectomorphic to each other.*

Step II in the proof of b.

We apply the same method used as in the proof of Theorem 1.3, i.e., when we apply the 'relative' Whitney theorem to glue the Milnor fibres corresponding to points $\sigma(s)$, we choose $|\sigma(s)| < \epsilon$, $|F_{x_0}(t)| < \epsilon$, $|F'_{x'_0}(t')| < \epsilon$ and , using the compactness of the interval $[-\frac{1}{2}, +\frac{1}{2}]$, there exists ϵ so small such that the tangent spaces of the submanifolds W_s are close to be complex, hence the W_s are symplectic submanifolds.

Then Moser's theorem gives the symplectomorphism of X with X'.

$Q.E.D.$for Theorem 1.2

Proof. **(of Theorem 1.4)** Let S be the minimal model of a surface of general type.

The assertion is rather clear in the case where we have more generally a smooth projective variety V whose canonical divisor K_V is ample.

In fact, let m be such that mK_V is very ample (any $m \geq 4$ does by Bombieri's theorem, cf. [Bom73] in the case of surfaces, for higher dimension we can use Matsusaka's big theorem, cf. [Siu93] for an effective version) thus the m-th pluricanonical map $\phi_m := \phi_{|mK_V|}$ is an embedding of V in a projective space $\mathbb{P}^{P_m - 1}$, where $P_m := dim H^0(\mathcal{O}_S(mK_V))$.

We define then ω_m as follows: $\omega_m := \frac{1}{m}\phi_m^*(FS)$ (where FS is the Fubini-Study form $\frac{1}{2\pi i}\partial\overline{\partial} log|z|^2$), whence ω_m yields a symplectic form as desired.

One needs to show that the symplectomorphism class of (V, ω_m) is independent of m. To this purpose, suppose that n is also such that ϕ_n yields an embedding of V: the same holds also for nm, whence it suffices to show that (V, ω_m) and (V, ω_{mn}) are symplectomorphic.

To this purpose we use first the well known and easy fact that the pull back of the Fubini-Study form under the n-th Veronese embedding v_n equals the n-th multiple of the Fubini-Study form. Second, since $v_n \circ \phi_m$ is a linear projection of ϕ_{nm}, by Moser's Theorem follows the desired symplectomorphism.

Assume now that S is a minimal surface of general type and that K_S is not ample: then for any $m \geq 5$ (by Bombieri's cited theorem) ϕ_m yields an embedding of the canonical model X of S, which is obtained by contracting the finite number of smooth rational curves with selfintersection number $= -2$ to a finite number of Rational Double Point singularities. For these, the base of the semiuniversal deformation is smooth and yields a smoothing of the singularity.

By Tjurina's theorem (cf. [Tju70]), S is diffeomorphic to any smoothing S' of X: however we have to be careful because there are many examples (cf. e.g. [Cat89]) where X does not admit any global smoothing.

Since however there are local smoothings, Tjurina's theorem tells us that S is obtained glueing the exterior $X \setminus B$ (B being the union of Milnor balls of radius η around the singular points of X) together with the respective Milnor fibres.

Argueing as in theorem 1.3 we find a symplectic submanifold W of projective space which is diffeomorphic to S, and by the previous remark W is unique up to symplectomorphism. Clearly moreover, if X admits a smoothing, we can then take S' sufficiently close to X as our approximation W. Then S' is a surface with ample canonical bundle, and, as we have seen, the symplectic structure induced by (a submultiple of) the Fubini Study form is the canonical symplectic structure.

The stability by deformation is again a consequence of Moser's theorem.

□

Proof. **(of Theorem 1.7)** In [Man01] Manetti constructs examples of surfaces S, S' of general type which are not deformation equivalent, yet with the property that there are flat families of normal surfaces $\mathcal{X} \subset \mathbb{P}^N \times \Delta$ and $\mathcal{X}' \subset \mathbb{P}^N \times \Delta'$

1) yielding a $\mathbb{Q}$-Gorenstein smoothings of the central fibres X_0, X'_0,

and such that

2) the fibres X_t, X'_t, for $t, t' \neq 0$ are smooth, and the canonical divisor of each fibre is ample

3) there are t_0, t'_0 with $S \cong X_{t_0}$, $S' \cong X'_{t'_0}$

4) there exists an equisingular family $\mathcal{Y} \to \Delta$ whose fibres have indeed only singularities of type $\frac{1}{4}(1,1)$, and such that $Y_0 \cong X_0$, $Y_1 \cong X'_0$.

There exists therefore a positive integer m such that for each X_t and $X'_{t'}$ the m-th multiple of the canonical (Weil-)divisor is Cartier and very ample, and therefore the relative m-pluricanonical maps yield three new projective families to which 1.2 applies.

By 1.2 and 1.3 it follows that S and S', endowed with their canonical symplectic structure, are symplectomorphic.

□

Acknowledgements.

I would like to thank the referee of [Cat03] for prompting me to write down this note, and Marco Manetti for observing that the result proven in [Cat02] was not sufficient for the application to Theorem 1.7, since his surfaces do not admit a direct degeneration to the same normal surface.

The research of the author was performed in the realm of the SCHWERPUNKT "Globale Methode in der komplexen Geometrie", and of the EAGER EEC Project.

Thanks to the IHES for hospitality in the month of march 2006, where the final version of the article was written; the visit was supported by contract nr RITA-CT-2004-505493 project acronym: IHES Euro-Programme 2.

Notes. a) Marco Manetti informed me that he was aware of a result similar to part a) of Theorem 1.2.

b) similar ideas to the ones of the present paper, but with different proofs and especially with different applications appear in [STY02] and [ST03].

References

[A-K00] D. Auroux, L. Katzarkov , *Branched coverings of* $\mathbb{CP}^2$ *and invariants of symplectic 4-manifolds* **Inv. Math. 142** (2000), 631-673.

[Aur02] D. Auroux, *Fiber sums of genus 2 Lefschetz fibrations.* **Turkish J. Math. 27, no. 1** (2003), 1-10 .

[ADK03] D. Auroux, S. Donaldson, L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves.* **Math. Ann. 326 no. 1** (2003), 185-203.

[ADKY04] D. Auroux, S. Donaldson, L. Katzarkov and M. Yotov, *Fundamental groups of complements of plane curves and symplectic invariants* **Topology 43, no. 6** (2004), 1285-1318 .

[Bom73] E. Bombieri, *Canonical models of surfaces of general type*, **Publ. Math. I.H.E.S., 42** (1973), 173-219.

[Cat84] F. Catanese, *On the Moduli Spaces of Surfaces of General Type*, **J. Diff. Geom 19** (1984) 483–515.

[Cat87] F. Catanese, *Automorphisms of Rational Double Points and Moduli Spaces of Surfaces of General Type*, **Comp. Math. 61** (1987) 81-102.

[Cat86] F. Catanese, *Connected Components of Moduli Spaces*, **J. Diff. Geom 24** (1986) 395-399.

[Cat89] F. Catanese, *Everywhere non reduced Moduli Spaces* , **Inv. Math. 98**, 293-310 (1989).

[Cat03] F. Catanese, *Moduli Spaces of Surfaces and Real Structures*, **Ann. Math.(2) 158, no.2** (2003),577-592.

[Cat02] F. Catanese, *Symplectic structures of algebraic surfaces and deformation*, 14 pages, math.AG/0207254.

[CW04] F. Catanese, B. Wajnryb, *Diffeomorphism of simply connected algebraic surfaces.* **math.AG/0405299** ,(2004).

[Don92] S.K. Donaldson, *Gauge theory and four-manifold topology.* [CA] Joseph, A. (ed.) et al., First European congress of mathematics (ECM), Paris, France, July 6-10, 1992. Volume I: Invited lectures (Part 1). **Basel: Birkhäuser, Prog. Math. 119** (1994) 121-151 .

[Don96] S.K. Donaldson, *The Seiberg-Witten Equations and 4-manifold topology.* **Bull. Am. Math. Soc., (N S) 33, 1** (1996) 45-70.

[Don96-2] S.K. Donaldson, *Symplectic submanifolds and almost-complex geometry.* **J. Diff. Geom. 44, 4** (1996) 666-705.

[D-K90] S.K. Donaldson,P.B. Kronheimer *The geometry of four-manifolds.* **Oxford Mathematical Monographs. Oxford: Clarendon Press,** (1990) ix, 440 p..

[Ehr43] C. Ehresmann, *Sur les espaces fibrés différentiables.* **C.R. Acad. Sci. Paris 224** (1947), 1611-1612.

[F-M88] R. Friedman and J.W.Morgan, *Algebraic surfaces and four-manifolds : some conjectures and speculations.*, **Bull. Amer. Math.Soc. 18** (1988) 1-19.

[F-M97] R. Friedman and J.W.Morgan, *Algebraic surfaces and Seiberg-Witten invariants.* **J. Algebr. Geom. 6, No.3** (1997) 445-479.

[KK02] V. M. Kharlamov, V.S. Kulikov, *On real structures of real surfaces.* Izv. Ross. Akad. Nauk Ser. Mat. **66**, no. 1, 133–152 (2002); translation in Izv. Math. **66**, no. 1, 133–150 (2002)

[K-SB88] J. Kollar and N.I. Shepherd Barron, *Threefolds and deformations of surface singularities.* **Inv. Math. 91** (1988) 299-338.

[L-W86] E. Looijenga and J. Wahl, *Quadratic functions and smoothing surface singularities* **Topology 25** (1986), 261-291.

[Man90] M. Manetti, $\mathbb{Q}$-*Gorenstein smoothings of quotient singularities*, **Preprint Scuola Normale Pisa** (1990).

[Man94] M. Manetti, *On some Components of the Moduli Space of Surfaces of General Type*, **Comp. Math. 92** (1994) 285-297.

[Man96] M. Manetti, *Degenerations of Algebraic Surfaces and Applications to Moduli Problems*, **Tesi di Perfezionamento Scuola Normale Pisa** (1996) 1-142.

[Man97] M. Manetti, *Iterated Double Covers and Connected Components of Moduli Spaces*, **Topology 36, 3** (1997) 745-764.

[Man01] M. Manetti, *On the Moduli Space of diffeomorphic algebraic surfaces*, **Inv. Math. 143** (2001), 29-76.

[Math70] J. Mather, *Notes on topological stability* , **Harvard University Math. Notes** (1970).

[Moi94] B. Moisezon, *The arithmetics of braids and a statement of Chisini*, in 'Geometric Topology, Haifa 1992' , **Contemp. Math. 164, A.M.S.** (1994), 151-175.

[Mor96] J. W. Morgan,*The Seiberg-Witten equations and applications to the topology of smooth four-manifolds.* **Mathematical Notes 44. Princeton Univ. Press** vi (1996).

[Mos65] J. Moser, *On the volume elements on a manifold.* **Trans. Amer. Math. Soc. 120** (1965), 286-294.

[Riem74] O. Riemennschneider, *Deformationen von Quotientensingularitäten (nach zyklischen Gruppen)*, **Math. Ann. 209** (1974), 211-248.

[Siu93] Y.T. Siu, *An effective Matsusaka big theorem*, **Ann. Inst. Fourier 43** (1993), 1387-1405.

[STY02] I. Smith, R. Thomas, S.T. Yau, *Symplectic conifold transitions* **J. Diff. Geom. 62** (2002), 209-242.

[ST03] I. Smith, R. Thomas, *Symplectic surgeries from singularities* **Turkish J. Math. 27** (2003), 231-250.

[Tju70] G.N. Tjurina, *Resolution of singularities of plane (= flat) deformations of double rational points* , **Funk.Anal. i Prilozen 4** (1970), 77-83.

[Wit94] E. Witten, *Monopoles and Four-Manifolds*, **Math. Res. Lett. 1** (1994) 809–822.

Author's address:

Prof. Fabrizio Catanese
Lehrstuhl Mathematik VIII
Universität Bayreuth, NWII
D-95440 Bayreuth, Germany

e-mail: Fabrizio.Catanese@uni-bayreuth.de