

Real Singular Del Pezzo Surfaces and 3-folds Fibred by Rational Curves, I

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0. Introduction

In complex algebraic geometry there is an established principle that the Kodaira dimension of a smooth complex projective variety W of dimension n strongly influences the topology of the set $W(\mathbb{C})$ of its complex points. This principle is clearly manifest already in dimension 1, and related to other points of view, as the uniformization theorem and the concept of curvature. This principle, although in a more difficult and complicated way, still goes on to hold in higher dimensions.

Let us assume now that W is a smooth real projective variety and let us consider the topology of the set $W(\mathbb{R})$ of its real points. In dimension 1, the connected components are just diffeomorphic to the circle S^1 , and their number is not dictated by the genus (there is only the Harnack inequality, which gives $g + 1$ as an upper bound for the number of connected components of $W(\mathbb{R})$).

So, there had been for some time the belief that the Kodaira dimension of W would not at all affect the topology of a connected component N of $W(\mathbb{R})$. This belief is contradicted already by the example of real algebraic surfaces of nonpositive Kodaira dimension (see e.g. [C; DIK; Ko5; Si]).

In a very interesting series of papers [Ko1; Ko2; Ko3; Ko4], Kollár used the recent progress on the minimal model program for 3-folds in order to understand the topology of the connected components $N \subset W(\mathbb{R})$, especially in the case where W has Kodaira dimension $-\infty$.

Our paper takes the origin from some questions that Kollár set in [Ko3], and we prove some optimal estimates that Kollár conjectured to hold.

This paper is mainly devoted to the proof of the following theorem.

THEOREM 0.1. *Let X be a projective surface defined over \mathbb{R} . Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of the topological normalization $\bar{X}(\mathbb{R})$ contains at most four Du Val singular points that are either not of type A^- or of type A^- but globally separating.*

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Applying this result to rational curve fibrations over rational surfaces, we obtain the answer to two of Kollár's three questions [Ko3, Rem. 1.2]. In a second paper, with slightly different methods, we plan to answer also the third question.

Let us now explain these applications in more detail.

Let $f: W \rightarrow X$ be a real smooth projective 3-fold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable. Then, by [Ko3, Thm. 1.1], a connected component $N \subset W(\mathbb{R})$ is a Seifert fibred manifold or a connected sum of lens spaces, or is obtained from one of these by taking connected sums with a finite number of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number of copies of $S^1 \times S^2$. Note that in [HM1] and [HM2] it was shown that all the manifolds N as just described do indeed occur.

Note also that the connected sum $N_1 \# N_2$ is taken in the category of oriented manifolds, where in general $N_1 \# N_2$ is not homeomorphic to $N_1 \# -N_2$. But for the particular choice $N_2 = \mathbb{P}^3(\mathbb{R})$ or $N_2 = S^1 \times S^2$, the connected sums $N_1 \# N_2$ and $N_1 \# -N_2$ are diffeomorphic (see e.g. [He]).

Take a decomposition $N = N' \#^a \mathbb{P}^3(\mathbb{R}) \#^b (S^1 \times S^2)$ with $a + b$ maximal and observe that this decomposition is unique by a theorem of Milnor [Mi]. We shall focus our attention on the integer $k := k(N)$ defined as follows:

- (a) if $g: N' \rightarrow F$ is a Seifert fibration, k denotes the number of multiple fibres of g ;
- (b) if N' is a connected sum of lens spaces, k denotes the number of lens spaces.

Observe that when N' is a connected sum of lens spaces, the number k is well-defined (again by Milnor's theorem).

We can then apply the result of Theorem 0.1 concerning singular rational surfaces in order to answer one question of Kollár [Ko3, Rem. 1.2(1)].

COROLLARY 0.2. *Let $W \rightarrow X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X . Suppose that $W(\mathbb{R})$ is orientable. Then for each connected component $N \subset W(\mathbb{R})$, $k(N) \leq 4$.*

Note that Kollár showed in [Ko3] the optimality of this estimate in case (a).

The following theorem answers another question of Kollár [Ko3, Rem. 1.2(3)].

THEOREM 0.3. *Let W be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X . Suppose that the fibration is defined over \mathbb{R} and that $W(\mathbb{R})$ is orientable. Let $N \subset W(\mathbb{R})$ be a connected component that admits a Seifert fibration $g: N \rightarrow S^1 \times S^1$. Then g has no multiple fibres. Furthermore, X is then rational over \mathbb{R} and $W(\mathbb{R})$ is connected.*

Section 1 is devoted to recalling the basic notions that come into play, especially the local and global separation properties of Du Val singularities of types A_μ . Hence the basic invariants m_i of a Du Val surface are defined.

Section 2 is the heart of the paper and contains a detailed description of the topological normalization of a real Du Val–Del Pezzo surface with more than four singular points. The description is based on the classical representation of the quadric cone on the plane that transforms the hyperplane sections of the cone into parabolas whose axes have a given direction.

Section 3 proves Corollary 0.2, and Section 4 is devoted to the proof of Theorem 0.3.

1. Real Du Val Surfaces

The aim of this section is to reduce the proof of Theorem 0.1 to the study of a certain kind of rational surface. The first part is close to the treatment in [Ko3, Sec. 9].

On a surface, a rational double point is called a *Du Val singularity*. Over \mathbb{C} , these singularities are classified by their Dynkin diagrams, namely A_μ ($\mu \geq 1$), D_μ ($\mu \geq 4$), E_6 , E_7 , and E_8 .

Over \mathbb{R} , there are more possibilities. In particular, a surface singularity will be said to be of type A_μ^+ if it is real-analytically equivalent to

$$x^2 + y^2 - z^{\mu+1} = 0, \quad \mu \geq 1,$$

and of type A_μ^- if it is real-analytically equivalent to

$$x^2 - y^2 - z^{\mu+1} = 0, \quad \mu \geq 1.$$

The type A_1^+ is real-analytically isomorphic to A_1^- ; otherwise, singularities with different names are not isomorphic. For μ odd, there is another real form of A_μ given by $x^2 + y^2 + z^{\mu+1} = 0$. We exclude this type of singular point because an isolated real point gives rise to \emptyset on the minimal resolution.

DEFINITION 1.1. Let X be a projective surface. The surface X is called a *Du Val surface* if X has only rational double points as singularities.

We want to use a suitable minimal model for X . In the minimal model program for real Du Val surfaces, the most useful statement for our purpose is the following description of the extremal contractions.

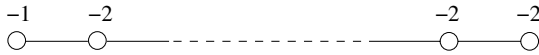
THEOREM 1.2 [Ko3, Thm. 9.6]. *Let X be a real Du Val surface, let $\overline{NE}(X)$ be its cone of curves, and let $R \subset \overline{NE}(X)$ be a K_X -negative extremal ray. Then R can be contracted. Furthermore, if $c: X \rightarrow Y$ is the contraction, c is one of the following:*

- Y is a Du Val surface, c is birational, and $\rho(Y) = \rho(X) - 1$;
- Y is a smooth curve, $\rho(X) = 2$, and $c: X \rightarrow Y$ is a conic bundle;
- Y is a point, $\rho(X) = 1$, and X is a Du Val–Del Pezzo surface (i.e., $-K_X$ is ample).

To apply the minimal model program for our purposes, we need to understand the behavior of c when c is a birational contraction. We begin with a typical example.

EXAMPLE 1.3. Let Y be a real Du Val surface, $x \in Y$ a smooth real point, and $\mu > 0$ an integer. Blow up Y at x , and denote by E_0 the exceptional curve of the blow-up $Y_0 \rightarrow Y$. Then take repeatedly the blow-up $Y_{l+1} \rightarrow Y_l$ at a general point on the exceptional curve E_l for $l = 0, 1, \dots, \mu - 1$. The exceptional divisor of the

composition of blow-ups $Y_\mu \rightarrow Y$ is a chain of rational curves whose configuration is of the following form.



Contracting the (-2) -curves $E_{\mu-l}, l = 0, 1, \dots, \mu - 1$, we get a surface X with a singularity of type A_μ^- .

The interesting fact is that the birational contractions of Theorem 1.2 involve only this kind of construction (see [Ko3, Thm. 9.6]).

As we shall see in Section 3, bounding the number of certain singularities on $X(\mathbb{R})$ yields a bigger upper bound for $k(N)$ than the one stated in Corollary 0.2. In order to obtain this finer estimate we have to bound this number on each component of the topological normalization $\overline{X(\mathbb{R})}$ of $X(\mathbb{R})$, which we shall now define.

DEFINITION 1.4. Let V be a simplicial complex with only a finite number of points $x \in V$ where V is not a manifold. Define the *topological normalization*

$$\bar{n}: \bar{V} \rightarrow V$$

as the unique proper continuous map such that \bar{n} is a homeomorphism over the set of points where V is a manifold and otherwise $\bar{n}^{-1}(x)$ is in one-to-one correspondence with the connected components of a good punctured neighborhood of x in V .

Observe that if V is pure of dimension 2, then \bar{V} is a topological manifold. Indeed, each point of \bar{V} has a neighborhood that is a cone over S^1 .

DEFINITION 1.5. Let X be a real algebraic surface with isolated singularities, and let $x \in X(\mathbb{R})$ be a singular point of type A_μ^\pm with μ odd. The topological normalization $\overline{X(\mathbb{R})}$ has two connected components locally near x . We will say that x is *globally separating* if these two local components are on different connected components of $\overline{X(\mathbb{R})}$ and *globally nonseparating* otherwise.

One can produce an arbitrarily high number of singular points of type A_μ^- by the construction of Example 1.3, but these singular points are globally nonseparating. Indeed, when μ is even, the singular point is in fact locally nonseparating, and when μ is odd, then the inverse image of the last $S^1 = E_\mu(\mathbb{R})$ yields a segment in $\overline{X(\mathbb{R})}$ connecting the two points. The key point for the sequel is Lemma 1.7.

DEFINITION 1.6. Let X be a real Du Val surface, and let

$$\mathcal{P}_X := \text{Sing } X \setminus \{x \text{ of type } A_\mu^-, \mu \text{ even}\} \setminus \{x \text{ of type } A_\mu^-, \mu \text{ odd, and } x \text{ globally nonseparating}\}. \quad (1)$$

LEMMA 1.7 [Ko3, Cor. 9.7]. *Let X be a real Du Val surface, let $\bar{n}: \overline{X(\mathbb{R})} \rightarrow X(\mathbb{R})$ be the topological normalization, and let M_1, M_2, \dots, M_r be the connected components of $\overline{X(\mathbb{R})}$. The unordered sequence of numbers $m_i := \#(\bar{n}^{-1}(\mathcal{P}_X) \cap M_i)$, $i = 1, 2, \dots, r$, is an invariant of extremal birational contractions of Du Val surfaces.*

By Theorem 1.2 and Lemma 1.7, it suffices to prove Theorem 0.1 in the case when X is a conic bundle or a Del Pezzo surface with $\rho(X) = 1$. Conic bundles were analyzed in [Ko3, Sec. 9]. The remaining case is when X is a Del Pezzo surface. We still slightly reduce the problem to the case where X is a degree-1 Del Pezzo surface.

LEMMA 1.8. *Let X be a real Du Val–Del Pezzo surface possessing a smooth real point and having $\rho(X) = 1$. Then there exists a blow-up of X in smooth points yielding Z , which is a conic bundle if $\deg X \geq 3$. Otherwise, we get Z a singular Del Pezzo surface of degree 1 with $\rho(Z) \leq 2$.*

Proof. Set $d := \deg X$. If $d \geq 3$, blow up $d - 3$ smooth points until you get a real cubic surface Z . The surface Z contains a real rational line L . We get $L \subset Z \subset \mathbb{P}^3$, and $\pi_L: \mathbb{P}^3 - L \rightarrow \mathbb{P}^2$ is a morphism and yields a real conic bundle. If $d = 2$, blow up a smooth real point: $\rho(X)$ increases by 1. □

2. Singular Del Pezzo Surfaces of Degree 1

Recall that a Del Pezzo surface X is by definition a surface whose anticanonical divisor is ample. We add the adjective “Du Val” to emphasize that we allow X to have Du Val singularities (observe that for a Du Val surface, the canonical divisor is a Cartier divisor). Let X be a real Du Val–Del Pezzo surface and let $p: S \rightarrow X$ be the minimal resolution of singularities. The smooth surface S has nef anticanonical divisor $-K_S = p^*(-K_X)$ and is called a *weak Del Pezzo surface* by many authors. As we saw in Section 1, we can assume the Del Pezzo surface X to have degree 1 by blowing up a finite number of pairs of conjugate imaginary smooth points and some real smooth point (there are several choices to do this); see Lemma 1.8. The anticanonical model of a Del Pezzo surface X of degree 1 is a ramified double covering $q: X \rightarrow Q$ of a quadric cone $Q \subset \mathbb{P}^3$ whose branch locus is the union of the vertex of the cone and a cubic section B not passing through the vertex (see e.g. [De, Exp. V]).

Note that the pull-back by q of the vertex of the cone is a smooth point of X and let X' be the singular elliptic surface obtained from X by blowing up this smooth point. We denote by $\bar{n}: \overline{X'(\mathbb{R})} \rightarrow X'(\mathbb{R})$ the topological normalization of the real part.

We shall now make a series of considerations that will later lead to a proof of the following.

PROPOSITION 2.1. *For each connected component $M \subset \overline{X'(\mathbb{R})}$,*

$$\#(\bar{n}^{-1}(\mathcal{P}_{X'}) \cap M) \leq 4.$$

Recall that Hirzebruch surfaces are the \mathbb{P}^1 -bundles over \mathbb{P}^1 . The surface X' is a ramified double covering of the Hirzebruch surface \mathbb{F}_2 whose branch curve is the union of the unique section of negative self-intersection, the section at infinity Σ_∞ , and a trisection B of the ruling $\mathbb{F}_2 \rightarrow \mathbb{P}^1$, which is disjoint from Σ_∞ .

The cone Q is the weighted projective plane $\mathbb{P}(1, 1, 2)$ whose coordinates are (x_0, x_1, y_2) , and X is the hypersurface in $\mathbb{P}(1, 1, 2, 3)$ with coordinates (x_0, x_1, y_2, z) defined by

$$z^2 = y_2^3 + p_4(x_0, x_1)y_2 + q_6(x_0, x_1).$$

We want to explain here the plane model of Q in which the hyperplane sections of Q embedded in \mathbb{P}^3 by $H^0(\mathcal{O}_Q(2))$ correspond to parabolas tangent to the line at infinity $L_\infty = \{w = 0\}$ at the point $O := \{w = x = 0\}$ of the projective plane with coordinates (x, y, w) . In other words, blow up O and then the infinitely near point O' to O corresponding to the tangent of the line at infinity L_∞ , and denote by \tilde{Q} the resulting surface. Denote by E, E' the respective total transforms of O, O' , and observe that $E = E' + E''$, where E'' is a (-2) -curve. The linear system $H^0(\mathcal{O}_{\tilde{Q}}(2H - E - E'))$ maps \tilde{Q} birationally onto the quadric cone $Q \subset \mathbb{P}^3$, contracting the proper transform \tilde{L}_∞ of L_∞ and E'' to points. Since \tilde{L}_∞ and E'' do not meet, first contracting \tilde{L}_∞ yields the Hirzebruch surface \mathbb{F}_2 , whose (-2) -section Σ_∞ is the image of the curve E'' . Let us write everything using the coordinates (x, y, w) in \mathbb{P}^2 : then $H^0(\mathcal{O}_Q(1))$ corresponds to $H^0(\mathcal{O}_{\tilde{Q}}(H - E))$ spanned by w, x , whereas $y_2 = yw$ completes w^2, wx, x^2 to a basis of $H^0(\mathcal{O}_{\tilde{Q}}(2H - E - E')) \cong H^0(\mathcal{O}_Q(2))$. Thus the morphism of \tilde{Q} to $\mathbb{P}(1, 1, 2)$ is given by $x_0 := w, x_1 := x, y_2 := yw$.

The elliptic surface X' is the double cover of \mathbb{F}_2 branched on Σ_∞ and on the curve B corresponding to the curve of Q of equation $y^3 + p_4(x_0, x_1)y + q_6(x_0, x_1) = 0$. Thus the curve B corresponds to the plane curve $w^3y^3 + p_4(w, x)yw + q_6(w, x) = 0$ whose affine part has equation

$$y^3 + p_4(1, x)y + q_6(1, x) = 0. \tag{2}$$

Note that any parabola as described here (i.e., a curve $C \in (2H - E - E')$) is disjoint from E'' (mapping to the vertex of the cone) unless it splits into two lines through the point O . In particular, we may always change coordinates in the affine plane so that C is transformed into the line $y = 0$. In order to understand with coordinates the geometry at infinity of such parabolas, let us observe that \mathbb{F}_2 has two open sets isomorphic to $\mathbb{C} \times \mathbb{P}^1$. They have respective coordinates $x/w \in \mathbb{C}, (w, y) \in \mathbb{P}^1$, while on the other chart we have $w/x \in \mathbb{C}$ and homogeneous coordinates $(x^2/w, y)$ (in fact, $(x^2/w)/w = (x/w)^2$). The section Σ_∞ at infinity corresponds to the curve $E'' \subset \tilde{X}$ and is defined by $w = 0$ and $x^2/w = 0$ on the respective charts. Then a parabola $yw = a_0w^2 + a_1xw + a_2x^2$ is given by the equation

$$\frac{1}{\eta} = a_0 + a_1 \frac{x}{w} + a_2 \left(\frac{x}{w}\right)^2$$

on the affine chart with coordinates $(x/w, w/y)$, where $w/y := \eta$. Using these coordinates at infinity it will be easy to see when some regions in the plane “meet” at infinity in \mathbb{F}_2 .

We shall now look for normal forms of equation (2). Singular points of $X'(\mathbb{R})$ are in one-to-one correspondence with singular points of $B(\mathbb{R})$. The different cases we shall now consider are distinguished by the number of irreducible components of the trisection B .

THREE COMPONENTS. In this case, we shall see that any connected component of the topological normalization of any real double cover ramified over B will have at most four singular points. Observe that at least one of the three components is real. Equation (2) becomes

$$(y - \alpha(x))(y - \beta(x))(y - \gamma(x)) = 0$$

and, changing real coordinates for $Q = \mathbb{P}(1, 1, 2)$, we may assume $\gamma = 0$. The case $\beta = \bar{\alpha}$ where two components are complex conjugate leads to at most two singular points: $\text{Re } \alpha(x) = 0$ and $y = \text{Im } \alpha(x)$. We can therefore assume that the three components are real. Thus equation (2) becomes $(y - \alpha)(y - \beta)y = 0$ where $\alpha(x) = \alpha_0 + \alpha_1x + \alpha_2x^2$ and $\beta(x) = \beta_0 + \beta_1x + \beta_2x^2$ are polynomials of degree 2.

Assume no two parabolas are tangent. Then, since we can permute the three curves, we can fix the one that is the lowest at infinity (i.e., if we write the curves as $y = a_0 + a_1x + a_2x^2$, the one with the smallest value of a_2). Changing coordinates, we get only the curve $y = 0$ and two convex parabolas—that is, with $\alpha_2 > 0$ and $\beta_2 > 0$.

The six intersection points are distinct and given by

$$y = \alpha(x)\beta(x) = 0, \quad \alpha(x) = \beta(x) = y.$$

The curve B is real; thus if one of these points is not real, then the number of real singular points is bounded above by 4 and we are done. From now on, we suppose that the six points are real. Set

$$\begin{aligned} \alpha(x) &= \alpha_2(x - a_1)(x - a_2), \quad a_1 < a_2; \\ \beta(x) &= \beta_2(x - b_1)(x - b_2). \end{aligned} \tag{3}$$

Multiplying y possibly by β_2 , we may assume $\beta_2 = 1$. We may reduce to the case $0 < \alpha_2 < 1$ by possibly exchanging the roles of α and β . We can further use a translation in the x -axis and assume $b_1 = -b_2$. Then (3) becomes:

$$\begin{aligned} \alpha(x) &= \alpha_2(x - a_1)(x - a_2), \quad a_1 < a_2, \quad 0 < \alpha_2 < 1; \\ \beta(x) &= (x^2 - b^2), \quad 0 < b. \end{aligned}$$

Up to reflection $x \leftrightarrow -x$, this leads to four possibilities—namely (see Figure 1)

$$b < a_1, \quad -b < a_1 < b < a_2, \quad a_1 < -b < b < a_2, \quad -b < a_1 < a_2 < b.$$

REMARK 2.2. Observe that in these figures two components are connected at infinity if their boundaries have two unbounded arcs belonging to the same pair of parabolas.

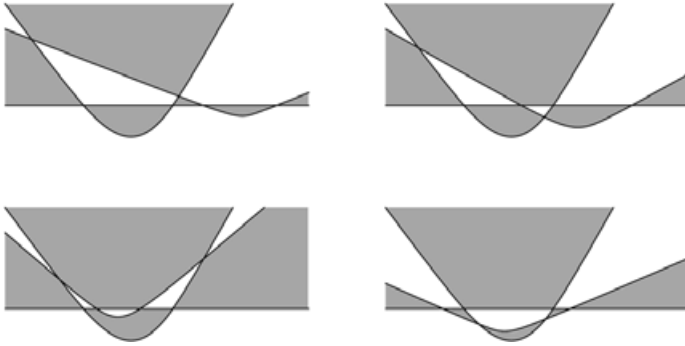


Figure 1 Three parabolas, six singular points

Now assume that two parabolas are tangent. Then we cannot arbitrarily permute the three curves, and we shall have to consider furthermore the cases $\alpha_2 > 1$ and $\alpha_2 < 0$.

Without loss of generality, the two tangent parabolas are given by $y = 0$ and $y = x^2$. The third parabola is

$$y = \alpha_2(x - a_1)(x - a_2), \quad a_1, a_2 \in \mathbb{R}^*, \quad a_1 < a_2.$$

If $\alpha_2 > 0$, again using the reflection $x \leftrightarrow -x$, we are led to only three possibilities, which are degenerate cases of the preceding ones (by possibly exchanging the roles of α and β). In fact, if a_1, a_2 have opposite signs and $\alpha_2 \leq 1$, then the two parabolas $y = x^2$ and $y = \alpha_2(x - a_1)(x - a_2)$ do not meet in real points. See Figure 2.

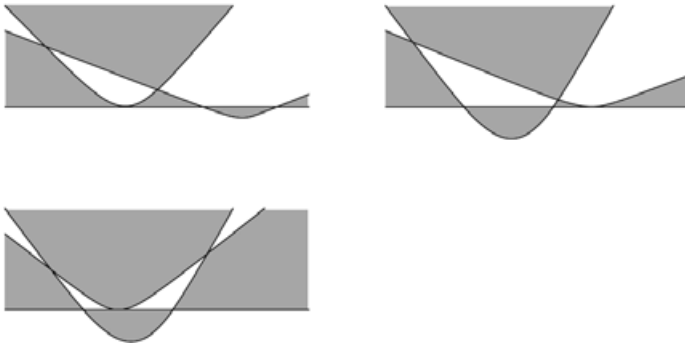


Figure 2 Three parabolas, five singular points

If $\alpha_2 < 0$ this leads to two possibilities, up to reflection $x \leftrightarrow -x$. See Figure 3.

TWO COMPONENTS. Here, we will see that in most cases, any connected component of the topological normalization of any real double cover ramified over B has



Figure 3 Three parabolas, five singular points

at most four singular points. There will remain two cases to examine separately (see Figures 6 and 7). Equation (2) becomes

$$(y - \alpha(x))(y^2 - \gamma(x)) = 0.$$

If the bisection $y^2 - \gamma(x) = 0$ is smooth, then the number of singular points is bounded from above by 4. Hence we assume the bisection to have a singular point O at $x = y = 0$. To ensure that the bisection and the parabola have four real intersection points, the polynomial $\alpha(x)^2 - \gamma(x)$ must have four distinct roots. These roots are all supposed to be real and nonvanishing in order for B to have five singular points. The singular point O is either nodal or cuspidal.

If O is an ordinary double point, the bisection is given by $y^2 - x^2h(x) = 0$, where the quadratic polynomial h is not a square since the bisection is irreducible. Changing coordinates, we can assume that the parabola is given by $y = 0$ and the bisection C by $(y + \alpha(x))^2 - x^2h(x) = 0$. Without loss of generality, $\alpha(0) > 0$. Observe that the leading coefficient of h is nonvanishing since the curve C does not pass through the vertex of Q .

The number of real singularities implies that h is not always negative and $h(0) \neq 0$. If $h(0) < 0$, then O will be isolated in $B(\mathbb{R})$ and gives rise to a globally non-separating point of the double covering X' , in view of the following.

REMARK 2.3. Let $\pi : X'(\mathbb{R}) \rightarrow F(\mathbb{R})$ be a double cover of a smooth connected real surface $F(\mathbb{R})$. If b is an isolated point of the real branch curve $B(\mathbb{R})$, then either $p = \pi^{-1}(b)$ is an isolated point of $X'(\mathbb{R})$ or p is a locally separating point of $X'(\mathbb{R})$. If however $B(\mathbb{R})$ has a component Γ of dimension 1, then p is globally nonseparating, as one can easily see by taking a path that connects b to Γ .

If $h(0) > 0$ and the function h is somewhere negative, observe that $y = 0$ disconnects the cylinder $\mathbb{F}_2(\mathbb{R}) - \Sigma_\infty(\mathbb{R})$. Since the polynomial $\alpha(x)^2 - x^2h(x)$ is assumed to have four distinct roots, up to taking a projectivity of $\mathbb{P}^1(\mathbb{R})$ sending ∞ to a finite point, we see that there is only one topological possibility, given by Figure 4.



Figure 4 Two irreducible components

If $h(x) > 0$ for all x , then C is a double cover of $\mathbb{P}^1(\mathbb{R})$ and we can write C as $C^u \cup C^l$, where C^u is the “upper part” and C^l the “lower part”. Because of our choice $\alpha(0) > 0$,

$$C^u \cap \{y = 0\} = \emptyset \implies C \cap \{y = 0\} = \emptyset.$$

Hence there are two cases: $\#(C^u \cap \{y = 0\}) = 2$, given by Figure 5, and $\#(C^u \cap \{y = 0\}) = 4$, given by Figure 6.



Figure 5 Two irreducible components



Figure 6 The point O is a globally nonseparating A_1 singular point

After we describe the branch curve B , observe that we obtain two different surfaces multiplying the equation of B by ± 1 . In Figure 5, any connected component of the topological normalization of any double cover will have at most four singular points. In Figure 6, for only one choice of sign, the topological normalization of the double cover will have a connected component with five singular points. For this double cover, however, the singular point O turns out to be a globally nonseparating A_1 singular point and hence does not belong to $\mathcal{P}_{X'}$.

If O is a cusp, the equation of B is

$$(y - \alpha(x))(y^2 - x^3 l(x)) = 0.$$

Using a dilatation $y \mapsto \lambda y$ and possibly the usual reflection $x \leftrightarrow -x$, we may assume $l(0) = 1$; then the equation of the bisection becomes

$$y^2 - x^3(1 + ax) = 0.$$

To ensure that the bisection and the parabola have four real intersection points, the equation $\alpha(x)^2 - x^3(1 + ax) = 0$ must have four distinct roots. Possibly changing the line $x = \infty$ via a projectivity, we may assume that $a > 0$ and indeed $a = 1$. It is easy then to see that the only possible configuration is given by Figure 7.

Recall that we obtain two different surfaces multiplying the equation of B by ± 1 . For only one choice of sign will the topological normalization of the double cover have a connected component with five singular points. For this double cover, however, the point O turns out to be of real type A_2^- , which does not belong to $\mathcal{P}_{X'}$.

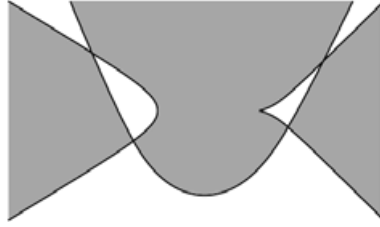


Figure 7 The cusp gives rise to a singular point of type A_2^-

ONE COMPONENT. If the trisection is irreducible, then it has at most four singular points, since $B(\mathbb{C})$ has genus 4.

Proof of Proposition 2.1. We proceed according to the number of irreducible components of B , recalling that the singular points of X correspond to the singular points of B .

If B is irreducible, we have already seen that B has at most four singular points. If instead B has two irreducible components and B has strictly more than four singular points, we have seen that B has exactly five singular points and that the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ has one of the topological configurations of Figures 4–7.

In the case of Figure 4, none of the connected components of the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ contains more than four points. The same occurs for the case of Figure 5, while for Figure 6 there is exactly one connected component D containing the five singular points. However, in this case the nodal point of the bisection yields a globally nonseparating singular point of X' for the choice of the positivity region that includes D .

Similarly, for the case of Figure 7 there is exactly one connected component D containing the five singular points. However, in this case the cuspidal point yields a point of type A_2^- , which does not belong to $\mathcal{P}_{X'}$.

Assume now that B has three irreducible components and at least five singular points. If there are six singular points, the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ has one of the topological configurations of Figure 1, and none of the connected components of the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ contains more than four points. An easy inspection of Figures 2 and 3 reveals that the same holds also in the remaining cases. \square

PROPOSITION 2.4 (Kollár). *Let X be a real conic bundle with X Du Val. Then $m_i \leq 4$, $i = 1, 2, \dots, r$. Moreover, if $m_i = 4$, then $\bar{n}(M_i) \cap \mathcal{P}_X$ contains four A_1 points, whereas if $m_i = 3$ then $\bar{n}(M_i) \cap \mathcal{P}_X$ contains at least two A_1 .*

Proof. The assertion $m_i \leq 4$ is the last assertion of the proof of [Ko3, Cor. 9.8]. But the same argument proves indeed what we have already stated. \square

Proof of Theorem 0.1. Recall that by Lemma 1.7 the numbers m_1, \dots, m_r of Du Val singular points on the connected components of $\overline{X(\mathbb{R})}$ that are not of type A^- and globally nonseparating are an invariant for extremal birational contractions. Hence,

by Theorem 1.2 it suffices to consider the case where X is either a conic bundle or a Del Pezzo surface.

The case of a conic bundle is settled by Proposition 2.4, and by virtue of Lemma 1.8 it suffices to consider the case where X is a Du Val–Del Pezzo surface of degree 1. Now it suffices to apply Proposition 2.1. □

3. Real Rationally Connected 3-folds

This section is devoted to the proof of Corollary 0.2. We first introduce the concept of a Werther fibration (cf. [HM2]), which allows us to set the integer k on an equal footing in cases (a) and (b) (see Section 0).

Let $S^1 \times D^2$ be the *solid torus*, where S^1 is the unit circle $\{u \in \mathbb{C} \mid |u| = 1\}$ and D^2 is the closed unit disc $\{z \in \mathbb{C}, |z| \leq 1\}$. A *Seifert fibration* of the solid torus is a differentiable map of the form

$$f_{p,q}: S^1 \times D^2 \rightarrow D^2, \quad (u, z) \mapsto u^q z^p,$$

where p and q are natural integers, with $p \neq 0$ and $\gcd(p, q) = 1$. Let N be a 3-manifold. A *Seifert fibration* of N is a differentiable map f from N into a differentiable surface S having the following property: Every point $P \in S$ has a closed neighborhood U such that the restriction of f to $f^{-1}(U)$ is diffeomorphic to a Seifert fibration of the solid torus.

Let A^2 be the half-open annulus $\{w \in \mathbb{C} \mid 1 \leq |w| < 2\}$. Let P be the differentiable 3-manifold defined by $P = \{(w, z) \in A^2 \times \mathbb{C} \mid |z|^2 = |w|^2 - 1\}$. Let $\omega: P \rightarrow A^2$ be the projection defined by $\omega(w, z) = w$. It is clear that ω is a differentiable map, that ω is a trivial circle bundle over the interior of A^2 , and that ω is a diffeomorphism over the boundary of A^2 .

DEFINITION 3.1. Let $g: N \rightarrow F$ be a differentiable map from a 3-manifold N without boundary into a differentiable surface F with boundary. The map g is a *Werther fibration* if

- the restriction of g over the interior of F is a Seifert fibration, and
- every point x in the boundary of F has an open neighborhood U such that the restriction of g to $g^{-1}(U)$ is diffeomorphic to the restriction of ω over an open neighborhood of 1 in A^2 .

This definition, introduced in [HM2], is motivated by the following theorem.

THEOREM 3.2 [HM2, Thm. 2.6]. *Let N be a 3-dimensional compact manifold without boundary. Then N is a Seifert fibred manifold or a connected sum of finitely many lens spaces if and only if there is a Werther fibration $g: N \rightarrow F$ over a compact connected differentiable surface F with boundary. Furthermore, N is Seifert fibred if and only if there exists such a map $g: N \rightarrow F$ with $\partial F = \emptyset$.*

Thanks to the minimal model program over \mathbb{R} (see [Ko2]), the original setting for $f: W \rightarrow X$ in Corollary 0.2 is replaced by the following: W is a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$, $W(\mathbb{R})$ is a

topological 3-manifold, and $f: W \rightarrow X$ is a rational curve fibration over \mathbb{R} such that $-K_W$ is f -ample.

The following result relates the connected components of $W(\mathbb{R})$ with the connected components of the topological normalization $\overline{X}(\mathbb{R})$.

PROPOSITION 3.3 [Ko3, Cor. 6.8]. *Let W be a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$. Let $f: W \rightarrow X$ be a rational curve fibration over \mathbb{R} such that $-K_W$ is f -ample. Let $N \subset W(\mathbb{R})$ be a connected component. Then $f(N)$ intersects only one of the connected components of $X(\mathbb{R}) \setminus \text{Sing } X$.*

Let $\bar{n}: \overline{W}(\mathbb{R}) \rightarrow W(\mathbb{R})$ be the topological normalization. The following is the key result that relates the integer $k(N)$ defined previously to the numbers m_i of the singularities in $\mathcal{P}_X \cap M_i$.

PROPOSITION 3.4 [Ko3, Thm. 8.1(6)]. *Let W be a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$. Let $f: W \rightarrow X$ be a rational curve fibration over \mathbb{R} such that $-K_W$ is f -ample. Let N be a connected component of the topological normalization $\overline{W}(\mathbb{R})$ and assume that N is an orientable topological 3-manifold. Then there exists a small perturbation $g: N \rightarrow F$ of $f|_{\bar{n}(N)}$ that is a Werther fibration. Furthermore, there is an injection from the set of multiple fibres of g to the set of singular points of X contained in $f(\bar{n}(N))$ that are of real type A_μ^+ , $\mu \geq 1$. If $\partial F = \emptyset$, then g is a Seifert fibration. If $\partial F \neq \emptyset$, then N is a connected sum of lens spaces and the number of lens spaces is equal to the number of multiple fibres of g .*

Proof of Corollary 0.2. We want to show that, for each component N of $\overline{W}(\mathbb{R})$, we have $k(N) \leq 4$. From Proposition 3.4 it follows that $k(N)$ is the number of multiple fibres of the Werther fibration; hence it suffices to bound the number of singular points of X contained in $f(\bar{n}(N))$ that are of real type A_μ^+ . If $f(\bar{n}(N))$ is not a connected component of $X(\mathbb{R}) \setminus \text{Sing } X$, then from [Ko3, 8.2], N is a connected sum of lens spaces and $f(\bar{n}(N))$ may contain some globally nonseparating singular points of type $A_1^+ \sim A_1^-$. These points produce double fibres for g , which however correspond to lens space summands $\mathbb{P}^3(\mathbb{R})$. These summands are excluded by the maximality of a in the definition of $k(N)$. Thus, by Proposition 3.3, it suffices to bound the number of singular points of X contained in $f(\bar{n}(N))$ that are of real type A_μ^+ or globally separating. But because these points are a subset of $\mathcal{P}_X \cap M_i$ for some $i \in \{1, \dots, r\}$, the desired inequality follows from Theorem 0.1. □

4. Seifert Fibrations over a Torus

This section is devoted to the proof of Theorem 0.3.

LEMMA 4.1. *Let $r: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ be a real conic bundle. Suppose that X is a Du Val surface. If $\overline{X}(\mathbb{R})$ has a connected component M diffeomorphic to $S^1 \times S^1$, then r is smooth along $X(\mathbb{R})$ and $X(\mathbb{R}) \sim \overline{X}(\mathbb{R}) \sim S^1 \times S^1$.*

Proof. We first want to show that $r_M := r \circ \bar{n}|_M$ is surjective and that $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Assume that r_M is not surjective. Then $\text{Im}(r_M)$ is homeomorphic to a segment $[a, b]$. The fibres $r_M^{-1}(a)$ and $r_M^{-1}(b)$ are the ends of $r_M^{-1}(a, b)$ and they have a (punctured) tubular neighborhood that is homeomorphic to an annulus. This shows that $r_M^{-1}(a)$ and $r_M^{-1}(b)$ are connected. The fibre $r_M^{-1}(a)$ is a simplicial complex of dimension ≤ 1 , and if $S^1 \subset r_M^{-1}(a)$ then S^1 has a (punctured) tubular neighborhood that is connected, contradicting the orientability of M . Hence $r_M^{-1}(a)$ and $r_M^{-1}(b)$ have Euler number 1.

It suffices to show that each fibre $r_M^{-1}(t)$, $t \in (a, b)$, has Euler number ≥ 0 , and we obtain a contradiction to $e(M) = 0$. Looking at the normal forms for singular points of type A_μ for conic bundles (given in the proof of [Ko3, Cor. 9.8]), we see that every fibre of $\bar{r} := r \circ \bar{n}$ is either a circle (and then r is smooth on the fibre) or a point or an interval. Thus r_M is surjective.

Again, the Euler number argument shows that all fibres of \bar{r} are circles; hence r is smooth on $X(\mathbb{R})$ and $M \sim \overline{X(\mathbb{R})} \sim X(\mathbb{R})$. □

PROPOSITION 4.2. *Let X be a real Du Val surface that is rational over \mathbb{C} . Assume that $X(\mathbb{R})$ contains only singularities of type A_μ^+ and that $\overline{X(\mathbb{R})}$ contains a connected component diffeomorphic to $S^1 \times S^1$. Then $\overline{X(\mathbb{R})}$ is connected and so $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Furthermore, there is a minimal model of X that is a real conic bundle over \mathbb{P}^1 .*

Proof. Let X be as before. The blow-up of a point of type A_μ^+ for $\mu \geq 2$ induces a homeomorphism between the real parts. Thus there is a surface Z such that all singular points are of type A_1 and $\overline{Z(\mathbb{R})}$ is homeomorphic to $\overline{X(\mathbb{R})}$. Let Z^* be a Du Val minimal model of Z . Then by [Ko3, Thm. 9.6], $\pi : Z \rightarrow Z^*$ is the composition of inverses of weighted blow-ups (of smooth points). Hence π is an isomorphism with the exception of a finite number of smooth points $p_1, \dots, p_s \in Z^*$ at which one takes the weighted blow-up that, in local coordinates (x, y) around p_j , has the form $\{xu - vy^2\} \rightarrow \{(x, y)\}$. Since this weighted blow-up produces globally non-separating points, there is a bijection between the connected components of $\overline{Z(\mathbb{R})}$ and the connected components $\overline{Z^*(\mathbb{R})}$. The weighted blow-up followed by topological normalization on a disc neighborhood of p_j has the effect of replacing p_j by a closed segment. Hence the connected component of $\overline{Z^*(\mathbb{R})}$ coming from the one of $\overline{Z(\mathbb{R})}$ diffeomorphic to $S^1 \times S^1$ is again diffeomorphic to $S^1 \times S^1$. Observe again that the singularities of Z^* are only of type A_1 .

We have two cases:

- (i) the minimal model Z^* is a real Del Pezzo surface of degree 1 or 2;
- (ii) Z^* is a real conic bundle.

In case (i), we have a realization of Z^* as a double cover, and the topological normalization $\overline{Z^*(\mathbb{R})}$ can be realized as the real part of a real perturbation Z_ϵ^* of Z^* (by Brusotti's theorem [Br]). The surface Z_ϵ^* is a smooth real Del Pezzo surface of degree 1 or 2. An orientable connected component of such a surface is a sphere (see e.g. [Si, Chap. 3]), so this case does not occur.

Case (ii) follows from Lemma 4.1. □

To prove Theorem 0.3, we need the conclusion of Proposition 4.2 in a more general setting. First, we give a partial generalization of Brusotti’s theorem in the case of a Du Val–Del Pezzo surface.

THEOREM 4.3. *Let X be a Du Val–Del Pezzo surface. One can obtain, by a global small deformation of X , all the possible smoothings of the singular points of X .*

Proof. The main theorem on deformations of compact complex spaces was proven in [G]. Good references are [P] and [Pi]. The tangent space to $\text{Def}(X)$ is given by $\text{Ext}^1(\Omega_X^1, \mathcal{O}_X)$; see [S, Cor. 1.1.11]. The obstruction space $\text{Ob}(X)$ is given by $\text{Ext}^2(\Omega_X^1, \mathcal{O}_X)$; see [S, Prop. 2.4.8].

By the local-to-global spectral sequence for Ext , we have exact sequence

$$H^1(X, \Theta_X) \rightarrow \text{Ext}^1(\Omega_X^1, \mathcal{O}_X) \rightarrow H^0(\mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X)) \rightarrow H^2(X, \Theta_X) \rightarrow 0$$

and $\text{Ob}(X) = H^2(X, \Theta_X)$. Therefore, the vanishing $H^2(X, \Theta_X) = 0$ implies that:

- the local deformation space is smooth;
- global deformations map onto local deformations.

We use a calculation by Burns and Wahl [BW, Prop. 1.2] to the effect that, if S is the minimal resolution of the Du Val singularities of X , then $p_*(\Theta_S) = \Theta_X$. Whence, $H^2(X, \Theta_X) = H^2(S, \Theta_S)$.

But the dual space of $H^2(S, \Theta_S)$ is $H^0(S, \Omega_S^1(K_S))$. The conclusion follows from $H^0(S, \Omega_S^1(K_S)) = 0$ since $H^0(S, \Omega_S^1) = 0$ and $H^0(S, \mathcal{O}_S(-K_S)) \neq 0$. \square

LEMMA 4.4. *Consider a real singular point of a surface X and of local equation $z^2 = f(x, y)$, where f vanishes at the origin and has there an isolated singular point that we assume to be a nonisolated real point. Then the topological normalization of $X(\mathbb{R})$ is locally homeomorphic to the real part $X_\varepsilon(\mathbb{R})$ of the surface X_ε with equation $z^2 = f(x, y) - \varepsilon$ for ε sufficiently small and positive.*

Proof. The real curve $f(x, y) = 0$ has $2m$ arcs entering into the singular point, ordered counterclockwise, and the region of positivity consists of m sectors, which alternate themselves to the m sectors of negativity. Furthermore, we have $m \neq 0$ because the origin is a nonisolated real point of the curve. The smooth curve $f(x, y) = \varepsilon$ determines m domains of positivity whose closure is homeomorphic to the closure of the corresponding sector of positivity of $f(x, y)$ (where it is contained). It follows right away that the double cover $z^2 = f(x, y) - \varepsilon$ replaces the singular point by m points, one for each connected component of $X(\mathbb{R}) \setminus \{0\}$. \square

PROPOSITION 4.5. *Let X be a real Du Val surface that is rational over \mathbb{C} . Assume that all locally separating singularities are gobally separating and that $\bar{X}(\mathbb{R})$ contains a connected component diffeomorphic to $S^1 \times S^1$. Then $\bar{X}(\mathbb{R})$ is connected and so $\bar{X}(\mathbb{R}) \sim S^1 \times S^1$. Furthermore, there is a minimal model of X that is a real conic bundle over \mathbb{P}^1 .*

Proof. The minimal resolution of a singular point of type A_μ^- , μ even, induces a homeomorphism between the real parts. Thus, as in the proof of Proposition 4.2, there is a surface Z such that:

- all singular points are of type A_1 , or of type A_μ^- ($\mu > 1$, μ odd), or not of type A ; and
- $\overline{Z(\mathbb{R})}$ is homeomorphic to $\overline{X(\mathbb{R})}$.

Let Z^* be a Du Val minimal model of Z . Suppose that the minimal model Z^* is a real Del Pezzo surface of degree 1 or 2. We have then a realization of Z^* as a double cover and we can apply Remark 2.3 to exclude singular points that are isolated real points of the branch curve. By Lemma 4.4 and Theorem 4.3, the topological normalization $\overline{Z^*(\mathbb{R})}$ can be realized as the real part of a real perturbation Z_ϵ^* of Z^* . The surface Z_ϵ^* is a smooth real Del Pezzo surface of degree 1 or 2. An orientable connected component of such a surface is a sphere, so this case does not occur.

Hence Z^* is a real conic bundle and the conclusion follows from Lemma 4.1. \square

Proof of Theorem 0.3. The component N of $W(\mathbb{R})$ is Seifert fibred; hence $f(N)$ is the closure of a connected component of $X(\mathbb{R}) \setminus \text{Sing } X$ (see the statement in [Ko3, 8.2]: “so we are in the case (4)”). In the proof of [Ko3, Cor. 6.8], Kollar uses [Ko3, 4.3] to claim that $f(N)$ cannot map to both local components of a locally separating singularity. Whence, this singularity must be globally separating. Thus all singularities of $f(N)$ that are locally separating are globally separating. We are now in the situation of Proposition 4.5, whence $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Furthermore, the minimal model Z^* is a real conic bundle and Lemma 4.1 gives that Z^* is smooth along $Z^*(\mathbb{R})$; thus $Z(\mathbb{R}) \cap \mathcal{P}_Z = \emptyset$ and hence $X(\mathbb{R}) \cap \mathcal{P}_X = \emptyset$.

Applying Proposition 3.4 on a minimal model of $W \rightarrow X$, we get the conclusion. \square

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