

# A REMARKABLE MODULI SPACE OF RANK 6 VECTOR BUNDLES RELATED TO CUBIC SURFACES

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ABSTRACT. We study the moduli space  $\mathfrak{M}^s(6; 3, 6, 4)$  of simple rank 6 vector bundles  $\mathcal{E}$  on  $\mathbb{P}^3$  with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$  and properties of these bundles, especially we prove some partial results concerning their stability. We first recall how these bundles are related to the construction of sextic nodal surfaces in  $\mathbb{P}^3$  having an even set of 56 nodes (cf. [Ca-To]). We prove that there is an open set, corresponding to the simple bundles with minimal cohomology, which is irreducible of dimension 19 and bimeromorphic to an open set  $\mathfrak{A}^0$  of the G.I.T. quotient space of the projective space  $\mathfrak{B} := \{B \in \mathbb{P}(U^\vee \otimes W \otimes V^\vee)\}$  of triple tensors of type  $(3, 3, 4)$  by the natural action of  $SL(W) \times SL(U)$ .

We give several constructions for these bundles, which relate them to cubic surfaces in 3-space  $\mathbb{P}^3$  and to cubic surfaces in the dual space  $(\mathbb{P}^3)^\vee$ . One of these constructions, suggested by Igor Dolgachev, generalizes to other types of tensors.

Moreover, we relate the so-called *cross-product involution* for  $(3, 3, 4)$ -tensors, introduced in [Ca-To], with the Schur quadric associated to a cubic surface in  $\mathbb{P}^3$  and study further properties of this involution.

## INTRODUCTION

A good motivation for the study of moduli spaces of vector bundles in  $\mathbb{P}^3$  comes from the classical problem concerning the geometry of nodal surfaces  $F$  in  $\mathbb{P}^3$ , and more specifically from the study of even sets  $\Delta$  of nodes on them (Beauville has shown in [Bea] that a surface of low degree and with many nodes contains necessarily several such even sets  $\Delta$ ).

In turn, as we recall in section 1 of the paper, every even set occurs as the second degeneracy locus of a symmetric map of a vector bundle in  $\mathbb{P}^3$ : more precisely, the main theorem of [Ca-Ca] asserts that, given  $\delta \in \{0, 1\}$  and a  $\delta$ -even set of nodes  $\Delta$  on a nodal surface  $F$  of degree  $d$  in  $\mathbb{P}^3$ , there is a vector bundle  $\mathcal{E}''$  on  $\mathbb{P}^3$  and a symmetric map  $\mathcal{E}''^\vee(-d -$

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*Date:* February 16, 2013.

$\delta) \xrightarrow{\varphi} \mathcal{E}''$  such that  $\Delta$  is precisely the locus where the corank of  $\varphi$  equals 2, while  $F \setminus \Delta$  is the locus of points where  $\text{corank}(\varphi) = 1$ .

The simplest case is the case where the vector bundle is a direct sum of line bundles (cf. [Cat1]): here  $\varphi$  is just a symmetric matrix with entries homogeneous polynomials of fixed degrees.

The problem of existence and classification of even sets of nodes is in general based on a preliminary analysis of certain moduli spaces of pairs  $(\mathcal{E}'', \varphi)$  as above. The main question being whether a pair as above defines loci  $F := \det(\varphi)$  and  $\Delta := \{x | \text{corank}(\varphi) \geq 2\}$  which have the desired singularities.

For instance, the classification of 0-even nodes on sextic surfaces was achieved in [Ca-To], showing the existence of sets of cardinality 56. In this case a remarkable feature was that the corresponding moduli space of vector bundles  $\mathcal{E} := \mathcal{E}''(3)$  was shown to be irreducible, yet computer experiments showed that for a general such bundle  $\mathcal{E}$  the determinant  $F := \det \varphi$  would yield a cubic surface  $G$  counted with multiplicity two and independent of the choice of  $\varphi$ .

This was the first motivation to try to understand the relation occurring between such bundles and cubic surfaces. This brought to a finer investigation of the moduli space of pairs  $(\mathcal{E}, \varphi)$  as above, which revealed that the latter moduli space is indeed reducible, with a second component of codimension 7 corresponding to cubic surfaces  $G$  reducible as the union of a plane and a smooth quadric intersecting transversally. Several explicit constructions for the vector bundles  $\mathcal{E}$  in question allowed finally to show that for a general pair  $(\mathcal{E}, \varphi)$  in the second component the determinant of the symmetric map  $\varphi$  yields a nodal surface  $F$  with an even set  $\Delta$  as wished for.

The purpose of the present paper is threefold: first of all we want to explain the beautiful geometry relating one of our vector bundles  $\mathcal{E}$  to a cubic surface  $G$  in  $\mathbb{P}^3$  and to another cubic surface  $G^*$  in the dual projective space.

This relation goes back to the classical discovery that a smooth cubic surface can be written as the determinant of a matrix of linear forms, and also as the image of the plane  $\mathbb{P}^2$  through the linear system of cubics passing through 6 given points.

Since our second purpose, following a suggestion of Igor Dolgachev, is to show how the above correspondence can be vastly generalized (this is done in section 2), we try to set up the classical story in a context of modern multilinear algebra.

Let  $B$  be a general tensor of type  $(3, 3, 4)$ , more precisely let  $B \in U^\vee \otimes W \otimes V^\vee$ , where  $U, W, V$  are complex vector spaces of respective dimensions  $3, 3, 4$ .

Now, it is classical that to a general  $3 \times 3 \times 4$  tensor  $B$  one can associate a cubic surface in  $\mathbb{P}^3$  by taking the determinant of the corresponding  $3 \times 3$  matrix of linear forms on  $\mathbb{P}^3$ . In this way we get a cubic surface  $G^*$  in the dual projective space  $\mathbb{P}^{3^\vee} = \mathbb{P}(V) := \text{Proj}(V^\vee)$ , together with two different realizations of  $G^*$  as a blow up of a projective plane  $\text{Proj}(U^\vee)$  (respectively  $\text{Proj}(W)$ ) in a set of 6 points. These are the points where the  $3 \times 4$  Hilbert–Burch matrix of linear forms on  $U$  (respectively on  $W^\vee$ ) drops rank by 1, and the rational map to  $\mathbb{P}^{3^\vee}$  is given by the system of cubics through the 6 points, a system which is generated by the determinants of the four  $3 \times 3$  minors of the Hilbert–Burch matrix. One passes from one realization to the other simply by transposing the tensor, and we shall call this the **trivial involution** for  $3 \times 3 \times 4$  tensors.

For a  $3 \times 3 \times 4$  tensor  $B \in U^\vee \otimes W \otimes V^\vee$ , besides this trivial involution, which consists in regarding  $B$  as element of  $W \otimes U^\vee \otimes V^\vee$ , there exists another involution, called the **cross-product involution** (see [Ca-To]). This second involution associates to a general tensor  $B \in U^\vee \otimes W \otimes V^\vee$  another tensor  $B' \in U'^\vee \otimes W \otimes V$ , where  $U'$  is defined as the kernel of the map  $\Lambda^2(W^\vee) \otimes V \rightarrow U^\vee \otimes W^\vee$  induced by contraction with  $B$ . The **reversing construction**<sup>1</sup> is then defined as the composition of the cross-product involution with the trivial involution, and associates to  $B \in U^\vee \otimes W \otimes V^\vee$  the tensor  $B' \in W \otimes U'^\vee \otimes V$ ,

In the paper [Ca-To] the authors give the following direct geometric construction of nodal sextic surfaces with an even sets of 56 nodes.

Consider the open set  $\mathfrak{B}^*$  of  $\mathfrak{B} = \mathbb{P}(U^\vee \otimes W \otimes V^\vee)$  given by the  $(3, 3, 4)$ -tensors  $B$  whose determinant (as  $3 \times 3$  matrices) defines a cubic surface  $G^* \subset \mathbb{P}^{3^\vee} = \text{Proj}(V^\vee)$ . To such a  $B$  we apply the reversing construction and we consider the following exact sequence induced by the  $(3, 3, 4)$ -tensor  $B'$  on  $\mathbb{P}(V^\vee) = \text{Proj}(V)$ :

$$0 \rightarrow W^\vee \otimes \mathcal{O}(-1) \xrightarrow{B'} U'^\vee \otimes \mathcal{O} \rightarrow \mathcal{G} \rightarrow 0.$$

Observe that if  $B'$  never drops rank by two then  $\mathcal{G}$  is an invertible sheaf on the cubic surface  $G$  associated to  $B'$ .

The direct construction produces a bundle  $\mathcal{E}$  as an extension of  $6\mathcal{O}$  by the sheaf  $\tau := \mathcal{G}^{\otimes 2}(-1)$ : it turns out that the extension  $\mathcal{E}$  is unique

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<sup>1</sup>We are pedantic with the order of the spaces of a multitensor, but this is essential for a correct correspondence between the various constructions we will develop from a multitensor.

(up to isomorphisms) if the cubic surface  $G$  is smooth, and also if it is reducible as the transversal union of a plane with a smooth quadric. In the latter case we obtain a sextic nodal surface  $F$  as the corank 2 degeneracy locus of a general symmetric map  $\varphi : \mathcal{E}^\vee \rightarrow \mathcal{E}$  (while, if  $G$  is smooth,  $F$  is the cubic surface  $G$  counted with multiplicity two) (cf. again [Ca-To]).

Concerning the cross-product involution, we show its relation to the Schur quadric. We fix an orientation of  $W$ , which allows us to identify  $\Lambda^2(W^\vee)$  with  $W$ . The Schur quadric  $Q$  is defined (up to scalars) as a generator of the kernel of the natural map  $S^2V \rightarrow \Lambda^2(U^\vee) \otimes \Lambda^2W$  obtained as the composition of  $S^2(B)$  with the projection of

$$S^2(U^\vee \otimes W) = (\Lambda^2(U^\vee) \otimes \Lambda^2W) \oplus (S^2(U^\vee) \otimes S^2W)$$

onto the first factor. Indeed,  $\dim S^2V = 10$ ,  $\dim (\Lambda^2(U^\vee) \otimes \Lambda^2W) = 9$ , and the kernel is 1-dimensional, cf. [Do-Ka, §0 and Thm 0.5].

Classically, the Schur quadric induces an isomorphism  $q$  between  $\mathbb{P}(V^\vee)$  and  $\mathbb{P}(V)$  sending the cubic surface  $G$  to the cubic surface  $G^*$ : here, we consider the tensor  $B$  as inducing an injective map  $U \rightarrow W \otimes V^\vee$ , and we show that the subspace  $U'$  is equal to the image inside  $W \otimes V$  of the composition of this inclusion with  $id_W \otimes q : W \otimes V^\vee \rightarrow W \otimes V$ . Hence we obtain the tensor  $B'$  associated to the inclusion  $U' \rightarrow W \otimes V$ .

Up to now we have been talking about moduli spaces, for instance about the moduli space of the bundles  $\mathcal{E}$  which we obtain from our tensors  $B$ . The trouble however is: does such a moduli space really exist? The answer is positive because we show that the bundles  $\mathcal{E}$  are simple rank 6 vector bundles with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$ , and from the holomorphic point of view one has a moduli space of simple vector bundles. We show that our construction leads then to the realization of an open set of the moduli space  $\mathfrak{M}^s(6; 3, 6, 4)$  of simple rank 6 vector bundles  $\mathcal{E}$  on  $\mathbb{P}^3$  with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$ . This open set is the biholomorphic image of an open set of the G.I.T. quotient space  $\mathfrak{B}/SL(U) \times SL(W)$  of  $\mathfrak{B} := \{B \in \mathbb{P}(U^\vee \otimes W \otimes V^\vee)\}$ . The above is a subset of the open set formed by the simple bundles with minimal cohomology, as explained in the following:

**Main Theorem** *Consider the moduli space  $\mathfrak{M}^s(6; 3, 6, 4)$  of rank 6 simple vector bundles  $\mathcal{E}$  on  $\mathbb{P}^3 := \text{Proj}(V)$  with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$ , and inside it the open set  $\mathfrak{A}$  corresponding to the*

simple bundles with minimal cohomology, i.e., those with

- (1)  $H^i(\mathcal{E}) = 0 \forall i \geq 1;$
- (2)  $H^i(\mathcal{E}(-1)) = 0 \forall i \neq 1;$
- (3)  $H^i(\mathcal{E}(-2)) = 0 \forall i \neq 1;$
- (4)  $H^i(\mathcal{E}(-3)) = 0 \forall i;$
- (5)  $H^i(\mathcal{E}(-4)) = 0 \forall i.$

Then  $\mathfrak{A}$  is irreducible of dimension 19 and it is bimeromorphic to  $\mathfrak{A}^0$ , where  $\mathfrak{A}^0$  is an open set of the G.I.T. quotient space of the projective space  $\mathfrak{B}$  of tensors of type  $(3, 4, 3)$ ,  $\mathfrak{B} := \{B \in \mathbb{P}(U^\vee \otimes W \otimes V^\vee)\}$  by the natural action of  $SL(W) \times SL(U)$  (recall that  $U, W$  are two fixed vector spaces of dimension 3, while  $V = H^0(\mathbb{P}^3, \mathcal{O}(1))$ ).

Let moreover  $[B] \in \mathfrak{A}^0$  be a general point: then to  $[B]$  corresponds a vector bundle  $\mathcal{E}_B$  on  $\mathbb{P}^3$  via the kernel construction, and also a vector bundle  $\mathcal{E}_B^*$  on  $\mathbb{P}^{3^\vee}$ , obtained from the direct construction applied to the tensor  $B \in U^\vee \otimes W \otimes V^\vee$  (cf. definition 3.9 applied to  $B$ , or equation (5.1)).  $\mathcal{E}_B^*$  is the vector bundle  $\mathcal{E}_{B'}$ , where  $B' \in W \otimes U'^\vee \otimes V$  is obtained from  $B$  via the reversing construction and  $[B'] \in \mathfrak{A}_*^0$ .

Moreover, in section 5 we address the question of Mumford-Takemoto (slope), respectively Gieseker (semi)stability of the bundles  $\mathcal{E}$ . This is an interesting but delicate question, since slope stability implies Gieseker stability, which implies slope semistability. We can prove slope semistability, but there remains the interesting question whether the general bundle  $\mathcal{E}$  associated to a tensor  $B$  is Gieseker stable.

## 1. QUADRATIC SHEAVES, NODAL SURFACES, AND RELATED VECTOR BUNDLES

The study of vector bundles of rank 6 is a slightly unusual topic of research, in the sense that a topic of this type is usually studied not for its own sake, but in view of applications to other problems. Since this is exactly the case here, in this section we want to explain how we got interested in our class of vector bundles and in which context we encountered them.

**Definition 1.1.** *Let  $F$  be a locally Cohen–Macaulay projective scheme. A locally Cohen–Macaulay, reflexive, coherent  $\mathcal{O}_F$ - sheaf  $\mathcal{F}$  is said to be  $\delta/2$ -quadratic ( $\delta \in \{0, 1\}$ ) if there exists a symmetric isomorphism  $\sigma : \mathcal{F}(\delta) \rightarrow \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F)$  (symmetric means that the associated bilinear map  $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{O}_F(-\delta)$  is symmetric).*

Let us now suppose that  $F$  is a hypersurface of degree  $d$  in a projective space  $\mathbb{P}$ . If  $\mathcal{F}$  is a coherent  $\mathcal{O}_F$ - sheaf, we have a natural isomorphism:

$$\mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(-\delta)) \cong \mathcal{E}xt_{\mathcal{O}_\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_\mathbb{P}(-d - \delta)).$$

Indeed, let  $f = 0$  be the equation defining  $F$  and consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d - \delta) \xrightarrow{f} \mathcal{O}_{\mathbb{P}}(-\delta) \rightarrow \mathcal{O}_F(-\delta) \rightarrow 0.$$

By applying the functor  $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, -)$ , we obtain

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(-\delta)) \rightarrow \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(-d - \delta)) \xrightarrow{f} \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(-\delta)),$$

where the last map is zero, since it is induced by multiplication by  $f$  as a morphism between  $\mathcal{O}_F$ -sheaves of modules.

Therefore, for a quadratic sheaf  $\mathcal{F}$  defined on  $F$ , we have

$$\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(-\delta)) \cong \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(-d - \delta)).$$

On the other hand, if  $\mathcal{E}''$  is a vector bundle on  $\mathbb{P}$  and  $\varphi : \mathcal{E}''^{\vee}(-d - \delta) \rightarrow \mathcal{E}''$  is a symmetric morphism, we define  $F$  as the locus where  $\text{rk}(\varphi) \leq \text{rk} \mathcal{E}'' - 1$  and set  $\mathcal{F} := \text{coker} \varphi$ . We then have the exact sequence

$$0 \rightarrow \mathcal{E}''^{\vee}(-d - \delta) \rightarrow \mathcal{E}'' \rightarrow \mathcal{F} \rightarrow 0$$

and, by applying  $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(-, \mathcal{O}(-d - \delta))$ ,

$$0 \rightarrow \mathcal{E}''^{\vee}(-d - \delta) \rightarrow \mathcal{E}'' \rightarrow \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(-d - \delta)) \rightarrow 0.$$

Thus  $\mathcal{F}$  is naturally isomorphic to  $\mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(-d - \delta))$ . By identifying  $\mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(-d - \delta))$  with  $\mathcal{H}om_{\mathcal{O}_F}(\mathcal{F}, \mathcal{O}_F(-\delta))$  via the natural isomorphism described above, we get a symmetric isomorphism  $\mathcal{F} \cong \text{Hom}(\mathcal{F}, \mathcal{O}_F(-\delta))$ , and we finally conclude that  $\mathcal{F}$  is a quadratic sheaf. Thus a symmetric morphism  $\varphi : \mathcal{E}''^{\vee}(-d - \delta) \rightarrow \mathcal{E}''$  induces a quadratic sheaf with support on the hypersurface  $F := \{\det \varphi = 0\}$ .

Assume now that the generic rank of  $\mathcal{F}$  is 1, and choose a section  $\beta \in H^0(\mathcal{O}_F(\delta))$ : then the bilinear pairing  $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{O}_F(-\delta)$  composed with multiplication by  $\beta$  yields a commutative ring structure on the module  $\mathcal{O}_F \oplus \mathcal{F}$ .

We can then consider the scheme  $\tilde{F} := \text{Spec}(\mathcal{O}_F \oplus \mathcal{F})$ , which yields a 2:1 covering  $\pi : \tilde{F} \rightarrow F$ , étale over the complement of  $\Delta \cup \{\beta = 0\}$ , where  $\Delta$  is the locus where  $\text{rk}(\varphi) \leq \text{rk} \mathcal{E} - 2$ .

In this way, a subscheme  $\Delta$  of a locally Cohen–Macaulay projective scheme  $F$  is said to be bundle–symmetric if there exist a bundle  $\mathcal{E}$  and a symmetric morphism  $\varphi$  such that  $F$  is the corank 1 locus of  $\varphi$  and  $\Delta$  is the corank 2 locus of  $\varphi$ , as in the above setting.

**Definition 1.2.** *Let  $F$  be a nodal surface in  $\mathbb{P}^3$  and  $\Delta$  be a set of nodes of  $F$ . Consider the resolution  $\tilde{F}$  of  $F$  along the singularities in  $\Delta$  and the corresponding  $(-2)$ -curves  $A_i$ . A set of nodes  $\Delta \subset F$  is called  $\delta/2$ -even if the corresponding divisor  $\sum A_i + \delta H$  is 2-divisible in the Picard*

group of  $\tilde{F}$ , where  $H$  is the divisor class corresponding to a hyperplane section of  $F$  in  $\mathbb{P}^3$  (cf. [Cat1]).

This property amounts to the existence of a  $2 : 1$  covering  $\tilde{S}$  of  $\tilde{F}$  ramified along the divisor  $\sum A_i + \delta H$ , or, equivalently, to the existence of a  $2 : 1$  covering  $S$  of  $F$  ramified on  $\Delta$  (and on a hyperplane section if  $\delta = 1$ ) ( $S$  is obtained by blowing down the  $(-1)$  rational curves on  $\tilde{S}$  which are the inverse images of the  $A_i$ 's).

Clearly, a bundle-symmetric set of nodes is a  $\delta/2$ -even set, but also the converse holds, as it was shown in [Ca-Ca]:

**Theorem 1.3.** [Ca-Ca, theorem 0.3 and corollary 0.4] *Let  $F \subset \mathbb{P}^3$  be a nodal surface of degree  $d$ . Then every  $\delta/2$ -even set of nodes  $\Delta$  on  $F$ ,  $\delta = 0, 1$ , is the degeneracy locus of a symmetric map of locally free  $\mathcal{O}_{\mathbb{P}^3}$ -sheaves  $\mathcal{E}''^\vee(-d-\delta) \xrightarrow{\varphi} \mathcal{E}''$  (i.e.,  $F$  is the locus where  $\text{rk}(\varphi) \leq \text{rk} \mathcal{E}'' - 1$ ,  $\Delta$  is the locus where  $\text{rk}(\varphi) = \text{rk} \mathcal{E}'' - 2$ ). Moreover, if  $p : S \rightarrow F$  is a  $2 : 1$  covering associated to  $\Delta$  and  $\mathcal{F}$  is defined as the anti-invariant part of  $p_*(\mathcal{O}_S) = \mathcal{O}_F \oplus \mathcal{F}$ , then  $\mathcal{F}$  fits into the exact sequence*

$$0 \rightarrow \mathcal{E}''^\vee(-d-\delta) \xrightarrow{\varphi} \mathcal{E}'' \rightarrow \mathcal{F} \rightarrow 0.$$

The authors of [Ca-Ca] also describe how to construct the bundle  $\mathcal{E}''$ . Recall that the first syzygy bundle  $\text{Syz}^1(M)$  associated to a graded module  $M$  is obtained as follows. Take a graded free resolution of the module  $M$ :

$$0 \rightarrow \mathcal{P}^4 \rightarrow \dots \rightarrow P^1 \xrightarrow{\alpha_1} P^0 \xrightarrow{\alpha_0} M \rightarrow 0.$$

Then the homomorphism  $\alpha_1 : P^1 \rightarrow P^0$  induces a corresponding homomorphism  $(\alpha_1)^\sim$  between the (Serre-) associated sheaves  $(P^1)^\sim$  and  $(P^0)^\sim$ , and the first syzygy bundle of  $M$  is defined as  $\text{Syz}^1(M) := \text{Ker}(\alpha_1^\sim)$ .

In [Ca-Ca] it is shown that, up to direct sum of line bundles,  $\mathcal{E}''$  is the first syzygy bundle of the module

$$M = U \oplus \left( \bigoplus_{m > (d-4+\delta)/2} H^1(F, \mathcal{F}(m)) \right),$$

where, if  $(d + \delta)$  is even,  $U$  is any maximal isotropic subspace of  $H^1(F, \mathcal{F}((d-4+\delta)/2))$  with respect to the non-degenerate alternating form on  $H^1(F, \mathcal{F}((d-4+\delta)/2))$  induced by Serre's duality, and, if  $(d + \delta)$  is odd,  $U$  is zero (cf. [Ca-Ca] for more details).

Even sets of nodes are classified and explicitly described for surfaces of degree up to 5. In [Ca-To], we studied the case of even (i.e., 0-even) sets on sextic surfaces.

Particularly interesting is the case of even sets of cardinality 56: this is the first case where the module  $M$  is relatively big. Concerning the possible dimensions of the various graded pieces of  $M$ , we showed that only two cases can occur: either  $h^1(F, \mathcal{F}(1)) = h^1(F, \mathcal{F}(2)) = 3$  or  $h^1(F, \mathcal{F}(1)) = 3, h^1(F, \mathcal{F}(2)) = 4$ . Then we studied the former case.

In the former case  $U$  is thus an (isotropic) 3-dimensional vector space in  $H^1(F, \mathcal{F}(1))$  and, if we denote by  $W$  the 3-dimensional vector space  $W := H^1(F, \mathcal{F}(2))$  and set  $V := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ , we have that  $M$  is completely determined by the multiplication tensor

$$(1.1) \quad B \in U^\vee \otimes W \otimes V^\vee.$$

We now describe Beilinson's table for  $\mathcal{E}''$ , up to direct sums of line bundles. Since  $H^0(\mathbb{P}^3, \mathcal{F}(2)) = 0$  (as shown in prop. 2.4 of [Ca-To], compare also Beilinson's table for  $\mathcal{F}$  given in section 3 of [Ca-To]), one has  $H^0(\mathbb{P}^3, \mathcal{E}''(2)) = 0$ . Since, up to direct sums of line bundles,  $\mathcal{E}''$  is the first syzygy bundle of  $M$ , we have that  $H_*^1(\mathbb{P}^3, \mathcal{E}) \cong M$  and all the other intermediate cohomology modules of  $\mathcal{E}''$  are zero. It follows that Beilinson's table  $h^i(\mathcal{E}''(j))$  for  $\mathcal{E}''$ , up to direct sums of line bundles, is

$\uparrow i$					
	0	0	0	0	0
	0	0	0	0	0
	0	0	3	3	0
	0	0	0	0	*
					$\rightarrow j$

If we denote from now on by  $\mathcal{E}$  the previous  $\mathcal{E}(3)$ , theorem 1.3 gives the exact sequence

$$0 \rightarrow \mathcal{E}^\vee \xrightarrow{\varphi} \mathcal{E} \rightarrow \mathcal{F}(3) \rightarrow 0,$$

and, by Beilinson's theorem and the above cohomology table,  $\mathcal{E}(-1)$  is obtained by adding a direct sum of line bundles to the bundle

$$\mathcal{E}'(-1) := \ker \left( U \otimes \Omega^1(1) \cong 3\Omega^1(1) \xrightarrow{\beta(-1)} W \otimes \mathcal{O} \cong 3\mathcal{O} \right).$$

Moreover, since Beilinson's complex has no cohomology in degree  $\neq 0$ , the above map is surjective: hence  $\mathcal{E}'$  is a vector bundle with  $rk(\mathcal{E}') = 6$ .

Recall the Euler sequence

$$(1.2) \quad 0 \rightarrow \Omega^1(1) \rightarrow V \otimes \mathcal{O} \cong 4\mathcal{O} \xrightarrow{\epsilon} \mathcal{O}(1) \rightarrow 0,$$

where  $V := H^0(\mathbb{P}^3, \mathcal{O}(1))$  is the space of linear forms on  $\mathbb{P}^3$ , and suppose that the map  $H^0(\beta)$ , induced in cohomology by  $\beta$ , is also surjective (as it happens for a general  $\beta$ ): then  $h^0(\mathbb{P}^3, \mathcal{E}') = 3h^1(\Omega^1(2)) - 12 = 6$ .



Since  $h^0(\mathbb{P}^3, \mathcal{F}) = 6$  (cf. again Beilinson's table for  $\mathcal{F}$  in section 3 of [Ca-To]), one can make the following generality assumption:

**FIRST ASSUMPTION:**  $\mathcal{F}$  is generated in degree 3 and the linear map  $H^0(\mathcal{E}') \rightarrow H^0(\mathcal{F}(3))$  is an isomorphism.

Under the above assumption the following holds:

**Proposition 1.4.** [Ca-To, Prop 3.3] *Notation being as above, if the first assumption holds true then  $\mathcal{E} = \mathcal{E}'$  or, equivalently,  $\mathbf{rank}(\mathcal{E}) = 6$ . More precisely, there exists a homomorphism  $\beta : U \otimes \Omega^1(2) \cong 3\Omega^1(2) \rightarrow W \otimes \mathcal{O}(1) \cong 3\mathcal{O}(1)$  with  $\mathcal{E} = \ker \beta$  and such that we have an exact sequence*

$$(1.3) \quad 0 \rightarrow \mathcal{E} \rightarrow U \otimes \Omega^1(2) \cong 3\Omega^1(2) \xrightarrow{\beta} W \otimes \mathcal{O}(1) \cong 3\mathcal{O}(1) \rightarrow 0.$$

*Conversely, if  $\mathcal{E}$  is obtained in this way it is a rank 6 bundle and, if the map  $H^0(\beta)$  is surjective, it has an intermediate cohomology module  $M := H_*^1(\mathcal{E})$  having the required Hilbert function of type (3, 3).*

We are now able to explain why we got interested in vector bundles  $\mathcal{E}$  of rank 6 with Chern polynomial  $c(\mathcal{E})(t) = 1 + 3t + 6t^2 + 4t^3$ :

**Lemma 1.5.** *Let  $\mathcal{E}$  be given as kernel of a surjective homomorphism  $\beta : U \otimes \Omega^1(2) \cong 3\Omega^1(2) \rightarrow W \otimes \mathcal{O}(1) \cong 3\mathcal{O}(1)$ , where  $U$  and  $W$  are 3-dimensional vector spaces:*

$$(1.4) \quad 0 \rightarrow \mathcal{E} \rightarrow U \otimes \Omega^1(2) \cong 3\Omega^1(2) \xrightarrow{\beta} W \otimes \mathcal{O}(1) \cong 3\mathcal{O}(1) \rightarrow 0.$$

*Then  $\mathcal{E}$  is a rank 6 bundle with total Chern class*

$$(1.5) \quad c(\mathcal{E})(t) = 1 + 3t + 6t^2 + 4t^3,$$

*and  $H^0(\mathcal{E}^\vee) = 0$ .*

*Moreover, if the map  $H^0(\beta)$  is surjective,  $\mathcal{E}$  is a bundle with minimal cohomology, more precisely, it satisfies:*

- (1)  $H^i(\mathcal{E}) = 0 \ \forall i \geq 1;$
- (2)  $H^i(\mathcal{E}(-1)) = 0 \ \forall i \neq 1;$
- (3)  $H^i(\mathcal{E}(-2)) = 0 \ \forall i \neq 1;$
- (4)  $H^i(\mathcal{E}(-3)) = 0 \ \forall i;$
- (5)  $H^i(\mathcal{E}(-4)) = 0 \ \forall i.$

*In this case, in particular,  $h^0(\mathcal{E}) = 6$  and the unique nonzero intermediate cohomology groups of  $\mathcal{E}$  are  $U = H^1(\mathbb{P}^3, \mathcal{E}(-2))$  and  $W = H^1(\mathbb{P}^3, \mathcal{E}(-1))$ .*

*Conversely, a bundle  $\mathcal{E}$  with total Chern class as above and with minimal cohomology as above admits a presentation of type (1.4), where  $U = H^1(\mathbb{P}^3, \mathcal{E}(-2))$  and  $W = H^1(\mathbb{P}^3, \mathcal{E}(-1))$  are 3-dimensional vector spaces and  $H^0(\beta)$  is surjective.*

*Proof.* If  $\mathcal{E}$  is given as in (1.4), then it is a rank 6 bundle and its Chern polynomial is  $c(\mathcal{E}) = c(\Omega^1(2))^3 c(\mathcal{O}(1))^{-3} = (c(\mathcal{O}(1))^4 c(\mathcal{O}(2))^{-1})^3 c(\mathcal{O}(1))^{-3} = (1+t)^9 (1+2t)^{-3} = (1+9t+36t^2+84t^3)(1-6t+24t^2-80t^3) = 1+3t+6t^2+4t^3$ .

Dualizing the sequence  $0 \rightarrow \mathcal{E} \rightarrow 3\Omega^1(2) \xrightarrow{\beta} 3\mathcal{O}(1) \rightarrow 0$  yields

$$0 \rightarrow 3\mathcal{O}(-1) \rightarrow 3T(-2) \rightarrow \mathcal{E}^\vee \rightarrow 0.$$

Thus  $h^0(\mathcal{E}^\vee) = 0$ .

Let us verify that a bundle given as in (1.4) satisfies properties (2) and (3). The exact cohomology sequences of the twists of (1.4) give:  $H^1(\mathcal{E}(-2)) \cong 3H^1(\Omega^1)$ ,  $H^1(\mathcal{E}(-1)) \cong 3H^0(\mathcal{O})$ . This also shows that all the other intermediate cohomology modules of the above twists of  $\mathcal{E}$  are zero. Considering the Euler sequence, it is also clear that all the negative twists of  $\mathcal{E}$  have no global sections, and that  $\mathcal{E}(-2)$  and  $\mathcal{E}(-1)$  have vanishing third cohomology groups. Therefore  $\mathcal{E}(-2)$  and  $\mathcal{E}(-1)$ , if  $\mathcal{E}$  is given as in (1.4), have only first cohomology group, and of dimension 3.

The exact cohomology sequence of (1.4) gives  $H^1(\mathcal{E}) = \text{coker}(H^0(\beta))$  and  $H^2(\mathcal{E}) = 0$ . If  $H^0(\beta)$  is assumed to be surjective, then also property (1) is satisfied and  $h^0(\mathcal{E}) = \chi(\mathcal{E}) = 6$  is determined by the Riemann-Roch theorem.

Moreover, if  $H^0(\beta)$  is surjective, then also  $H^0(\beta(k))$  is surjective for positive twists  $k$ . Since  $\Omega^1(k)$  has no global section for  $k \leq 0$ , it follows easily that the intermediate cohomology group  $H_*^1(\mathcal{E})$  has nonzero degree parts only in degree -2 and -1 and that  $H_*^2(\mathcal{E}) = 0$ .

It remains to show the vanishing of the groups  $H^3(\mathcal{E}(k))$ , for  $-4 \leq k \leq 0$ . This follows from the vanishing of  $H^3(\Omega^1(k))$ , for  $-2 \leq k \leq 2$ , which is a straightforward computation:  $H^3(\Omega^1(k)) \cong H^0(\mathcal{T}^1(-k-4)) = 0$  for  $-2 \leq k \leq 2$ .

We now prove the converse. Assume that we have a vector bundle with such a Chern polynomial and minimal cohomology as described above. Then the Euler characteristics of  $\mathcal{E}$  (or its twists) are the same as the Euler characteristic of a bundle (or its twists) given as in (1.4). Therefore the first cohomology groups  $U = H^1(\mathcal{E}(-2))$  and  $W = H^1(\mathcal{E}(-1))$  are both 3-dimensional vector spaces.

By applying Beilinson's theorem to  $\mathcal{E}(-1)$ , it follows that  $\mathcal{E}$  fits in an exact sequence as in (1.4). Condition (1) implies that  $H^0(\beta)$  is surjective.  $\square$

2. SHEAVES ASSOCIATED TO TENSORS

Let  $W_1, \dots, W_r$  be vector spaces of respective dimensions  $\dim_{\mathbb{K}} W_j = d_j + 1$ , where  $\mathbb{K}$  is  $\mathbb{C}$  or any algebraically closed field. Assume that we have a tensor

$$(2.1) \quad B \in W_1 \otimes \cdots \otimes W_r.$$

To  $B$  one can associate a collection of subschemes of products of projective spaces, namely

**Definition 2.1.** *Let  $1 \leq i_1 < \cdots < i_h \leq r$  be a strictly increasing  $h$ -tuple of indexes between 1 and  $r$ . Then we define*

$$\Gamma_{i_1, \dots, i_h}(B) := \left\{ u = u_{i_1} \otimes \cdots \otimes u_{i_h} \mid u_{i_j} \in W_{i_j}^\vee, B \lrcorner u = 0 \right\};$$

where  $B \lrcorner u$  denotes the contraction of the tensor  $B$  with  $u$ .

**Remark 2.2.** 1) *Without loss of generality, we may assume  $i_1 = 1, \dots, i_h = h$ , and  $r = h$  or  $r = h + 1$  (it suffices to replace the vector spaces  $W_{h+1}, \dots, W_r$  with their tensor product  $W_{h+1} \otimes \cdots \otimes W_r$ ).*

2) *Observe that  $\Gamma_{i_1, \dots, i_h}(B) \subset \mathbb{P}(W_{i_1}^\vee) \times \cdots \times \mathbb{P}(W_{i_h}^\vee)$  is the intersection of hypersurfaces of multidegree  $(1, \dots, 1)$ .*

3) *Is  $h = r$ , then we have a single hypersurface of multidegree  $(1, \dots, 1)$ . Otherwise, we shall make the following assumption of generality:  $\Gamma_{1, \dots, r-1}(B)$  is the complete intersection of  $d_r + 1$  hypersurfaces of multidegree  $(1, \dots, 1)$ .*

4) *The case  $r = 1$  is empty, while the case  $r = 2$  is not very interesting, since we get corresponding linear maps  $A : W_1^\vee \rightarrow W_2$  and  ${}^tA : W_2^\vee \rightarrow W_1$  and loci  $\mathbb{P}(\ker A), \mathbb{P}(\ker {}^tA), \{(x, y) \in \mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee) \mid \langle y, Ax \rangle = 0\}$*

5) *Observe finally that it suffices to treat the case  $r = h + 1$ . In fact, if  $r = h$ , we take  $W_{r+1} = \mathbb{C}$ , whereas the case  $r > h + 1$  can be reduced to the case  $r = h + 1$ , as observed in part (1).*

We now fix a tensor  $B \in W_1 \otimes \dots \otimes W_{h+1}$  as above. In order to study the sheaves associated to the tensor  $B$ , the following assumption is fundamental.

**Main Assumption:** *consider  $\mathbb{P} := \mathbb{P}(W_1^\vee) \times \cdots \times \mathbb{P}(W_h^\vee)$  and the variety  $\Gamma := \Gamma_{1, \dots, h} \subset \mathbb{P}$ . We assume that  $\Gamma := \Gamma_{1, \dots, h} \subset \mathbb{P}$  is a complete intersection of  $d_{h+1} + 1$  hypersurfaces of multidegree  $(1, \dots, 1)$ .*

*We further assume that  $\Gamma \neq \emptyset$  (under the above assumption, this happens if and only if  $d_1 + \cdots + d_h \geq d_{h+1} + 1$ ).*

If the main assumption holds we have then a Koszul exact sequence

$$(2.2) \quad \dots \rightarrow \wedge^2 W^\vee \otimes \mathcal{O}_{\mathbb{P}}(-2, \dots, -2) \rightarrow W^\vee \otimes \mathcal{O}_{\mathbb{P}}(-1, \dots, -1) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\Gamma} \rightarrow 0,$$

where

$$\begin{cases} W := W_{h+1} \\ \mathbb{P} := \mathbb{P}(W_1^\vee) \times \cdots \times \mathbb{P}(W_h^\vee) \quad ; \\ \mathbb{P}(W_i^\vee) := \text{Proj}(\text{Sym}(W_i)) \end{cases}$$

or, equivalently,

$$(2.3) \quad \dots \rightarrow \wedge^2 W^\vee \otimes \mathcal{O}_{\mathbb{P}}(-2, \dots, -2) \rightarrow W^\vee \otimes \mathcal{O}_{\mathbb{P}}(-1, \dots, -1) \rightarrow \mathcal{I}_\Gamma \rightarrow 0,$$

**Definition 2.3.** *Assume now that  $s < h$ : consider  $\mathbb{P}' := \mathbb{P}(W_1^\vee) \times \cdots \times \mathbb{P}(W_s^\vee)$ ,  $\mathbb{P}'' = \mathbb{P}(W_{s+1}^\vee) \times \cdots \times \mathbb{P}(W_h^\vee)$ , and let  $p: \mathbb{P} \rightarrow \mathbb{P}'$  be the projection of the product  $\mathbb{P} = \mathbb{P}' \times \mathbb{P}'' \rightarrow \mathbb{P}'$  onto the first factor.*

*For  $\vec{t} = (t_{s+1}, \dots, t_h)$ , we define  $\mathcal{G}_{\vec{t}} := p_* \mathcal{O}_\Gamma(0, \dots, 0, t_{s+1}, \dots, t_h)$ .*

We aim at giving a resolution of  $\mathcal{G}_{\vec{t}}$ . The above situation is quite general, but in any case the object  $\mathcal{O}_\Gamma(0, \dots, 0, t_{s+1}, \dots, t_h)$  can be replaced, as an object in the derived category of coherent sheaves on  $\mathbb{P}$ , by the twisting by  $\mathcal{O}_{\mathbb{P}}(0, \dots, 0, t_{s+1}, \dots, t_h)$  of the resolution (2.2). By applying  $p_*$  to the exact sequence obtained in this way, we get a spectral sequence converging to  $R^k p_* \mathcal{O}_\Gamma(0, \dots, 0, t_{s+1}, \dots, t_h)$  (if  $h = s + 1$  we get a complex, as in Beilinson's theorem (cf. [Bei]), whose  $k$ -th cohomology group is  $R^k p_* \mathcal{O}_\Gamma(0, \dots, 0, t_{s+1}, \dots, t_h)$ ).

The advantage of using the twisted Koszul complex (2.2) is that a line bundle  $\mathcal{O}(a_{s+1}, \dots, a_h)$  on  $\mathbb{P}''$  is an external tensor product  $\mathcal{O}(a_{s+1}) \boxtimes \cdots \boxtimes \mathcal{O}(a_h)$ , hence its total cohomology  $H^*(\mathbb{P}'', \mathcal{O}(a_{s+1}, \dots, a_h))$  is the tensor product  $H^*(\mathbb{P}(W_{s+1}^\vee), \mathcal{O}(a_{s+1})) \otimes \cdots \otimes H^*(\mathbb{P}(W_h^\vee), \mathcal{O}(a_h))$ .

On the other hand,  $H^*(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(a))$  contains at most one term:  $H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(a))$  if  $a \geq 0$ ,  $H^d(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(a))$  if  $a \leq -d - 1$ , none if  $-d \leq a \leq -1$ . Whence, fixed  $i$ ,  $R^j p_* \mathcal{O}_{\mathbb{P}}(0, \dots, 0, t_{s+1} - i, \dots, t_h - i) = 0$  with only one possible exception  $j$ .

We thus obtain the following proposition.

**Proposition 2.4.** *There is a spectral sequence with  $E_1$  term  $E_1(-i, j)$  given by*

$$(2.4) \quad \begin{aligned} R^j p_* (\wedge^i W^\vee \otimes \mathcal{O}_{\mathbb{P}}(-i, \dots, -i, t_{s+1} - i, \dots, t_h - i)) = \\ = \wedge^i W^\vee \otimes H^j(\mathcal{O}_{\mathbb{P}''}(t_{s+1} - i, \dots, t_h - i)) \otimes \mathcal{O}_{\mathbb{P}'}(-i, \dots, -i) \end{aligned}$$

*which converges to the direct image sheaves  $R^k p_* \mathcal{O}_\Gamma(0, \dots, 0, t_{s+1}, \dots, t_h)$ .*

*Proof.* This is a standard spectral sequence argument, compare pages 149-150 of [Wei]. Consider the complex given by (2.2) (without the last

term at the right) and tensor it by  $\mathcal{O}_{\mathbb{P}}(0, \dots, 0, t_{s+1}, \dots, t_h)$ . The sequence obtained, say  $C^\cdot$ , remains exact, with the exception of the rightmost term, where the cohomology group is  $\mathcal{O}_{\Gamma}(0, \dots, 0, t_{s+1}, \dots, t_h)$ .

If we take an injective resolution of the complex and apply the functor  $p_*$ , we obtain a double complex with two associated spectral sequences. The horizontal spectral sequence degenerates at the  $E_1$  term, and yields the direct image sheaves  $E_1^{hor}(0, k) = R^k p_* \mathcal{O}_{\Gamma}(0, \dots, 0, t_s, \dots, t_h)$ .

The vertical spectral sequence, instead, yields an  $E_1$  term of the form  $E_1^{vert}(-i, j) = R^j p_* (\wedge^i W^\vee \otimes \mathcal{O}_{\mathbb{P}}(-i, \dots, -i, t_{s+1} - i, \dots, t_h - i))$ . This is precisely the spectral sequence which we choose, and which converges to  $R^k p_* \mathcal{O}_{\Gamma}(0, \dots, 0, t_{s+1}, \dots, t_h)$  as claimed.  $\square$

Consider the differential  $d_1$  at the  $E_1$  level of the vertical spectral sequence: it is a horizontal differential given at the place  $(-i, j)$  by a map

$$(2.5) \quad \wedge^i W^\vee \otimes H^j(\mathcal{O}_{\mathbb{P}''}(t_{s+1} - i, \dots, t_h - i)) \otimes \mathcal{O}_{\mathbb{P}'}(-i, \dots, -i) \rightarrow \\ \wedge^{i-1} W^\vee \otimes H^j(\mathcal{O}_{\mathbb{P}''}(t_{s+1} - i + 1, \dots, t_h - i + 1)) \otimes \mathcal{O}_{\mathbb{P}'}(-i + 1, \dots, -i + 1)$$

induced by 2.2.

By the above discussion on the cohomology groups  $H^*(\mathbb{P}'', \mathcal{O}(a_{s+1}, \dots, a_h))$ , first of all it follows that the term  $E_1(-i, j)$  is nonzero, for fixed  $i$ , only for at most one value of  $j$ .

More precisely, if  $H^j(\mathcal{O}_{\mathbb{P}''}(t_{s+1} - i, \dots, t_h - i)) \neq 0$ , then there is an expression  $j = j_{s+1} + \dots + j_h$  such that the above group is an external tensor product of cohomology groups  $H^{j_c}(\mathbb{P}^{d_c}, \mathcal{O}(t_c - i))$ . Since each term of the external tensor product must be nonzero, it follows that  $j_c = 0$  or  $j_c = d_c$  and that  $t_c - i \geq 0$  if  $j_c = 0$ , else  $t_c - i \leq -d_c - 1$ .

Moreover, we conclude also that  $H^{j-p}(\mathcal{O}_{\mathbb{P}''}(t_{s+1} - i + p + 1, \dots, t_h - i + p + 1)) = 0$  unless there is some  $j_c = d_c$  such that  $H^{d_c}(\mathbb{P}^{d_c}, \mathcal{O}(t_c - i)) \neq 0$  and  $H^0(\mathbb{P}^{d_c}, \mathcal{O}(t_c - i + p + 1)) \neq 0$ : this is only possible if  $t_c - i \leq -d_c - 1$  and  $t_c - i + p + 1 \geq 0$ . This implies  $-d_c - 1 \geq t_c - i \geq -p - 1$ , in particular,  $p \geq d_c$ .

We want now to consider an easier situation, first of all we would like to have

$$R^j p_*(\mathcal{O}_{\Gamma}(0, \dots, 0, t_{s+1}, \dots, t_h)) = 0 \text{ for } j \geq 1,$$

so that the given spectral sequence converges then to  $R^j p_*(\mathcal{O}_{\Gamma}(0, \dots, 0, t_{s+1}, \dots, t_h))$ .

To achieve this property, we assume  $s = h - 1$ .

**Lemma 2.5.** *If  $s = h - 1$ ,  $R^j p_*(\mathcal{O}_\Gamma(0, \dots, 0, t))$  for  $j \geq 1$ , assuming that  $t \geq -1$ .*

*Proof.* This follows from the base change theorem, since the fibres of  $\Gamma \rightarrow \mathbb{P}' = \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_{h-1}}$  are linear subspaces of  $\mathbb{P}^{d_h}$ , and since  $H^j(\mathcal{O}_{\mathbb{P}^d}(t)) = 0$  for any  $d$ ,  $t \geq -1$ , and  $j \geq 1$ .  $\square$

**Corollary 2.6.** *If  $s = r - 2$  and  $h = r - 1$ , then for  $t \geq -1$   $\mathcal{G}_t = p_* \mathcal{O}_\Gamma(0, \dots, 0, t)$  has a resolution given by an exact complex of vector bundles on  $\mathbb{P}'$  whose  $k$ -th term is*

$$(2.6) \quad \bigoplus_{j-i=k} \wedge^i W^\vee \otimes H^j(\mathcal{O}_{\mathbb{P}^{d_h}}(t-i)) \otimes \mathcal{O}_{\mathbb{P}'}(-i, \dots, -i).$$

*Proof.* In this case the spectral sequence degenerates at the  $E_2$  level if all the nonzero terms of  $E_1$  occur only for  $j = 0$ .

If there is a nonzero term with  $j = d_h$ , then, as we observed, for  $p > 0$  we have  $H^{j-p}(\mathcal{O}_{\mathbb{P}^{d_h}}(t-i+p+1)) \neq 0$  iff  $p = d_h$  and  $t-i = -d_h - 1$ .

In terms of differentials of the spectral sequence, this implies that on this nonzero term  $d_1 = d_2 = d_{p-1} = 0$ , and then also in the corresponding place  $d_{p+1} = \dots = 0$ . The result now follows.  $\square$

We want now to restrict ourselves to the case where we obtain sheaves on projective spaces, i.e., we restrict to the case  $r = 3$  of tritensors.

We have  $\mathbb{P} := \mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$ ,  $\mathbb{P}' := \mathbb{P}^{d_1}$  and  $p : \mathbb{P}^{d_1} \times \mathbb{P}^{d_2} \rightarrow \mathbb{P}^{d_1}$ . Recall that under the main assumption that  $\Gamma \subset \mathbb{P}$  be a complete intersection we have:

$$(2.7) \quad \Gamma \neq \emptyset \iff d_1 + d_2 \geq d_3 + 1.$$

By applying the above corollary (2.6) we get:

**Corollary 2.7.** *Suppose that  $\Gamma$  is not empty, c.f. (2.7). Assume that  $t - d_3 - 1 \geq -d_2$  (i.e.,  $t \geq d_3 + 1 - d_2$ ) and  $t \geq -1$ .*

*Then  $\mathcal{G}_t$  has a resolution given by an Eagon–Northcott type complex:*

$$(2.8) \quad \begin{array}{ccccccc} & \wedge^{d_3+1} W_3^\vee & & \wedge^2 W_3^\vee & & W_3^\vee & \\ & \otimes & & \otimes & & \otimes & S^t W_2 \\ 0 \rightarrow & S^{t-d_3-1} W_2 & \rightarrow \dots \rightarrow & S^{t-2} W_2 & \rightarrow & S^{t-1} W_2 & \rightarrow \otimes \rightarrow \mathcal{G}_t \rightarrow 0. \\ & \otimes & & \otimes & & \otimes & \mathcal{O}_{\mathbb{P}'} \\ & \mathcal{O}_{\mathbb{P}'}(-d_3-1) & & \mathcal{O}_{\mathbb{P}'}(-2) & & \mathcal{O}_{\mathbb{P}'}(-1) & \end{array}$$

**Remark 2.8.** 1) *Note that  $t \geq d_1$  suffices in the above corollary.*

2) Observe that  $S^t(W_2) = 0$  for  $t < 0$ , and  $S^0(W_2) = \mathbb{C}$ . We may then assume  $t \geq 0$ . For  $t = 0$  we need  $d_2 \geq d_3 + 1$ , and then we get  $\mathcal{G}_0 \cong \mathcal{O}_{\mathbb{P}'}$ .

**Example 2.1.** For  $t = 1$  we need  $d_2 \geq d_3$ , and then we get the “standard” matricial resolution

$$(2.9) \quad 0 \rightarrow W_3^\vee \otimes \mathcal{O}_{\mathbb{P}'}(-1) \rightarrow W_2 \otimes \mathcal{O}_{\mathbb{P}'} \rightarrow \mathcal{G}_1 \rightarrow 0,$$

where  $B \in \text{Hom}(W_3^\vee \otimes \mathcal{O}_{\mathbb{P}'}(-1), W_2 \otimes \mathcal{O}_{\mathbb{P}'}) \cong W_3 \otimes W_1 \otimes W_2$ .

**Example 2.2.** For  $t = 2$ , in the case where  $d_2 \geq d_3 - 1$  we get:

$$(2.10) \quad 0 \rightarrow \wedge^2 W_3^\vee \otimes \mathcal{O}_{\mathbb{P}'}(-2) \rightarrow W_3^\vee \otimes W_2^\vee \otimes \mathcal{O}_{\mathbb{P}'}(-1) \rightarrow S^2 W_2 \otimes \mathcal{O}_{\mathbb{P}'} \rightarrow \mathcal{G}_2 \rightarrow 0.$$

This case is the one we are particularly interested in, because if we set  $\widehat{\mathcal{E}} := p_* \mathcal{I}_\Gamma(0, 2)$ , then we obtain

$$(2.11) \quad 0 \rightarrow \wedge^2 W_3^\vee \otimes \mathcal{O}_{\mathbb{P}'}(-2) \rightarrow W_3^\vee \otimes W_2^\vee \otimes \mathcal{O}_{\mathbb{P}'}(-1) \rightarrow \widehat{\mathcal{E}} \rightarrow 0$$

$$(2.12) \quad 0 \rightarrow \widehat{\mathcal{E}} \rightarrow S^2 W_2 \otimes \mathcal{O}_{\mathbb{P}'} \rightarrow \mathcal{G}_2 \rightarrow 0.$$

**Remark 2.9.** The sheaves  $\mathcal{G}_t$  are supported on

$$\begin{aligned} p(\Gamma) &= \{u_1 \in W_1^\vee \mid \exists u_2 \in W_2^\vee \text{ s.t. } B \neg(u_1 \otimes u_2) = 0\} \\ &= \{u_1 \in W_1^\vee \mid B \neg u_1 \text{ has a nontrivial kernel}\} \\ &= \{u_1 \in W_1^\vee \mid \text{rk}(B \neg u_1) \leq d_2\}. \end{aligned}$$

In particular, the expected codimension of  $p(\Gamma)$  equals  $d_3 - d_2 + 1 = d_1 - \dim \Gamma$ .

**Remark 2.10.** If  $p(\Gamma)$  has codimension  $\geq 1$ , then  $\widehat{\mathcal{E}}$  is a vector bundle if and only if  $p(\Gamma)$  is a hypersurface and  $\mathcal{G}_2$  is Cohen-Macaulay.

*Proof.* Notice that (2.12) implies that, if  $\widehat{\mathcal{E}}$  is locally free, then  $\mathcal{G}_2$  has local projective dimension at most 1 (over the local ring  $\mathcal{O}_{\mathbb{P}'}$ ). Whence the codimension of  $p(\Gamma)$  (the support of  $\mathcal{G}_2$ ) is at most 1.

Thus, if  $p(\Gamma) \neq \mathbb{P}'$ ,  $p(\Gamma)$  is a hypersurface. Conversely, if  $p(\Gamma)$  is a hypersurface, then  $\mathcal{G}_2$  is Cohen-Macaulay iff it has projective dimension 1.

We dualize the exact sequence (2.12), obtaining:

$$0 \rightarrow S^2 W_2 \otimes \mathcal{O}_{\mathbb{P}'} \rightarrow (\widehat{\mathcal{E}})^\vee \rightarrow \mathcal{E}xt^1(\mathcal{G}_2, \mathcal{O}_{\mathbb{P}'}) \rightarrow 0;$$

$$0 \rightarrow \mathcal{E}xt^m(\widehat{\mathcal{E}}, \mathcal{O}_{\mathbb{P}'}) \rightarrow \mathcal{E}xt^{m+1}(\mathcal{G}_2, \mathcal{O}_{\mathbb{P}'}) \rightarrow 0, \quad \forall m \geq 1.$$

We have now that  $\text{pd } \mathcal{G}_2 = 1$  if and only if  $\mathcal{E}xt^m(\mathcal{G}_2, \mathcal{O}_{\mathbb{P}'}) = 0, \forall m > 1$ . Thus  $\mathcal{E}xt^m(\widehat{\mathcal{E}}, \mathcal{O}_{\mathbb{P}'}) = 0 \forall m > 0$ , equivalently  $\text{pd}(\widehat{\mathcal{E}}) = 0$  and  $\widehat{\mathcal{E}}$  is locally free.

□

We are now going to describe the case where  $d_1 = 3$ ,  $d_2 = d_3 = 2$ , and relate the above constructions (considering also all the possible permutations of the spaces  $W_1, W_2, W_3$ ) to the geometry of cubic surfaces in  $\mathbb{P}^3$ . We consider now a tensor

$$(2.13) \quad \widehat{B} \in V \otimes (\widehat{U})^\vee \otimes \widehat{W}; \quad \dim V = 4, \dim \widehat{U} = \dim \widehat{W} = 3.$$

Observe that we have 6 permutations of the three vector spaces, inducing 6 distinct product projections. Moreover, we may vary the twisting factor  $t$ .

We consider the exact order given above of the three vector spaces. On  $\mathbb{P}^3 := \mathbb{P}(V^\vee) = \text{Proj}(\text{Sym}(V))$ , for  $t = 1$  we get the sheaf  $\widehat{\mathcal{G}}_1 = (\mathcal{G}_1)_{\widehat{B}}$ :

$$(2.14) \quad 0 \rightarrow \widehat{W}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\widehat{B}_{\widehat{W}^\vee, \widehat{U}^\vee}} \widehat{U}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \widehat{\mathcal{G}}_1 \rightarrow 0$$

For  $t = 2$  we get a vector bundle  $\widehat{\mathcal{E}} := \widehat{\mathcal{E}}_{\widehat{B}}$ , fitting in the two exact sequences

$$(2.15) \quad 0 \rightarrow \wedge^2(\widehat{W}^\vee) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \widehat{W}^\vee \otimes \widehat{U}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \widehat{\mathcal{E}} \rightarrow 0$$

$$(2.16) \quad 0 \rightarrow \widehat{\mathcal{E}} \rightarrow S^2(\widehat{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \widehat{\mathcal{G}}_2 \rightarrow 0.$$

For  $t \geq 3$  it is pointless to proceed further, since indeed one finds that  $\widehat{\mathcal{G}}_t$  is the  $t$ -th symmetric power of  $\widehat{\mathcal{G}}_1$ . In fact, by Corollary 2.7,  $\widehat{\mathcal{G}}_t$  has resolution:

$$(2.17) \quad \begin{array}{ccccccc} & \wedge^3 \widehat{W}^\vee & \wedge^2 \widehat{W}^\vee & \widehat{W}^\vee & & & \\ & \otimes & \otimes & \otimes & S^t \widehat{U} & & \\ 0 \rightarrow & S^{t-3} \widehat{U} & \rightarrow S^{t-2} \widehat{U} & \rightarrow S^{t-1} \widehat{U} & \rightarrow \otimes & \rightarrow \widehat{\mathcal{G}}_t \rightarrow 0, & \\ & \otimes & \otimes & \otimes & \mathcal{O}_{\mathbb{P}^3} & & \\ & \mathcal{O}_{\mathbb{P}^3}(-3) & \mathcal{O}_{\mathbb{P}^3}(-2) & \mathcal{O}_{\mathbb{P}^3}(-1) & & & \end{array}$$

which is the third symmetric power of (2.14).

We now consider the order  $V \otimes \widehat{W} \otimes (\widehat{U})^\vee$  and repeat the same construction: this is equivalent to consider the above construction applied to the tensor

$$\sigma(\widehat{B}) \in V \otimes \widehat{W} \otimes (\widehat{U})^\vee,$$

where  $\sigma$  is the involution permuting  $\widehat{W}$  with  $(\widehat{U})^\vee$ .

We obtain the sheaves  $(\widehat{\mathcal{G}}_1)_{\sigma(\widehat{B})}$ ,  $(\widehat{\mathcal{G}}_2)_{\sigma(\widehat{B})}$ , and another vector bundle  $\widehat{\mathcal{E}}_{\sigma(\widehat{B})}$ , sitting in exact sequences equals to (2.14)–(2.17) with the roles



of  $\widehat{W}$  and  $(\widehat{U})^\vee$  exchanged:

$$(2.18) \quad 0 \rightarrow \widehat{U} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\widehat{B}_{\widehat{U}, \widehat{W}}} \widehat{W} \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow (\widehat{\mathcal{G}}_1)_{\sigma(\widehat{B})} \rightarrow 0$$

$$(2.19) \quad 0 \rightarrow \wedge^2 \widehat{U} \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \widehat{U} \otimes \widehat{W} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \widehat{\mathcal{E}}_{\sigma(\widehat{B})} \rightarrow 0$$

$$(2.20) \quad 0 \rightarrow \widehat{\mathcal{E}}_{\sigma(\widehat{B})} \rightarrow S^2 \widehat{W} \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \widehat{\mathcal{G}}_2 \rightarrow 0.$$

As we shall see, these two sheaves, supported on the same cubic surface  $G = \{\det(\widehat{B}_{\widehat{W}^\vee, \widehat{U}^\vee}) = 0\} (= \{\det(\widehat{B}_{\widehat{U}, \widehat{W}}) = 0\})$ , correspond to two plane representations of  $G$  as the blow-down of a sextuple of lines in a Double-6 configuration. One passes from one plane representation to the other by exchanging the roles of  $\widehat{W}$  and  $(\widehat{U})^\vee$ , i.e., by applying the trivial involution  $\sigma$  to the tensor  $\widehat{B}$ .

Considering the ordering  $\widehat{B} \in \widehat{U}^\vee \otimes V \otimes \widehat{W}$ , similar in spirit to the one  $\widehat{B} \in \widehat{W} \otimes V \otimes \widehat{U}^\vee$  we obtain a different geometric picture. Recall that  $\mathbb{P}^3 = \mathbb{P}(V^\vee)$ . We are considering the projection  $p: \mathbb{P}(\widehat{U}) \times \mathbb{P}^3 \rightarrow \mathbb{P}(\widehat{U})$  and  $\Gamma \subset \mathbb{P}(\widehat{U}) \times \mathbb{P}^3$  is the graph of the contraction morphism from  $G \subset \mathbb{P}^3$  to  $\mathbb{P}(\widehat{U})$ .

For  $t = 1$ , corollary (2.7) provides the resolution

$$0 \rightarrow \widehat{W}^\vee \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-1) \rightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})} \rightarrow p_*(\mathcal{O}_G(1)) \rightarrow 0,$$

which is the Hilbert–Burch resolution of  $\mathcal{I}_\zeta(3)$ , a twist of the ideal sheaf of a length 6 0-dimensional subscheme  $\zeta := p(\Gamma)$  of  $\mathbb{P}(\widehat{U})$ . Thus  $p_*(\mathcal{O}_G(1)) = \mathcal{I}_\zeta(3)$ , that is the linear forms on  $G$  correspond to the cubics in  $\mathbb{P}(\widehat{U})$  which are in the ideal sheaf  $\mathcal{I}_\zeta$ .

For  $t = 2$  we get a resolution of  $p_*(\mathcal{O}_G(2))$

$$(2.21) \quad \wedge^2 \widehat{W}^\vee \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-2) \rightarrow \widehat{W}^\vee \otimes V \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-1) \rightarrow S^2 V \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})} \rightarrow p_*(\mathcal{O}_G(2)),$$

and we find again the symmetric square of the previous resolution, thus a resolution for  $\mathcal{I}_\zeta^2(6)$ .

Similarly for the cases where  $t \geq 2$ .

Quite interesting is instead the ordering  $\widehat{B} \in \widehat{U}^\vee \otimes \widehat{W} \otimes V^\vee$ . similar in spirit to the one  $\widehat{B} \in \widehat{W} \otimes \widehat{U}^\vee \otimes V^\vee$ . In this case  $p(\Gamma) = \{u \in \widehat{U}^\vee \mid \widehat{B} \cdot u \text{ has a kernel}\}$  and  $\Gamma \subset \mathbb{P}(\widehat{U}) \times \mathbb{P}(\widehat{W}^\vee)$  is the complete intersection of 4 hypersurfaces of bidegree  $(1, 1)$ . Let  $H_1$  be the hyperplane class in  $\mathbb{P}(\widehat{U})$  and  $H_2$  the one in  $\mathbb{P}(\widehat{W}^\vee)$ : since  $(H_1 + H_2)^4 = 6H_1^2 H_2^2$ , we conclude that in general  $\Gamma$  consists of 6 points (but for our purposes it suffices that  $\Gamma$  is a complete intersection 0-dimensional subscheme of length equal to 6). Let again be  $\zeta$  the length-6 subscheme  $\zeta := p(\Gamma)$ .

For  $t = 2$  corollary (2.7) is still applicable and we get a non-classical resolution for  $\mathcal{O}_\zeta = p_*\mathcal{O}_\Gamma(0, 2)$ :

$$0 \rightarrow \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-2) \rightarrow V \otimes W \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-1) \rightarrow S^2 W \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})} \rightarrow p_*\mathcal{O}_\Gamma(0, 2) \rightarrow 0.$$

For  $t = 1$ , the complex

$$V \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-1) \rightarrow \widehat{W} \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})} \rightarrow \mathcal{O}_\zeta \rightarrow 0$$

is no longer necessarily exact, in the sense that corollary (2.7) does not apply. We shall now show (cf. next corollary) that we get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(\widehat{U})}(-4) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-1) \rightarrow \widehat{W} \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})} \rightarrow \mathcal{O}_\zeta \rightarrow 0.$$

This exact sequence may also be obtained, using  $\mathcal{O}_\zeta \cong \mathcal{E}xt^2(\mathcal{O}_\zeta, \mathcal{O}_{\mathbb{P}(\widehat{U})}(-4)) \cong \mathcal{E}xt^2(\mathcal{O}_\zeta, \mathcal{O}_{\mathbb{P}(\widehat{U})})$ , as the dual of the Hilbert–Burch resolution of  $\mathcal{O}_\zeta$ :

$$0 \rightarrow \widehat{W}^\vee \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-4) \rightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}(\widehat{U})}(-3) \rightarrow \mathcal{O}_{\mathbb{P}(\widehat{U})} \rightarrow \mathcal{O}_\zeta \rightarrow 0.$$

The following corollary spells out in detail corollary 2.6.

**Corollary 2.11.** *Suppose that  $\Gamma$  is not empty, c.f. (2.7). Assume that  $t > 0$  but  $t < d_3 + 1 - d_2$ .*

*Then  $\mathcal{G}_t$  has a resolution given by an Eagon–Northcott type complex:*

$$(2.22) \quad \begin{array}{ccccccc} & \wedge^{d_3+1} W_3^\vee & & \wedge^{t+d_2+2} W_3^\vee & & \wedge^{t+d_2+1} W_3^\vee & \\ & \otimes & & \otimes & & \otimes & \\ 0 \rightarrow & S^{d_3-d_2-t} W_2^\vee & \rightarrow \dots \rightarrow & W_2^\vee & \rightarrow & \mathcal{O}_{\mathbb{P}'}(-t-d_2-1) & \rightarrow \\ & \otimes & & \otimes & & & \\ & \mathcal{O}_{\mathbb{P}'}(-d_3-1) & & \mathcal{O}_{\mathbb{P}'}(-t-d_2-2) & & & \\ & & & & & \wedge^1 W_3^\vee & \\ & & & \wedge^t W_3^\vee & & \otimes & S^t W_2 \\ & & & \otimes & \rightarrow \dots \rightarrow & S^{t-1} W_2 & \rightarrow \otimes \rightarrow \mathcal{G}_t \rightarrow 0. \\ & & & \mathcal{O}_{\mathbb{P}'}(-t) & & \otimes & \mathcal{O}_{\mathbb{P}'} \\ & & & & & \otimes & \\ & & & & & \mathcal{O}_{\mathbb{P}'}(-1) & \end{array}$$

*Proof.* Of course, we have

$$H^j(\mathcal{O}_{\mathbb{P}^{d_2}}(a)) = 0 \text{ except for } \begin{cases} a \geq 0 & \text{if } j = 0; \\ a \leq -d_2 - 1 & \text{if } j = d_2. \end{cases}$$

Suppose that we have a free resolution on  $\mathbb{P}$  of  $\mathcal{O}_\Gamma(t)$  with terms

$$0 \rightarrow L_r \rightarrow \dots \rightarrow L_{t+d_2+1} \rightarrow \dots \rightarrow L_t \rightarrow \dots \rightarrow L_0 \rightarrow \mathcal{O}_\Gamma(t),$$

with  $\deg L_j = t - j$ . By applying the functor  $p_*(-)$  to an injective resolution, we obtain a double complex whose vertical spectral term

has an  $E_1$  term of the form

$$\begin{array}{cccccccc} R^{d_2}p_*(L_r) & * \cdots * & R^{d_2}p_*(L_{t+d_2+1}) & \cdots & 0 & \cdots & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & & & & \\ 0 & \cdots & \cdots & \cdots & R^0p_*(L_t) & * \cdots * & R^0p_*(L_0) & \end{array},$$

whose non-zero terms are only the ones indicated explicitly or with  $* \cdots *$ .

Moreover  $R^{d_2}p_*(L_{t+a}) = \wedge^{t+a}W_3^\vee \otimes H^{d_2}(\mathbb{P}(W_2), \mathcal{O}(-a)) \otimes \mathcal{O}_{\mathbb{P}^r}(-t-a)$  and  $H^{d_2}(\mathbb{P}(W_2), \mathcal{O}(-a)) \cong H^0(\mathbb{P}(W_2), \mathcal{O}(-d_2-1+a))$  by Serre's duality. As proven in corollary 2.6, we obtain therefore complex, which is exact by lemma (2.5) since  $t > 0$ , and is a resolution of  $\mathcal{G}_t$ .  $\square$

3. VECTOR BUNDLES  $\mathcal{E}$  ON  $\mathbb{P}^3$  WITH CHERN POLYNOMIAL  $1 + 3t + 6t^2 + 4t^3$ : GENERAL FEATURES, THEIR CONSTRUCTION, AND CUBIC SURFACES

In this section we study some general features of vector bundles  $\mathcal{E}$  on  $\mathbb{P}^3$  with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$ . Recall lemma 1.5 of section 1: under the open condition of having minimal cohomology, these bundles have quite a simple copresentation in terms of their intermediate cohomology modules. Indeed, we have seen that  $H_*^2(\mathbb{P}^3, \mathcal{E}) = 0$  and that  $H_*^1(\mathbb{P}^3, \mathcal{E})$  has only two nonzero graded pieces, namely the vector spaces  $U = H^1(\mathbb{P}^3, \mathcal{E}(-2))$  and  $W = H^1(\mathbb{P}^3, \mathcal{E}(-1))$ . Recall moreover that  $V := H^0(\mathbb{P}^3, \mathcal{O}(1))$  is the space of linear forms on  $\mathbb{P}^3$ .

We will see that there are three ways to construct such bundles:

- (1) as syzygy bundles starting from a tensor

$$(3.1) \quad B \in U^\vee \otimes W \otimes V^\vee,$$

which will be our natural choice to parametrize  $\mathcal{E}$  (we shall call this the **kernel construction**);

- (2) as extensions, starting from another tensor  $B' \in W'^\vee \otimes U'^\vee \otimes V$  (we shall call this the **direct construction**)

- (3) as a **direct image sheaf**, starting from a third tensor  $\widehat{B} \in V \otimes \widehat{U}^\vee \otimes \widehat{W}$  and using the construction described in section 2.

The relation occurring between  $B$  and  $B'$  will lead to the definition of the cross-product involution, while the relation occurring between  $B$  and  $\widehat{B}$  will be investigated in the section 4 after we introduce the cross-product involution.

We now explain the first construction. Suppose we have a bundle  $\mathcal{E}$  as in (1.4). Applying  $\text{Hom}(-, \mathcal{O})$  to the Euler sequence (1.2) and tensoring

by  $\mathcal{O}(1)$  yields, since  $\mathrm{Hom}(\mathcal{O}(2), \mathcal{O}(1)) = 0$ ,  $\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(1)) = 0$ ,

$$(3.2) \quad \mathrm{Hom}(\Omega^1(2), \mathcal{O}(1)) \cong \mathrm{Hom}(V \otimes \mathcal{O}(1), \mathcal{O}(1)).$$

Thus the map  $\beta$  factors through a map  $B : U \otimes (V \otimes \mathcal{O}(1)) \rightarrow W \otimes \mathcal{O}(1)$  and the sheaf map  $B$  is surjective. This surjectivity is obviously equivalent to  $H^0(B(-1)) : U \otimes V \rightarrow W$  being surjective.

In the sequel we shall often identify the sheaf map  $B$  with the corresponding tensor  $H^0(B(-1)) \in U^\vee \otimes W \otimes V^\vee$ .

Let  $\epsilon$  be the tensor product of the identity map of  $U$  with the evaluation map  $V \otimes \mathcal{O} \rightarrow \mathcal{O}(1)$ . Then one sees easily that  $\mathcal{E} = \ker \beta = \ker B \cap \ker \epsilon$  and that the short exact sequence (1.4) can be replaced by

$$(3.3) \quad 0 \rightarrow \mathcal{E} \rightarrow U \otimes V \otimes \mathcal{O}(1) \xrightarrow{B \oplus \epsilon} (W \otimes \mathcal{O}(1)) \oplus (U \otimes \mathcal{O}(2)) \rightarrow 0,$$

**Remark 3.1.** *The cohomology exact sequence associated to the following twist of (3.3), namely:*

$$0 \rightarrow \mathcal{E}(-2) \rightarrow U \otimes V \otimes \mathcal{O}(-1) \xrightarrow{(B \oplus \epsilon)(-2)} (W \otimes \mathcal{O}(-1)) \oplus (U \otimes \mathcal{O}) \rightarrow 0$$

*yields a canonical isomorphism  $U \cong H^1(\mathcal{E}(-2))$ .*

*Since there is a canonical isomorphism*

$$H^0(\epsilon(-1)) : U \otimes V \rightarrow U \otimes H^0(\mathcal{O}(1)),$$

*the projection of  $W \oplus (U \otimes V) \rightarrow W$  induces an isomorphism of  $H^1(\mathcal{E}(-1)) = \mathrm{Coker} H^0((B \oplus \epsilon)(-1))$  with  $W$ , such that the map  $B : U \otimes V \rightarrow W$  corresponds to the multiplication map of the cohomology module  $H_*^1(\mathcal{E})$ .*

**Definition 3.2.** *The kernel construction of the bundle  $\mathcal{E}$  is as follows. Consider a  $3 \times 3 \times 4$  tensor*

$$(3.4) \quad B \in U^\vee \otimes W \otimes V^\vee.$$

*Such  $B$  induces a linear map  $B : V \otimes U \rightarrow W$  and a homomorphism  $B : V \otimes U \otimes \mathcal{O} \rightarrow W \otimes \mathcal{O}$  of vector bundles on  $\mathbb{P}^3 := \mathbb{P}(V^\vee)$ , which induces a homomorphism  $\beta = B \oplus \epsilon : U \otimes \Omega^1(2) \rightarrow W \otimes \mathcal{O}(1)$  as described above.*

*If  $\beta$  is surjective,  $\mathcal{E} := \ker(\beta)$  as in (1.4) is a vector bundle.*

Moreover, lemma 1.5 shows that such an  $\mathcal{E}$  is a vector bundle with total Chern class  $c(\mathcal{E})(t) = 1 + 3t + 6t^2 + 4t^3$  and, if  $H^0(\beta)$  is surjective, with minimal cohomology (i.e., the conditions (1)–(5) of lemma 1.5 are satisfied), and, moreover,  $U = H^1(\mathbb{P}^3, \mathcal{E}(-2))$ ,  $W = H^1(\mathbb{P}^3, \mathcal{E}(-1))$  and the multiplication tensor is exactly  $B$ .

We now proceed by illustrating the direct construction, our second construction. Recall again that, by lemma 1.5, a tensor  $B$  such that  $H^0(B \oplus \epsilon)$  is surjective gives an  $\mathcal{E}$  with  $H^i(\mathcal{E}) = 0 \forall i \geq 1$ , and therefore  $h^0(\mathcal{E}) = 6$ . It seems therefore natural to introduce the so-called

**SECOND ASSUMPTION:**

1)  $\iota : 6\mathcal{O} \rightarrow \mathcal{E}$  is injective, hence we get an exact sequence:

$$(3.5) \quad 0 \rightarrow 6\mathcal{O} \rightarrow \mathcal{E} \rightarrow \tau \rightarrow 0,$$

2) the torsion sheaf  $\tau$  is  $\mathcal{O}_G$ -invertible, where  $G$  is the divisor of  $\Lambda^6(\iota)$ .

If for a vector bundle given as in (1.4) the second assumption is satisfied, then the divisor  $G$  is a cubic surface, and  $\mathcal{E}$  may be reconstructed as an extension of  $6\mathcal{O}$  by  $\tau$ .

We first analyse the geometry and cohomology of the  $\mathcal{O}_G$ -invertible sheaf  $\tau$ . Let  $\tau = \mathcal{O}_G(D)$  and let  $H$  denote the hyperplane divisor in  $\mathbb{P}^3$ . We reproduce here remark 4.3 of [Ca-To]

**Remark 3.3.** *Notation being as 1) above (even without assuming  $\tau$  to be  $\mathcal{O}_G$ -invertible) set  $\tau' = \mathcal{E}xt^1(\tau, \mathcal{O})$ : then the dual of the previous exact sequence (3.5) gives*

$$(3.6) \quad 0 \rightarrow \mathcal{E}^\vee \rightarrow 6\mathcal{O} \rightarrow \tau' \rightarrow 0.$$

and we have:

- (1) By (3.5) clearly  $H^0(\tau) = H^1(\tau) = H^2(\tau) = 0$ .
- (2) By (3.6) and since  $h^i(\mathcal{E}^\vee) \cong h^{3-i}(\mathcal{E}(-4))$  we get  $h^0(\tau') = 6$ ,  $H^1(\tau') = H^2(\tau') = 0$ .
- (3) Since by definition  $\tau' = \mathcal{E}xt^1(\tau, \mathcal{O})$ , applying the functor  $\mathcal{H}om(\tau, -)$  to the exact sequence  $0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(3) \rightarrow \mathcal{O}_G(3) \rightarrow 0$  we get  $\tau' = \mathcal{H}om(\tau, \mathcal{O}_G(3))$ . Therefore, if  $\tau = \mathcal{O}_G(D)$ , then  $\tau' = \mathcal{O}_G(3H - D)$ .

Since  $h^i(D) = 0 \forall i$ ,  $h^0(3H - D) = 6$ ,  $h^i(3H - D) = 0$  for  $i = 1, 2$ , by Riemann Roch follows that  $D^2 + DH = -2$  and  $10 = 36 - 7DH + D^2$ . Therefore  $HD = 3$ ,  $D^2 = -5$ .

Setting  $\Delta := D + H$ , it turns out that  $\Delta H = 6$ ,  $\Delta^2 + \Delta K_G = -2$ , i.e.,

$$\Delta H = 6, \Delta^2 = 4.$$

**Lemma 3.4.** *Assume that  $G$  is a smooth cubic surface: then there exists a realization of  $G$  as the image of the plane under the system  $|3L - \sum_1^6 E_i|$  of plane cubics through six points, such that either  $\Delta \equiv 2L$ , i.e.,  $\Delta$  corresponds to the conics in the plane, or (up to permutations of the six points)  $\Delta \equiv 3L - 2E_1 - E_2$ .*

*Proof.* Observe preliminarily that if  $|H| = |3L - \sum_1^6 E_i|$  is such a planar realization of a cubic surface, then another one is obtained via a standard Cremona transformation centered at three of the points  $P_i$  corresponding to the  $(-1)$ -curves  $E_i$ .

In fact, if  $L' := 2L - E_1 - E_2 - E_3$ , then

$$H = 3L' - (L - E_1 - E_2) - (L - E_1 - E_3) - (L - E_2 - E_3) - E_4 - E_5 - E_6.$$

We have  $0 = H^2(D) = H^0(-D - H)$  and a fortiori  $H^2(\Delta) = H^0(-D - 2H) = 0$ . It follows that  $|\Delta|$  has  $h^0(\Delta) \geq 6$ ,  $\Delta^2 = 4$ , and the arithmetic genus  $p_a(\Delta) = 0$ .

Since  $0 = H^1(D) = H^1(K_G - D) = H^1(-H - D) = H^1(-\Delta)$ , it follows that  $\Delta$  is connected.

Hence we have a representation  $\Delta \equiv nL - \sum_1^6 a_i E_i$ , where the  $a_i$ 's are non negative and we assume  $a_1 \geq a_2 \geq \dots \geq a_6$ .

We have:  $\Delta^2 = 4 = n^2 - \sum_1^6 a_i^2$ ,  $\Delta \cdot H = 6 = 3n - \sum_1^6 a_i$ , i.e.

$$(3.7) \quad n^2 = \sum_1^6 a_i^2 + 4, \quad 3n = \sum_1^6 a_i + 6.$$

We want to show that, after a suitable sequence of standard Cremona transformations,  $\Delta \equiv 2L$  or  $\Delta \equiv 3L - 2E_1 - E_2$ . By (3.7), we have  $n \geq 2$  and for  $n = 2, 3$   $\Delta$  is as claimed. Hence the claim is that there exists a sequence of standard Cremona transformations which makes  $|\Delta|$  have degree  $n \leq 3$ .

By applying  $|2L - E_1 - E_2 - E_3|$  we get a new system  $\Delta'$  with degree  $n' = 2n - a_1 - a_2 - a_3$ .

By our ordering choice for the  $a_i$ 's, we have

$$a_1 + a_2 + a_3 \geq \left( \sum_{i=1}^6 a_i \right) / 2 = 3n/2 - 3,$$

with strict inequality unless all  $a_i$ 's are equal. We study this latter case first:

**Sublemma.** In the previous setting,  $a_1 = a_2 = \dots = a_6$  if and only if  $n = 2$  and  $a_1 = a_2 = \dots = a_6 = 0$  or  $n = 10$  and  $a_1 = a_2 = \dots = a_6 = 4$ .

*Proof.* The statement follows immediately by defining  $a := a_1 = a_2 = \dots = a_6$  and using both conditions of (3.7):  $n = 2a + 2$ ,  $n^2 = 6a^2 + 4$ , which imply  $8a = 2a^2$ .  $\square$

The previous inequality gives:

$$n' \leq \frac{n}{2} + 3 \leq n \quad \text{for } n \geq 6,$$

and  $n' < n$  for  $n \geq 6$  unless  $n' = n = 6$  and  $a_1 = a_2 = \dots = a_6$ , which has no solution by the above sublemma. We conclude that after suitable Cremona transformations  $n \leq 5$ .

If  $n = 5$ , then  $n' \leq 5/2 + 3$ , i.e.,  $n' \leq 5$ . Moreover, if also  $n' = 5$ , then  $a_1 + a_2 + a_3 = 5$  and using again 3.7 we obtain  $a_4 + a_5 + a_6 = 4$ . But then  $a_1 = a_6 + 1$  and we easily get a contradiction since then  $a_2 = a_3 = a_4 = a_5$  and they either equal  $a_1$  or  $a_6$ . Hence, after a suitable Cremona transformation, we can always reduce to the case  $n \leq 4$ .

Let now  $n = 4$ . Using (3.7) we get  $\sum_{i=1}^6 a_i = 6$  and  $\sum_{i=1}^6 a_i(a_i - 1) = 6$ . We have the following two possibilities:  $a_1 = 3, a_2 = a_3 = a_4 = 1, a_5 = a_6 = 0$  or  $a_1 = a_2 = a_3 = 2, a_4 = a_5 = a_6 = 0$ . In both cases we have that  $a_1 + a_2 + a_3 \geq 5$ , and therefore  $n' \leq 3$ .

*Q.E.D.*

**Remark 3.5.** *The complete linear system  $\Delta$  has as image in  $\mathbb{P}^5$  either the Veronese embedding of  $\mathbb{P}^2$ , or the embedding of  $\mathbb{P}^1 \times \mathbb{P}^1$  through  $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2))$ . In both case we have a surface of minimal degree (=4).*

Thus we have concluded that either  $D = 2L - H = -L + \sum E_i$ , or  $D = 3L - 2E_1 - E_2 - H = -E_1 + \sum_3^6 E_i$ . The latter case does not occur, because an extension of  $6\mathcal{O}$  by such a  $\tau$  will not have the required cohomology table. The former case is instead possible. We refer to [Ca-To], Lemma 4.12 and Lemma 4.13 for the proof of these facts.

We are now able to explain the second way to construct vector bundles  $\mathcal{E}$  such that  $H^i(\mathcal{E}) = 0 \forall i \geq 1$  and  $h^0(\mathcal{E}) = 6$ , as required in Lemma (1.5): we construct them as extensions of  $6\mathcal{O}$  by  $\tau$ , where  $\tau$  is the sheaf corresponding to  $-L + \sum_1^6 E_i$  on a smooth cubic surface  $G$  (cf. [Ca-To, Lemma 4.12]).

Before we give this second construction, we study these extensions. Setting  $\tau' := \mathcal{E}xt^1(\tau, \mathcal{O})$  and recalling remark (3.3), we see that such extensions are parametrized by  $Ext^1(\tau, 6\mathcal{O}) = H^0(6\mathcal{E}xt^1(\tau, \mathcal{O})) \cong \mathbb{C}^{36}$ .

**Lemma 3.6.** [Ca-To, Lemma 4.10] *Assume that  $h^0(\mathcal{E}^\vee) = 0$  and that  $\mathcal{E}$  is an extension as in (3.5): then the extension class in  $Ext^1(\tau, 6\mathcal{O}) = H^0(6\mathcal{E}xt^1(\tau, \mathcal{O})) \cong \mathbb{C}^6 \otimes \mathbb{C}^6$  is a rank 6 tensor (we shall refer to this statement by saying that **the extension does not partially split**).*

*In particular,  $\mathcal{E}$  is then uniquely determined up to isomorphism.*

*Proof.* The extensions which yield vector bundles form an open set.

We canonically view the space of these extension classes as  $\mathrm{Hom}(H^0(\tau'), H^0(6\mathcal{O})) = \mathrm{Hom}(H^0(\tau'), \mathbb{C}^6)$ , through the coboundary map of the corresponding exact sequence. We have then an action of  $GL(6, \mathbb{C})$  as a group of automorphisms of  $6\mathcal{O}$ , which induces an action on  $\mathrm{Hom}(H^0(\tau'), H^0(6\mathcal{O})) = \mathrm{Hom}(H^0(\tau'), \mathbb{C}^6)$  which corresponds to the composition of the corresponding linear maps.

The extensions which yield vector bundles form an open set, which contains an open dense orbit, on which this action is free, namely, the tensors of rank = 6.

If the rank of the tensor corresponding to an extension is  $= r < 6$ , it follows that the extension is obtained from an extension  $0 \rightarrow r\mathcal{O} \rightarrow \mathcal{E}'' \rightarrow \tau \rightarrow 0$  taking then a direct sum with  $(6-r)\mathcal{O}$ : but then  $(6-r)\mathcal{O}$  is a direct summand of  $\mathcal{E}^\vee$ , contradicting  $h^0(\mathcal{E}^\vee) = 0$ . □

**Corollary 3.7.**  *$\mathcal{E}$  as above (3.6) is a vector bundle if  $H^0(\tau')$  has no base points.*

*Proof.* Our hypothesis shows that in each point of  $G$  the local extension class is non zero, hence it yields a locally free sheaf. □

The second assumption yields a cubic surface  $G \subset \mathrm{Proj}(V)$  and an invertible sheaf  $\tau$  on  $G$ . If  $G$  is smooth, the invertible sheaf  $\tau(1) = 2L$  yields then a birational morphism onto a Veronese surface, whence represents  $G$  as the blow up of a projective plane  $\mathbb{P}^2$  in a subscheme  $\zeta$  consisting of six points (distinct if the cubic  $G$  is smooth), and as the image of  $\mathbb{P}^2$  through the linear system of cubic curves passing through  $\zeta$ . The Hilbert-Burch theorem allows us to make an explicit construction which goes in the opposite direction.

**Remark 3.8.** *Let  $U', W'$  be 3-dimensional vector spaces and set  $\mathbb{P}^2 := \mathbb{P}(U')$ . Consider a  $3 \times 3 \times 4$  tensor*

$$(3.8) \quad B' \in W'^{\vee} \otimes U'^{\vee} \otimes V$$

*and assume that the induced sheaf homomorphism  $W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^2}$ , which we call again  $B'$ , yields an exact sequence*

$$(3.9) \quad 0 \rightarrow W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{B'} V \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\Lambda^3(B')} \mathcal{O}_{\mathbb{P}^2}(3) \rightarrow \mathcal{O}_{\zeta}(3) \rightarrow 0$$

*which is the Hilbert Burch resolution of a codimension 2 subscheme  $\zeta$  of length 6.*



We obtain a canonical isomorphism  $V \cong H^0(I_\zeta(3))$  and we let  $G \subset \text{Proj}(V)$  be the image of  $\mathbb{P}^2$  via the rational map  $\psi$  associated to  $V$ . Under the above assumption on  $B'$ , if moreover  $\zeta$  is a local complete intersection,  $G$  is a normal cubic surface and, if we set  $\mathcal{G} := (\psi_*(\mathcal{O}(1)))$ , there is an exact sequence on  $\text{Proj}(V)$ :

$$(3.10) \quad 0 \rightarrow W' \otimes \mathcal{O}(-1) \xrightarrow{B'} U'^{\vee} \otimes \mathcal{O} \rightarrow \mathcal{G} \rightarrow 0.$$

Under the more general assumption that the sheaf map  $B'$  in (3.10) never drops rank by 2,  $\mathcal{G}$  is an invertible sheaf on the cubic surface  $G$ , and there is a Cartier divisor  $L$  on  $G$  such that  $\mathcal{G} = \mathcal{O}_G(L)$  (and  $h^0(\mathcal{O}_G(L)) = 3$ ).

**Definition 3.9.** We define now the **direct construction** of the bundle  $\mathcal{E}$  relying on our results above.

Consider a  $3 \times 3 \times 4$  tensor

$$(3.11) \quad B' \in W'^{\vee} \otimes U'^{\vee} \otimes V$$

such that the sheaf  $\mathcal{G}$  defined by the exact sequence (3.10) is invertible on the cubic surface  $G$  (i.e., at each point  $y \in \mathbb{P}^3$   $\text{rank}(\mathcal{G} \otimes \mathbb{C}_y) \leq 1$ ).

Define  $\tau := \mathcal{G}^{\otimes 2}(-1)$  and let  $\mathcal{E}$  be a vector bundle which is an extension of  $6\mathcal{O}$  by  $\tau$  as in (3.5) (here and elsewhere,  $\mathcal{O} := \mathcal{O}_{\text{Proj}(V)}$ ).

We then have the following results:

**Proposition 3.10.** [Ca-To, Proposition 4.17]  $\mathcal{E}$  as above is unique up to isomorphism in the following cases:

- (1) if  $G$  is a smooth cubic surface.
- (2) if  $G$  is the reduced union of a plane  $T$  and a smooth quadric  $Q$  intersecting transversally.

**Remark 3.11.** The case where  $G$  is a linear projection of the cubic scroll  $Y$  (birational embedding of  $\mathbb{P}^2$  in  $\mathbb{P}^4$  by the system  $|2L - E|$ ) yields two sheaves  $\mathcal{G}$  which are not invertible.

As it is well known, every point in  $\mathbb{P}^4$  lies in one of the planes spanned by the conics of the system  $L$ . If we project from a point in  $\mathbb{P}^4 \setminus Y$ , this conic maps two to one to the double line of the cubic  $G$ .

Such a plane is said to be **special** if the conic splits into two lines  $E + F, F \equiv L - E$ .

In the non special case, we may assume without loss of generality that the conic corresponds to the line  $z = 0$  in the plane, that the blown up point is the point  $x = y = 0$ , and that the linear system mapping to  $G$  is generated by  $(zx := x_0, zy := x_1, x^2 := x_2, y^2 := x_3)$ . In this case

one sees that the matrix  $B'$  is

$$B' = \begin{pmatrix} x_0 & 0 & x_1 \\ 0 & x_1 & -x_0 \\ -x_2 & -x_3 & 0 \end{pmatrix},$$

hence the rank of  $B'$  drops by 2 on the line  $x_0 = x_1 = 0$  ( $G$  is then the cubic of equation  $-x_0^2x_3 + x_2x_1^2 = 0$ ).

In the special case, we may again assume that the blown up point is the point  $x = y = 0$ , we assume that the line  $F$  is the proper transform of  $x = 0$ , and that the linear system mapping to  $G$  is generated by  $(y^2 + zx := x_0, x^2 := x_1, yz := x_2, xy := x_3)$  (in the projective embedding given by  $(zx, yz, x^2, xy, y^2)$  it corresponds to projection from the point  $(1, 0, 0, 0, -1) \in \mathbb{P}^4 \setminus Y$ ).

In this case one sees that the matrix  $B'$  is

$$B' = \begin{pmatrix} x_3 & -x_2 & -x_0 \\ -x_1 & 0 & x_3 \\ 0 & x_3 & -x_1 \end{pmatrix},$$

hence the rank of  $B'$  drops by 2 on the line  $x_3 = x_1 = 0$  ( $G$  is then the cubic of equation  $-x_3^3 + x_1^2x_2 + x_1x_2x_3 = 0$ ).

#### 4. THE CROSS-PRODUCT-INVOLUTION AND SCHUR'S QUADRIC

In the previous section we have seen that to a vector bundle as in (1.4) satisfying the second assumption one can associate two tri-tensors: the tri-tensor  $B \in U^\vee \otimes W \otimes V^\vee$  and the tri-tensor  $B' \in W'^\vee \otimes U'^\vee \otimes V$ . The first corresponds to the unique nonzero multiplication matrix of the intermediate cohomology module  $H_*^1(\mathcal{E})$ , the second, according to the direct construction, defines on a cubic surface the invertible sheaf  $\mathcal{G}$  such that  $\mathcal{E}$  is an extension of  $\tau = \mathcal{G}^{\otimes 2}(-1)$  and  $6\mathcal{O}$ .

What is the relation between them? In this section we will show that there is indeed a strict relation between such tri-tensors: a birational involution, which the authors call cross-product-involution.

In [Ca-To] the authors, after having discovered these two tensors, relate them by constructing a not necessarily minimal resolution of a bundle  $E$  constructed by means of the tri-tensors  $B$  and  $B'$ .

Indeed, given  $B$ , Beilinson's complex for  $\mathcal{E}$  yields a short exact sequence

$$0 \rightarrow U \otimes \Omega^2(2) \rightarrow W \otimes \Omega^1(1) \oplus 6\mathcal{O} \rightarrow \mathcal{E} \rightarrow 0,$$

where  $U = H^1(\mathcal{E}(-2))$  and  $W = H^1(\mathcal{E}(-1))$ . We get:

$$(4.1) \quad \begin{array}{ccccccc} & U \otimes \mathcal{O}(-2) & & U \otimes V \otimes \mathcal{O}(-1) & & 6\mathcal{O} & \\ 0 \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow \mathcal{E} \rightarrow 0. \\ & W \otimes \mathcal{O}(-3) & & W \otimes \Lambda^3 V \otimes \mathcal{O}(-2) & & W \otimes \Lambda^2 V \otimes \mathcal{O}(-1) & \end{array}$$

On the other side,  $B'$  gives a resolution of  $\mathcal{G}$ , from which it is possible to compute a resolution of  $\tau = \mathcal{G}^{\otimes 2}(-1)$ . From this one, by applying the mapping cone, it is possible again to get a resolution of  $\mathcal{E}$ :

$$(4.2) \quad 0 \rightarrow \Lambda^2 W' \otimes \mathcal{O}(-3) \rightarrow U'^{\vee} \otimes W' \otimes \mathcal{O}(-2) \rightarrow \begin{array}{c} 6\mathcal{O} \\ \oplus \\ S^2 U'^{\vee} \otimes \mathcal{O}(-1) \end{array} \rightarrow \mathcal{E} \rightarrow 0.$$

Comparing the two resolutions, the authors obtained the following identifications:

- (1)  $W' \cong \Lambda^2 W$ ,  $U \cong \ker[-B' : \Lambda^2 W' \otimes V^{\vee} \rightarrow U'^{\vee} \otimes W']$ ,
- (2)  $U'^{\vee} \otimes W' \cong (W \otimes \Lambda^3 V)/U$ ,  $S^2 U'^{\vee} \cong (W \otimes \Lambda^2 V)/(U \otimes V)$ ;

where  $-B'$  is the contraction given by the composition of the natural inclusion from  $(\Lambda^2 W') \otimes V^{\vee}$  to  $(W' \otimes W') \otimes V^{\vee}$  with the map  $B' \otimes id_{W'}(-1)$ .

Based on the above considerations we state the following correspondence for a pair of tri-tensors as above.

#### Definition-Proposition 4.1.

*The Cross-Product Involution on Tensors of type 3 x 3 x 4 is given as follows.*

*Consider a 4-tuple  $(U, W, V^{\vee}, B)$ , where:*

- (1)  $U, W$  are 3-dimensional vector spaces and  $V$  is a 4-dimensional vector space;
- (2) each vector space is equipped with a given orientation, identifying respectively  $\wedge^3 U$ ,  $\wedge^3 W$ , and  $\wedge^4(V^{\vee})$  with  $\mathbb{C}$ ;
- (3)  $B$  is a tensor

$$B \in U^{\vee} \otimes W \otimes V^{\vee} \cong \text{Hom}(U, W) \otimes V^{\vee}.$$

*We remark that the ordering of the three vector spaces yields a sheaf homomorphism which is canonically associated to the tensor  $B$ , namely*

$$U \otimes \mathcal{O}_{\text{Proj}(V^{\vee})}(-1) \xrightarrow{[B]} W \otimes \mathcal{O}_{\text{Proj}(V^{\vee})}.$$

*The **trivial involution** associates to the 4-tuple  $(U, W, V^{\vee}, B)$  the 4-tuple  $(W^{\vee}, U^{\vee}, V^{\vee}, B)$ .*

*The **reversing construction** associates to the 4-tuple  $(U, W, V^{\vee}, B)$  the 4-tuple  $(W', U'^{\vee}, V, B')$ , where:*

- (1)  $W' := \Lambda^2(W)$ . Since  $W$  is equipped with an orientation, the duality  $W \otimes \Lambda^2 W \rightarrow \mathbb{C}$  induces a canonical isomorphism of  $W'$  with  $W^\vee$ .  
 $U' := \ker[\neg B : \Lambda^2(W^\vee) \otimes V \rightarrow U^\vee \otimes W^\vee]$ , where  $\neg B$  is the contraction with the tensor  $B$ ; in particular,  $U'$  is canonically isomorphic to a subspace of  $W \otimes V$ .
- (2) the three vector spaces  $W'$ ,  $U'$ , and  $V$  are equipped with the orientations induced from the orientations of  $U$ ,  $W$ ,  $V^\vee$ , under the 'main assumption' that the contraction map  $\neg B$  be surjective.
- (3) the tensor

$$B' \in W'^\vee \otimes U'^\vee \otimes V = \text{Hom}(W', U'^\vee) \otimes V,$$

which corresponds to the natural inclusion  $U' \rightarrow W \otimes V$ , in view of the isomorphism  $W \cong (W')^\vee$ .

The dimension of  $U'$  is equal to 3 if we make the

**MAIN ASSUMPTION:** The contraction  $\neg B$  is surjective.

The **cross-product involution** is the involution, defined for the 4-tuples  $(U, W, V^\vee, B)$ , where  $B$  is assumed to belong to the open set of tensors satisfying the main assumption, which is given by the composition of the reversing construction with the trivial involution.

The cross-product involution associates thus to the 4-tuple  $(U, W, V^\vee, B)$  the 4-tuple  $(U', W = W'^\vee, V, B')$ .

*Proof.* We only need to show that the cross-product involution is an involution, i.e., that applying it twice, we obtain the identity.

We present here a different proof from the one given in [Ca-To], and based on the following

**Fact:** if the main assumption holds, then there is an element  $Q \in S^2(V)$ , called Schur's quadric, such that if we denote by  $q : V^\vee \rightarrow V$  the corresponding linear map, then  $q$  is an isomorphism and  $id_W \otimes q$  carries  $U \subset W \otimes V^\vee$  to  $U' \subset W \otimes V$ .

Indeed, by the construction of the Schur quadric, it follows that the inverse  $q^{-1}$  of the linear map  $q$  is obtained from the Schur quadric  $Q^\vee \in S^2(V^\vee)$  associated to  $B'$ , and therefore  $U'' = id_W \otimes q^{-1}(U') = U$ .  $\square$

**Remark 4.2.** The two tensors considered in remark 3.11, whose respective determinants yield the two non normal irreducible cubics (which are not projectively equivalent) satisfy the main assumption. But the cross-product involution constructs out of them two tensors which do no longer satisfy the main assumption, and which are projectively equivalent:

$$\begin{pmatrix} x_2^\vee & x_3^\vee & 0 \\ 0 & x_2^\vee & x_3^\vee \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ x_0^\vee & x_2^\vee & 0 \\ x_2^\vee & 0 & x_3^\vee \end{pmatrix}.$$

Let us now explain how the Schur quadric is obtained.

Let  $U, W, V$  be complex vector spaces of respective dimensions 3,3,4 and let  $B \in U^\vee \otimes W \otimes V^\vee$  be a tensor of type (3,3,4), as in (3.1). Following the notation of [Do-Ka], to  $B$  are associated then 3 maps:

$$(4.3) \quad g_{V^\vee} : V \rightarrow U^\vee \otimes W,$$

(we think of it as a  $3 \times 3$  matrix of linear forms on  $V$ ), and similarly we view

$$g_{U^\vee} : U \rightarrow W \otimes V^\vee,$$

$$g_W : W^\vee \rightarrow U^\vee \otimes V^\vee,$$

as  $3 \times 4$  matrices of linear forms (respectively on  $U$  and  $W^\vee$ ).

For a general  $B$ , the determinant of the  $3 \times 3$  matrix  $g_{V^\vee}$  of linear forms on  $V$  gives a smooth cubic surface  $G^*$  in the dual projective space  $\mathbb{P}^{3^\vee} = \text{Proj}(V^\vee)$ , together with two different realizations of  $G^*$  as a blow up of a projective plane  $\text{Proj}(U^\vee)$  (respectively  $\text{Proj}(W)$ ) in a set of six points  $Z$ . These are the points where the  $3 \times 4$  Hilbert–Burch matrix of linear forms on  $U$  (respectively on  $W^\vee$ ) drops rank by 1, and the rational map to  $\mathbb{P}^{3^\vee}$  is given by the system of cubics through the 6 points, a system which is generated by the determinants of the four minors of order 3 of the Hilbert–Burch matrix, the matrix  $g_{U^\vee}$  (resp.  $g_W$ ). One passes from one realization to the other one simply by applying the trivial involution to the tensor  $B$ , i.e., replacing  $g_{U^\vee}$  with  $g_W$ .

Also the 12 lines of the double–six configuration can be obtained from the original tensor  $B$ , as the union of the 6 lines  $A_z = \text{Ker}(g_{U^\vee})$  with the 6 lines  $A'_z = \text{Ker}(g_W)$  for  $z \in Z$ , cf. [Do-Ka, § 0]. According to this notation, Dolgachev and Kapranov give the following modern formulation of Schur’s classical theorem in [Schu]:

**Theorem 4.3.** [Do-Ka, Theorem 0.5] *Given a smooth cubic there exists a symmetric bilinear form  $Q(x, y)$  on  $V$ , unique up to a scalar factor, which satisfies the following property:  $Q(x, y) = 0$  whenever  $x \in A_z$  and  $y \in A'_z$  for some  $z \in Z$  (i.e., the corresponding lines of the double–six are orthogonal with respect to  $Q$ ).  $Q$  is nondegenerate.*

The bilinear form  $Q \in S^2(V)$  is called the Schur quadric, and it is obtained as follows.

Given a tri-tensor  $B \in U^\vee \otimes W \otimes V^\vee$ , consider the second symmetric power of the linear map  $g = g_{V^\vee}$ ,

$$S^2(g): \rightarrow S^2(U^\vee \otimes W)$$

and compose it with the projection of

$$S^2(U^\vee \otimes W) = (\Lambda^2 U^\vee \otimes \Lambda^2 W) \oplus (S^2 U^\vee \otimes S^2 W)$$

onto the first factor.

Since  $\dim S^2 V = 10$ ,  $\dim(\Lambda^2 U^\vee \otimes \Lambda^2 W) = 9$ , the kernel is 1-dimensional for a general tensor (cf. [Do-Ka, §0 and Thm 0.5]).

Recall once more that the cross-product involution associates to a general tensor  $B \in U^\vee \otimes W \otimes V^\vee$  another tensor  $B' \in U'^\vee \otimes W \otimes V$ , where  $U'$  is defined as the kernel of the map  $\Lambda^2(W^\vee) \otimes V \rightarrow U^\vee \otimes W^\vee$  induced by contraction with  $B$ .

Associate to the Schur quadric  $Q \in S^2 V$  a linear map  $q: V^\vee \rightarrow V$ . The map  $q$  then relates  $B$  and  $B'$  as follows.

**Proposition 4.4.** *Let  $B \in U^\vee \otimes W \otimes V^\vee$  be a tri-tensor such that the associated cubic surface  $G^* \subset \mathbb{P}^{3^\vee}$  is smooth (in particular,  $B$  and  $B'$  lie in the open set of the tri-tensors where the cross-product involution is defined).*

*Then the composition of  $g_{U^\vee}: U \rightarrow W \otimes V^\vee$  with*

$$(4.4) \quad id_W \otimes q: W \otimes V^\vee \rightarrow W \otimes V$$

*maps  $U$  to  $U'$ , where  $U'$  is the vector space associated to  $U$  via the cross-product involution.*

*In particular, the tensor  $B'$ , corresponding to the inclusion  $U' \rightarrow W \otimes V$ , is determined in this way by the tensor  $B$  and by the Schur quadric  $Q$ .*

*Proof.* According to the definition of the cross-product involution, we can identify  $\Lambda^2 W^\vee$  with  $W$ ,  $W^\vee$  with  $\Lambda^2 W$ , and moreover

$$(4.5) \quad \begin{aligned} U' &= Ker(W \otimes V \rightarrow U^\vee \otimes W^\vee), \\ U &= Ker(W \otimes V^\vee \rightarrow U'^\vee \otimes W^\vee), \end{aligned}$$

and both spaces have dimension equal to 3.

Therefore, since  $q$  is invertible, in order to show that  $(id_W \otimes q)(g_{U^\vee}(U)) = U'$ , it suffices to show that  $(id_W \otimes q)(g_{U^\vee}(U))$  is contained in  $U'$ , i.e., this space maps to zero in  $U^\vee \otimes W^\vee$ .

Recall that the first map in (4.5) is the composition

$$W \otimes V \xrightarrow{\otimes B} (W \otimes W) \otimes (V \otimes V^\vee) \otimes U^\vee \rightarrow U^\vee \otimes W^\vee,$$

where the second map is naturally obtained by the projection  $p_{W^\vee} : W \otimes W \rightarrow \Lambda^2 W = W^\vee$  and the contraction  $V \otimes V^\vee \rightarrow \mathbb{C}$  corresponding to the identity of  $V$ . Then we have to show that  $U$  maps to 0 in  $U^\vee \otimes W^\vee$  via the composition

$$(4.6) \quad U \xrightarrow{g_{U^\vee}} W \otimes V^\vee \xrightarrow{id_W \otimes q} W \otimes V \xrightarrow{\otimes B} (W \otimes W) \otimes (V \otimes V^\vee) \otimes U^\vee \rightarrow U^\vee \otimes W^\vee.$$

One sees easily that the above assertion is equivalent to the property that  $B \otimes B$  maps to 0 via the map

$$(U^\vee \otimes U^\vee) \otimes (W \otimes W) \otimes (V^\vee \otimes V^\vee) \xrightarrow{id \otimes p_{W^\vee} \otimes (-Q)} (U^\vee \otimes U^\vee) \otimes (W^\vee).$$

The above map factors through

$$(4.7) \quad (U^\vee \otimes U^\vee) \otimes (\Lambda^2 W) \otimes S^2(V^\vee),$$

and we have to show that the image of  $B \otimes B$  in this space maps to 0 via  $id \otimes id \otimes (-Q)$ .

Write  $U^\vee \otimes U^\vee$  as a direct sum  $S^2(U^\vee) \oplus \Lambda^2(U^\vee)$ . By the definition of  $Q$  we get 0 for the contraction  $-Q$  with  $Q$  of the component in  $(\Lambda^2 U^\vee) \otimes (\Lambda^2 W) \otimes S^2(V^\vee)$  of the image of  $B \otimes B$ .

On the other side, the component of the image of  $B \otimes B$  in  $(S^2 U^\vee) \otimes (\Lambda^2 W) \otimes S^2(V^\vee) = Hom(S^2 V, (S^2 U^\vee) \otimes (\Lambda^2 W))$  is also zero, because  $S^2(V)$  maps to  $S^2(U^\vee \otimes W) = (\Lambda^2 U^\vee \otimes \Lambda^2 W) \oplus (S^2 U^\vee \otimes S^2 W)$ .  $\square$

We now want to relate the method to construct such bundles  $\mathcal{E}$  as kernels with the direct image method illustrated in section 2.

Consider therefore a tensor

$$\widehat{B} \in V \otimes \widehat{U}^\vee \otimes \widehat{W},$$

and apply to it the direct image method of section 2 with twist  $t = 2$  (assuming of course that  $\widehat{B}$  defines a complete intersection  $\Gamma \subset \mathbb{P}(V) \times \mathbb{P}(U')$ ). Exact sequence (2.15) gives

$$(4.8) \quad 0 \rightarrow \wedge^2(\widehat{W}^\vee) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \widehat{W}^\vee \otimes \widehat{U}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \widehat{\mathcal{E}}_{\widehat{B}} \rightarrow 0$$

$$(4.9) \quad 0 \rightarrow \widehat{\mathcal{E}}_{\widehat{B}} \rightarrow S^2(\widehat{U}^\vee) \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \widehat{\mathcal{G}}_{2\widehat{B}} \rightarrow 0.$$

and  $\widehat{\mathcal{E}}$  is a vector bundle on  $\mathbb{P}^3$ .

Denote  $\widehat{\mathcal{E}}_{\widehat{B}}$  simply by  $\widehat{\mathcal{E}}$ , and consider  $\widehat{\mathcal{E}}^\vee$ : we want to show that there is a tensor  $B$  such that  $\widehat{\mathcal{E}}^\vee = \mathcal{E}_B$ .

Indeed we can dualize the first exact sequence above, obtaining

$$(4.10) \quad 0 \rightarrow \widehat{\mathcal{E}}^\vee \rightarrow \widehat{W} \otimes \widehat{U} \otimes \mathcal{O}_{\mathbb{P}'}(1) \rightarrow \widehat{W}^\vee \otimes \mathcal{O}_{\mathbb{P}'}(2) \rightarrow 0.$$

Thus  $\hat{\mathcal{E}}^\vee$  is a rank 6 vector bundle and, by looking at the long exact cohomology sequences associated to the twists of the previous exact sequence, we obtain that the only non-vanishing intermediate cohomology groups of  $\hat{\mathcal{E}}$  are the two groups

$$H^1(\hat{\mathcal{E}}^\vee(-2)) = \widehat{W}^\vee$$

$$H^1(\hat{\mathcal{E}}^\vee(-1)) = \text{coker} \left( \widehat{W} \otimes \widehat{U} \rightarrow \widehat{W}^\vee \otimes V \right) \cong \left( \ker(\widehat{W} \otimes V^\vee \rightarrow \widehat{W}^\vee \otimes \widehat{U}^\vee) \right)^\vee$$

Thus first of all  $\hat{\mathcal{E}}^\vee$  is again a vector bundle on  $\mathbb{P}^3$  with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$  and minimal cohomology.

Observe that, in terms of the cross-product involution applied to the 4-tuple

$$(\widehat{W}^\vee, \widehat{U}^\vee, V, \widehat{B}),$$

$$\text{we have: } \left( \ker(\widehat{W} \otimes V^\vee \rightarrow \widehat{W}^\vee \otimes \widehat{U}^\vee) \right)^\vee = (\widehat{U}')^\vee.$$

Hence, if we set

$$U := \widehat{W}^\vee, W := (\widehat{U}')^\vee, B := (\widehat{B})',$$

the bundle  $\mathcal{E}_B$  associated to  $B$  via the kernel construction will be isomorphic to the bundle  $\hat{\mathcal{E}}^\vee$ , as we wanted.

## 5. SEMISTABILITY AND MODULI SPACE

In this section we shall show that the explicit geometric construction we gave before lends itself to construct a natural moduli space  $\mathfrak{A}^0$  for the vector bundles considered in this paper.

Since moduli space for vector bundles have been constructed in great generality by Maruyama, it seems natural to investigate their Gieseker stability (we refer to [O-S-S] and especially to [Hu-Le] as general references). We conjecture that our bundles are Gieseker stable, but unfortunately for the time being we only managed to prove their slope (Mumford-Takemoto) semistability.

We are however able to prove that our vector bundles are simple, and we observe then (cf. Theorem 2.1 of [Kob]) that moduli spaces of simple vector bundles exist as (possibly non Hausdorff) complex analytic spaces.

We show indeed that the above moduli space exists as an algebraic variety. More precisely, we show that, under a suitable open condition, we can construct a G.I.T. quotient  $\mathfrak{A}^0$  which is a coarse moduli space.

Recall lemma 1.5: it will lead to a characterization of the vector bundles obtained from the kernel construction as an open set in any family of vector bundles with the above Chern polynomial.



**Proposition 5.1.** *Consider a rank 6 vector bundle of  $\mathcal{E}$  with total Chern class  $1 + 3t + 6t^2 + 4t^3$ , such that*

- (1)  $h^0(\mathcal{E}) = 6$
- (2) *the 6 sections generate a rank 6 trivial subsheaf with quotient  $\tau$*
- (3)  $h^0(\mathcal{E}^\vee) = 0$
- (4)  $\mathcal{E}$  *is a subbundle of  $3\Omega^1(2)$ .*

*Then  $\mathcal{E}$  is slope-semistable.*

*Proof.* Let  $\mathcal{E}''$  be a destabilizing subsheaf of rank  $r \leq 5$  and maximal slope  $\mu = d/r$ : without loss of generality we may assume that  $\mathcal{E}''$  is a saturated reflexive subsheaf, and similarly  $\tilde{\mathcal{E}} := \mathcal{E}'' \cap 6\mathcal{O}$  is a saturated reflexive subsheaf of  $6\mathcal{O}$ .

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & 6\mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & \tau & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & \mathcal{E}'' & \longrightarrow & \tau'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & & 
 \end{array}$$

The slope  $\mu(\mathcal{E})$  equals  $1/2$ . On the other hand, by hypothesis 4 and since  $\Omega^1(2)$  is a stable bundle (cf. 1.2.6 b, page 167 of [O-S-S]), the slope of  $\mathcal{E}''$  is at most  $2/3$ , and  $< 2/3$  unless  $\mathcal{E}'' \cong \Omega^1(2)$ .

**CLAIM:**  $\mathcal{E}$  contains no subsheaf isomorphic to  $\Omega^1(2)$ .

*Proof of the claim:*  $h^0(\Omega^1(2)) = 6 = h^0(\mathcal{E})$ , thus this calculation contradicts hypothesis 2.  $\square$

We have that  $d := c_1(\mathcal{E}'') = c_1(\tilde{\mathcal{E}}) + c_1(\tau'')$ , and  $\tau'' \subset \tau$  is a coherent subsheaf supported on a divisor, thus,  $c_1(\tau'') \leq c_1(\tau) = 3$ .

On the other hand,  $c_1(\tilde{\mathcal{E}}) \leq 0$ , and if equality holds, then  $\tilde{\mathcal{E}} \cong r\mathcal{O}$ .

Hence,  $1 \leq d \leq 3$ , and we have

$$2/3 > \mu = d/r > 1/2 \Leftrightarrow 4d > 2r > 3d.$$

These inequalities leave open only the case  $d = 3, r = 5$ .

We show that this case does not exist.

In fact, otherwise we consider the quotient by the subbundle  $\tilde{\mathcal{E}} \cong r\mathcal{O}$ . By hypothesis 3, and the proof of lemma 3.6 we see that  $\mathcal{E}/\tilde{\mathcal{E}}$  is an extension corresponding to a tensor of maximal rank, hence it yields a vector bundle  $\mathcal{V}$  (cf. corollary 3.7).

Since the torsion sheaf  $\tau'' \subset \mathcal{V}$ , we obtain  $\tau'' = 0$ , hence  $d \leq 0$ , absurd.

*Q.E.D.*

**Remark 5.2.** *The possible exceptions to slope-stability, in view of the inequalities*

$$2/3 > \mu = d/r \geq 1/2 \Leftrightarrow 4d \geq 2r > 3d$$

are:

1.  $d = 1, r = 2$
2.  $d = 2, r = 4$ .

*Matei Toma pointed out how case 2. could be excluded using Bogomolov's inequality for stable bundles, as done in Lemma 3.1 of his paper [Toma]. The case  $r = 2, c_1(\tilde{\mathcal{E}}) = -2$  seems as of now the most difficult case.*

*Observe that slope-stability of  $\mathcal{E}$  implies Gieseker stability of  $\mathcal{E}$ , which in turn implies that there is a point in the moduli space of Gieseker semistable bundles corresponding to the isomorphism class of  $\mathcal{E}$ .*

**Lemma 5.3.** *Let  $\mathcal{E}$  be a vector bundle as in (1.4) with  $h^0(\mathcal{E}) = 6$  (equivalently,  $h^1(\mathcal{E}) = 0$ ) and verifying the second assumption. Then  $\text{hom}(\mathcal{E}, \mathcal{E}) = 1$ , i.e.,  $\mathcal{E}$  is simple.*

*Proof.* We consider the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{E}, 6\mathcal{O}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Hom}(\mathcal{E}, \tau) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}).$$

We have  $\text{Ext}^1(\mathcal{E}, \mathcal{O}) \cong H^1(\mathcal{E}^\vee) \cong H^2(\mathcal{E}(-4))$  and from the exact sequence (1.4) we infer  $H^2(\mathcal{E}(-4)) = 0$ . Since  $\text{Hom}(\mathcal{E}, 6\mathcal{O}) = 0$  by proposition 3.2, it follows that  $\text{Hom}(\mathcal{E}, \mathcal{E}) \cong \text{Hom}(\mathcal{E}, \tau)$ .

We compute  $\text{hom}(\mathcal{E}, \tau)$  by considering the exact sequence

$$0 \rightarrow \text{Hom}(\tau, \tau) \rightarrow \text{Hom}(\mathcal{E}, \tau) \rightarrow \text{Hom}(6\mathcal{O}, \tau).$$

Indeed  $\text{hom}(\mathcal{O}, \tau) = h^0(\tau) = 0$  (since  $h^0(\mathcal{E}) = 6$ ) and, since  $\tau$  is  $\mathcal{O}_G$ -invertible, we have  $\text{hom}(\tau, \tau) = 1$ .  $\square$

**Lemma 5.4.** *Let  $\mathcal{E}$  be a simple vector bundle of rank 6, with Chern classes  $c_1(\mathcal{E}) = 3, c_2(\mathcal{E}) = 6, c_3(\mathcal{E}) = 4$ . Then the local dimension of the moduli space  $\mathfrak{M}^s(6; 3, 6, 4)$  of simple vector bundles at the point corresponding to  $\mathcal{E}$  is at least 19.*

*Proof.* The moduli space of simple vector bundles exists (cf. [Kob], Theorem 2.1) and it is well known that the local dimension is at least equal to the expected dimension  $h^1(\mathcal{E}^\vee \otimes \mathcal{E}) - h^2(\mathcal{E}^\vee \otimes \mathcal{E})$ . On the other hand,  $\mathcal{E}$  simple means that  $h^0(\mathcal{E}^\vee \otimes \mathcal{E}) = 1$ , hence follows also that  $h^3(\mathcal{E}^\vee \otimes \mathcal{E}) = 0$ , since by Serre duality  $h^3(\mathcal{E}^\vee \otimes \mathcal{E}) = h^0(\mathcal{E}^\vee \otimes \mathcal{E}(-4)) = 0$ .

Thus the expected dimension equals  $-\chi(\mathcal{E}^\vee \otimes \mathcal{E}) + 1$  and there remains to calculate  $-\chi(\mathcal{E}^\vee \otimes \mathcal{E})$ . This can be easily calculated in the case where we have an exact sequence  $0 \rightarrow \mathcal{E} \rightarrow 9\mathcal{O}(1) \rightarrow 3\mathcal{O}(2)$ . We omit the rest of the easy calculation.  $\square$

In the following theorem we shall phrase the geometric meaning of the cross-product involution in terms of a birational duality of moduli space of vector bundles,  $\mathfrak{A}^0$  on  $\mathbb{P}^3$ ,  $\mathfrak{A}_*^0$  on  $\mathbb{P}^{3^\vee}$ .

**Main Theorem** *Consider the moduli space  $\mathfrak{M}^s(6; 3, 6, 4)$  of rank 6 simple vector bundles  $\mathcal{E}$  on  $\mathbb{P}^3 := \text{Proj}(V)$  with Chern polynomial  $1 + 3t + 6t^2 + 4t^3$ , and inside it the open set  $\mathfrak{A}$  corresponding to the simple bundles with minimal cohomology, i.e., those with*

- (1)  $H^i(\mathcal{E}) = 0 \ \forall i \geq 1$ ;
- (2)  $H^i(\mathcal{E}(-1)) = 0 \ \forall i \neq 1$ ;
- (3)  $H^i(\mathcal{E}(-2)) = 0 \ \forall i \neq 1$ ;
- (4)  $H^i(\mathcal{E}(-3)) = 0 \ \forall i$ ;
- (5)  $H^i(\mathcal{E}(-4)) = 0 \ \forall i$ .

*Then  $\mathfrak{A}$  is irreducible of dimension 19 and it is bimeromorphic to  $\mathfrak{A}^0$ , where  $\mathfrak{A}^0$  is an open set of the G.I.T. quotient space of the projective space  $\mathfrak{B}$  of tensors of type  $(3, 4, 3)$ ,  $\mathfrak{B} := \{B \in \mathbb{P}(U^\vee \otimes W \otimes V^\vee)\}$  by the natural action of  $SL(W) \times SL(U)$  (recall that  $U, W$  are two fixed vector spaces of dimension 3, while  $V = H^0(\mathbb{P}^3, \mathcal{O}(1))$ ).*

*Let moreover  $[B] \in \mathfrak{A}^0$  be a general point: then to  $[B]$  corresponds a vector bundle  $\mathcal{E}_B$  on  $\mathbb{P}^3$  via the kernel construction, and also a vector bundle  $\mathcal{E}_B^*$  on  $\mathbb{P}^{3^\vee}$ , obtained from the direct construction applied to the tensor  $B \in U^\vee \otimes W \otimes V^\vee$  (cf. definition 3.9 applied to  $B$ , or equation (5.1)).  $\mathcal{E}_B^*$  is the vector bundle  $\mathcal{E}_{B'}$ , where  $B' \in W \otimes U'^\vee \otimes V$  is obtained from  $B$  via the reversing construction and  $[B'] \in \mathfrak{A}_*^0$ .*

*Proof.* To any such tensor  $B$  we tautologically associate two linear maps which we denote by the same symbol,

$$B : U \otimes V \rightarrow W, \quad B : U \otimes V \otimes \mathcal{O}(1) \rightarrow W \otimes \mathcal{O}(1)$$

and using the Euler sequence we define a coherent sheaf  $\mathcal{E}$  on  $\mathbb{P}^3$  as a kernel, exactly as in the exact sequence (3.3) (except that surjectivity holds only for  $B$  general), following what we called the kernel construction.

As we already saw in (3.2), this is equivalent to giving  $\mathcal{E}$  as the kernel of a homomorphism  $\beta$  as in (1.4). Observe that  $GL(W) \times GL(U)$  acts on the vector space of such tensors, preserving the isomorphism class of the sheaf thus obtained.

We define  $\mathfrak{B}'$  as the open set in  $\mathfrak{B}$  where  $\beta$  is surjective (thus  $\mathcal{E}$  is a rank 6 bundle) and  $h^0(\mathcal{E}) = 6$ . Both conditions amount to the surjectivity of  $h^0(\beta) = h^0(B \oplus \epsilon)$ , cf. (3.3), and imply that  $\mathcal{E}$  is a bundle with minimal cohomology, in the sense of lemma 1.5. We further define  $\mathfrak{B}''$  as the smaller open set where the second assumption is verified, and we observe then that lemma 5.3 ensures the existence of a morphism  $\mathfrak{B}'' \rightarrow \mathfrak{A}$  which factors through the action of  $SL(W) \times SL(U)$ .

Since we want to construct a G.I.T. quotient of an open set of  $\mathfrak{B}$ , we let  $\mathfrak{B}^*$  the open set of tensors  $B$  whose determinant defines a cubic surface  $G^* \subset \mathbb{P}^{3^\vee}$ , i.e., we have an exact sequence on  $\mathbb{P}^{3^\vee}$  of the form (set  $\mathcal{O}_* := \mathcal{O}_{\mathbb{P}^{3^\vee}}$ )

$$(5.1) \quad 0 \rightarrow U \otimes \mathcal{O}_*(-1) \xrightarrow{B} W \otimes \mathcal{O}_* \rightarrow \mathcal{G}^* \rightarrow 0.$$

Since the determinant map is obviously  $SL(W) \times SL(U)$ -invariant, the tensors in  $\mathfrak{B}^*$  are automatically semistable points for the  $SL(W) \times SL(U)$ -action, by virtue of the criterion of Hilbert-Mumford.

Observe now that the maximal torus  $\mathbb{C}^* \times \mathbb{C}^*$  of  $GL(W) \times GL(U)$  acts trivially on  $\mathfrak{B}$ , thus we get an effective action of  $SL(W) \times SL(U)$  only upon dividing by a finite group  $K' \cong (\mathbb{Z}/3)^2$ .

We claim that  $(SL(W) \times SL(U))/K'$  acts freely on the open subset  $\mathfrak{B}^{**} \subset \mathfrak{B}^*$ ,  $\mathfrak{B}^{**} = \{B \in \mathfrak{B}^* | \text{End}(\mathcal{G}^*) = \mathbb{C}\}$ .

This is clear since the stabilizer of  $B$  corresponds uniquely to the group of automorphisms of  $\mathcal{G}^*$ , and any such automorphism acts on  $W \cong H^0(\mathcal{G}^*)$ , and induces a unique automorphism of  $U$  in view of the exact sequence (5.1). But every automorphism is multiplication by a constant, thus it yields an element in  $K'$ .

We want to show that the orbits are closed. But the orbits are contained in the fibres of the determinant map: thus, it suffices to show that, fixed the cubic surface  $G^*$ , if we have a 1-parameter family where  $\mathcal{G}_t \cong \mathcal{G}_1$  for  $t \neq 0$ , then also  $\mathcal{G}_0 \cong \mathcal{G}_1$ .

This holds on the smaller open set  $\mathfrak{B}^{***} \subset \mathfrak{B}^{**}$  consisting of the tensors such that the cubic surface  $G^*$  is smooth: since then  $\mathcal{G}_0$  is invertible, and the Picard group of  $G^*$  is discrete.

We have proven that  $\mathfrak{B}^{***}$  consists of stable points, and observe that the condition  $\text{End}(\mathcal{G}^*) = \mathbb{C}$  holds if  $\mathcal{G}^*$  is  $\mathcal{O}_{G^*}$ -invertible, or it is torsion free and  $G^*$  is normal. Therefore the open set  $\mathfrak{B}^{st}$  of stable points is nonempty.

We define  $\mathfrak{A}^0$  as the open set of the G.I.T. quotient corresponding to  $\mathfrak{B}^{st} \cap \mathfrak{B}''$ .

The fact that  $\mathfrak{A}$  is irreducible follows since every bundle  $\mathcal{E}$  in  $\mathfrak{A}$  has a cohomology table which (by Beilinson's theorem, as explained in lemma 1.5) implies that  $\mathcal{E}$  is obtained from a tensor  $B$  in the open subset  $\mathfrak{B}'^0 \subset \mathfrak{B}'$  consisting of those  $B$  for which the corresponding bundle  $\mathcal{E}$  is simple (note that  $\mathfrak{B}'^0 \supset \mathfrak{B}''$ ).

Now,  $\dim \mathfrak{A}^0 = 19$ , while  $\dim \mathfrak{A} \geq 19$  by 5.4; we only need to observe that if  $[B], [B'] \in \mathfrak{A}^0$  and two bundles  $\mathcal{E}_B$  and  $\mathcal{E}_{B'}$  are isomorphic, then the corresponding tensors  $B, B'$  are  $GL(U) \times GL(W)$  equivalent, since

they express the multiplication matrix for the intermediate cohomology module  $H_*^1(\mathcal{E})$ . Thus  $[B] = [B'] \in \mathfrak{A}^0$ .

It follows on the one side that  $\mathfrak{A}^0$  parametrizes isomorphism classes of bundles, and on the other side that  $\mathfrak{A}^0$  maps bijectively to an open set in  $\mathfrak{A}$ , in particular  $\dim \mathfrak{A} = 19$ , since  $\mathfrak{A}$  is irreducible.

*Q.E.D.*

**Acknowledgement** The authors would like to thank I. Dolgachev for pointing out how the Schur quadric could be related to the cross-product involution, for suggesting a third construction of the bundles  $\mathcal{E}$ , and for hospitality and the nice atmosphere during the visit of the first author in Ann Arbor in January 2006. The present research was carried out in the realm of the DFG Schwerpunkt “Globale Methoden in der komplexen Geometrie”. The second author also profited from a travel grant from a DAAD-VIGONI program.

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