The Rationality of Certain Moduli Spaces of Curves of Genus 3

Ingrid Bauer and Fabrizio Catanese

Mathematisches Institut
Universität Bayreuth, NW II
D-95440 Bayreuth, Germany
Ingrid.Bauer@uni-bayreuth.de, Fabrizio.Catanese@uni-bayreuth.de

Summary. We prove rationality of the moduli space of pairs of curves of genus three together with a point of order three in their Jacobian.

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1 Introduction

The aim of this paper is to give an explicit geometric description of the birational structure of the moduli space of pairs (C, η), where C is a general curve of genus 3 over an algebraically closed field k of arbitrary characteristic and $\eta \in Pic^0(C)_3$ is a nontrivial divisor class of 3-torsion on C.

As was observed in [B-C04, Lemma (2.18)], if C is a general curve of genus 3 and $\eta \in Pic^0(C)_3$ is a nontrivial 3-torsion divisor class, then we have a morphism $\varphi_\eta := \varphi_{|K_C+\eta|} \times \varphi_{|K_C-\eta|} : C \to \mathbb{P}^1 \times \mathbb{P}^1$, corresponding to the sum of the linear systems $|K_C + \eta|$ and $|K_C - \eta|$, which is birational onto a curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4, 4)$. Moreover, Γ has exactly six ordinary double points as singularities, located in the six points of the set $\mathcal{S} := \{(x, y) | x \neq y, \ x, y \in \{0, 1, \infty\}\}$.

In [B-C04] we only gave an outline of the proof (and there is also a minor inaccuracy). Therefore we dedicate the first section of this article to a detailed geometrical description of such pairs (C, η), where C is a general curve of genus 3 and $\eta \in Pic^0(C)_3 \setminus \{0\}$.

The main result of the first section is the following:

Theorem 1.1. *Let C be a general (in particular, nonhyperelliptic) curve of genus 3 over an algebraically closed field k (of arbitrary characteristic) and $\eta \in Pic^0(C)_3 \setminus \{0\}$.*

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Then the rational map $\varphi_\eta : C \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$\varphi_\eta := \varphi_{|K_C+\eta|} \times \varphi_{|K_C-\eta|} : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

is a morphism, birational onto its image Γ, which is a curve of bidegree $(4,4)$ having exactly six ordinary double points as singularities. We can assume, up to composing φ_η with a transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}GL(2,k)^2$, that the singular set of Γ is the set

$$\mathcal{S} := \{(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1 | x \neq y \; ; x, y \in \{0,1,\infty\}\}.$$

Conversely, if Γ is a curve of bidegree $(4,4)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, whose singularities consist of exactly six ordinary double points at the points of $\mathcal{S}$, its normalization C is a curve of genus 3, such that $\mathcal{O}_C(H_2 - H_1) =: \mathcal{O}_C(\eta)$ (where H_1, H_2 are the respective pullbacks of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$) yields a nontrivial 3-torsion divisor class, and $\mathcal{O}_C(H_1) \cong \mathcal{O}_C(K_C+\eta)$, $\mathcal{O}_C(H_2) \cong \mathcal{O}_C(K_C-\eta)$.

From Theorem 1.1 it follows that

$$\mathcal{M}_{3,\eta} := \{(C,\eta) : C \text{ is a general curve of genus 3, } \eta \in Pic^0(C)_3 \setminus \{0\}\}$$

is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/\mathfrak{S}_3$, where

$$V(4,4,-\mathcal{S}) := H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)(-2 \sum_{a \neq b, a,b \in \{\infty,0,1\}} (a,b))).$$

In fact, the permutation action of the symmetric group $\mathfrak{S}_3 := \mathfrak{S}(\{\infty,0,1\})$ extends to an action on $\mathbb{P}^1$, so $\mathfrak{S}_3$ is naturally a subgroup of $\mathbb{P}GL(2,k)$. We consider then the diagonal action of $\mathfrak{S}_3$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and observe that $\mathfrak{S}_3$ is exactly the subgroup of $\mathbb{P}GL(2,k)^2$ leaving the set $\mathcal{S}$ invariant. The action of $\mathfrak{S}_3$ on $V(4,4,-\mathcal{S})$ is naturally induced by the diagonal inclusion $\mathfrak{S}_3 \subset \mathbb{P}GL(2,k)^2$.

On the other hand, if we consider only the subgroup of order three of $Pic^0(C)$ generated by a nontrivial 3-torsion element η, we see from Theorem 1.1 that we have to allow the exchange of η with $-\eta$, which corresponds to exchanging the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore

$$\mathcal{M}_{3,\langle\eta\rangle} := \{(C,\langle\eta\rangle) : C \text{ general curve of genus 3, } \langle\eta\rangle \cong \mathbb{Z}/3\mathbb{Z} \subset Pic^0(C)\}$$

is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/(\mathfrak{S}_3 \times \mathbb{Z}/2)$, where the action of the generator σ (of $\mathbb{Z}/2\mathbb{Z}$) on $V(4,4,-\mathcal{S})$ is induced by the action on $\mathbb{P}^1 \times \mathbb{P}^1$ obtained by exchanging the two coordinates.

Our main result is the following:

Theorem 1.2. *Let k be an algebraically closed field of arbitrary characteristic. We have:*

1) the moduli space $\mathcal{M}_{3,\eta}$ is rational;

2) the moduli space $\mathcal{M}_{3,\langle\eta\rangle}$ is rational.

One could obtain the above result abstractly from the method of Bogomolov and Katsylo (cf. [B-K85]), but we prefer to prove the theorem while explicitly calculating the field of invariant functions. It mainly suffices to decompose the vector representation of $\mathfrak{S}_3$ on $V(4,4,-\mathcal{S})$ into irreducible factors. Of course, if the characteristic of k equals two or three, it is no longer possible to decompose the $\mathfrak{S}_3$-module $V(4,4,-\mathcal{S})$ as a direct sum of irreducible submodules. Nevertheless, we can write down the field of invariants and see that it is rational.

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2 The geometric description of pairs (C, η)

In this section we give a geometric description of pairs (C, η), where C is a general curve of genus 3 and η is a nontrivial element of $Pic^0(C)_3$, and we prove Theorem 1.1.

Let k be an algebraically closed field of arbitrary characteristic. We recall the following observation from [B-C04, p. 374].

Lemma 2.1. *Let C be a general curve of genus 3 and $\eta \in Pic^0(C)_3$ a nontrivial divisor class (i.e., η is not linearly equivalent to 0). Then the linear system $|K_C + \eta|$ is base point free. This holds more precisely under the assumption that the canonical system $|K_C|$ does not contain two divisors of the form $Q + 3P$, $Q + 3P'$, and where the 3-torsion divisor class $P - P'$ is the class of η. This condition for all such η is in turn equivalent to the fact that C is either hyperelliptic or it is nonhyperelliptic but the canonical image Σ of C does not admit two inflexional tangents meeting in a point Q of Σ.*

Proof. Note that P is a base point of the linear system $|K_C + \eta|$ if and only if

$$H^0(C, \mathcal{O}_C(K_C + \eta)) = H^0(C, \mathcal{O}_C(K_C + \eta - P)).$$

Since $\dim H^0(C, \mathcal{O}_C(K_C + \eta)) = 2$ this is equivalent to

$$\dim H^1(C, \mathcal{O}_C(K_C + \eta - P)) = 1.$$

Since $H^1(C, \mathcal{O}_C(K_C + \eta - P)) \cong H^0(C, \mathcal{O}_C(P - \eta))^*$, this is equivalent to the existence of a point P' such that $P - \eta \equiv P'$ (note that we denote linear equivalence by the classical notation "$\equiv$"). Therefore $3P \equiv 3P'$ and $P \neq P'$, whence in particular $H^0(C, \mathcal{O}_C(3P)) \geq 2$. By Riemann–Roch we have

$$\dim H^0(C, \mathcal{O}_C(K_C - 3P)) =$$

$$\deg(K_C - 3P) + 1 - g(C) + \dim H^0(C, \mathcal{O}_C(3P)) \geq 1.$$

In particular, there is a point Q such that $Q \equiv K_C - 3P \equiv K_C - 3P'$.

Going backwards, we see that this condition is not only necessary, but sufficient. If C is hyperelliptic, then $Q + 3P, Q + 3P' \in |K_C|$, hence P, P' are Weierstrass points, whence $2P \equiv 2P'$, hence $P - P'$ yields a divisor class η of 2-torsion, contradicting the nontriviality of η.

Consider now the canonical embedding of C as a plane quartic Σ. Our condition means, geometrically, that C has two inflection points P, P', such that the tangent lines to these points intersect in $Q \in C$.

We shall show now that the (nonhyperelliptic) curves of genus 3 whose canonical image is a quartic Σ with the above properties are contained in a five-dimensional family, whence are special in the moduli space $\mathcal{M}_3$ of curves of genus 3.

Let now p, q, p' be three noncollinear points in $\mathbb{P}^2$. The quartics in $\mathbb{P}^2$ form a linear system of dimension 14. Imposing that a plane quartic contains the point q is one linear condition. Moreover, the condition that the line containing p and q has intersection multiplicity equal to 3 with the quartic in the point p gives three further linear conditions. Similarly for the point p', and it is easy to see that the above seven linear conditions are independent. Therefore the linear subsystem of quartics Σ having two inflection points p, p', such that the tangent lines to these points intersect in $q \in \Sigma$, has dimension $14-3-3-1 = 7$. The group of automorphisms of $\mathbb{P}^2$ leaving the three points p, q, p' fixed has dimension 2 and therefore the above quartics give rise to a five-dimensional algebraic subset of $\mathcal{M}_3$.

Finally, if the points P, P', Q are not distinct, we have (w.l.o.g.) $P = Q$ and a similar calculation shows that we have a family of dimension $7 - 3 = 4$. □

Consider now the morphism

$$\varphi_\eta (:= \varphi_{|K_C+\eta|} \times \varphi_{|K_C-\eta|}) : C \to \mathbb{P}^1 \times \mathbb{P}^1,$$

and denote by $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ the image of C under φ_η.

Remark 2.2.

1) Since η is nontrivial, either Γ is of bidegree $(4, 4)$, or $\deg \varphi_\eta = 2$ and Γ is of bidegree $(2, 2)$. In fact, deg $\varphi_\eta = 4$ implies $\eta \equiv -\eta$.

2) We shall assume in the following that φ_η is birational, since otherwise C is either hyperelliptic (if Γ is singular) or C is a double cover of an elliptic curve Γ (branched in 4 points).

In both cases C lies in a five-dimensional subfamily of the moduli space $\mathcal{M}_3$ of curves of genus 3.

Let $P_1, \ldots, P_m$ be the (possibly infinitely near) singular points of Γ, and let r_i be the multiplicity in P_i of the proper transform of Γ. Then, denoting by H_1, respectively H_2, the divisors of a vertical, respectively of a horizontal line in $\mathbb{P}^1 \times \mathbb{P}^1$, we have that $\Gamma \in |4H_1 + 4H_2 - \sum_{i=1}^m r_i P_i|$. By adjunction, the canonical system of Γ is cut out by $|2H_1 + 2H_2 - \sum_{i=1}^m (r_i - 1)P_i|$, and therefore

$$4 = \deg K_C = \Gamma \cdot (2H_1 + 2H_2 - \sum_{i=1}^{m}(r_i - 1)P_i) = 16 - \sum_{i=1}^{m} r_i(r_i - 1).$$

Hence $\sum_{i=1}^m r_i(r_i - 1) = 12$, and we have the following possibilities:

	m	$(r_1, \ldots, r_m)$
i)	1	(4)
ii)	2	(3,3)
iii)	4	(3,2,2,2)
iv)	6	(2,2,2,2,2,2)

We will show now that for a general curve only the last case occurs, i.e., Γ has exactly 6 singular points of multiplicity 2.

We denote by S the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in $P_1, \ldots, P_m$, and let E_i be the exceptional divisor of the first kind, total transform of the point P_i.

We shall first show that the first case (i.e., $m = 1$) corresponds to the case $\eta \equiv 0$.

Proposition 2.3. *Let $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree $(4, 4)$ having a point P of multiplicity 4, such that its normalization $C \in |4H_1 + 4H_2 - 4E|$ has genus 3 (here, E is the exceptional divisor of the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in P). Then*

$$\mathcal{O}_C(H_1) \cong \mathcal{O}_C(H_2) \cong \mathcal{O}_C(K_C).$$

In particular, if $\Gamma = \varphi_\eta(C)$ (i.e., we are in the case $m = 1$), then $\eta \equiv 0$.

Remark 2.4. Let Γ be as in the proposition. Then the rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ given by $|H_1 + H_2 - E|$ maps Γ to a plane quartic. Vice versa, given a plane quartic C', blowing up two points $p_1, p_2 \in (\mathbb{P}^1 \times \mathbb{P}^1) \setminus C'$, and then contracting the strict transform of the line through p_1, p_2, yields a curve Γ of bidegree $(4, 4)$ having a singular point of multiplicity 4.

Proof (of the proposition). Let H_1 be the full transform of a vertical line through P. Then there is an effective divisor H_1' on the blowup S of $\mathbb{P}^1 \times \mathbb{P}^1$ in P such that $H_1 \equiv H_1' + E$. Since $H_1 \cdot C = E \cdot C = 4$, H_1' is disjoint from C, whence $\mathcal{O}_C(H_1) \cong \mathcal{O}_C(E)$. The same argument for a horizontal line through P obviously shows that $\mathcal{O}_C(H_2) \cong \mathcal{O}_C(E)$. If $h^0(C, \mathcal{O}_C(H_1)) = 2$, then the two projections $p_1, p_2 : \Gamma \to \mathbb{P}^1$ induce the same linear series on C, thus $\varphi_{|H_1|}$ and $\varphi_{|H_2|}$ are related by a projectivity of $\mathbb{P}^1$, hence Γ is the graph of a projectivity of $\mathbb{P}^1$, contradicting the fact that the bidegree of Γ is $(4, 4)$.

Therefore we have a smooth curve of genus 3 and a divisor of degree 4 such that $h^0(C, \mathcal{O}_C(H_1)) \geq 3$. Hence $h^0(C, \mathcal{O}_C(K_C - H_1)) \geq 1$, which implies that $K_C \equiv H_1$. Analogously, $K_C \equiv H_2$. $\square$

The next step is to show that for a general curve C of genus 3, cases ii) and iii) do not occur. In fact, we show:

Lemma 2.5. *Let C be a curve of genus 3 and $\eta \in Pic^0(C)_3 \setminus \{0\}$ such that φ_η is birational and the image $\varphi_\eta(C) = \Gamma$ has a singular point P of multiplicity 3. Then C belongs to an algebraic subset of $\mathcal{M}_3$ of dimension ≤ 5.*

Proof. Let S again be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in P, and denote by E the exceptional divisor. Then $\mathcal{O}_C(E)$ has degree 3 and arguing as in Proposition 2.3, we see that there are points Q_1, Q_2 on C such that $\mathcal{O}_C(H_i) \cong \mathcal{O}_C(Q_i + E)$. Therefore $\mathcal{O}_C(Q_2 - Q_1) \cong \mathcal{O}_C(H_2 - H_1) \cong \mathcal{O}_C(K_C - \eta - (K_C + \eta)) \cong \mathcal{O}_C(\eta)$, whence $3Q_1 \equiv 3Q_2$, $Q_1 \neq Q_2$. This implies that there is a morphism $f : C \to \mathbb{P}^1$ of degree 3, having double ramification in Q_1 and Q_2. By Hurwitz' formula the degree of the ramification divisor R is 10 and since $R \geq Q_1 + Q_2$ f has at most eight branch points in $\mathbb{P}^1$. Fixing three of these points to be $\infty, 0, 1$, we obtain (by Riemann's existence theorem) a finite number of families of dimension at most 5. $\square$

From now on, we shall make the following

Assumptions.
C is a curve of genus 3, $\eta \in Pic^0(C)_3 \setminus \{0\}$, and

1) $|K_C + \eta|$ and $|K_C - \eta|$ are base point free;
2) $\varphi_\eta : C \to \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ is birational;
3) $\Gamma \in |4H_1 + 4H_2|$ has only double points as singularities (possibly infinitely near).

Remark 2.6. By the considerations so far, we know that a general curve of genus 3 fulfills the assumptions for any $\eta \in Pic^0(C)_3 \setminus \{0\}$.

We use the notation introduced above: we have $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ and $C \subset S$, $C \in |4H_1 + 4H_2 - 2\sum_{i=1}^{6} E_i|$.

Remark 2.7. Since S is a regular surface, we have an easy case of Ramanujam's vanishing theorem: if D is an effective divisor which is 1-connected (i.e., for every decomposition $D = A + B$ with $A, B > 0$, we have $A \cdot B \geq 1$), then $H^1(S, \mathcal{O}_S(-D)) = 0$.

This follows immediately from Ramanujam's lemma ensuring $H^0(D, \mathcal{O}_D) = k$, and from the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

In most of our applications we shall show that D is linearly equivalent to a reduced and connected divisor (this is a stronger property than 1-connectedness).

We know now that $\mathcal{O}_C(H_1 + H_2) \cong \mathcal{O}_C(2K_C)$, i.e.,

$$\mathcal{O}_C \cong \mathcal{O}_C(3H_1 + 3H_2 - \sum_{i=1}^{6} 2E_i).$$

Since $h^1(S, \mathcal{O}_S(-H_1 - H_2)) = 0$, the exact sequence

$$0 \to \mathcal{O}_S(-H_1 - H_2) \to \mathcal{O}_S(3H_1 + 3H_2 - \sum_{i=1}^{6} 2E_i)$$
$$\to \mathcal{O}_C(3H_1 + 3H_2 - \sum_{i=1}^{6} 2E_i) \cong \mathcal{O}_C \to 0, \quad (1)$$

is exact on global sections.

In particular, $h^0(S, \mathcal{O}_S(3H_1 + 3H_2 - \sum_{i=1}^{6} 2E_i)) = 1$. We denote by G the unique divisor in the linear system $|3H_1 + 3H_2 - \sum_{i=1}^{6} 2E_i|$. Note that $C \cap G = \emptyset$ (since $\mathcal{O}_C \cong \mathcal{O}_C(G)$).

Remark 2.8. There is no effective divisor $\tilde{G}$ on S such that $G = \tilde{G} + E_i$, since otherwise $\tilde{G} \cdot C = -2$, contradicting that $\tilde{G}$ and C have no common component.

This means that $G + 2\sum_{i=1}^{6} E_i$ is the total transform of a curve $G' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (3,3).

Lemma 2.9. $h^0(G, \mathcal{O}_G) = 3$, $h^1(G, \mathcal{O}_G) = 0$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + G) \to \mathcal{O}_G(K_G) \to 0.$$

Since $h^0(S, \mathcal{O}_S(K_S)) = h^1(S, \mathcal{O}_S(K_S)) = 0$, we get

$$h^0(S, \mathcal{O}_S(K_S + G)) = h^0(G, \mathcal{O}_G(K_G)).$$

Now, $K_S + G \equiv H_1 + H_2 - \sum_{i=1}^{6} E_i$, therefore $(K_S + G) \cdot C = -4$, whence $h^0(G, \mathcal{O}_G(K_G)) \cong h^0(S, \mathcal{O}_S(K_S + G)) = 0$.

Moreover, $h^1(G, \mathcal{O}_G(K_G)) = h^1(S, \mathcal{O}_S(K_S + G)) + 1$, and by Riemann–Roch we infer that, since $h^1(S, \mathcal{O}_S(K_S + G)) = h^0(S, \mathcal{O}_S(-G)) = 0$, that $h^1(S, \mathcal{O}_S(K_S + G)) = 2$. $\square$

We will show now that G is reduced, hence, by the above lemma, we shall obtain that G has exactly three connected components.

Proposition 2.10. *G is reduced.*

Proof. By Remark 2.8 it is sufficient to show that the image of G in $\mathbb{P}^1 \times \mathbb{P}^1$, which we denoted by G', is reduced.

Assume that there is an effective divisor A' on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $3A' \leq G'$. We clearly have $A' \cap \Gamma \neq \emptyset$ but, after blowing up the six points $P_1, \ldots, P_6$, the strict transforms of A' and of Γ are disjoint, whence A' and G' must intersect in one of the P_i's, contradicting Remark 2.8.

If G' is not reduced, we may uniquely write $G' = 2D_1 + D_2$ with D_1, D_2 reduced and having no common component. Up to exchanging the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we have the following two possibilities:

i) $D_1 \in |H_1 + H_2|$;
ii) $D_1 \in |H_1|$.

In the first case also $D_2 \in |H_1 + H_2|$ and its strict transform is disjoint from C. Remark 2.8 implies that D_2 meets Γ in points which do not belong to D_1, whence D_2 has double points where it intersects Γ. Since $D_2 \cdot \Gamma = 8$ we see that D_2 has two points of multiplicity 2, a contradiction (D_2 has bidegree $(1,1)$).

Assume now that $D_1 \in |H_1|$. Then, since $2D_1 \cdot \Gamma = 8$, D_1 contains four of the P_i's and D_2 passes through the other two, say P_1, P_2. This implies that for the strict transform of D_2 we have: $\hat{D}_2 \equiv H_1 + 3H_2 - 2E_1 - 2E_2$, whence $\hat{D}_2 \cdot C = 8$, a contradiction. $\square$

We write now $G = G_1 + G_2 + G_3$ as a sum of its connected components, and accordingly $G' = G'_1 + G'_2 + G'_3$.

Lemma 2.11. *The bidegree of G'_j ($j \in \{1,2,3\}$) is $(1,1)$. Up to renumbering $P_1, \ldots, P_6$ we have*

$$G'_1 \cap G'_2 = \{P_1, P_2\},\ G'_1 \cap G'_3 = \{P_3, P_4\} \quad \textit{and} \quad G'_2 \cap G'_3 = \{P_5, P_6\}.$$

More precisely,

$$\begin{aligned} G_1 &\in |H_1 + H_2 - E_1 - E_2 - E_3 - E_4|, \\ G_2 &\in |H_1 + H_2 - E_1 - E_2 - E_5 - E_6|, \\ G_3 &\in |H_1 + H_2 - E_3 - E_4 - E_5 - E_6|. \end{aligned}$$

Proof. Assume for instance that G'_1 has bidegree $(1,0)$. Then there is a subset $I \subset \{1, \ldots, 6\}$ such that $G_1 = H_1 - \sum_{i \in I} E_i$. Since $G_1 \cdot C = 0$, it follows that $|I| = 2$. But then $G_1 \cdot (G - G_1) = 1$, contradicting the fact that G_1 is a connected component of G.

Let (a_j, b_j) be the bidegree of G_j: then $a_j, b_j \geq 1$ since a reduced divisor of bidegree $(m, 0)$ is not connected for $m \geq 2$. Since $\sum a_j = \sum b_j = 3$, it follows that $a_j = b_j = 1$.

Writing now $G_j \equiv H_1 + H_2 - \sum_{i=1}^6 \mu(j,i) E_i$ we obtain

$$\sum_{j=1}^{3} \mu(j,i) = 2,\ \sum_{i=1}^{6} \mu(j,i) = 4,\ \sum_{i=1}^{6} \mu(k,i)\mu(j,i) = 2$$

since $G_j \cdot C = 0$ and $G_k \cdot G_j = 0$. We get the second claim of the lemma provided that we show: $\mu(j,i) = 1, \forall i, j$.

The first formula shows that if $\mu(j,i) \geq 2$, then $\mu(j,i) = 2$ and $\mu(h,i) = 0$ for $h \neq j$. Hence the second formula shows that

$$\sum_{h,k \neq j} \sum_{i=1}^{6} \mu(j,i)(\mu(h,i) + \mu(k,i)) \leq 2,$$

contradicting the third formula. □

In the remaining part of the section we will show that each G'_i consists of the union of a vertical and a horizontal line in $\mathbb{P}^1 \times \mathbb{P}^1$.

Since $\mathcal{O}_C(K_C + \eta) \cong \mathcal{O}_C(H_1)$ and $\mathcal{O}_C(K_C - \eta) \cong \mathcal{O}_C(H_2)$ we get:

$$\mathcal{O}_C(2H_2 - H_1) \cong \mathcal{O}_C(K_C) \cong \mathcal{O}_C(2H_1 + 2H_2 - \sum_{i=1}^{6} E_i),$$

whence the exact sequence

$$0 \to \mathcal{O}_S(-H_1 - 4H_2 + \sum_{i=1}^{6} E_i) \to \mathcal{O}_S(3H_1 - \sum_{i=1}^{6} E_i) \\ \to \mathcal{O}_C(3H_1 - \sum_{i=1}^{6} E_i) \cong \mathcal{O}_C \to 0. \quad (2)$$

Proposition 2.12. $H^1(S, \mathcal{O}_S(-(H_1 + 4H_2 - \sum_{i=1}^{6} E_i))) = 0$.

Proof. The result follows immediately by Ramanujam's vanishing theorem, but we can also give an elementary proof using Remark 2.7.

It suffices to show that the linear system $|H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$ contains a reduced and connected divisor.

Note that $G_1 + |3H_2 - E_5 - E_6| \subset |H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$, and that $|3H_2 - E_5 - E_6|$ contains $|H_2 - E_5 - E_6| + |2H_2|$, if there is a line H_2 containing P_1, P_2, else it contains $|H_2 - E_5| + |H_2 - E_6| + |H_2|$. Since

$$G_1 \cdot H_2 = G_1 \cdot (H_2 - E_5) = G_1 \cdot (H_2 - E_6) = G_1 \cdot (H_2 - E_5 - E_6) = 1,$$

we have obtained in both cases a reduced and connected divisor. □

Remark 2.13. One can indeed show, using

$$G_2 + |3H_2 - E_3 - E_4| \subset |H_1 + 4H_2 - \sum_{i=1}^{6} E_i|,$$

$$G_3 + |3H_2 - E_1 - E_2| \subset |H_1 + 4H_2 - \sum_{i=1}^{6} E_i|,$$

that $|H_1 + 4H_2 - \sum_{i=1}^6 E_i|$ has no fixed part, and then by Bertini's theorem, since $(H_1 + 4H_2 - \sum_{i=1}^6 E_i)^2 = 8 - 6 = 2 > 0$, a general curve in $|H_1 + 4H_2 - \sum_{i=1}^6 E_i|$ is irreducible.

In view of Proposition 2.12 the above exact sequence (and the one where the roles of H_1, H_2 are exchanged) yields the following:

Corollary 2.14. *For $j \in \{1, 2\}$ there is exactly one divisor $N_j \in |3H_j - \sum_{i=1}^6 E_i|$.*

By the uniqueness of G, we see that $G = N_1 + N_2$. Denote by N'_j the curve in $\mathbb{P}^1 \times \mathbb{P}^1$ whose total transform is $N_j + \sum_{i=1}^6 E_i$.

We have just seen that G is the strict transform of three vertical and three horizontal lines in $\mathbb{P}^1 \times \mathbb{P}^1$. Hence each connected component G_j splits into the strict transform of a vertical and a horizontal line. Since G is reduced, the lines are distinct (and there are no infinitely near points).

We can choose coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $G'_1 = (\{\infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{\infty\})$, $G'_2 = (\{0\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{0\})$, and $G'_3 = (\{1\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{1\})$.

Remark 2.15. The points $P_1, \dots, P_6$ are then the points of the set $\mathcal{S}$ previously defined.

Conversely, consider in $\mathbb{P}^1 \times \mathbb{P}^1$ the set

$$\mathcal{S} := \{P_1, \dots, P_6\} = (\{\infty, 0, 1\} \times \{\infty, 0, 1\}) \setminus \{(\infty, \infty), (0, 0), (1, 1)\}.$$

Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blowup of the points $P_1, \dots, P_6$ and suppose (denoting the exceptional divisor over P_i by E_i) that $C \in |4H_1 + 4H_2 - \sum 2E_i|$ is a smooth curve. Then C has genus 3, $\mathcal{O}_C(3H_1) \cong \mathcal{O}_C(\sum E_i) \cong \mathcal{O}_C(3H_2)$. Setting $\mathcal{O}_C(\eta) := \mathcal{O}_C(H_2 - H_1)$, we obtain therefore $3\eta \equiv 0$.

It remains to show that $\mathcal{O}_C(\eta)$ is not isomorphic to $\mathcal{O}_C$.

Lemma 2.16. *η is not trivial.*

Proof. Assume $\eta \equiv 0$. Then $\mathcal{O}_C(H_1) \cong \mathcal{O}_C(H_2)$ and, since Γ has bidegree $(4, 4)$, we argue as in the proof of Proposition 2.3 that $h^0(\mathcal{O}_C(H_i)) \geq 3$, whence $\mathcal{O}_C(H_i) \cong \mathcal{O}_C(K_C)$.

The same argument shows that the two projections of Γ to $\mathbb{P}^1$ yield two different pencils in the canonical system. It follows that the canonical map of C factors as the composition of $C \to \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ with the rational map $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ which blows up one point and contracts the vertical and horizontal line through it. Since Γ has six singular points, the canonical map sends C birationally onto a singular quartic curve in $\mathbb{P}^2$, contradiction. $\square$

3 Rationality of the moduli spaces

In this section we will use the geometric description of pairs (C,η), where C is a genus 3 curve and η a nontrivial 3-torsion divisor class, and study the birational structure of their moduli space.

More precisely, we shall prove the following:

Theorem 3.1.
1) The moduli space

$$\mathcal{M}_{3,\eta} := \{(C,\eta) : C \ \textit{a general curve of genus } 3,\ \eta \in Pic^0(C)_3 \setminus \{0\}\}$$

is rational.
2) The moduli space

$$\mathcal{M}_{3,\langle\eta\rangle} := \{(C,\langle\eta\rangle) : C \ \textit{a general curve of genus } 3, \langle\eta\rangle \cong \mathbb{Z}/3\mathbb{Z} \subset Pic^0(C)\}$$

is rational.

Remark 3.2. By the result of the previous section, and since any automorphism of $\mathbb{P}^1\times\mathbb{P}^1$ which sends the set $\mathcal{S}$ to itself belongs to the group $\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z}$, it follows immediately that, if we set

$$V(4,4,-\mathcal{S}) := H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)(-2\sum_{i\neq j, i,j\in\{\infty,0,1\}} P_{ij})),$$

then $\mathcal{M}_{3,\eta}$ is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/\mathfrak{S}_3$, while $\mathcal{M}_{3,\langle\eta\rangle}$ is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/(\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z})$, where the generator σ of $\mathbb{Z}/2\mathbb{Z}$ acts by coordinate exchange on $\mathbb{P}^1\times\mathbb{P}^1$, whence on $V(4,4,-\mathcal{S})$.

In order to prove the above theorem we will explicitly calculate the respective subfields of invariants of the function field of $\mathbb{P}(V(4,4,-\mathcal{S}))$ and show that they are generated by purely transcendental elements.

Consider the following polynomials of $\mathbb{V} := V(4,4,-\mathcal{S})$, which are invariant under the action of $\mathbb{Z}/2\mathbb{Z}$:

$$f_{11}(x,y) := x_0^2x_1^2y_0^2y_1^2,$$

$$f_{\infty\infty}(x,y) := x_1^2(x_1-x_0)^2y_0^2(y_1-y_0)^2,$$

$$f_{00}(x,y) := x_0^2(x_1-x_0)^2y_0^2(y_1-y_0)^2.$$

Let $ev : \mathbb{V} \to \bigoplus_{i=0,1,\infty} k_{(i,i)} =: \mathbb{W}$ be the evaluation map at the three standard diagonal points, i.e., $ev(f) := (f(0,0), f(1,1), f(\infty,\infty))$.

Since $f_{ii}(j,j) = \delta_{i,j}$, we can decompose $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, where $\mathbb{U} := \ker(ev)$ and $\mathbb{W}$ is the subspace generated by the three above polynomials, which is easily shown to be an invariant subspace using the following formulae $(*)$:

- $(1,3)$ exchanges x_0 with x_1, multiplies $x_1 - x_0$ by -1,
- $(1,2)$ exchanges $x_1 - x_0$ with x_1, multiplies x_0 by -1,
- $(2,3)$ exchanges $x_0 - x_1$ with x_0, multiplies x_1 by -1.

In fact, 'the permutation' representation $\mathbb{W}$ of the symmetric group splits (in characteristic $\neq 3$) as the direct sum of the trivial representation (generated by $e_1 + e_2 + e_3$) and the standard representation, generated by $x_0 := e_1 - e_2, x_1 := -e_2 + e_3$, which is isomorphic to the representation on $V(1) := H^0(\mathcal{O}_{\mathbb{P}^1}(1))$.

Note that $\mathbb{U} = x_0 x_1 (x_1 - x_0) y_0 y_1 (y_0 - y_1) H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$. We write

$$V(1,1) := H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)) = V(1) \otimes V(1),$$

where $V(1) := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, is as above the standard representation of $\mathfrak{S}_3$.

Now $V(1) \otimes V(1)$ splits, in characteristic $\neq 2, 3$, as a sum of irreducible representations $\mathbb{I} \oplus \mathfrak{A} \oplus W$, where the three factors are the *trivial*, the *alternating* and the *standard* representation of $\mathfrak{S}_3$.

Explicitly, $V(1) \otimes V(1) \cong \wedge^2(V(1)) \oplus Sym^2(V(1))$, and $Sym^2(V(1))$ is isomorphic to $\mathbb{W}$, since it has the following basis: $x_0 y_0, x_1 y_1, (x_1 - x_0)(y_1 - y_0)$. We observe for further use that $\mathbb{Z}/2\mathbb{Z}$ acts as the identity on $Sym^2(V(1))$, while it acts on $\wedge^2(V(1))$, spanned by $x_1 y_0 - x_0 y_1$ via multiplication by -1.

We have thus seen

Lemma 3.3. *If $char(k) \neq 2, 3$, then the $\mathfrak{S}_3$-module $\mathbb{V}$ splits as a sum of irreducible modules as follows:*

$$\mathbb{V} \cong 2(\mathbb{I} \oplus W) \oplus \mathfrak{A}.$$

Choose now a basis $(z_1, z_2, z_3, w_1, w_2, w_3, u)$ of $\mathbb{V}$, such that the z_i's and the w_i's are respective bases of $\mathbb{I} \oplus W$ consisting of eigenvectors of $\sigma = (123)$, and u is a basis element of $\mathfrak{A}$. The eigenvalue of z_i, w_i with respect to $\sigma = (123)$ is ϵ^{i-1}, u is σ-invariant and $(12)(u) = -u$.

Note that if (v_1, v_2, v_3) is a basis of $\mathbb{I} \oplus W$, such that $\mathfrak{S}_3$ acts by permutation of the indices, then $z_1 = v_1 + v_2 + v_3$, $z_2 = v_1 + \epsilon v_2 + \epsilon^2 v_3$, $z_3 = v_1 + \epsilon^2 v_2 + \epsilon v_3$, where ϵ is a primitive third root of unity.

Remark 3.4. Since z_1, w_1 are $\mathfrak{S}_3$-invariant, $\mathbb{P}(V(4,4,-\mathcal{S}))/\mathfrak{S}_3$ is birational to a product of the affine line with $Spec(k[z_2, z_3, w_2, w_3, u]^{\mathfrak{S}_3})$, and therefore it suffices to compute $k[z_2, z_3, w_2, w_3, u]^{\mathfrak{S}_3}$.

Part 1 of the theorem follows now from the following

Proposition 3.5. *Let $T := z_2 z_3$, $S := z_2^3$, $A_1 := z_2 w_3 + z_3 w_2$, $A_2 := z_2 w_3 - z_3 w_2$. Then*

$$k(z_2, z_3, w_2, w_3, u)^{\mathfrak{S}_3} \supset K := k(A_1, T, S + \frac{T^3}{S}, u(S - \frac{T^3}{S}), A_2(S - \frac{T^3}{S})),$$

and $[k(z_2, z_3, w_2, w_3, u) : K] = 6$, *hence* $k(z_2, z_3, w_2, w_3, u)^{\mathfrak{S}_3} = K$.

Proof. We first calculate the invariants under the action of $\sigma = (123)$, i.e., $k(z_2, z_3, w_2, w_3, u)^{\sigma}$. Note that u, z_2z_3, z_2w_3, w_2w_3,z_2^3 are σ-invariant, and $[k(z_2, z_3, w_2, w_3, u) : k(u, z_2z_3, z_2w_3, w_2w_3, z_2^3)] = 3$. In particular,

$$k(z_2, z_3, w_2, w_3, u)^{\sigma} = k(u, z_2z_3, z_2w_3, w_2w_3, z_2^3) =: L.$$

Now, we calculate L^{τ}, with $\tau = (12)$. Observe that $L = k(T, A_1, A_2, S, u)$. Since $\tau(z_2) = \epsilon z_3$, $\tau(z_3) = \epsilon^2 z_2$ (and similarly for w_2, w_3), we see that $\tau(A_1) = A_1$ and $\tau(T) = T$. On the other hand, $\tau(u) = -u$, $\tau(A_2) = -A_2$, $\tau(S) = \frac{T^3}{S}$.

Claim.
$L^{\tau} = k(A_1, T, S + \frac{T^3}{S}, u(S - \frac{T^3}{S}), A_2(S - \frac{T^3}{S})) =: E.$

Proof of the Claim. Obviously A_1,T,$S+\frac{T^3}{S}$,$u(S-\frac{T^3}{S})$, $A_2(S-\frac{T^3}{S})$ are invariant under τ, whence $E \subset L^{\tau}$. Since $L = E(S)$, using the equation $B \cdot S = S^2 + T^3$ for $B := S + \frac{T^3}{S}$, we get that $[E(S) : E] \leq 2$.

This proves the claim and the proposition. $\square$

It remains to show the second part of the theorem. We denote by τ' the involution on $k(z_1, z_2, z_3, w_1, w_2, w_3, u)$ induced by the involution $(x, y) \mapsto (y, x)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. It suffices to prove the following

Proposition 3.6. $E^{\tau'} = k(A_1, T, S + \frac{T^3}{S}, (u(S - \frac{T^3}{S}))^2, A_2(S - \frac{T^3}{S}))$.

Proof. Since $[E : k(A_1, T, S + \frac{T^3}{S}, (u(S - \frac{T^3}{S}))^2, A_2(S - \frac{T^3}{S}))] \leq 2$, it suffices to show that the five generators A_1, T,$S + \frac{T^3}{S}$,$(u(S - \frac{T^3}{S}))^2$, $A_2(S - \frac{T^3}{S})$ are τ'-invariant. This will now be proven in Lemma 3.7. $\square$

Lemma 3.7. τ' *acts as the identity on* $(z_1, z_2, z_3, w_1, w_2, w_3)$ *and sends* $u \mapsto -u$.

Proof. We note first that τ' acts trivially on the subspace $\mathbb{W}$ generated by the polynomials f_{ii}.

Since $\mathbb{U} = x_0x_1(x_1 - x_0)y_0y_1(y_1 - y_0)V(1, 1)$ and $x_0x_1(x_1 - x_0)y_0y_1(y_1 - y_0)$ is invariant under exchanging x and y, it suffices to recall that the action of τ' on $V(1, 1) = V(1) \otimes V(1)$ is the identity on the subspace $Sym^2(V(1))$, while the action on the alternating $\mathfrak{S}_3$-submodule $\mathfrak{A}$ sends the generator u to $-u$. $\square$

3.1 *Char(k) = 3*

In order to prove Theorem 3.1 if the characteristic of k is equal to 3, we describe the $\mathfrak{S}_3$-module $\mathbb{V}$ as follows:

$$\mathbb{V} \cong 2\mathbb{W} \oplus \mathfrak{A},$$

where $\mathbb{W}$ is the (three-dimensional) permutation representation of $\mathfrak{S}_3$.

Let now $z_1, z_2, z_3, w_1, w_2, w_3, u$ be a basis of $\mathbb{V}$ such that the action of $\mathfrak{S}_3$ permutes z_1, z_2, z_3 (resp. w_1, w_2, w_3), and $(123) : u \mapsto u$, $(12)u \mapsto -u$. Then we have:

Proposition 3.8. *The $\mathfrak{S}_3$-invariant subfield $k(\mathbb{V})^{\mathfrak{S}_3}$ of $k(\mathbb{V})$ is rational.*

More precisely, the seven $\mathfrak{S}_3$-invariant functions

$$\sigma_1 = z_1 + z_2 + z_3,$$

$$\sigma_2 = z_1z_2 + z_1z_3 + z_2z_3,$$

$$\sigma_3 = z_1z_2z_3,$$

$$\sigma_4 = z_1w_1 + z_2w_2 + z_3w_3,$$

$$\sigma_5 = w_1z_2z_3 + w_2z_1z_3 + w_3z_1z_2,$$

$$\sigma_6 = w_1(z_2 + z_3) + w_2(z_1 + z_3) + w_3(z_1 + z_2),$$

$$\sigma_7 = u(z_1(w_2 - w_3) + z_2(w_3 - w_1) + z_3(w_1 - w_2))$$

form a basis of the purely transcendental extension over k.

Proof. $\sigma_1, \dots, \sigma_7$ determine a morphism $\psi : \mathbb{V} \to \mathbb{A}^7_k$. We will show that ψ induces a birational map $\bar{\psi} : \mathbb{V}/\mathfrak{S}_3 \to \mathbb{A}^7_k$, i.e., for a Zariski open set of $\mathbb{V}$ we have: $\psi(x) = \psi(x')$ if and only if there is a $\tau \in \mathfrak{S}_3$ such that $x = \tau(x')$. By [Cat, Lemma 2.2] we can assume (after acting on x with a suitable $\tau \in \mathfrak{S}_3$) that $x_i = x'_i$ for $1 \leq i \leq 6$, and we know that (setting $u := x_7$, $u' := x'_7$)

$$u(x_1(x_5 - x_6) + x_2(x_6 - x_4) + x_3(x_4 - x_5)) = \\ u'(x_1(x_5 - x_6) + x_2(x_6 - x_4) + x_3(x_4 - x_5)).$$

Therefore, if $B(x_1, \dots, x_6) := x_1(x_5 - x_6) + x_2(x_6 - x_4) + x_3(x_4 - x_5) \neq 0$, this implies that $u = u'$. $\square$

Therefore, we have shown part 1 of Theorem 3.1.

We denote again by τ' the involution on $k(z_1, z_2, z_3, w_1, w_2, w_3, u)$ induced by the involution $(x, y) \mapsto (y, x)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. In order to prove part 2 of Theorem 3.1, it suffices to observe that $\sigma_1, \dots, \sigma_6, \sigma_7^2$ are invariant under τ' and $[k(\sigma_1, \dots, \sigma_7) : k(\sigma_1, \dots, \sigma_7^2)] \leq 2$, whence $(k(\mathbb{V})^{\mathfrak{S}_3})^{(\mathbb{Z}/2\mathbb{Z})} = k(\sigma_1, \dots, \sigma_7^2)$. This proves Theorem 3.1.

3.2 *Char*$(k) = 2$

Let k be an algebraically closed field of characteristic 2. Then we can describe the $\mathfrak{S}_3$-module $\mathbb{V}$ as follows:

$$\mathbb{V} \cong \mathbb{W} \oplus V(1,1),$$

where $\mathbb{W}$ is the (three-dimensional) permutation representation of $\mathfrak{S}_3$. We denote a basis of $\mathbb{W}$ by z_1, z_2, z_3. As in the beginning of the chapter, $V(1,1) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$. We choose the following basis of $V(1,1)$: $w_1 := x_1y_1$, $w_2 := (x_0+x_1)(y_0+y_1)$, $w_3 := x_0y_0$, $w := x_0y_1$. Then $\mathfrak{S}_3$ acts on w_1, w_2, w_3 by permutation of the indices and

$$(1,2) : w \mapsto w + w_3,$$

$$(1,2,3) : w \mapsto w + w_2 + w_3.$$

Let $\epsilon \in k$ be a nontrivial third root of unity. Then Theorem 3.1 (in characteristic 2) follows from the following result:

Proposition 3.9. *Let k be an algebraically closed field of characteristic* 2. *Let $\sigma_1, \ldots, \sigma_6$ be as defined in (3.6) and set*

$$v := (w + w_2)(w_1 + \epsilon w_2 + \epsilon^2 w_3) + (w + w_1 + w_3)(w_1 + \epsilon^2 w_2 + \epsilon w_3),$$

$$t := (w + w_2)(w + w_1 + w_3).$$

Then

1) $k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3} = k(\sigma_1, \ldots, \sigma_6, v)$*;*

2) $k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z}} = k(\sigma_1, \ldots, \sigma_6, t)$*.*

In particular, the respective invariant subfields of $k(\mathbb{V})$ are generated by purely transcendental elements, and this proves Theorem 3.1.

Proof (of Proposition 3.9). 2) We observe that $\mathbb{Z}/2\mathbb{Z}$ ($x_i \mapsto y_i$) acts trivially on $z_1, z_2, z_3, w_1, w_2, w_3$ and maps w to $w + w_1 + w_2 + w_3$. It is now easy to see that t is invariant under the action of $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$. Therefore $k(\sigma_1, \ldots, \sigma_6, t) \subset K := k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z}}$. By [Cat, Lemma 2.8], $[k(z_1, z_2, z_3, w_1, w_2, w_3, t) : k(\sigma_1, \ldots, \sigma_6, t)] = 6$, and obviously, $[k(z_1, z_2, z_3, w_1, w_2, w_3, w) : k(z_1, z_2, z_3, w_1, w_2, w_3, t)] = 2$. Therefore $[k(z_1, z_2, z_3, w_1, w_2, w_3, w) : k(\sigma_1, \ldots, \sigma_6, t)] = 12$, whence $K = k(\sigma_1, \ldots, \sigma_6, t)$.

1) Note that for $W_2 := w_1 + \epsilon w_2 + \epsilon^2 w_3$, $W_3 := w_1 + \epsilon^2 w_2 + \epsilon^3 w_3$, we have: W_2^3 and W_3^3 are invariant under $(1,2,3)$ and are exchanged under $(1,2)$. Therefore v is invariant under the action of $\mathfrak{S}_3$ and we have seen that $k(\sigma_1, \ldots, \sigma_6, v) \subset L := k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3}$, in particular $[k(z_1, z_2, z_3, w_1, w_2, w_3, w) : k(\sigma_1, \ldots, \sigma_6, v)] \geq 6$. On the other hand, note that $k(z_1, z_2, z_3, w_1, w_2, w_3, w) = k(z_1, z_2, z_3, w_1, w_2, w_3, v)$ (since v is linear in w) and again, by [Cat, Lemma 2.8], $[k(z_i, w_i, v) : k(\sigma_1, \ldots, \sigma_6, v)] = 6$. This implies that $L = k(\sigma_1, \ldots, \sigma_6, v)$. $\square$

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