ON VARIETIES WHOSE UNIVERSAL COVER IS A PRODUCT OF CURVES

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Abstract. We investigate a necessary condition for a compact complex manifold X of dimension n in order that its universal cover be the Cartesian product C^n of a curve $C = \mathbb{P}^1$ *or* $\mathbb{H}$: the existence of a semispecial tensor ω.

A semispecial tensor is a non zero section $0 \neq \omega \in H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta))$, where η is an invertible sheaf of 2-torsion (i.e., $\eta^2 \cong \mathcal{O}_X$). We show that this condition works out nicely, as a sufficient condition, when coupled with some other simple hypothesis, in the case of dimension $n = 2$ or $n = 3$; but it is not sufficient alone, even in dimension 2.

In the case of Kähler surfaces we use the above results in order to give a characterization of the surfaces whose universal cover is a product of two curves, distinguishing the 6 possible cases.

1. Introduction

The beauty of the theory of algebraic curves is deeply related to the manifold implications of the:

Theorem 1.1 (Uniformization theorem of Koebe and Poincaré)**.** *Let C be a smooth (connected) compact complex curve of genus g, and let $\tilde{C}$ be its universal cover. Then*

$$\tilde{C} \cong \begin{cases} \mathbb{P}^1 & \text{if } g = 0 \\ \mathbb{C} & \text{if } g = 1 \\ \mathbb{H} & \text{if } g \geq 2 \end{cases}$$

($\mathbb{H}$ denotes as usual the Poincaré upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$, but we shall often refer to it as the 'disk' since it is biholomorphic to $\mathbb{D} := \{z \in \mathbb{C} : ||z|| < 1\}$).

Hence a smooth (connected) compact complex curve C of genus $g \geq 1$ admits a uniformization in the strong sense *(ii)* of the following definition (for $g = 0$, only (i) holds):

Definition 1.2. *A connected complex space X of complex dimension n admits a* **Galois uniformization** *if :*

(i) there is a connected open set $\Omega \subset \mathbb{C}^n$ and a properly discontinuous group $\Gamma \subset Aut(\Omega)$ such that $\Omega/\Gamma \cong X$

If X is a complex manifold, there is the stronger property where we require the action of Γ to be free:

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*(ii) there is a connected open set $\Omega \subset \mathbb{C}^n$ biholomorphic to the universal cover of X (**strong uniformization**).*

Observe that a result of Fornaess and Stout (cf. [F-S77]) says that, if X is an n-dimensional complex manifold, then there is a connected open set $\Omega \subset \mathbb{C}^n$ and a surjective holomorphic submersion $f \colon \Omega \to X$; i.e., every complex manifold admits an 'étale (but not Galois) uniformization'.

On the contrary, the condition that the universal cover be biholomorphic to a bounded domain $\Omega \subset\subset \mathbb{C}^n$ tends to be quite exceptional in dimension $n \geq 2$, where plenty of simply connected manifolds exist.

An important remark is that if Ω is bounded and Γ acts freely on Ω with compact quotient, then the complex manifold $X := \Omega/\Gamma$ has ample canonical bundle K_X (see [Sieg73]): in particular it is a projective manifold of general type.

Even more exceptional is the case where the universal cover is biholomorphic to a bounded symmetric domain Ω, or where there is a Galois uniformization with source a bounded symmetric domain, and there is already a vast literature on a characterization of these properties (cf. [Yau77], [Yau88], [Yau93], [Bea00]). The basic result in this direction is S.T. Yau's uniformization theorem (explained in [Yau88] and [Yau93]), and for which a very readable exposition is contained in the first section of [V-Z05], emphasizing the role of polystability of the cotangent bundle for varieties of general type. One would wish nevertheless for more precise or simple characterizations of the various possible cases.

The paper [B-P-T06], which extends work of Yau and Beauville, especially [Bea00], gives a nice sufficient condition in order that the universal cover of a compact Kähler manifold X be biholomorphic to a product of curves. If the tangent bundle T_X splits as a sum of line subbundles, $T_X = L_1 \oplus \cdots \oplus L_n$, then its universal cover $\tilde{X}$ is biholomorphic to a product of curves:

$$\tilde{X} \cong (\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t,$$

for suitable $r, s, t \in \mathbb{N}$.

The above result is not a characterization, in the sense that the splitting condition is not a necessary one, even if we weaken it to the condition that there is a finite étale covering $X' \to X$ such that the tangent bundle of X' splits.

The purpose of this work is to investigate to which extent one can find a simple characterization of the above property in terms of some necessary and sufficient conditions which a compact complex (respectively, Kähler) manifold X must fulfill in order that its universal cover be biholomorphic to a product of curves.

If we require that the universal cover $\tilde{X}$ be biholomorphic to $(\mathbb{P}^1)^n$ or $\mathbb{H}^n$ we have the following necessary condition (the case of Kodaira surfaces, cf. [Bea00], shows that $\tilde{X} \cong \mathbb{C}^n$ without the Kähler assumption does not imply this condition):

Definition 1.3. *Let X be a complex manifold of complex dimension n.*

Then a **special tensor** *is a non zero section* $0 \neq \omega \in H^0(X, S^n\Omega^1_X(-K_X))$, *while a* **semi special tensor** *is a non zero section* $0 \neq \omega \in H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta)$, *where* η *is an invertible sheaf such that* $\eta^2 \cong \mathcal{O}_X$.

We shall say that the semi special tensor is of unique type if moreover $dim(H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta)) = 1$.

We have in fact:

Proposition 1.4. *Let* X *be a compact complex manifold whose universal cover is biholomorphic to* $(\mathbb{P}^1)^n$ *or to* $\mathbb{H}^n$*: then* X *admits a semi special tensor.*

As we shall see considering the two dimensional case, the existence of a semispecial tensor is not sufficient in order to guarantee a totally split universal cover, and one has to look for further complementary assumptions, one such can be for instance the condition of ampleness of the canonical divisor K_X.

Let us discuss first the case of a smooth compact complex surface.

Here, a famous uniformization result is the characterization, due to Miyaoka and Yau, of complex surfaces whose universal cover is the two dimensional ball $\mathbb{B}_2$. It is given purely in terms of certain numbers which are either bimeromorphic or topological invariants.

Theorem 1.5 (Miyaoka-Yau)**.** *Let* X *be a compact complex surface. Then* $X \cong \mathbb{B}_2/\Gamma$ *(with* Γ *a cocompact discrete subgroup of* $\mathrm{Aut}(\mathbb{B}_2)$ *acting freely on* $\mathbb{B}_2$*) if and only if*

(1) $K_X^2 = 9\chi(S) > 0$;
(2) *the second plurigenus* $P_2(X) > 0$.

The theorem follows combining Miyaoka's result ([Miy82]), that these two conditions imply the ampleness of K_X, with Yau's uniformization result ([Yau77]) which proves the existence of a Kähler-Einstein metric.

In the case where $X = (\mathbb{H} \times \mathbb{H})/\Gamma$, with Γ a discrete cocompact subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ acting freely, one has $K_X^2 = 8\chi(X)$.

But Moishezon and Teicher in [MT87] showed the existence of a simply connected surface of general type (hence with $P_2(X) > 0$) having $K_X^2 = 8\chi(X)$, so that the above conditions are necessary, but not sufficient. Our contribution here is a by-product of our attempt to answer the still open question whether there exists a minimal surface of general type with $p_g(X) = 0, K_X^2 = 8$ which is not uniformized by $\mathbb{H} \times \mathbb{H}$ (one has the same question for $\chi(X) = 1, K_X^2 = 8$).

The first result of this note is a precise characterization of compact complex surfaces whose universal cover is the bidisk, respectively the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, discussing whether some hypotheses can be dispensed with. We have the following result giving a refinement of a theorem of S.T. Yau (theorem 2.5 of [Yau93]), giving sufficient conditions for (ii) to hold.

Theorem 1.6. *Let* X *be a compact complex surface.*

X *is strongly uniformized by the bidisk (* $X \cong (\mathbb{H} \times \mathbb{H})/\Gamma$ *, where* Γ *is a cocompact discrete subgroup of* $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ *acting freely) if and only if*

(1*) X *admits a semi special tensor of unique type;*

(2) $K_X^2 > 0$;
(3) *the second plurigenus* $P_2(X) \geq 1$.

X is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if

(1**) *X admits a unique special tensor;*
(2) $K_X^2 = 8$;
(3**) *the second plurigenus* $P_2(X) = 0$;
(4) $h^0(\Omega^1_X(-K_X)) = 6$

In the above theorem one can replace condition (3) by :

(3*)$P_2(X) \geq 2$,

it is moreover interesting to see that none of the above hypotheses can be dispensed with. The most intriguing examples are provided by

Proposition 1.7. *There do exist properly elliptic surfaces X satisfying*

- *(1) X admits a special tensor;*
- *(3*) the second plurigenus $P_2(X) \geq 2$;*
- $q(X) := dim(H^1(\mathcal{O}_X)) > 0$;
- $K_X^2 = 0$;
- *X is not birational to a product.*

In this respect, we would like to pose the following question, which will be discussed in a later section.

Question. *Let X be a surface with $q(X) = 0$ and satisfying* (1*) *and* (3*)*: is then X strongly uniformized by the bidisk?*

Our final result concerning algebraic surfaces whose universal cover is a product of two curves follows combining the previous Theorem 1.6 with the following

Theorem 1.8. *Let S be a smooth compact Kähler surface S. Then the universal cover of S is biholomorphic to*

(1) $\mathbb{P}^1 \times \mathbb{C} \Leftrightarrow P_{12} := P_{12}(S) = 0,\ q := q(S) = 1,\ K_S^2 = 0$.
(2) $\mathbb{P}^1 \times \mathbb{H} \Leftrightarrow P_{12} = 0,\ q \geq 2,\ K_S^2 = 8(1-q)$.
(3) $\mathbb{C}^2 \Leftrightarrow P_{12} = 1,\ q = 1$ *or* $q = 2,\ K_S^2 = 0$.
(4) $\mathbb{C} \times \mathbb{H} \Leftrightarrow P_{12} \geq 2,\ e(S) = 0$.

Concerning the higher dimensional cases, we restrict our attention here to the case of manifolds with ample canonical divisor K_X which, by Yau' s theorem ([Yau77]) admit a canonical Kähler-Einstein metric.

Assume now that X admits a semi special tensor $\omega \in H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta)$. Then by [Yau88, p.272] and by [Yau93, p.479] (see also [V-Z05, p.10]) ω induces on the tangent bundle T_X a homogeneous hypersurface F_X of relative degree n which is parallel with respect to the Kähler-Einstein metric.

In particular, take a point $x \in X$, and consider the hypersurface of the projectivized tangent bundle induced by F_X: its fibre over x is a projective hypersurface $F_{X,x}$ of degree n which is invariant for the action of the (restricted) holonomy group $H \subset U(n)$ (H is the connected component of the identity in the holonomy group).

In this situation, assume that we can prove (possibly passing to a finite étale covering of X) that the holonomy leaves invariant a complete flag. Then, since the holonomy is unitary, it follows that $H \subset U(1)^n$ and we can conclude, either by Berger's classical theorem ([Ber53]), or by [B-P-T06], that the universal cover of X turns out to be $\mathbb{H}^n$.

In the three dimensional case the existence of a special tensor is enough in order to guarantee such a splitting.

Theorem 1.9. *Let X be a compact complex manifold of dimension $n \leq 3$. Then the following two conditions:*

(1) *X admits a semi special tensor;*
(2*) *K_X is ample*

hold if and only if $X \cong (\mathbb{H}^n)/\Gamma$ (where Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\mathbb{H}^n)$ *acting freely).*

In dimension ≥ 4, the above conditions are no longer sufficient. The natural category which is relevant to consider is the category of Hermitian symmetric spaces of noncompact type, since by the theorem of Berger-Simons an irreducible (in the sense of De Rham's theorem) Kähler manifold X of dimension n with ample canonical divisor K_X has holonomy $H \neq U(n)$ if and only if X is a Hermitian symmetric space of rank ≥ 2 (see [Yau88], and [V-Z05], section 1, page 300).

One has the Cartan realization of a Hermitian symmetric space of noncompact type as a bounded symmetric domain, and by the classical result of Borel on compact Clifford-Klein forms (see [Bor63]) any bounded symmetric domain X of dimension n admits a compact complex analytic Clifford-Klein form, that is a compact complex manifold X' whose universal covering is isomorphic to X.

The above results translate the question whether a compact complex manifold X admitting a semi special tensor and with ample canonical divisor K_X has the polydisk as universal cover into a purely Lie theoretic problem, the problem of existence of holonomy invariant hypersurfaces of degree n.

We leave aside for the moment this more general investigation, for which some partial results are contained in the appendix, due to A.J. Di Scala, who answered some of our questions.

For the bounded domain $\Omega \subset \mathbb{C}^4 \cong Mat(2,2,\mathbb{C}) := M_{2,2}(\mathbb{C})$, $\Omega = \{Z \in M_{2,2}(\mathbb{C}) \, : \, \mathrm{I}_2 - {}^tZ \cdot \overline{Z} > 0\}$, the Cartan realization of the Hermitian symmetric space $SU(2,2)/S(U(2) \times U(2))$, Di Scala pointed out that the holonomy action of $(A, D) \in S(U(2) \times U(2)$ is given by $Z \mapsto AZD^{-1}$. Hence the square of the determinant yields an invariant hypersurface of degree 4 which is twice a smooth quadric (and this is indeed the only other possible case).

Using this simple but important observation, we get the following

Theorem 1.10. *There exist compact Kähler manifolds X, for each dimension $n \geq 4$, such that*

(1) *X admits a special tensor;*
(2*) *K_X is ample*

and whose universal cover $\tilde{X}$ is not $\cong \mathbb{H}^n$ (i.e., is not a product of curves).

2. Preliminaries and remarks

2.1. **Notation.** X denotes throughout the paper a smooth compact complex manifold of dimension n.

We use the standard notation of algebraic geometry: Ω^1_X is the cotangent bundle (locally free sheaf), T_X is the holomorphic tangent bundle, $c_1(X)$, $c_2(X)$ are the Chern classes of X. K_X is a canonical divisor on X, i.e., $\Omega^n_X = \mathcal{O}_X(K_X)$ and the m-th plurigenus is defined as $P_m(X) := h^0(X, mK_X)$.

In particular, for $m = 1$, we have the geometric genus of X $p_g(X) := h^0(X, K_X)$, while $q(X) := h^1(X, \mathcal{O}_X)$ is classically called the **irregularity** of X.

Finally, $\chi(X) := \chi(\mathcal{O}_X)$ is the holomorphic Euler Poincaré characteristic of X, whereas $e(X)$ denotes the topological Euler Poincaré characteristic of X.

In the surface case ($n = 2$), $\chi(X) = 1 + p_g(X) - q(X)$.

With a slight abuse of notation, we do not distinguish between invertible sheaves, line bundles and divisors, while the symbol $\equiv$ denotes linear equivalence of divisors.

2.2. **Necessary conditions.**
First of all notice that the existence of a semi special tensor corresponds to the existence of a special tensor on an étale double cover of our manifold:

Remark 2.1. *A complex manifold X admits a semi special tensor if and only if it has an unramified cover X' of degree at most two which admits a special tensor.*

Proof. Assume that we have an invertible sheaf η such that $\eta^2 \cong \mathcal{O}_X$, $\eta \not\cong \mathcal{O}_X$. Take the corresponding double connected étale covering $\pi : X' \to X$ such that $\pi_* \mathcal{O}_{X'} \cong \mathcal{O}_X \oplus \eta$ and observe that

$$H^0(X', S^n\Omega^1_{X'}(-K_{X'})) \cong H^0(X, S^n\Omega^1_X(-K_X)) \oplus H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta).$$

Whence, there is a special tensor on X' if and only if there is a semi special tensor on X. □

Let us now show that if X is isomorphic to $(\mathbb{P}^1)^m/\Gamma$ or $(\mathbb{H})^m/\Gamma$ then X admits a semi special tensor.

Proof of Prop. 1.4 . Let us remark first that for a simply connected curve C , with $C \cong \mathbb{P}_1$, or $C \cong \mathbb{H}$, and any integer m, the group of automorphism of C^m, $\mathrm{Aut}(C^m)$, is the semidirect product of $(\mathrm{Aut}(C))^m$ with the symmetric group $\mathfrak{S}_m$, hence for every subgroup Γ_C of $\mathrm{Aut}(C^n)$ we have a diagram:

$$\begin{array}{ccccccccc} 1 \to & (\mathrm{Aut}(C))^m & \to & \mathrm{Aut}(C^m) & \to \mathfrak{S}_m & \to 1 \\ & \bigcup & & \bigcup & \bigcup & \\ 1 \to & \Gamma^0_C & \hookrightarrow & \Gamma_C & \to H_C & \to 1. \end{array}$$

Let now $X \cong (C^n)/\Gamma$ be a compact complex manifold whose universal cover $\tilde{X}$ is isomorphic to C^n. Then X admits a semi special tensor, induced by the following special tensor:

$$\tilde{\omega} := \frac{\mathrm{d}\, z_1 \otimes \cdots \otimes \mathrm{d}\, z_n}{\mathrm{d}\, z_1 \wedge \cdots \wedge \mathrm{d}\, z_n},$$

where $(z_1, \ldots, z_n)$ is the standard system of coordinates on $C = \mathbb{H}^n$, respectively on the standard open set $\mathbb{C}^n \subset (\mathbb{P}^1)^n$ (observe that $\tilde{\omega}$ is in this case everywhere regular).

$\tilde{\omega}$ is clearly invariant for $(\mathrm{Aut}(C))^n$ and for the alternating subgroup $\mathfrak{A}_n$. Let η be the 2-torsion invertible sheaf on X associated to the signature character of $\mathfrak{S}_n$ restricted to H_C: then clearly $\tilde{\omega}$ descends to a semi special tensor $\omega \in H^0(X, S^n\Omega^1_X(-K_X) \otimes \eta)$.

□

In the more general case where the universal cover is a product of curves, we have the following proposition:

Proposition 2.2. *We have a homomorphism*

$$\Phi : \mathrm{Aut}((\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t) \to \mathrm{Aut}(\mathbb{C}^s \times \mathbb{H}^t)$$

which is injective on any subgroup Γ which acts freely. Moreover, if $\Gamma_2 \subset \mathrm{Aut}(\mathbb{C}^s \times \mathbb{H}^t)$ is the image of Γ under Φ, Γ_2 acts also freely, and Γ_2 acts properly discontinuosly if Γ is properly discontinuos.

In particular, if $X \cong ((\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t)/\Gamma$, with Γ a cocompact discrete subgroup of $\mathrm{Aut}((\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t)$ which acts freely, then the natural projection

$$((\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t)/\Gamma \to (\mathbb{C}^s \times \mathbb{H}^t)/\Gamma_2$$

inherits a $(\mathbb{P}^1)^r-$bundle structure.

Before giving the proof let us point out the following:

Lemma 2.3. *Let $\psi \in \mathrm{Aut}((\mathbb{P}^1)^r)$ be an automorphism. Then ψ has a fixed point.*

Proof. For $r = 1$ this is well known, since there exists an eigenvector for each $A \in GL(2, \mathbb{C})$.

For $r \geq 2$ any automorphism $\psi \in \mathrm{Aut}((\mathbb{P}^1)^r)$ is of the form

$$(\psi(x))_i = \psi_i(x_{\sigma(i)})$$

for a suitable permutation σ of $\{1, \ldots r\}$. Therefore a fixed point is a solution to the system of equations

$$x_i = \psi_i(x_{\sigma(i)}) \ (i = 1, \ldots r).$$

Using the cycle decomposition of σ we easily reduce to the case where $\sigma = (1, 2, \ldots r)$ and it suffices to find a solution to $x_1 = \psi_1 \circ \ldots \psi_r(x_1)$.

□

Proof. of Prop 2.2. Let $\phi \in \mathrm{Aut}((\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t)$.

Let Φ_2 be the composition $p_2 \circ \phi$, where

$p_2 : (\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t \to \mathbb{C}^s \times \mathbb{H}^t$

is the second projection.

Now, for every point $p \in \mathbb{C}^s \times \mathbb{H}^t$, Φ_2 is constant on $(\mathbb{P}^1)^r \times \{p\}$ since $(\mathbb{P}^1)^r$ is compact.

Hence ϕ induces $\phi_2 \in \mathrm{Aut}(\mathbb{C}^s \times \mathbb{H}^t)$.

Assume that ϕ acts freely, and that ϕ_2 has a fixed point p. Then ϕ acts on $(\mathbb{P}^1)^r \times \{p\}$ and it has a fixed point there by the previous lemma: whence ϕ is the identity.

If the action of Γ is properly discontinuous, then for any compact $K \subset (\mathbb{C}^s \times \mathbb{H}^t)$, also $(\mathbb{P}^1)^r \times K$ is compact; hence the set $\Gamma_2(K,K) = \Gamma((\mathbb{P}^1)^r \times K, (\mathbb{P}^1)^r \times K)$ is finite. Therefore Γ_2 is also properly discontinuous.

□

Remark 2.4. *We also have a homomorphism*

$$\Phi : \mathrm{Aut}((\mathbb{C}^1)^r \times \mathbb{H}^t) \to \mathrm{Aut}(\mathbb{H}^t)$$

However, as shown by the case of Inoue surfaces, if $X \cong ((\mathbb{C}^r \times \mathbb{H}^t)/\Gamma$, where Γ is a cocompact discrete subgroup of $\mathrm{Aut}((\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{H}^t)$ which acts freely, then the image group $\Gamma_2 \subset \mathrm{Aut}(\mathbb{H}^t)$ does not necessarily act properly discontinuously. One needs for this the assumption that X be Kähler.

3. Surfaces whose universal cover is a product of curves

In the case of surfaces the existence of a special tensor, as we are now going to explain, is equivalent to the existence of a trace zero endomorphism of the tangent bundle: and if this endomorphism is not nilpotent, one obtains a splitting of the tangent bundle.

Let us recall a result of Beauville which characterizes compact complex surfaces whose universal cover is a product of two complex curves (cf. [Bea00, Thm. C]).

Theorem 3.1 (Beauville)**.** *Let X be a compact complex surface. The tangent bundle T_X splits as a direct sum of two line bundles if and only if either X is a special Hopf surface or the universal covering space of X is a product $U \times V$ of two complex curves and the group $\pi_1(X)$ acts diagonally on $U \times V$.*

Given a direct sum decomposition of the cotangent bundle $\Omega^1_X \cong L_1 \oplus L_2$, Beauville shows moreover that $(L_1)^2 = (L_2)^2 = 0$ (cf. [Bea00, 4.1, 4.2]) hence

$$K_X \equiv L_1 + L_2 \qquad c_1(X)^2 = 2 \cdot (L_1 \cdot L_2) = 2 \cdot c_2(X), \ \textit{i.e.}, K_X^2 = 8\chi(X).$$

Let us now consider the bundle $\mathrm{End}(T_X)$ of endomorphisms of the tangent bundle. We can write $\mathrm{End}(T_X) = \Omega^1_X \otimes T_X$ and since from the nondegenerate bilinear map

$$\Omega^1_X \times \Omega^1_X \longrightarrow \Omega^2_X \cong K_X$$

we get $T_X = (\Omega^1_X)^\vee \cong \Omega^1_X(-K_X)$ we have an isomorphism

$\mathrm{End}(T_X) \cong \Omega^1_X \otimes \Omega^1_X(-K_X)$.

Let us see how this isomorphism works in local coordinates (z_1, z_2). I.e., let us see how an element $\frac{\mathrm{d}\, z_i \otimes \mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$ in $\Omega^1_X \otimes \Omega^1_X(-K_X)$ acts on a vector of the form $\frac{\partial}{\partial z_h}$. We have

$$\frac{\mathrm{d}\, z_i \otimes \mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}\left(\frac{\partial}{\partial z_h}\right) = \begin{cases} \frac{\mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2} & \text{if } h = i \\ 0 & \text{if } h \neq i \end{cases}$$

In turn, $\dfrac{\mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$ evaluated on $\mathrm{d}\, z_k$ gives $\dfrac{\mathrm{d}\, z_j \wedge \mathrm{d}\, z_k}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$.

Therefore a generic element $\sum_{i,j} a_{ij} \dfrac{\mathrm{d}\, z_i \otimes \mathrm{d}\, z_j}{\mathrm{d}\, z_1 \wedge \mathrm{d}\, z_2}$ corresponds to an endomorphism, which, with respect to the basis $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$ is expressed by the matrix

$$\begin{pmatrix} -a_{12} & -a_{22} \\ a_{11} & a_{21} \end{pmatrix}$$

In particular for the symmetric tensors (i.e., $a_{12} = a_{21}$), respectively for the skewsymmetric tensors (i.e., $a_{12} = -a_{21}, a_{11} = a_{22} = 0$) the following isomorphisms hold:

$$S^2(\Omega^1_X)(-K_X) \cong \left\{ \begin{pmatrix} -a & -a_{22} \\ a_{11} & a \end{pmatrix} \right\}; \qquad \bigwedge^2(\Omega^1_X)(-K_X) \cong \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \right\}$$

We can summarize the above discussion in the following

Lemma 3.2. *If X is a complex surface there is a natural isomorphism between the sheaf $S^2(\Omega^1_X)(-K_X)$ and the sheaf of trace zero endomorphisms of the (co)tangent sheaf $\mathrm{End}^0(T_X) \cong \mathrm{End}^0(\Omega^1_X)$.*

A special tensor $\omega \in H^0(S^2(\Omega^1_X)(-K_X))$ with nonzero determinant $det(\omega) \in \mathbb{C}$ yields an eigenbundle splitting $\Omega^1_X \cong L_1 \bigoplus L_2$ of the cotangent bundle.

If instead $det(\omega) = 0 \in \mathbb{C}$, the corresponding endomorphism ϵ is nilpotent and yields an exact sequence of sheaves

$$0 \to L \to \Omega^1_X \to \mathcal{I}_Z L(-\Delta) \to 0$$

where $L := ker(\epsilon)$ is invertible, Δ is an effective divisor, and Z is a 0-dimensional subscheme(which is a local complete intersection).

We have in particular $K_X \equiv 2L - \Delta$ and $c_2(X) = length(Z) + L \cdot (L - \Delta)$.

Proof. We need only to observe that $det(\omega)$ is a constant, since $det(\mathrm{End}(T_X)) = det(\mathrm{End}(\Omega^1_X)) \cong \mathcal{O}_X$.

If $det(\omega) \neq 0$, there is a constant $c \in \mathbb{C} \setminus \{0\}$ such that $det(\omega) = c^2$, hence at every point of X the endomorphism ϵ corresponding to the special tensor ω has two distinct eigenvalues $\pm c$.

Let $\omega \in H^0(S^2\Omega^1_X(-K_X))$, $\omega \neq 0$, be such that $\det(\omega) = 0$. Then the corresponding endomorphism ϵ is nilpotent of order 2, and there exists an open nonempty subset $U \subseteq X$ such that $\mathrm{Ker}(\epsilon_{|U}) = \mathrm{Im}(\epsilon_{|U})$. At a point p where $\mathrm{rank}(\epsilon) = 0$, in local coordinates the endomorphism ϵ may be expressed by

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad a, b, c \text{ regular functions such that } a^2 = -b \cdot c$$

Let $\delta := \mathrm{G.C.D.}(a,b,c)$. After dividing by δ, every prime factor of a is either not in b, or not in c, thus we can write

$$-b = \beta^2 \qquad c = \gamma^2 \qquad a = \beta \cdot \gamma$$

Therefore we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \operatorname{Ker} \epsilon \iff \begin{cases} a \cdot u + b \cdot v = 0 \\ c \cdot u - a \cdot v = 0 \end{cases} \iff \gamma \cdot u - \beta \cdot v = 0 \iff \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \beta \cdot f \\ \gamma \cdot f \end{pmatrix}$$

and, writing our endomorphism ϵ as $\epsilon = \delta \cdot \alpha$, we have

$$\operatorname{Im}(\alpha) = \begin{cases} \beta \cdot \gamma \cdot u - \beta^2 \cdot v = \beta \cdot (\gamma \cdot u - \beta \cdot v) \\ \gamma^2 \cdot u - \gamma \cdot \beta \cdot v = \gamma \cdot (\gamma \cdot u - \beta \cdot v) \end{cases}$$

Let Z be the 0-dimensional scheme defined by $\{\beta = \gamma = 0\}$ and Δ be the Cartier divisor defined by $\{\delta = 0\}$.

From the above description we deduce that the kernel of ϵ is a line bundle L which fits in the following exact sequence:

$$0 \to L \to \Omega^1_X \to \mathcal{I}_Z L(-\Delta) \to 0.$$

Taking the total Chern classes we infer that: $K_X \equiv 2L - \Delta$ as divisors on X and $c_2(X) = length(Z) + L \cdot (L - \Delta)$. □

Lemma 3.3. *Let X be a complex surface and let X' be the blow up of X at a point p. Then a special tensor ω' on X' induces a special tensor ω on X, and the converse only holds if and only if ω vanishes at p (in particular, it must hold : $det(\omega) = 0$).*

Proof. First of all, ω' induces a special tensor on $X \setminus \{p\}$, and by Hartogs' theorem the latter extends to a special tensor ω on X.

Conversely, choose local coordinates (x, y) for X around p and take a local chart of the blow up with coordinates (x, u) where $y = ux$.

Locally around p we can write

$$\omega = \frac{a(\mathrm{d}\, x)^2 + b(\mathrm{d}\, y)^2 + c(\mathrm{d}\, x \,\mathrm{d}\, y)}{\mathrm{d}\, x \wedge \mathrm{d}\, y}$$

The pull back ω' of ω is given by the following expression:

$$\frac{a(\mathrm{d}\, x)^2 + b(u \,\mathrm{d}\, x + x \,\mathrm{d}\, u)^2 + c(u \,\mathrm{d}\, x + x \,\mathrm{d}\, u)\,\mathrm{d}\, x}{x \,\mathrm{d}\, x \wedge \mathrm{d}\, u} =$$

$$= \frac{\mathrm{d}\, x^2(a + bu^2 + cu) + bx^2 \,\mathrm{d}\, u^2 + (2bux + cx)\,\mathrm{d}\, x \,\mathrm{d}\, u}{x \,\mathrm{d}\, x \wedge \mathrm{d}\, u},$$

hence ω' is regular if and only if $\frac{a+bu^2+cu}{x}$ is a regular function.

This is obvious if a, b, c vanish at p, since then their pull back is divisible by x. Assume on the other side that a, b, c are constant: then we get a rational function which is only regular if $a = b = c = 0$. □

Lemma 3.4. *Let X be a compact minimal rational surface admitting a special tensor ω. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X \cong \mathbb{F}_n, n \geq 2$. If moreover the special tensor is unique, then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X \cong \mathbb{F}_2$.*

Proof. Assume that X is a $\mathbb{P}^1$ bundle over a curve $B \cong \mathbb{P}^1$, i.e., a ruled surface $\mathbb{F}_n$ with $n \geq 0$. Let $\pi \colon X \to B$ the projection.

By the exact sequence

$$0 \to \pi^* \Omega^1_B \to \Omega^1_X \to \Omega^1_{X|B} \to 0$$

and since on a general fibre F the subsheaf $\pi^*\Omega^1_B$ is trivial, while the quotient sheaf $\Omega^1_{X|B}$ is negative, we conclude that any endomorphism ϵ carries $\pi^*\Omega^1_B$ to itself. If it has non zero determinant we can conclude by Theorem 3.1 that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Otherwise, ϵ is nilpotent and we have a nonzero element in $\mathrm{Hom}(\Omega^1_{X|B}, \pi^*\Omega^1_B)$.

Since these are invertible sheaves, it suffices to see when

$$H^0(\mathcal{O}_X(2\pi^* K_B - K_X)) \neq 0.$$

But, letting Σ be the section with selfintersection $\Sigma^2 = -n$, our vector space equals $H^0(\mathcal{O}_X(2\Sigma + (n-2)F))$. Intersecting this divisor with Σ we see that (since each time the intersection number with Σ is negative) $H^0(\mathcal{O}_X(2\Sigma + (n-2)F)) = H^0(\mathcal{O}_X(\Sigma + (n-2)F)) = H^0(\mathcal{O}_X(+(n-2)F))$. This space has dimension $n-1$, whence our claim follows for the surfaces $\mathbb{F}_n$.

There remains the case where X is $\mathbb{P}^2$.

In this case ϵ must be a nilpotent endomorphism by Theorem 3.1, and it cannot vanish at any point by our previous result on $\mathbb{F}_1$. Therefore the rank of ϵ equals 1 at each point. By lemma 3.2 it follows that there is a divisor L such that $K_X = 2L$, a contradiction. □

3.1. **Proof of Theorem 1.6.**

Proof. If X is strongly uniformized by the bidisk, then K_X is ample, in particular $K_X^2 \geq 1$ and, since by Castelnuovo's theorem $\chi(X) \geq 1$, by the vanishing theorem of Kodaira and Mumford it follows that $P_2(X) \geq 2$ (see [Bom73]).

Thus one direction follows from proposition 1.4, except that we shall show only later that (1*) holds.

Assume conversely that (1), (2) hold. Without loss of generality we may assume by lemma 3.3 that X is minimal, since K_X^2 can only decrease via a blowup and the bigenus is a birational invariant.

$K_X^2 \geq 1$ implies that either the surface X is of general type, or it is a rational surface.

These two cases are distinguished by the respective properties (3) (obviously implied by (3*)), guaranteeing that X is of general type, and (3**) ensuring that X is rational.

Let us first assume that X is of general type and, passing to an étale double cover if necessary, that X admits a special tensor.

By the cited Theorem 3.1 of [Bea00] it suffices to find a decomposition of the cotangent bundle Ω^1_X as a direct sum of two line bundles L_1 and L_2.

The two line bundles L_1, L_2 will be given as eigenbundles of a diagonizable endomorphism $\epsilon \in \mathrm{End}(\Omega^1_X)$.

Our previous discussion shows then that it is sufficient to show that any special tensor cannot yield a nilpotent endomorphism.

Otherwise, by lemma 3.2, we can write $2L \equiv K_X + \Delta$ and then deduce that L is a big divisor since Δ is effective by construction and K_X is big because X is of general type. This assertion gives the required contradiction since by the Bogomolov-Castelnuovo-de Franchis Theorem (cf. [Bog77]) for an invertible subsheaf L of Ω^1_X it is $h^0(X, mL) \leq O(m)$, contradicting the bigness of L.

There remains to show (1*). But if $h^0(X, S^2\Omega^1_X(-K_X)) \geq 2$ then, given a point $p \in X$, there is a special tensor which is not invertible in p, hence a special tensor with vanishing determinant, a contradiction.

If X is a rational surface we use the hypothesis $K_X^2 = 8$, ensuring that X is a surface $\mathbb{F}_n$; then, by lemma 3.4 we conclude that either $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X \cong \mathbb{F}_2$. In the former case $h^0(\Omega^1_X(-K_X)) = 6$, in the latter case $h^0(\Omega^1_X(-K_X)) = 7$. □

4. Elliptic surfaces with a special tensor not birational to a product of curves

In this section we are going to prove proposition 1.7.

We consider surfaces X with bigenus $P_2(X) \geq 2$ (property (3*)), therefore their Kodaira dimension equals 1 or 2, hence either they are properly (canonically) elliptic, or they are of general type.

Since we took already care of the latter case in the main theorem 1.6, we restrict our attention here to the former case, and try to see when does a properly elliptic surface admit a special tensor (we can reduce to this situation in view of remark 2.1). We can moreover assume that the associated endomorphism ϵ is nilpotent by theorem 3.1.

Again without loss of generality we may assume that X is minimal by virtue of lemma 3.3.

Proof. Let X be a minimal properly elliptic surface and let $f : X \to B$ be its (multi)canonical elliptic fibration. Write any fibre $f^{-1}(p)$ as $F_p = \sum_{i=1}^{h_p} m_i C_i$ and, setting $n_p := G.C.D.(m_i)$, $F_p = n_p F'_p$, we say that a fibre is multiple if $n_p > 1$. By Kodaira's classification ([Kod60]) of the singular fibres we know that in this case $m_i = n_p, \forall i$.

Assume that the multiple fibres of the elliptic fibration are $n_1 F'_1, \ldots, n_r F'_r$, and consider the divisorial part of the critical locus

$$\mathcal{S}_p := \sum_{i=1}^{h_p} (m_i - 1) C_i, \quad \mathcal{S} := \sum_{p \in B} \mathcal{S}_p$$

so that we have then the exact sequence

$$0 \to f^*\Omega^1_B(\mathcal{S}) \to \Omega^1_X \to \mathcal{I}_{\mathcal{C}}\ \omega_{X|B} \to 0,$$

where $\mathcal{C}$ is a 0-dimensional (l.c.i.) subscheme.

For further calculations we separate the divisorial part of the critical locus as the sum of two disjoint effective divisors, the multiple fibre contribution and the rest:

$$\mathcal{S}_m := \sum_{i=1}^{r} (n_i - 1)F_i', \ \hat{\mathcal{S}} := \mathcal{S} - \mathcal{S}_m.$$

Let us assume that we have a nilpotent endomorphism corresponding to another exact sequence

$$0 \to L \to \Omega^1_X \to \mathcal{I}_Z L(-\Delta) \to 0,$$

in turn determined by a homomorphism

$$\epsilon' : \mathcal{I}_Z L(-\Delta) \to L,$$

i.e., by a section

$$s \in H^0(\mathcal{O}_X(\Delta)) =$$
$$= H^0(\mathcal{O}_X(2L - K_X)) = H^0(S^2(L)(-K_X)) \subset H^0(S^2(\Omega^1_X)(-K_X)).$$

Observe by the way that, if $L \neq L'$, where we set $L' := f^*\Omega^1_B(\mathcal{S})$, we get a non trivial homomorphism $L' \to \mathcal{I}_Z L(-\Delta)$, hence $L - \Delta \geq L'$.

Since $2L \equiv K_X + \Delta$, it follows that, if F is a general fibre, then (use $K_X \cdot F = 0 = L' \cdot F$)

$$L \cdot F = \Delta \cdot F = 0,$$

hence the effective divisor Δ is contained in a finite union of fibres.

The first candidate we try with is then the choice of $L = L' = f^*\Omega^1_B(\mathcal{S})$.

To this purpose we recall Kodaira's canonical bundle formula:

$$K_X \equiv \mathcal{S}_m + f^*(\delta) = \sum_{i=1}^{r} (n_i - 1)F_i' + f^*(\delta), \ deg(\delta) = \chi(X) - 2 + 2b,$$

where b is the genus of the base curve B.

Then $H^0(\mathcal{O}_X(2L' - K_X)) = H^0(\mathcal{O}_X(f^*(2K_B - \delta) + 2\mathcal{S} - \mathcal{S}_m)$, and we search for an effective divisor linearly equivalent to

$$f^*(2K_B - \delta) + 2\mathcal{S} - \mathcal{S}_m = f^*(2K_B - \delta) + 2\hat{\mathcal{S}} + \mathcal{S}_m.$$

We claim that $H^0(\mathcal{O}_X(2L' - K_X)) = H^0(\mathcal{O}_X(f^*(2K_B - \delta))$: it will then suffice to have examples where $|2K_B - \delta| \neq \emptyset$.

Proof of the claim

It suffices to show that $f_*\mathcal{O}_X(2\hat{\mathcal{S}} + \mathcal{S}_m) = \mathcal{O}_B$. Since the divisor $2\hat{\mathcal{S}} + \mathcal{S}_m$ is supported on the singular fibres, and it is effective, we have to show that, for each singular fibre $F_p = \sum_{i=1}^{h_p} m_i C_i$, neither $2\hat{\mathcal{S}}_p \geq F_p$ nor $\mathcal{S}_{m,p} \geq F_p$.

The latter case is obvious since $\mathcal{S}_{m,p} = (n_p - 1)F_p' < F_p = n_p F_p'$.

In the former case, $2\hat{\mathcal{S}}_p = \sum_{i=1}^{h_p} 2(m_i - 1)C_i$, but it is not possible that $\forall i$ one has $2(m_i - 1) \geq m_i$, since there is always an irreducible curve C_i with multiplicity $m_i = 1$.

*Q.E.D.*for the claim

Assume that the elliptic fibration is not a product (in this case there is no special tensor with vanishing determinant): then the irregularity of X equals the genus of B, whence our divisor on the curve B has degree equal to $2b - 2 - (1 - b + p_g(X)) = 3b - 3 - p_g$.

Since $\chi(X) \geq 1$, $p_g := p_g(X) \geq b$, and there exist an elliptic surface X with any $p_g \geq b$ ([Cat07]).

Since any divisor on B of degree $\geq b$ is effective, it suffices to choose $b \leq p_g \leq 2b-3$ and we get a special tensor with trivial determinant, provided that $b \geq 3$.

Take now a Jacobian elliptic surface in Weierstrass normal form

$$ZY^2 - 4X^3 - g_2XZ^2 - g_3Z^3 = 0,$$

where $g_2 \in H^0(\mathcal{O}_B(4M))$, $g_3 \in H^0(\mathcal{O}_B(6M))$, and assume that all the fibres are irreducible.

Then the space of special tensors corresponding to our choice of L corresponds to the vector space $H^0(\mathcal{O}_B(2K_B - \delta)) = H^0(\mathcal{O}_B(K_B - 6M))$. It suffices now to take a hyperelliptic curve B of genus $b = 6h+1$, and, denoting by H the hyperelliptic divisor, set $M := hH$, so that $K_B - 6M \equiv 0$ and we have $h^0(\mathcal{O}_X(2L - K_X)) = 1$. We leave aside for the time being the question whether the surface X admits a unique special tensor.

$\square$

Already in the introduction, we posed the following

Question. *Let X be a surface with $q(X) = 0$ and satisfying (1*) and (3*): is then X strongly uniformized by the bidisk?*

Concerning the above question, recall the following

Definition 4.1. $\Gamma \subset \mathrm{Aut}(\mathbb{H}^n)$ *is said to be reducible if there exists a subgroup of finite index $\Gamma^0 < \Gamma$ such that $\gamma(z_1, ..., z_n) = (\gamma_1(z_1), ..., \gamma_n(z_n))$ for every $\gamma \in \Gamma^0$) and a decomposition $\mathbb{H}^n = \mathbb{H}^k \times \mathbb{H}^h$ (with $h > 0$) such that the action of Γ^0 on $\mathbb{H}^k$ is properly discontinuous.*

For $n = 2$ there are only two alternatives:

Remark 4.2. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{H}^2)$ be a discrete cocompact subgroup acting freely and let $X = \mathbb{H}^2/\Gamma$. Then*

- Γ *is reducible if and only if X is isogenous to a product of curves, i.e., there is a finite group G and two curves of genera at least 2 such that $X \cong C_1 \times C_2/G$. Both cases $q(X) \neq 0$, $q(X) = 0$ can occur here.*
- Γ *is irreducible: then $q(X) = 0$ (this result holds in all dimensions and is a well-known result of Matsushima* [Ma62]*).*

5. Other surfaces whose universal cover is a product of curves

For the sake of completeness, using the Enriques classification of surfaces, we give here a characterization of the Kähler surfaces S whose universal cover is a product of curves, other than $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{H} \times \mathbb{H}$, which was treated in section 3. We already mentioned in the introduction the following theorem.

Theorem 1.8 *Let S be a smooth compact Kähler surface S. Then the universal cover of S is biholomorphic to*

(1) $\mathbb{P}^1 \times \mathbb{C} \Leftrightarrow P_{12} = 0,\ q = 1,\ K_S^2 = 0.$

(2) $\mathbb{P}^1 \times \mathbb{H} \Leftrightarrow P_{12} = 0,\ q = g \geq 2,\ K_S^2 = 8(1-q)$.
(3) $\mathbb{C}^2 \Leftrightarrow P_{12} = 1,\ q = 1$ *or* $q = 2,\ K_S^2 = 0$.
(4) $\mathbb{C} \times \mathbb{H} \Leftrightarrow P_{12} \geq 2,\ e(S) = 0$.

Proof. We consider the several possible cases separately:

1) $\mathbb{P}^1 \times \mathbb{C}$: by proposition 2.2 these are the $\mathbb{P}^1$- bundles over an elliptic curve. They are characterized for instance by the properties $P_{12} = 0$, which implies that the surface is ruled, $q = 1$, which implies that it is ruled over an elliptic curve, and $K^2 = 0$, which implies that all the fibres are smooth, hence we have a $\mathbb{P}^1$ bundle.

2) $\mathbb{P}^1 \times \mathbb{H}$: these are the $\mathbb{P}^1$-bundles over a curve B of genus $g \geq 2$, hence characterized for instance by the properties $P_{12} = 0$, $q = g \geq 2$, $K^2 = 8(1-q)$. The argument is here identical to the one given above.

3) $\mathbb{C}^2$: these, by the celebrated theorem of Enriques-Severi and Bagnera-de Franchis, are the tori or the hyperelliptic surfaces, characterized (see for instance [Cat08] page 65), by the properties: $P_{12} = 1$, $q = 1$ or $q = 2$, $K^2 = 0$ (more precisely, $p_g = 1$, $q = 2$, $K^2 = 0$ for tori, $P_{12} = 1$, $q = 1$, $K^2 = 0$ for the hyperelliptic surfaces).

4) $\mathbb{C} \times \mathbb{H}$: in this case, by the same argument as in proposition 2.2, the action of $\gamma \in \Gamma$ is as follows:

$$(z, \tau) \mapsto (a_\gamma(\tau) z + b_\gamma(\tau), f_\gamma(\tau)),$$

since for fixed τ we get an automorphism of $\mathbb{C}$.

The cocycle $a_\gamma(\tau)$ induces a line bundle L which is trivial on the leaves $F_\tau := (\mathbb{C} \times \{\tau\})/\Gamma$, and its dual yields a subbundle of the tangent bundle of S.

Moreover, the canonical divisor K_S corresponds to the cocycle $a_\gamma(\tau) \cdot \frac{\partial}{\partial \tau} f_\gamma(\tau)$. Therefore the canonical divisor is also trivial on the leaves F_τ, and the extension class of

$$0 \to \mathcal{O}_S(K_S - L) \to \Omega^1_S \to L \to 0$$

is given by a group cocycle involving only the function τ.

If the action of Γ on $\mathbb{H}$ is properly discontinuous, then $\mathbb{H}/\Gamma$ is a compact complex curve B, and the fibres of $f : S \to B$ are elliptic curves. There exists an étale cover S' of S, such that S' admits an elliptic fibration with smooth fibres onto a compact complex curve B' of genus at least 2, hence this is an elliptic bundle (the period map is constant and S' is Kähler).

If the action is not properly discontinuous, then the leaves F_τ are not compact. The sections of multiples of the canonical divisor yield bounded functions on the leaves, hence by Liouville's theorem these are constant. Since the leaves are not compact, the conclusion is that the Kodaira dimension of S is negative or zero. It cannot be negative, else the universal cover would contain a family of $\mathbb{P}^1$'s. If the Kodaira dimension is zero, we know by surface classification that either the universal cover is $\mathbb{C}^2$ or the fundamental group has order at most two, and in all cases we have derived a contradiction.

Hence we concluded that our surfaces S are the elliptic quasi- bundles S over a curve B of genus $g \geq 2$; more precisely, these are the quotients of a product $(E \times C)/G$, where E is an elliptic curve, C is a curve of genus $g' \geq 2$, and G is a finite group acting diagonally on the product $E \times C$. These are characterized then by the properties: $P_{12} \geq 2$, $e(S) = 0$.

In fact $P_{12} \geq 2$ ensures that the Kodaira dimension is ≥ 1, a surface of general type has $e(S) \geq 1$, whereas for an elliptic fibration $e(S) = 0$ holds if and only if we have a quasi-bundle, i.e., all the fibres are either smooth or multiple of a smooth curve.

Since $K_S^2 = e(S) = 0$, then $\chi(S) = 0$, and Kodaira's canonical bundle formula says that K_S is the pull back of a $\mathbb{Q}$-divisor on the base curve B of degree equal to the degree of $K_B + \sum_{i=1}^r (n_i - 1)F_i'$. This means that the base orbifold is of hyperbolic type, and by the fundamental exact sequence $\pi_1(E) \to \pi_1(S) \to \pi_1^{orb}(B) \to 0$ (see [CKO03] and also chapter 5 of [Cat08]), the universal cover of S is the product $\mathbb{C} \times \mathbb{H}$.

□

6. 3-dimensional Kähler manifolds whose universal cover is the polydisk

In this section we are going to prove theorem 1.9.

Let X be a smooth compact Kähler manifold of general type of dimension 3. Assume that the canonical divisor K_X is ample and consider the canonical Kähler-Einstein metric provided by the theorem of Aubin and Yau (cf. [Yau77]).

As shown in the introduction, if X admits a special tensor $\omega \in H^0(X, S^3\Omega^1_X(-K_X))$, then by [Yau88, p.272] and [Yau93, p.479] (see also [V-Z05, p.10]) ω induces on the tangent bundle T_X a homogeneous hypersurface F_X of relative degree 3 which is parallel with respect to the Levi-Civita connection associated to the Kähler-Einstein metric.

In particular, taking a point $x \in X$, and considering the projectivized tangent bundle, we obtain a cubic curve $C_x \subset \mathbb{P}(T_X, x) \cong \mathbb{P}^2$, invariant for the action of the holonomy.

By the theorem of De Rham, the universal cover $\tilde{X}$ splits as a product of irreducible factors, $\tilde{X} = \tilde{X}_1 \times \tilde{X}_2 \times \cdots \times \tilde{X}_k$ with $\dim(\tilde{X}_i) = n_i$. The restricted holonomy group also splits as $H = H_1 \times H_2 \times \cdots \times H_k$, where the action of H_i on $T_{\tilde{X}_i, x_i}$ is irreducible ($x_i \in \tilde{X}_i$ being an arbitrary point).

Moreover by the classical theorem of Berger-Simons either $H_i \cong U(n_i)$ or H_i is the holonomy of an irreducible Hermitian symmetric space of rank > 1.

The idea of our proof consists in pointing out how the existence of such a cubic projective curve (possibly singular or reducible) forces a complete splitting for the action of the holonomy group (i.e., it implies the isomorphism $H \cong U(1)^3$). Consequently we obtain that $\tilde{X} \cong (\mathbb{H})^3$.

Proof. of theorem 1.9.

Let X be a smooth Kähler manifold of general type of dimension 3, with K_X ample. Fix a point $x \in X$ and let $\omega \in H^0(X, S^3\Omega^1_X(-K_X))$ be a non zero section. Then ω induces a projective cubic curve $C_x \subset \mathbb{P}(T_{X,x}) \cong \mathbb{P}^2$ invariant for the action of the (restricted) holonomy H.

In particular C_x is invariant for the action of the minimal linear algebraic group which contains H, and which we denote by $\hat{H}$. Observe that $\hat{H}$ is connected.

On the other side, by the description given above, we have $H = H_1 \times H_2 \times \cdots \times H_k$, where either $H_i \cong U(n_i)$ or H_i is the holonomy of an irreducible Hermitian symmetric space of rank > 1.

Let $\mathrm{Lin}(C_x)$ be the linear algebraic group of projectivities leaving C_x invariant. We shall analyse all the possible cases for C_x, including the study of its singularities and the description of $\mathrm{Lin}(C_x)$, keeping in mind that we have $\mathbb{P}(\hat{H}) \subset \mathrm{Lin}(C_x)$.

(a) C_x irreducible and smooth. In this case $\mathrm{Lin}(C_x)$ is finite, which contradicts $\mathbb{P}(\hat{H}) \subset \mathrm{Lin}(C_x)$, since dim $\hat{H}$ is at least 3.

(b) C_x irreducible with a node p. In this case $\hat{H}$ fixes the node p and the pair of tangent lines of C_x at p. Since $\hat{H}$ is connected, it fixes both tangent lines.

Therefore H fixes the point p and a line L through p, i.e. H fixes a flag. Since H is a subgroup of the unitary subgroup it acts diagonally for a suitable unitary basis, hence we conclude that $H = U(1)^3$.

Therefore there exists an étale covering X' of X such that $T_{X'}$ decomposes as the direct sum of 3 line bundles (the eigenbundles of the action), and the universal cover of X is biholomorphic to $\mathbb{H}^3$.

(c) C_x irreducible with a cusp p. In this case we can choose coordinates on $\mathbb{P}^2$ so that $p = (1 : 0 : 0)$ and on the affine chart $\{x_0 = 1\}$ the curve C_x is parametrized by $t \mapsto (1, t^2, t^3)$.

Now we have : $\mathbb{C}^* \cong \mathrm{Lin}(C_x)$ and in the affine chart $\{x_0 = 1\}$ $\lambda \in \mathbb{C}^*$ yields the automorphism

$$\begin{array}{ccc} C_x & \to & C_x \\ (1, t^2, t^3) & \mapsto & (1, \lambda^2 t^2, \lambda^3 t^3) \end{array}$$

Whence even in this case the action of $\hat{H}$ is diagonal and we conclude as before.

(d) C_x decomposes as the union of a line L and an irreducible conic Q. In this case $\hat{H}$ fixes the intersection set $L \cap Q$, which consists of one or two points. By connectedness of $\hat{H}$, $\hat{H}$ fixes a point $P \in L$ and the line L, and we conclude as before.

(e) C_x decomposes as the union of a double line $2L_1$ and a line L_2. In this case $\hat{H}$ fixes the point $L_1 \cap L_2 = \{p\}$ and the line L_2 and we are done.

(f) C_x decomposes as the union of 3 distinct lines $C_x = L_1 \cup L_2 \cup L_3$. There are two possibilities: the three lines are concurrent in the same point p or $L_1 \cap L_2 \cap L_3 = \emptyset$ and there are three singular points $p_{ij} = L_i \cap L_j$ $(1 \leq i < j \leq 3)$.

In both cases, since $\hat{H}$ is connected it fixes each singular point and each line. Hence there is a flag fixed by $\hat{H}$ and we are done.

(g) C_x decomposes as a triple line $3L$. We are going to show that this case cannot happen.

Assume the contrary and consider the line subbundle $\mathcal{L} \subset \Omega^1_X$ corresponding to L. We have a section

$$\mathcal{O}_X \to \mathcal{O}_X(3\mathcal{L} - K_X) \subset S^3\Omega^1_X(-K_X)$$

(indeed, cf. [Yau93] or [V-Z05], this section has no zeros).

Therefore we have $3\mathcal{L} \equiv K_X + D$, with D effective (in fact D is a trivial divisor). This in particular implies $\mathcal{L}$ big because K_X is ample by our assumption. This assertion, as in the proof of theorem 1.6, contradicts the theorem of Bogomolov (cf. [Bog77]).

Conversely, if $X \cong \mathbb{H} \times \mathbb{H} \times \mathbb{H}/\Gamma$, with Γ a cocompact discrete subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H} \times \mathbb{H})$ acting freely, then by [Sieg73] it is immediately seen that K_X is ample and by Prop. 1.4 X admits a semi special tensor.

□

7. 4-dimensional Kähler manifolds of general type with a special tensor whose universal cover is not a product of curves

One of the consequences of the theorem of Berger-Simons is that an irreducible Kähler manifold X of dimension n and with K_X ample (irreducible in the sense of De Rham's theorem) has as holonomy group a proper subgroup $H \subset U(n)$ if and only if $\tilde{X}$ is a Hermitian symmetric space of rank ≥ 2 (see [Yau88], and especially [V-Z05], 1.4 and 1.5).

Since we are interested in the case where K_X is ample we look for the Cartan realization of a Hermitian symmetric space of noncompact type as a bounded complex symmetric domain.

We shall find first such a bounded symmetric domain such that it has a holonomy invariant hypersurface of degree n, and then we shall apply the classical result of Borel on complex analytic Clifford-Klein forms. A complex analytic Clifford-Klein form is simply a compact quotient $X = \tilde{X}/\Gamma$, where the group Γ acts freely (thus X is a projective manifold with ample canonical bundle).

Borel's theorem (cf. [Bor63]) states that any bounded symmetric domain $\tilde{X}$ of dimension n admits infinitely many compact complex analytic Clifford-Klein forms, whose arithmetic genus $1 - \chi(X)$ can be arbitrarily large in absolute value.

We shall prove Theorem 1.10 considering a Clifford-Klein form X associated to the noncompact Hermitian symmetric space of complex dimension 4 $\tilde{X} := SU(2,2)/S(U(2) \times U(2))$. In higher dimensions, it clearly suffices to take the product of such a projective manifold X with $n-4$ projective curves $C_1, \dots C_{n-4}$ of genus at least 2.

Proof. of Theorem 1.10

Let $\tilde{X} = SU(2,2)/S(U(2)\times U(2))$. $\tilde{X}$ is a noncompact Hermitian symmetric space of dimension 4 and rank 2. Recall that a 4×4 matrix $g \in SU(2,2)$ can be written as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $\det(g) = 1$ and A, B, C, D are 2×2 complex matrices satisfying

$$(\star)\quad {}^t\overline{A}\cdot A -{}^t\,\overline{C}\cdot C = \mathrm{Id};\ {}^t\overline{B}\cdot B -{}^t\,\overline{D}\cdot D = -\,\mathrm{Id};\ {}^t\overline{B}\cdot A -{}^t\,\overline{D}\cdot C = 0,$$

whereas the subgroup $S(U(2)\times U(2))$ can be identified with the matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad (\text{with}\ \ A, D \in U(2)\ ,\ \det(A)\cdot\det(D) = 1).$$

Let $\mathfrak{su}(2,2)$ be the Lie algebra of $SU(2,2)$. The Cartan decomposition $\mathfrak{su}(2,2) = \mathfrak{k}\oplus\mathfrak{p}$ can be written down explicitly by means of

$$\mathfrak{p} \cong \begin{pmatrix} 0 & B \\ {}^t\overline{B} & 0 \end{pmatrix}\ ,\ \mathfrak{k} \cong \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\quad (\text{with}\ \ {}^t\overline{A} = -A\ ,\ {}^t\overline{D} = -D)$$

and for $x \in \tilde{X}$ we have a canonical isomorphism $\mathfrak{p} \cong T_{X,x}$.

The holonomy action coincides with the adjoint representation of $S(U(2)\times U(2))$ on $\mathfrak{p}$, given for every matrix $M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in S(U(2)\times U(2))$ by the map $\mathrm{Ad}_M : \mathfrak{p}\to\mathfrak{p}$ described by

$$\begin{pmatrix} 0 & B \\ {}^t\overline{B} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & A\cdot B\cdot D^{-1} \\ D\cdot({}^t\overline{B})\cdot A^{-1} & 0 \end{pmatrix}$$

Let us now consider the Cartan realization of $\tilde{X}$. It is obtained by the Siegel domain in the space of 2×2 matrices $M_{2,2}(\mathbb{C})$ (see [Hel78, p.527]):

$$X \cong \{Z \in M_{2,2}(\mathbb{C})\ :\ \mathrm{Id} -{}^tZ\cdot\overline{Z} > 0\}$$

and the action of $SU(2,2)$ on X is given by:

$$Z \mapsto (AZ+B)\cdot(CZ+D)^{-1}$$

Considering the tangent space at 0, the action of $S(U(2)\times U(2))$ on an "infinitesimal" 2×2 matrix Z becomes

$$Z \mapsto AZD^{-1}$$

and in particular we recover the above description of the adjoint representation of $S(U(2)\times U(2))$.

Notice that, since $\det(A)\cdot\det(D) = 1$, we have $\det AZD^{-1} = \det(A)^2\cdot\det Z$. This exactly means that the determinant is a semi-invariant for the action of $S(U(2)\times U(2))$ on $T_{X,0}$.

Therefore, identifying $T_{X,0}$ with $M_{2,2}$, and considering the projectivized tangent bundle at 0, $\mathbb{P}(T_{X,0}) \cong \mathbb{P}^3$, $\{\det(Z) = 0\}$ defines a quadric surface, invariant for the action of $S(U(2)\times U(2))$, and of course we obtain an invariant quartic projective surface given by $\{Z \in M_{2,2}\ : (\det(Z))^2 = 0\}$.

Applying now the theorem of Borel cited above we obtain a compact complex analytic Clifford-Klein form $X \cong \tilde{X}/\Gamma$ of $\tilde{X}$. We shall exhibit a semispecial tensor $\tilde{\omega}$ on $\tilde{X}$ which will descend to X yielding a semispecial tensor. Since $\tilde{X}$ is irreducible, our proof will be complete.

We want to show how this invariant surface defines a special tensor.

Write, for $\gamma \in \Gamma$,

$$\gamma(Z) = (AZ + B) \cdot (CZ + D)^{-1} \Leftrightarrow \gamma(Z) \cdot (CZ + D) = (AZ + B).$$

Differentiating the above equality, we obtain

$$d\gamma(Z) \cdot (CZ + D) = (A - (AZ + B) \cdot (CZ + D)^{-1}C) \cdot dZ.$$

Taking determinants, we obtain

$$det(d\gamma(Z)) \cdot det(CZ + D) = det(A - (AZ + B) \cdot (CZ + D)^{-1}C) \cdot det(dZ) =$$

$$= det((CZ + D)^{-1})det(C)det(AC^{-1}D - B) \cdot det(dZ).$$

Observe now that, setting ${}^*B :={}^t \bar{B}$, equations $(\star)$ yield

$$det(AC^{-1}D - B) = det({}^*B^{-1*}DD - B) = det({}^*B^{-1}).$$

An easy calculation using the above equations yields then $det(C)det(AC^{-1}D - B) = det(A)\det({}^*D)^{-1} = det(A)\det(D)$.

If we restrict to the isotropy subgroup $H = S(U(2) \times U(2))$, we get $det(A) \cdot det(D) = 1$. We have now a character of the group which is trivial on H. This character is then trivial since the homogeneous domain is contractible, whence the group $G := S(U(2, 2))$ is homotopically equivalent to H.

Since finally $det((CZ + D)^{-4})$ is the Jacobian determinant of the transformation γ, $\tilde{\omega} := det(dZ)^2$ is a Γ-invariant section of $H^0(\tilde{X}, S^n\Omega^1_{\tilde{X}}(-K_{\tilde{X}}))$, thus a special tensor which descends to X.

□

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