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The moduli space of Keum–Naie surfaces

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Alles Gute zum 60. Geburtstag, Fritz![†]

Abstract. Using a new description of the surfaces discovered by Keum and later investigated by Naie, and of their fundamental group, we prove the following main result.

Let S be a smooth complex projective surface which is homotopically equivalent to a Keum–Naie surface. Then S is a Keum–Naie surface. The connected component of the Gieseker moduli space corresponding to Keum–Naie surfaces is irreducible, normal, unirational of dimension 6.

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Introduction

In the nineties Y. H. Keum and D. Naie (cf. [Nai94], [Ke]) constructed a family of minimal surfaces of general type with $K_S^2 = 4$ and $p_g = 0$ as double covers of an Enriques surface with eight nodes.

They calculated the fundamental group of the constructed surfaces, but they did not address the problem of determining the moduli space of their surfaces.

The motivation for the present paper comes from our joint work [BCGP09] together with F. Grunewald and R. Pignatelli. In that article, among other results, we constructed several series of new surfaces of general type with $p_g = 0$ as minimal resolutions of quotients of a product of two curves (of respective genera g_1, g_2 at least two) by the action of a finite group G. This construction produced many interesting examples of new fundamental groups (of surfaces of general type with $p_g = 0$) but in general yields proper subfamilies and not full irreducible components of the respective moduli spaces of surfaces of general type (see also [BC10], [BC11]).

*The present work took place in the realm of the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds".

[†]Our wishes did not come through, and we regret very much the loss of an excellent mathematician, an exceptional person and a dear friend.

Obviously, when two such families yield surfaces with non-isomorphic fundamental groups, then clearly the two families lie in distinct connected components of the moduli space. But what happens if the fundamental groups are isomorphic (and the value of K_S^2 is the same)?

In particular, two of the families we constructed in [BCGP09] corresponded to surfaces having the same fundamental group as the Keum–Naie surfaces.[1]

We reproduce below an excerpt of the classification table (of quotients as above by a non free action of G, but with canonical singularities) in [BCGP09].

K^2	T_1	T_2	g_1	g_2	G	dim	$\pi_1(S)$
4	$2^2, 4^2$	$2^2, 4^2$	3	3	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^2$
4	2^5	2^5	3	3	$(\mathbb{Z}/2\mathbb{Z})^3$	4	$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^2$

This excerpt shows the 2 families, of respective dimensions 2 and 4, that we constructed as $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$- resp. $(\mathbb{Z}/2\mathbb{Z})^3$-coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ and branched on a divisor of type $(4, 4)$ resp. $(5, 5)$ which are the union of horizontal and vertical lines (T_1, T_2 stand for the type of branching on each line).

Once we found out that their fundamental groups were isomorphic to the fundamental groups of the surfaces constructed by Keum and Naie, the most natural question was whether all these surfaces would belong to a unique irreducible component of the moduli space.

A straightforward computation showed that our family of dimension 4 was equal to the family constructed by Keum, and that both families were subfamilies of the family constructed by Naie. To be more precise, each surface of our family of $(\mathbb{Z}/2\mathbb{Z})^3$-coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ has four nodes. These nodes can be smoothened simultaneously, thus obtaining a 5-dimensional family of $(\mathbb{Z}/2\mathbb{Z})^3$-Galois coverings of $\mathbb{P}^1 \times \mathbb{P}^1$. The full 6-dimensional component is obtained then as the family of natural deformations (see [Cat08]) of the family of such Galois coverings.

A somewhat lengthy but essentially standard computation in local deformation theory showed that the 6-dimensional family of natural deformations of smooth $(\mathbb{Z}/2\mathbb{Z})^3$-Galois coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ is an irreducible component of the moduli space. We will not give the details of this calculation, since we get a stronger result by a different method.

The following theorem is the main result of this article:

Theorem 0.1. *Let S be a smooth complex projective surface which is homotopically equivalent to a Keum–Naie surface. Then S is a Keum–Naie surface. The connected component of the Gieseker moduli space $\mathfrak{M}_{1,4}^{\mathrm{can}}$ corresponding to Keum–Naie surfaces is irreducible, normal, unirational of dimension* 6.

[1] Observe however that the correct description of the fundamental group is only to be found in [Nai94].

Observe that for surfaces of general type we have two moduli spaces: one is the moduli space $\mathfrak{M}^{\min}_{\chi,K^2}$ for minimal models S having $\chi(\mathcal{O}_S) = \chi$, $K_S^2 = K^2$, the other is the moduli space $\mathfrak{M}^{\mathrm{can}}_{\chi,K^2}$ for canonical models X having $\chi(\mathcal{O}_X) = \chi$, $K_X^2 = K^2$; the latter is called the Gieseker moduli space and is a quasi projective scheme by Gieseker's theorem ([Gie77]). Moreover, there is a natural morphism $\mathfrak{M}^{\min}_{\chi,K^2} \to \mathfrak{M}^{\mathrm{can}}_{\chi,K^2}$ which is a bijection. The local structure of $\mathfrak{M}^{\mathrm{can}}_{\chi,K^2}$ as complex analytic space is the quotient of the base of the Kuranishi family by the action of the finite group $\mathrm{Aut}(S) = \mathrm{Aut}(X)$.

Usually the structure as analytic space of $\mathfrak{M}^{\min}_{\chi,K^2}$ tends to be more singular than the one of $\mathfrak{M}^{\mathrm{can}}_{\chi,K^2}$ (see e.g. [Cat89]).

In order to achieve our main result, we resort first of all to a slightly different construction of Keum–Naie surfaces.

We start with a $(\mathbb{Z}/2\mathbb{Z})^2$-action on the product of two elliptic curves $E_1' \times E_2'$.

This action has 16 fixed points and the quotient is an 8-nodal Enriques surface. Instead of constructing S as the double cover of the Enriques surface, we consider an étale $(\mathbb{Z}/2\mathbb{Z})^2$-covering $\widehat{S}$ of S, whose existence is guaranteed from the structure of the fundamental group of S. $\widehat{S}$ is obtained as a double cover of $E_1' \times E_2'$ branched in a $(\mathbb{Z}/2\mathbb{Z})^2$-invariant divisor of type $(4,4)$, and S is recovered as the quotient of $\widehat{S}$ by the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on it.

The structure of this $(\mathbb{Z}/2\mathbb{Z})^2$ action and the geometry of the covering $\widehat{S}$ of S is essentially encoded in the fundamental group $\pi_1(S)$, which is described as an affine group $\Gamma \in \mathbb{A}(2,\mathbb{C})$. In particular, it follows that the Albanese map of $\widehat{S}$ is the above double cover $\hat{\alpha}\colon \widehat{S} \to E_1' \times E_2'$.

If S' is now homotopically equivalent to a Keum–Naie surface S, then we have a corresponding étale $(\mathbb{Z}/2\mathbb{Z})^2$-covering $\hat{S}'$ which is homotopically equivalent to $\widehat{S}$. Since we know that the degree of the Albanese map of $\widehat{S}$ is equal to two (by construction), we can conclude the same for the Albanese map of $\hat{S}'$ and this allows to deduce that also $\hat{S}'$ is a double cover of a product of elliptic curves branched in a $(\mathbb{Z}/2\mathbb{Z})^2$-invariant divisor of type $(4,4)$.

Our paper is organized as follows: in Section 1 we study a certain $(\mathbb{Z}/2\mathbb{Z})^2$-action on a product of two elliptic curves $E_1' \times E_2'$ and explain our construction of Keum–Naie surfaces.

In Section 2 we use elementary representation theory to calculate the dimension of the space of $(\mathbb{Z}/2\mathbb{Z})^2$-invariant divisors of type $(4,4)$ on $E_1' \times E_2'$, and show that the Gieseker moduli space of Keum–Naie surfaces is a normal, irreducible, unirational variety of dimension six.

In Section 3 we conclude the proof of our main result 0.1

The brief Section 4 is devoted to the bicanonical image of Keum–Naie surfaces: we show that the map has degree 4 and that the image is always the same 4-nodal Del Pezzo surface of degree 4.

We stick to the traditional ('old fashioned'?) notation $\equiv$ to denote linear equivalence.

1. A $(\mathbb{Z}/2\mathbb{Z})^2$-action on a product of elliptic curves and Keum–Naie surfaces

Let (E, o) be any elliptic curve, with an action of the group

$$G := (\mathbb{Z}/2\mathbb{Z})^2 = \{0, g_1, g_2, g_3 := g_1 + g_2\}$$

given by

$$g_1(z) := z + \eta, \quad g_2(z) = -z,$$

where $\eta \in E$ is a 2-torsion point of E.

Remark 1.1. The effective divisor $[o] + [\eta] \in \mathrm{Div}^2(E)$ is invariant under G, hence the invertible sheaf $\mathcal{O}_E([o] + [\eta])$ carries a natural G-linearization.

In particular, G acts on the vector space $H^0(E, \mathcal{O}_E([o] + [\eta]))$ which then splits as a direct sum

$$H^0(E, \mathcal{O}_E([o] + [\eta])) = \bigoplus_{\chi \in G^*} H^0(E, \mathcal{O}_E([o] + [\eta]))^\chi$$

of the eigenspaces corresponding to the characters χ of G. We shall use the self-explanatory notation $H^0(E, \mathcal{O}_E([o] + [\eta]))^{+-}$ for the eigenspace corresponding to the character χ such that $\chi(g_1) = 1$, $\chi(g_2) = -1$.

We have the following:

Lemma 1.1. *In the above setting we have*

$$H^0(E, \mathcal{O}_E([o] + [\eta]))^{+-} = H^0(E, \mathcal{O}_E([o] + [\eta]))^{-+} = 0,$$

and we have a splitting as a sum of two 1*-dimensional eigenspaces:*

$$H^0(E, \mathcal{O}_E([o] + [\eta])) = H^0(E, \mathcal{O}_E([o] + [\eta]))^{++} \oplus H^0(E, \mathcal{O}_E([o] + [\eta]))^{--}.$$

Proof. Obviously, since the G linearization is obtained by considering the vector space of rational functions with polar divisor at most $[o] + [\eta]$, the subspace $H^0(E, \mathcal{O}_E([o] + [\eta]))^{++}$ has dimension at least 1. On the other hand, there are exactly two G invariant divisors in the linear system $|[o] + [\eta]|$.

If $[P]+[Q] \in |[o]+[\eta]|$ is G invariant, then $g_1([P]+[Q]) = [P+\eta]+[Q+\eta] = [P]+[Q]$, hence $[P+\eta] = [Q]$. Since $[P]+[Q] \equiv [o]+[\eta]$, P, Q are 2-torsion points of E (which automatically implies $g_2([P] + [Q]) = [-P] + [-Q] = [P] + [Q]$), and we have shown that there are exactly two G-invariant divisors.

Therefore $H^0(E, \mathcal{O}_E([o] + [\eta]))$ splits as the direct sum of two 1-dimensional eigenspaces, one of which is $H^0(E, \mathcal{O}_E([o] + [\eta]))^{++}$.

It suffices now to show that

$$H^0(E, \mathcal{O}_E([o] + [\eta]))^{+-} = H^0(E, \mathcal{O}_E([o] + [\eta]))^{-+} = 0.$$

In fact, if this were not the case, all the divisors in the linear system $|[o] + [\eta]|$ would be invariant by either g_1 or by g_2.

The first possibility was already excluded above, while the second one means that, for each point P, $[P] + [\eta - P] \in |[o] + [\eta]|$ satisfies $g_2([P] + [\eta - P]) = [-P] + [P - \eta] = [P] + [\eta - P]$, which implies $[P] = [-P]$, a contradiction. $\square$

Consider now two complex elliptic curves E_1', E_2', which can be written as quotients $E_i' := \mathbb{C}/\Lambda_i'$, $i = 1, 2$, with $\Lambda_i' := \mathbb{Z}e_i \oplus \mathbb{Z}e_i'$.

We consider the affine transformations $\gamma_1, \gamma_2 \in \mathbb{A}(2, \mathbb{C})$, defined as follows:

$$\gamma_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} z_1 + \frac{e_1}{2} \\ -z_2 \end{pmatrix}, \quad \gamma_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} -z_1 \\ z_2 + \frac{e_2}{2} \end{pmatrix},$$

and let $\Gamma \leq \mathbb{A}(2, \mathbb{C})$ be the affine group generated by γ_1, γ_2 and by the translations e_1, e_1', e_2, e_2'.

Remark 1.2. i) Γ contains the lattice $\Lambda_1' \oplus \Lambda_2'$, hence Γ acts on $E_1' \times E_2'$ inducing a faithful action of $G := (\mathbb{Z}/2\mathbb{Z})^2$ on $E_1' \times E_2'$.

ii) While γ_1, γ_2 have no fixed points on $E_1' \times E_2'$, the involution $\gamma_1\gamma_2$ has 16 fixed points on $E_1' \times E_2'$. It is easy to see that the quotient $Y := (E_1' \times E_2')/G$ is an Enriques surface having eight nodes, with canonical double cover the Kummer surface $(E_1' \times E_2')/\langle\gamma_1\gamma_2\rangle$.

We will in the sequel lift the G-action on $E_1' \times E_2'$ to an appropriate ramified double cover $\widehat{S}$ and in such a way that G acts freely on $\widehat{S}$.

Consider the geometric line bundle $\mathbb{L}$ on $E_1' \times E_2'$, whose invertible sheaf of sections is given by

$$\mathcal{O}_{E_1' \times E_2'}(\mathbb{L}) := p_1^* \mathcal{O}_{E_1'}\big([o_1] + \big[\tfrac{e_1}{2}\big]\big) \otimes p_2^* \mathcal{O}_{E_2'}\big([o_2] + \big[\tfrac{e_2}{2}\big]\big),$$

where $p_i \colon E_1' \times E_2' \to E_i'$ is the projection onto the i-th factor.

Remark 1.3. By Remark 1.1, the divisor $[o_i] + [\frac{e_i}{2}] \in \mathrm{Div}^2(E_i')$ is invariant under G. Whence, we get a natural G-action on $\mathbb{L}$. But this is not the G-action on $\mathbb{L}$ that we shall consider.

In fact, any two G-actions on $\mathbb{L}$ differ by a character $\chi \colon G \to \mathbb{C}^*$. We shall twist the above natural action of $\mathbb{L}$ by the character such that $\chi(\gamma_1) = 1$, $\chi(\gamma_2) = -1$. We shall call this twisted G-action the canonical one.

Definition 1.2. Consider the canonical G-action on $\mathbb{L}$ and on all its tensor powers, and let

$$f \in H^0\big(E_1' \times E_2', p_1^* \mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^* \mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^G$$

be a G-invariant section of $\mathbb{L}^{\otimes 2}$.

Denoting by w a fibre coordinate of $\mathbb{L}$, let $\widehat{X}$ be the double cover of $E_1' \times E_2'$ branched in $\{f = 0\}$, i.e., set

$$\widehat{X} = \{w^2 = f(z_1, z_2)\} \subset \mathbb{L}.$$

Then $\widehat{X}$ is a G-invariant hypersurface in $\mathbb{L}$, and we define the canonical model of a *Keum–Naie surface* to be the quotient of $\widehat{X}$ by the G-action.

More precisely, we define S to be a *Keum–Naie surface*, if

- G acts freely on $\widehat{X}$, and
- $\{f = 0\}$ has only *non-essential singularities*, i.e., $\widehat{X}$ has canonical singularities (at most rational double points);
- S is the minimal resolution of singularities of $X := \widehat{X}/G$.

Remark 1.4. One might also call the above surfaces 'primary Keum–Naie surfaces'. In fact a similar construction, applied to the case where the action of G has fixed points at some nodal singularities of some special $\widehat{X}$, produces other surfaces, which could appropriately be named 'secondary Keum–Naie surfaces'.

Lemma 1.3. *If*

$$f \in H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^G$$

is such that $\{(z_1, z_2) \in E_1' \times E_2' \mid f(z_1, z_2) = 0\} \cap \mathrm{Fix}(\gamma_1\gamma_2) = \emptyset$, *then* G *acts freely on* $\widehat{X}$.

Proof. Recall that γ_1, γ_2 do not have fixed points on $E_1' \times E_2'$, whence they have no fixed points on $\widehat{X}$. Since, by 1.3, $(\gamma_1\gamma_2)(w) = -w$, it follows that G acts freely on $\widehat{X}$ if and only if $\{f = 0\}$ does not intersect the fixed points of $\gamma_1\gamma_2$ on $E_1' \times E_2'$. □

Proposition 1.4. *Let* S *be a Keum–Naie surface. Then* S *is a minimal surface of general type with*

i) $K_S^2 = 4$,
ii) $p_g(S) = q(S) = 0$,
iii) $\pi_1(S) = \Gamma$.

Proof. i) Let $\pi\colon \widehat{X} \to E_1' \times E_2'$ be the above double cover branched on $\{f = 0\}$. Then $K_{\widehat{X}} \equiv \pi^*(K_{E_1' \times E_2'} + p_1^*([o_1] + [\frac{e_1}{2}]) + p_2^*([o_2] + [\frac{e_2}{2}]))$, whence $K_{\widehat{X}}^2 = 2\cdot(p_1^*([o_1]+[\frac{e_1}{2}])+p_2^*([o_2]+[\frac{e_2}{2}]))^2 = 2\cdot 8 = 16$. Therefore $K_S^2 = K_X^2 = \frac{K_{\widehat{X}}^2}{|G|} = 4$.

ii) Let $\sigma\colon \widehat{S} \to \widehat{X}$ be the minimal resolution of singularities of $\widehat{X}$. Then $S = \widehat{S}/G$, and

$$H^0(S, \Omega_S^1) = H^0(\widehat{S}, \Omega_{\widehat{S}}^1)^G.$$

Since $\pi \circ \sigma \colon \widehat{S} \to E_1' \times E_2'$ has degree 2, it is the Albanese map of $\widehat{S}$, and we have that $H^0(\widehat{S}, \Omega^1_{\widehat{S}}) = H^0(E_1' \times E_2', \Omega^1_{E_1' \times E_2'}) \cong \mathbb{C}dz_1 \oplus \mathbb{C}dz_2$. Hence

$$H^0(S, \Omega^1_S) = H^0(\widehat{S}, \Omega^1_{\widehat{S}})^G = 0,$$

i.e., $q(S) = 0$.

Observe that since G acts freely

$$H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^G = H^0(X, \mathcal{O}(K_X)) = H^0(S, \Omega^2_S).$$

Consider now the decomposition of

$$V := H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}})) = H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^+ \oplus H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^-$$

in the invariant and anti-invariant part for the action of the involution σ of the double cover $\pi \colon \widehat{X} \to E_1' \times E_2'$ ($\sigma(z_1, z_2, w) = (z_1, z_2, -w)$).

Note that

a) $H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^+ = H^0(E_1' \times E_2', \Omega^2_{E_1' \times E_2'}) = \mathbb{C}(dz_1 \wedge dz_2)$,

b) $H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^- \cong H^0(E_1' \times E_2', \Omega^2_{E_1' \times E_2'}(\mathbb{L}))$.

In the uniformizing coordinates the first summand a) is generated by $dz_1 \wedge dz_2$, which is an eigenvector for the G-action, with character χ such that $\chi(\gamma_1) = \chi(\gamma_2) = -1$. We shall call this eigenspace V^{--}.

Each vector y in the addendum b) can be written as

$$y = \frac{\varphi_1(z_1)\varphi_2(z_2)}{w} dz_1 \wedge dz_2,$$

where $\varphi_i \in H^0(E_i', \mathcal{O}_{E_i'}([o_i] + [\frac{e_i}{2}]))$.

Recall that (cf. Lemma 1.1) $H^0(E_i', \mathcal{O}_{E_i'}([o_i] + [\frac{e_i}{2}])) =: H_i$ splits as $H_i^{++} \oplus H_i^{--}$ (observe that exchanging the roles of g_1 and g_2 in Lemma 1.1 makes fortunately no difference).

Using that $\gamma_1(w) = w$, $\gamma_2(w) = -w$ and that $dz_1 \wedge dz_2 \in V^{--}$, we get:

$$\frac{\varphi_1(z_1)\varphi_2(z_2)}{w} dz_1 \wedge dz_2 \in V^{+-} \iff \varphi_1 \in H_1^{++} \wedge \varphi_2 \in H_2^{--} \text{ or } \varphi_1 \in H_1^{--} \wedge \varphi_2 \in H_2^{++}; \tag{1}$$

$$\frac{\varphi_1(z_1)\varphi_2(z_2)}{w} dz_1 \wedge dz_2 \in V^{-+} \iff \varphi_1 \in H_1^{++} \wedge \varphi_2 \in H_2^{++} \text{ or } \varphi_1 \in H_1^{--} \wedge \varphi_2 \in H_2^{--}. \tag{2}$$

The above calculations show that both eigenspaces V^{-+}, V^{+-} are 2-dimensional. Since the summand b) has dimension 4, we obtain then:

i) $H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^{--} = \mathbb{C}(dz_1 \wedge dz_2)$,

ii) $H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^{+-} = \{\frac{\varphi_1(z_1)\varphi_2(z_2)}{w} dz_1 \wedge dz_2 \mid (\varphi_1 \in H_1^{++}$ and $\varphi_2 \in H_2^{--})$ or $(\varphi_1 \in H_1^{--}$ and $\varphi_2 \in H_2^{++})\}$ has dimension 2;

iii) $H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^{-+} = \{\frac{\varphi_1(z_1)\varphi_2(z_2)}{w} dz_1 \wedge dz_2 \mid (\varphi_1 \in H_1^{++}$ and $\varphi_2 \in H_2^{++})$ or $(\varphi_1 \in H_1^{--}$ and $\varphi_2 \in H_2^{--})\}$ has dimension 2;

iv) $H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^{++} = 0$.

In particular, we get $p_g(S) = \dim H^0(\widehat{X}, \mathcal{O}(K_{\widehat{X}}))^{++} = 0$.

iii) it suffices to show that the fundamental group of $\widehat{S}$ maps isomorphically to the fundamental group of $E_1' \times E_2'$. By the theorem of Brieskorn–Tyurina ([Brie68], [Brie71], [Tju70]) we can reduce to the case where $\widehat{X}$ is smooth, since the parameter space is connected, and there is a non empty open set of smooth branch curves D.

When D is smooth, we conclude by the Lefschetz type theorem of Mandelbaum and Moishezon ([M-M80], p. 218), since D is ample. □

2. The moduli space of Keum–Naie surfaces

The aim of this section is to prove the following result.

Theorem 2.1. *The connected component of the Gieseker moduli space corresponding to Keum–Naie surfaces is normal, irreducible, unirational of dimension equal to* 6. *Moreover, the base of the Kuranishi family of the canonical model X of a Keum–Naie surface is smooth.*

In order to describe the moduli space of Keum–Naie surfaces we shall preliminarily describe the vector space

$$H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^G.$$

We consider E_1' (resp. E_2') as a bidouble cover of $\mathbb{P}^1$ ramified in 4 points $\{0, 1, \infty, P\}$ (resp. $\{0, 1, \infty, Q\}$), where $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{0, g_1, g_2, g_3 := g_1 + g_2\}$ acts as follows:

$$\begin{aligned} g_1(z) &= z + \tfrac{e_1}{2}, \quad & g_2(z) &= -z \quad \text{on } E_1', \\ g_1(z) &= -z, & g_2(z) &= z + \tfrac{e_2}{2} \quad \text{on } E_2'. \end{aligned}$$

We denote the respective bidouble covering maps from E_i' to $\mathbb{P}^1$ by π_i. Observe moreover that the quotient of E_1' by the action of g_1 is an elliptic curve E_1, while the quotient of E_1' by the action of g_2 (resp. g_3) is isomorphic to $\mathbb{P}^1$.

Remark 2.1. It is immediate from the above remark that the character eigensheaves of the direct image sheaf $\pi_{1*}\mathcal{O}_{E_1'}$ for the bidouble cover $\pi_1 \colon E_1' \to \mathbb{P}^1$ are

$$\mathcal{L}_1^{-+} = \mathcal{O}_{\mathbb{P}^1}(1), \quad \mathcal{L}_1^{+-} = \mathcal{O}_{\mathbb{P}^1}(2), \quad \mathcal{L}_1^{--} = \mathcal{O}_{\mathbb{P}^1}(1).$$

In fact, for instance, the direct image on $\mathbb{P}^1$ of the sheaf of functions on E_1'/g_1 must be $\cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and it equals $\mathcal{O}_{\mathbb{P}^1} \oplus (\mathcal{L}_1^{+-})^{-1}$.

Similarly for $\pi_2\colon E_2' \to \mathbb{P}^1$ we have the character sheaves

$$\mathcal{L}_2^{-+} = \mathcal{O}_{\mathbb{P}^1}(2), \quad \mathcal{L}_2^{+-} = \mathcal{O}_{\mathbb{P}^1}(1), \quad \mathcal{L}_2^{--} = \mathcal{O}_{\mathbb{P}^1}(1).$$

Since $\mathcal{O}_{E_i'}(2[o_i] + 2[\frac{e_i}{2}]) = \pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$ we get

$$H^0\big(E_i', \mathcal{O}_{E_i'}\big(2[o_i] + 2\big[\tfrac{e_i}{2}\big]\big)\big) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes (\pi_i)_*\mathcal{O}_{E_i'}),$$

and therefore

i) $V_1^{++} := H^0(E_1', \mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]))^{++} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2$;

ii) $V_1^{+-} := H^0(\mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]))^{+-} = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{L}_1^{+-})^{-1}) = 0$;

iii) $V_1^{-+} := H^0(\mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]))^{-+} = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{L}_1^{-+})^{-1}) \cong \mathbb{C}$;

iv) $V_1^{--} := H^0(\mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]))^{--} = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{L}_1^{--})^{-1}) \cong \mathbb{C}$;

v) $V_2^{++} := H^0(E_2', \mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^{++} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2$;

vi) $V_2^{+-} := H^0(\mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^{+-} = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{L}_2^{+-})^{-1}) \cong \mathbb{C}$;

vii) $V_2^{-+} := H^0(\mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^{-+} = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{L}_2^{-+})^{-1}) = 0$;

viii) $V_2^{--} := H^0(\mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^{--} = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{L}_2^{--})^{-1}) \cong \mathbb{C}$.

As a consequence of the above remark, we get

Lemma 2.2.

1) $$H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^{++} = (V_1^{++} \otimes V_2^{++}) \oplus (V_1^{--} \otimes V_2^{--}) \cong \mathbb{C}^5;$$

2) $$H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^{--} = (V_1^{++} \otimes V_2^{--}) \oplus (V_1^{--} \otimes V_2^{++}) \oplus (V_1^{-+} \otimes V_2^{+-}) \cong \mathbb{C}^5;$$

Proof. This follows immediately from the above remark since

1) $$\begin{aligned} &H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^G \\ &= \bigoplus_{\chi \in G^*} \big(H^0\big(\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big)\big)^{\chi} \otimes H^0\big(E_2', \mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^{\chi^{-1}}\big) \\ &= (V_1^{++} \otimes V_2^{++}) \oplus (V_1^{--} \otimes V_2^{--}) \cong \mathbb{C}^4 \oplus \mathbb{C}; \end{aligned}$$

and

$$
\begin{aligned}
2)\ \ & H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^{--} \\
&= \bigoplus_{\chi \in G^*} \big(H^0(\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big)\big)^{\chi} \otimes H^0\big(E_2', \mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^{\chi^{-1}\chi'}\big) \\
&= (V_1^{++} \otimes V_2^{--}) \oplus (V_1^{--} \otimes V_2^{++}) \oplus (V_1^{-+} \otimes V_2^{+-}) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C},
\end{aligned}
$$

where $\chi'(g_1) = -1$, $\chi'(g_2) = -1$. □

Now we can conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. Note that V_i^{++} is without base points, whence also $V_1^{++} \otimes V_2^{++}$ has no base points. Therefore a generic

$$
f \in H^0\big(E_1' \times E_2', p_1^*\mathcal{O}_{E_1'}\big(2[o_1] + 2\big[\tfrac{e_1}{2}\big]\big) \otimes p_2^*\mathcal{O}_{E_2'}\big(2[o_2] + 2\big[\tfrac{e_2}{2}\big]\big)\big)^G
$$

has smooth and irreducible zero divisor D (observe that D is ample).

We obtain a 6-dimensional rational family parametrizing all the Keum–Naie surfaces simply by varying the two points P, Q in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and varying f in an open set of the bundle of 4-dimensional projective spaces associated to the rank five vector bundle with fibre

$$
(V_1^{++} \otimes V_2^{++}) \oplus (V_1^{--} \otimes V_2^{--}).
$$

We obtain an irreducible unirational algebraic subset of the moduli space which, by the results of the forthcoming section, is indeed a connected component of the Gieseker moduli space (cf. Theorem 3.1). The dimension of this component is equal to 6, since if two surfaces S, S' are isomorphic, then this isomorphism lifts to a G-equivariant isomorphism between $\widehat{S}$ and $\hat{S}'$, and we get in particular an isomorphism of the corresponding Albanese surfaces carrying one branch locus D to the other D'. It is now easy to see that, since we have normalized the line bundle $\mathbb{L}$, the morphism of the base of the rational family to the moduli space is quasi finite.

We shall show that for each canonical model X the base $\mathfrak{B}_X$ of the Kuranishi family of deformations of X is smooth of dimension 6. For this it suffices to show that the dimension of the Zariski tangent space to $\mathfrak{B}_X$ is at most 6, since we already saw that $\dim(\mathfrak{B}_X) \geq 6$.

In fact we could also show that for each canonical model X the above 6-dimensional family induces a morphism ψ of the smooth rational base whose Kodaira–Spencer map is an isomorphism, whence ψ yields an isomorphism of the base with $\mathfrak{B}_X$.

Observe moreover that the assertion about the normality of this component of the Gieseker moduli space follows right away from the fact that the moduli space $\mathfrak{M}_{\chi,K^2}$ is locally analytically isomorphic to the quotient of the base of the Kuranishi family by the action of the finite group $\mathrm{Aut}(X)$. Indeed, a quotient of a normal

space is normal, and the local ring of a complex algebraic variety is normal if its corresponding analytic algebra is normal.

Let now $X = \hat{X}/G$ be the canonical model of a Keum–Naie surface. Note that

$$\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) = \mathrm{Ext}^1(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^G$$

and that

$$\mathfrak{B}_X = \mathfrak{B}_{\hat{X}} \cap \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) = \mathfrak{B}_{\hat{X}} \cap \mathrm{Ext}^1(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^G.$$

In order to conclude the proof, it suffices therefore to show that

$$\mathfrak{B}_{\hat{X}} = \mathrm{Ext}^1(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}}),$$

which shows that $\mathfrak{B}_{\hat{X}}$ is smooth.

We consider then $\hat{X}$ as a double cover of its Albanese variety A, and observe that the family of such double covers of a principally polarized abelian surface has dimension equal to $18 = 3 + 15$, since abelian surfaces depend on three moduli, and the branch divisor D varies in a linear system of projective dimension $\frac{1}{2}D^2 - 1 = 16 - 1 = 15$ (observe that changing the divisor class to an algebraically equivalent one can be achieved by a translation, which does not change the isomorphism class of the double cover).

Hence we are done once we show that $\dim \mathrm{Ext}^1_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}}) = 18$. This is the content of the following proposition, where we split all the relevant cohomology groups in eigenspaces for the action of the group $\mathbb{Z}/2\mathbb{Z}$ generated by the covering involution for the Albanese morphism.

Proposition 2.3. (1) $\dim \mathrm{Ext}^1_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^+ = 18$;
(2) $\dim \mathrm{Ext}^1_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^- = 0$;
(3) $\dim \mathrm{Ext}^2_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^+ = 2$;
(4) $\dim \mathrm{Ext}^2_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^- = 8$.

Proof. Consider the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \to H^0(\mathcal{O}_D(D)) \to \mathrm{Ext}^1_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^+ \to \mathrm{Ext}^1_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \\ \to H^1(\mathcal{O}_D(D)) \to \mathrm{Ext}^2_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^+ \to \mathrm{Ext}^2_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \to 0, \end{aligned} \tag{3}$$

for which a convenient reference is [Man01], where the following is proven:

Proposition 2.4. *For every locally simple normal flat $(\mathbb{Z}/2\mathbb{Z})^r$-cover $f : X \to Y$ there is a $(\mathbb{Z}/2\mathbb{Z})^r$-equivariant exact sequence of sheaves*

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \bigoplus_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^r} \mathcal{O}_{R_\sigma}(-R_\sigma) \to 0, \tag{4}$$

where R_σ is the divisorial part of $\mathrm{Fix}(\sigma)$.

Moreover, for each $\sigma \in (\mathbb{Z}/2\mathbb{Z})^r$ and $i \geq 1$, we have

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_{R_\sigma}(-R_\sigma), \mathcal{O}_X) \cong \bigoplus_{\{\chi|\chi(\sigma)=0\}} H^{i-1}(\mathcal{O}_{D_\sigma}(D_\sigma - \mathcal{L}_\chi)).$$

Observe that

- $\mathrm{Hom}_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \cong \mathbb{C}^2$,
- $\mathrm{Ext}^1_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \cong \mathbb{C}^4$,
- $\mathrm{Ext}^2_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \cong \mathbb{C}^2$,
- $H^1(\mathcal{O}_D(D))(= H^1(\mathcal{O}_D(K_D))) \cong \mathbb{C}$,
- $H^0(\mathcal{O}_D(D))(= H^0(\mathcal{O}_D(K_D))) \cong \mathbb{C}^{17}$, since D has genus $g = 17$ (in fact $2(g-1) = D^2 = 32$).

Note that the map $\lambda\colon \mathrm{Ext}^1_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A) \to H^1(\mathcal{O}_D(D))$ is the Serre dual of

$$\mathbb{C} \cong H^0(D, \mathcal{O}_D) \to H^1(A, \Omega^1_A), \quad 1 \mapsto c_1(D),$$

which is injective. Therefore λ is surjective, and part 1) and 2) of the claim follow.

In order to calculate the antiinvariant parts of $\mathrm{Ext}^i_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})$, $i = 1, 2$, observe that

$$\mathrm{Ext}^i_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^- \cong \mathrm{Ext}^i_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A(-L)).$$

But

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{O}_A}(\Omega^1_A, \mathcal{O}_A(-L)) &\cong H^i(A, \Theta_A(-L)) \\ &\cong H^i(A, \mathcal{O}_A(-L))^{\oplus 2} \cong (H^{2-i}(A, \mathcal{O}_A(L))^{\oplus 2})^\vee. \qquad \square\end{aligned}$$

3. The fundamental group of Keum–Naie surfaces

In the previous sections we proved that Keum–Naie surfaces form a normal unirational irreducible component of dimension 6 of the Gieseker moduli space. In this section we shall prove that indeed they form a connected component. More generally, we shall prove the following:

Theorem 3.1. *Let S be a smooth complex projective surface which is homotopically equivalent to a Keum–Naie surface. Then S is a Keum–Naie surface.*

Let S be a smooth complex projective surface with $\pi_1(S) = \Gamma$ (Γ being the fundamental group of a Keum–Naie surface).

Recall that $\gamma_i^2 = e_i$ for $i = 1, 2$. Therefore $\Gamma = \langle \gamma_1, e_1', \gamma_2, e_2' \rangle$ and recall that, as we observed in Section 1, we have the exact sequence

$$1 \to \mathbb{Z}^4 \cong \langle e_1, e_1', e_2, e_2' \rangle \to \Gamma \to (\mathbb{Z}/2\mathbb{Z})^2 \to 1,$$

where $\gamma_1 \mapsto (1, 0)$, $\gamma_2 \mapsto (0, 1)$.

We have set $\Lambda_i' := \mathbb{Z}e_i \oplus \mathbb{Z}e_i'$, so that $\pi_1(E_1' \times E_2') = \Lambda_1' \oplus \Lambda_2'$.

We define also the two lattices $\Lambda_i := \mathbb{Z}\frac{e_i}{2} \oplus \mathbb{Z}e_i'$.

Remark 3.1. 1) Γ acts as a group of affine transformations on the lattice $\Lambda_1 \oplus \Lambda_2$.

2) We have an étale double cover $E_i' = \mathbb{C}/\Lambda_i' \to E_i := \mathbb{C}/\Lambda_i$, which is the quotient by a semiperiod of E_i', namely $e_i/2$.

Γ has two subgroups of index two:

$$\Gamma_1 := \langle \gamma_1, e_1', e_2, e_2' \rangle, \quad \Gamma_2 := \langle e_1, e_1', \gamma_2, e_2' \rangle,$$

corresponding to two étale covers of S: $S_i \to S$, for $i = 1, 2$.

Lemma 3.2. *The Albanese variety of S_i is E_i. In particular, $q(S_1) = q(S_2) = 1$.*

Proof. Denoting the translation by e_i by $t_{e_i} \in \mathbb{A}(2, \mathbb{C})$ we see that

$$\gamma_1 t_{e_2} = t_{e_2}^{-1}\gamma_1, \quad \gamma_1 t_{e_2'} = t_{e_2'}^{-1}\gamma_1, \quad \gamma_1 t_{e_1'} = t_{e_1'}\gamma_1.$$

This implies that $t_{e_2}^2, t_{e_2'}^2 \in [\Gamma_1, \Gamma_1]$, and we get a surjective homomorphism

$$\Gamma_1' := \Gamma_1/2\langle e_2, e_2' \rangle \cong \Gamma_1/2\mathbb{Z}^2 \to \Gamma_1^{\mathrm{ab}} = \Gamma_1/[\Gamma_1, \Gamma_1].$$

Since γ_1 and e_1' commute, we have that Γ_1' is commutative, hence

$$\Gamma_1' \cong \langle \gamma_1, e_1' \rangle \oplus (\mathbb{Z}/2\mathbb{Z})^2 \cong \mathbb{Z}\tfrac{e_1}{2} \oplus \mathbb{Z}e_1' \oplus (\mathbb{Z}/2\mathbb{Z})^2 = \Lambda_1 \oplus (\mathbb{Z}/2\mathbb{Z})^2.$$

Since Γ_1' is abelian $\Gamma_1' = \Gamma_1^{\mathrm{ab}} = H_1(S_1, \mathbb{Z})$. This implies that $\mathrm{Alb}(S_1) = \mathbb{C}/\Lambda_1 = E_1$.

The same calculation shows that $\Gamma_2^{\mathrm{ab}} = H_1(S_2, \mathbb{Z}) = \Lambda_2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$, whence $\mathrm{Alb}(S_2) = \mathbb{C}/\Lambda_2 = E_2$. □

For the sake of completeness we prove the following

Lemma 3.3. $H_1(S, \mathbb{Z}) = \Gamma^{\mathrm{ab}} = \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$.

Proof. We have already seen in the proof of Lemma 3.2 that

$$\gamma_1 t_{e_2} = t_{e_2}^{-1}\gamma_1, \quad \gamma_1 t_{e_2'} = t_{e_2'}^{-1}\gamma_1,$$
$$\gamma_2 t_{e_1} = t_{e_1}^{-1}\gamma_2, \quad \gamma_2 t_{e_1'} = t_{e_1'}^{-1}\gamma_2,$$

and moreover, for $i = 1, 2$, we have that γ_i commutes with e_i, e'_i.

This shows that we have a surjective homomorphism

$$\Gamma' := \Gamma/\langle 2e_1, 2e'_1, 2e_2, 2e'_2\rangle \cong \Gamma/2\mathbb{Z}^2 \to \Gamma/[\Gamma, \Gamma].$$

Since $\gamma_2\gamma_1 = t_{e_2}t_{e_1}^{-1}\gamma_1\gamma_2$, it follows that $e_2 - e_1 \in [\Gamma, \Gamma]$, whence we have a surjective homomorphism

$$\Gamma'' := \Gamma'/\langle e_1 - e_2\rangle \to \Gamma/[\Gamma, \Gamma],$$

and it is easy to see that the homomorphism $\psi\colon \Gamma'' \to \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$, given by

$$\psi(\bar{\gamma}_1) = (1, 0, 0, 0), \quad \psi(\bar{\gamma}_2) = (1, 1, 0, 0),$$
$$\psi(\bar{e'}_1) = (0, 0, 1, 0), \quad \psi(\bar{e'}_2) = (0, 0, 0, 1).$$

is well defined and is an isomorphism. This shows the claim. □

Let $\widehat{S} \to S$ be the étale $(\mathbb{Z}/2\mathbb{Z})^2$-covering which is associated to $\Lambda'_1 \oplus \Lambda'_2 = \langle e_1, e'_1, e_2, e'_2\rangle \triangleleft \Gamma$. Since $\widehat{S} \to S_i \to S$, and S_i maps to E_i (via the Albanese map), we get a morphism

$$f\colon \widehat{S} \to E_1 \times E_2 = \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2.$$

Then f factors through the Albanese map of $\widehat{S}$: but, since the fundamental group of $\widehat{S}$ equals $\Lambda'_1 \oplus \Lambda'_2$, and the covering of $E_1 \times E_2$ associated to $\Lambda'_1 \oplus \Lambda'_2 \leq \Lambda_1 \oplus \Lambda_2$ is $E'_1 \times E'_2$, we see that f factors through $E'_1 \times E'_2$ and that the Albanese map of $\widehat{S}$ is $\hat{\alpha}\colon \widehat{S} \to E'_1 \times E'_2$.

We will conclude the proof of Theorem 3.1 with the following

Proposition 3.4. *Let S be a smooth complex projective surface, which is homotopically equivalent to a Keum–Naie surface. Let $\widehat{S} \to S$ be the étale $(\mathbb{Z}/2\mathbb{Z})^2$-cover associated to $\langle e_1, e'_1, e_2, e'_2\rangle \triangleleft \Gamma$ and let*

$$\begin{tikzcd} \widehat{S} \arrow[r, "\hat{\alpha}"] \arrow[dr] & E'_1 \times E'_2 \\ & Y \arrow[u, "\varphi"'] \end{tikzcd}$$

be the Stein factorization of the Albanese map of $\widehat{S}$.

Then φ has degree 2 and Y is a canonical model of $\widehat{S}$.

Corollary 3.5. *Y is a finite double cover of $E'_1 \times E'_2$ branched on a divisor of type* $(4, 4)$.

This completes the proof of Theorem 3.1.

Proof of Proposition 3.4. Consider the Albanese map $\hat{\alpha}\colon \widehat{S} \to E_1' \times E_2'$. Then we calculate the degree of the Albanese map as the index of a certain subgroup of $H^4(\widehat{S}, \mathbb{Z})$, namely:

$$\begin{aligned}\deg(\hat{\alpha}) &= [H^4(\widehat{S}, \mathbb{Z}) : \hat{\alpha}^* H^4(E_1' \times E_2', \mathbb{Z}) = \wedge^4 \hat{\alpha}^* H^1(E_1' \times E_2', \mathbb{Z})] \\ &= [H^4(\widehat{S}, \mathbb{Z}) : \wedge^4 H^1(\widehat{S}, \mathbb{Z})].\end{aligned}$$

But since S is homotopically equivalent to a Keum–Naie surface S', also $\widehat{S}$ is homotopically equivalent to the étale $(\mathbb{Z}/2\mathbb{Z})^2$-covering $\widehat{S}'$ of S'. Since the $[H^4(\widehat{S}, \mathbb{Z}) : \wedge^4 H^1(\widehat{S}, \mathbb{Z})]$ is a homotopy invariant, and the degree of the Albanese map of $\widehat{S}'$ is two, it follows that $\deg(\hat{\alpha}) = 2$.

It remains to show that Y has only rational double points. This follows from the following lemma. □

Lemma 3.6. *Let A be an abelian surface and let $\widehat{S}$ be a surface with $K_{\widehat{S}}^2 = 16$ and $\chi(\widehat{S}) = 4$. Moreover, let $\varphi\colon \widehat{S} \to A$ be a generically finite morphism of degree 2. Then the branch divisor of φ has only non essential singularities (i.e., the local multiplicities of the singular points are ≤ 3, and for each infinitely near point we have multiplicity at most two, cf.* [Hor78]); *equivalently, if*

$$\begin{tikzcd} \widehat{S} \arrow[r, "\varphi"] \arrow[dr] & A \\ & Y \arrow[u, "\delta"'] \end{tikzcd}$$

is the Stein factorization, then Y has at most rational double points as singularities.

Proof. We use the notation and results on double covers due to E. Horikawa (cf. [Hor78]). Consider the following diagram:

$$\begin{tikzcd} \widehat{S} \arrow[r, "\varphi"] & A \\ S^* \arrow[u, "\sigma"] \arrow[r] & \tilde{A}, \arrow[u, "\delta"'] \end{tikzcd}$$

where $S^* \to A$ is the so-called *canonical resolution* in the terminology of Horikawa.

This means that $\tilde{A} \to A$ is a minimal sequence of blow ups such that the reduced transform of the branch divisor of φ is smooth, so that $S^* \to \tilde{A}$ is a finite double cover with S^* smooth, and $S^* \to \widehat{S}$ is a sequence of blow ups of smooth points.

Then we have the following formulae:

$$K_{S^*}^2 = K_{\widehat{S}}^2 - t = 2(K_A + \mathcal{L})^2 - 2\sum \left(\left[\tfrac{m_i}{2}\right] - 1\right)^2, \tag{5}$$

$$\chi(S^*) = \chi(\widehat{S}) = \tfrac{1}{2}\mathcal{L}(K_A + \mathcal{L}) - \tfrac{1}{2}\sum \left[\tfrac{m_i}{2}\right]\left(\left[\tfrac{m_i}{2}\right] - 1\right), \tag{6}$$

where t is the number of points on $\widehat{S}$ blown up by σ, $\mathcal{O}_A(2\mathcal{L}) \cong \mathcal{O}_A(B)$, where B is the branch divisor of the (singular) double cover $Y \to A$. Finally $m_i \geq 2$ is the multiplicity of the branch curve in the i-th center of the successive blow up of A. For details we refer to [Hor78].

Notice that Y has R.D.P.s if and only if $\xi_i := [\frac{m_i}{2}] = 1$ for each singular point (and for all infinitely near points).

In our situation, the above two equations read:

$$K_{S^*}^2 = 16 - t = 2\mathcal{L}^2 - 2\sum(\xi_i - 1)^2,$$
$$\chi(\widehat{S}) = 4 = \tfrac{1}{2}\mathcal{L}^2 - \tfrac{1}{2}\sum \xi_i(\xi_i - 1).$$

This implies that

$$2\mathcal{L}^2 - 2\sum(\xi_i - 1)^2 + t = 16 = 2\mathcal{L}^2 - 2\sum \xi_i(\xi_i - 1),$$

or, equivalently,

$$t = -2\sum(\xi_i - 1).$$

Since $\xi_i \geq 1$ this is only possible iff $\xi_i = 1$ for all i and $t = 0$. □

Remark 3.2. Note that the above equations also imply that in the case $A = E_1' \times E_2'$, $\mathcal{L}$ has to be of type $(2, 2)$ or $(1, 4)$ (resp. $(4, 1)$). But a divisor of type $(1, 4)$ cannot be $(\mathbb{Z}/2\mathbb{Z})^2$ invariant. This proves the above corollary.

In fact, we conjecture the following to hold true:

Conjecture 3.7. *Let S be a minimal smooth projective surface such that*

i) $K_S^2 = 4$,
ii) $\pi_1(S) \cong \Gamma$.

Then S is a Keum–Naie surface.

In fact, we can prove

Theorem 3.8. *Let S be a minimal smooth projective surface such that*

i) $K_S^2 = 4$,
ii) $\pi_1(S) \cong \Gamma$,
iii) *there is a deformation of S having ample canonical bundle.*

Then S is a Keum–Naie surface.

Before proving the above theorem, we recall the following results:

Theorem 3.9 (Severi's conjecture, [Par05]). *Let S be a minimal smooth projective surface of maximal Albanese dimension (i.e., the image of the Albanese map is a surface): then $K_S^2 \geq 4\chi(S)$.*

M. Manetti proved Severi's inequality under the stronger assumption that K_S is ample, but he also gave a description of the limit case $K_S^2 = 4\chi(S)$, which will be crucial for our result.

Theorem 3.10 (M. Manetti,[Man03]). *Let S be a minimal smooth projective surface of maximal Albanese dimension with K_S ample: then $K_S^2 \geq 4\chi(S)$, and equality holds if and only if $q(S) = 2$, and the Albanese map $\alpha\colon S \to \mathrm{Alb}(S)$ is a finite double cover.*

Proof of Theorem 3.8. We know that there is an étale $(\mathbb{Z}/2\mathbb{Z})^2$-cover $\widehat{S}$ of S with Albanese map $\hat{\alpha}\colon \widehat{S} \to E_1' \times E_2'$. The Albanese map of $\widehat{S}$ must be surjective, otherwise the Albanese image, by the universal property of the Albanese map, would be a curve C of genus 2. But then we would have a surjection $\pi_1(\widehat{S}) \to \pi_1(C)$, which is a contradiction since $\pi_1(\widehat{S})$ is abelian and $\pi_1(C)$ is not abelian.

Note that $K_{\widehat{S}}^2 = 4K_S^2 = 16$. By Severi's inequaltiy, it follows that $\chi(\widehat{S}) \leq 4$, but since $1 \leq \chi(S) = \frac{1}{4}\chi(\widehat{S})$, we have $\chi(\widehat{S}) = 4$. Since S deforms to a surface with K_S ample, we can apply Manetti's result and obtain that $\hat{\alpha}\colon \widehat{S} \to E_1' \times E_2'$ has degree 2, and we conclude as before. □

It seems reasonable to conjecture (cf. [Man03]) the following, which would obviously imply our conjecture 3.7.

Conjecture 3.11. *Let S be a minimal smooth projective surface of maximal Albanese dimension. Then $K_S^2 = 4\chi(S)$ if and only if $q(S) = 2$, and the Albanese map has degree 2.*

Remark 3.3. 1) In [Ke] the author proves that Bloch's conjecture holds, i.e., $A_0(S) = \mathbb{Z}$, for the family of surfaces he constructs. Since Keum constructs only a 4-dimensional subfamily of the connected component of the moduli space, this does not imply that Bloch's conjecture holds for all Keum–Naie surfaces. Nevertheless, exactly the same proof holds in the general case, thereby showing that Bloch's conjecture holds true for all Keum–Naie surfaces.

4. The bicanonical map of Keum–Naie surfaces

It is shown in [Nai94] that the bicanonical map of a Keum–Naie surface is base point free and has degree 4. Moreover, in [ML-P02], the authors show that the bicanonical image of a Keum–Naie surface is a rational surface, and the bicanonical morphism

factors through the double cover $S \to Y$, where $Y = (E_1' \times E_2')/(\mathbb{Z}/2\mathbb{Z})^2$ is an 8-nodal Enriques surface. More precisely they show the following (cf. [ML-P02], 5.2.): minimal surfaces S of general type with $p_g = 0$ and $K^2 = 4$ having an involution σ such that

i) S/σ is birational to an Enriques surface and
ii) the bicanonical map is composed with σ

are precisely the Keum–Naie surfaces.

As a corollary of our very explicit description of Keum–Naie surfaces we prove the following

Theorem 4.1. *The bicanonical map of a Keum–Naie surface is a finite iterated double covering of the* 4*-nodal Del Pezzo surface* $\Sigma \subset \mathbb{P}^4$ *of degree* 4*, the complete intersection of the following two quadric hypersurfaces in* $\mathbb{P}^4$:

$$Q_1 = \{z_0 z_3 - z_1 z_2 = 0\}, \quad Q_2 = \{z_4^2 - z_0 z_3 = 0\}.$$

Proof. Observe first of all that $H^0(2K_S) \cong H^0(2K_{\widehat{S}})^{++}$. The standard formulae for the bicanonical system of a double cover allow to decompose $H^0(2K_{\widehat{S}})$ as the direct sum of the invariant part U and the anti-invariant part U' for the covering involution of the Albanese map.

We have then $H^0(2K_{\widehat{S}}) = U \oplus U'$, where

$$U := \left\{\Phi(z_1, z_2)\frac{(dz_1 \wedge dz_2)^{\otimes 2}}{w^2}\right\} = \hat{\alpha}^* H^0(\mathcal{O}_{E_1' \times E_2'}(D))$$

and

$$U' := \left\{\Psi(z_1, z_2)\frac{(dz_1 \wedge dz_2)^{\otimes 2}}{w}\right\} = \hat{\alpha}^* H^0(\mathcal{O}_{E_1' \times E_2'}(L)).$$

Here $\Phi(z_1, z_2) = \Phi_1(z_1)\Phi_2(z_2)$ is a section of $\mathbb{L}^{\otimes 2} = \mathcal{O}_{E_1' \times E_2'}(D)$, whereas $\Psi(z_1, z_2) = \Psi_1(z_1)\Psi_2(z_2)$ is a section of $\mathbb{L} = \mathcal{O}_{E_1' \times E_2'}(L)$.

Since however w is an eigenvector for G with character of type $(-, +)$, it follows that w^2 is a G-invariant, and, moreover, $U^{++} = \hat{\alpha}^* H^0(\mathcal{O}_{E_1' \times E_2'}(D))^{++}$, while $U'^{++} = \hat{\alpha}^* H^0(\mathcal{O}_{E_1' \times E_2'}(L))^{-+}$.

By the formulae that we developed in Lemma 1.1 the second space is equal to 0, while the formulae developed in Section 2 show that

$$U^{++} = (V_1^{++} \otimes V_2^{++}) \oplus (V_1^{--} \otimes V_2^{--}) \cong \mathbb{C}^4 \oplus \mathbb{C}.$$

The first consequence of this calculation is that the composition of $\widehat{S} \to S$ with the bicanonical map of S factors through the product $E_1' \times E_2'$.

Moreover, these sections are invariant for the action of the group G, and further for the action of the automorphism

$$g'(z_1, z_2) := \left(-z_1 + \tfrac{e_1}{2}, z_2\right)$$

(observe that G and g' are contained in $(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$).

Whence the above composition factors through the $(\mathbb{Z}/2\mathbb{Z})$ quotient Σ of the Enriques surface $(E_1' \times E_2')/G$ by the action of g'.

Σ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified in the union of two vertical plus two horizontal lines. The subspace $(V_1^{++} \otimes V_2^{++})$ is the pull back of the hyperplane series of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, thus we get a basis of sections z_0, z_1, z_2, z_3 satisfying $z_0 z_3 - z_1 z_2 = 0$.

We can complete these to a basis of $H^0(2K_S)$ by choosing z_4 such that $z_4^2 = z_0 z_3$.

Since $H^0(\mathcal{O}_{E_1' \times E_2'}(D))^{++}$ is base point free, the bicanonical map is a morphism, factoring through the double cover $S \to Y$ and the double cover $Y \to \Sigma$.

It is immediate to see that $(z_0, z_1, z_2, z_3, z_4)$ yield an embedding of Σ. We get a complete intersection of degree 4, hence a Del Pezzo surface of degree 4. The four nodes, which correspond to the 4 points where the 4 lines of the branch locus meet, are seen to be the 4 points

$$z_4 = z_1 = z_2 = z_3 = 0,$$

$$z_4 = z_1 = z_2 = z_0 = 0,$$

$$z_4 = z_0 = z_3 = z_1 = 0,$$

$$z_4 = z_0 = z_3 = z_2 = 0. \qquad \square$$

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