

Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces

I. Bauer, F. Catanese*

Abstract. This is the first in a series of articles on the moduli spaces of Burniat surfaces. We prove that Burniat surfaces are Inoue surfaces and we calculate their fundamental groups. As a main result of this paper we prove that every smooth projective surface which is homotopically equivalent to a primary Burniat surface (i.e., a Burniat surface with $K^2 = 6$) is indeed a primary Burniat surface.

2010 Mathematics Subject Classification. 14J29, 14J25, 14J10, 14D22, 14H30, 20F34, 32G05, 32Q30

Keywords. Moduli of surfaces of general type, fundamental groups of algebraic varieties.

Introduction

In a recent joint paper ([BCGP09]) with Fritz Grunewald and Roberto Pignatelli we constructed many new families of surfaces of general type with $p_g = 0$, hence we got interested about the current status of the classification of such surfaces, in particular about the structure of their moduli spaces.

For instance, in the course of deciding which families were new and which were not new, we ran into the problem of determining whether surfaces with $K^2 = 4$ and with fundamental group equal to the one of Keum-Naie surfaces were indeed Keum-Naie surfaces. This problem was solved in [BC09a], where we showed that any surface homotopically equivalent to a Keum-Naie surface is a Keum-Naie surface, whence we got a complete description of a connected irreducible component of the moduli space of surfaces of general type.

We soon realized that similar methods would apply to the 'primary' Burniat surfaces, the ones with $K^2 = 6$; hence we got interested about the components of the moduli space containing the Burniat surfaces.

This article is the first of a series of articles devoted to the so called Burniat surfaces. These are several families of surfaces of general type with $p_g = 0$, $K^2 = 6, 5, 4, 3, 2$, first constructed by P. Burniat in [Bu66] as 'bidouble covers' (i.e., $(\mathbb{Z}/2\mathbb{Z})^2$ Galois covers) of the plane \mathbb{P}^2 branched on certain configurations of nine lines.

These surfaces were later considered by Peters in [Pet77], who gave an account of Burniat's construction in the modern language of double covers. He missed

*The present work took place in the realm of the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds".

however one of the two families with $K^2 = 4$, the 'non nodal' one. He also calculated (ibidem) the torsion group $H_1(S, \mathbb{Z})$ for Burniat's surfaces (observe that a surface of general type with $p_g = 0$ has first Betti number $b_1 = 0$). He asserted that $H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{K^2}$. This result is however correct only for $K^2 \neq 2$, as we shall see.

Later, following a suggestion by Miles Reid, another construction of these surfaces was given by Inoue in [In94], who constructed 'surfaces closely related to Burniat's surfaces' with a different technique as $G^2 := (\mathbb{Z}/2\mathbb{Z})^3$ -quotients of a G^2 -invariant hypersurface \hat{X} of multidegree $(2, 2, 2)$ in a product of three elliptic curves.

Another description of the Burniat surfaces as 'singular bidouble covers' was later given in [Cat99], where also other examples were proposed of 'Burniat type surfaces'. These however turn out to give no new examples.

The important feature of the Burniat surfaces S is that their bicanonical map is a bidouble cover of a normal Del Pezzo surface of degree K_S^2 (obtained as the anticanonical model of the blow up of the plane in the points of multiplicity at least 3 of the divisor given by the union of the lines of the configuration).

Burniat surfaces with $K^2 = 6$ were studied from this point of view by Mendes-Lopes and Pardini in [MLP01].

Although Burniat surfaces had been known for a long time, we found that their most important properties were yet to be discovered, and we devote two articles to show in particular that the four families of Burniat surfaces, the ones with $K^2 = 6, 5$ respectively, and the two ones with $K^2 = 4$ (the nodal and the non nodal one) are irreducible connected components of the moduli space of surfaces of general type.

Since there is no reference known where it is proved that Burniat's surfaces are exactly Inoue's surfaces, we start by giving in the present paper a proof of this fact.

This is crucial in order to calculate the fundamental groups of Burniat's surfaces with $K^2 = 6, 5, 4, 3, 2$. Our proof confirms the results stated by Inoue without proof in his beautiful paper, except for $K^2 = 2$ where Inoue's claim turns out to be wrong.

Our proof combines the 'transcendental' description given by Inoue with delicate algebraic calculations, which are based on explicit algebraic normal forms for the 2-torsion of elliptic curves, described in the first section.

We first prove the following:

Theorem 0.1. *Let S be the minimal model of a Burniat surface.*

- i) $K_S^2 = 6 \implies \pi_1(S) = \Gamma, H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^6$;
- ii) $3 \leq K_S^2 \leq 5 \implies \pi_1(S) = \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^{K^2-2}, H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{K^2}$;
- iii) $K^2 = 2 \implies \pi_1(S) = H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^3$.

Here \mathbb{H} denotes the quaternion group of order 8, while Γ is a group of affine transformations on \mathbb{C}^3 , explicitly described in section 3.

The main result of this article is however the following theorem:

Theorem 0.2. *Let S be a smooth complex projective surface which is homotopically equivalent to a primary Burniat surface. Then S is a Burniat surface.*

We can then use this result to give an alternative, and less involved proof of the following result due to Mendes-Lopes and Pardini ([MLP01]).

Theorem 0.3. *The subset of the Gieseker moduli space corresponding to primary Burniat surfaces is an irreducible connected component, normal, rational and of dimension equal to 4.*

In [MLP01] openness is shown (using Burniat's description of a primary Burniat surface) by standard local deformation theory of bidouble covers. We give an alternative proof of this result using Inoue's description.

For the closedness, Mendes Lopes and Pardini use their characterization of primary Burniat surfaces as exactly those surfaces with $p_g = 0$, $K^2 = 6$ such that the bicanonical map has degree 4.

Our proof is much less involved. It only uses the description of the fundamental group as an affine group of transformations of \mathbb{C}^3 .

Moreover we prove the rationality of the moduli space of primary Burniat surfaces.

In the second article we shall show that the Burniat surfaces with $K^2 = 5$ yield an irreducible connected component of dimension 3 in the moduli space of surfaces of general type. Instead, there are two different configurations in the plane giving Burniat surfaces with $K^2 = 4$. We shall show that, in fact, both yield an irreducible connected component of dimension 2 in the moduli space of surfaces of general type.

This is interesting, since it follows that the bicanonical map of S is a bidouble cover of a Del Pezzo surface of degree K_S^2 for all the surfaces in the connected component.

There is only one Burniat surface with $K^2 = 2$, and since its fundamental group is $(\mathbb{Z}/2\mathbb{Z})^3$, it turns out to be a surface in the 6 dimensional family of standard Campedelli surfaces ([Miy77]), for which the bicanonical map is then a degree 8 covering of the plane.

We analyse it briefly in the last section. In fact, at the moment of completing the paper we became aware of the article [Ku04] where the author had already pointed out and corrected the errors of [In94] and [Pet77] on the fundamental group and the first homology of the Burniat surface with $K^2 = 2$.

1. The Legendre and other normal forms for 2-torsion of elliptic curves

This section reviews classical mathematics which will be reiteratedly used in the sequel.

The Legendre form of an elliptic curve is given by an equation of the form

$$y^2 = (\xi^2 - 1)(\xi^2 - a^2).$$

It yields a curve \mathcal{E}' of genus 1 as a double cover of \mathbb{P}^1 branched on the 4 points $\xi = \pm 1, \xi = \pm a$.

These 4 points yield 4 points on \mathcal{E}' , $P'_1, P'_{-1}, P'_a, P'_{-a}$, which correspond to the 2-torsion points, once any of them is fixed as the origin, as we shall more amply now illustrate.

We consider now 3 automorphisms of order 2 of \mathcal{E}' defined by:

$$g'_1(\xi, y) := (-\xi, -y), \quad g'_2(\xi, y) := (\xi, -y), \quad g'_3(\xi, y) := (-\xi, y).$$

We get in this way an action of $(\mathbb{Z}/2\mathbb{Z})^2$ on \mathcal{E}' such that the quotient is \mathbb{P}^1 , with coordinate $u := \xi^2$.

Clearly the quotient by g'_2 is the original \mathbb{P}^1 with coordinate ξ , hence g'_2 corresponds to multiplication by -1 on the elliptic curve, once we fix one of the above points as the origin.

The quotient of \mathcal{E}' by g'_3 is instead the smooth curve of genus 0, given by the conic $y^2 = (u - 1)(u - a^2)$.

What is more interesting is the quotient of \mathcal{E}' by g'_1 : the invariants are u and $r := \xi y$, thus we obtain as quotient the elliptic curve

$$\mathcal{E} := \{r^2 = u(u - 1)(u - a^2)\}$$

in Weierstrass normal form.

This shows that g'_1 is the translation by a 2-torsion element η' . By looking at the action of g'_1 on the 4 above points, we see that η' is the class of the degree zero divisor $[P'_1] - [P'_{-1}]$. In other words, the divisor classes of degree 0

$$\eta' := [P'_1] - [P'_{-1}] = [P'_a] - [P'_{-a}], \quad \eta'' := [P'_1] - [P'_a] = [P'_{-1}] - [P'_{-a}]$$

generate $\text{Pic}^0(\mathcal{E}') [2] \cong (\mathbb{Z}/2\mathbb{Z})^2$.

We can now also understand that the automorphism g'_3 , which is the product $g'_1 g'_2 = g'_2 g'_1$, has as fixed points the 4 points lying over $\xi = 0$ and $\xi = \infty$. These correspond to the 4-torsion points whose associated translation has as square the translation by the 2-torsion point η' .

More importantly, we want to give now a nicer form for the group of translations of order 2 of an elliptic curve (this leads to the theory of 2-descent on an elliptic curve). This form will be used in the sequel, but in another coordinate system for the line \mathbb{P}^1 with coordinate ξ .

To this purpose, let us consider the curve C defined by

$$v^2 = (\xi^2 - 1), \quad w^2 = (\xi^2 - a^2).$$

We shall show that this curve is the same elliptic curve \mathcal{E} which we had above.

In fact, setting $y := vw$, we see that we obtain C as a double cover of \mathcal{E}' , which is unramified (as we see by calculating the ramification of the $(\mathbb{Z}/2\mathbb{Z})^2$ -Galois cover of \mathbb{P}^1 with coordinate ξ).

The transformations of order 2

$$g_1 : (\xi, v, w) \mapsto (\xi, -v, -w), \quad f_2 : (\xi, v, w) \mapsto (-\xi, v, -w),$$

$$f_3 : (\xi, v, w) \mapsto (-\xi, -v, w)$$

generate a group $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ such that the quotient curve is the elliptic curve \mathcal{E} , since the invariants are $\xi^2 = u$, $v^2 = u - 1$, $w^2 = u - a^2$, $\xi vw = \xi y = r$.

The quotients by the above involutions are respectively \mathcal{E}' , the elliptic curve

$$\mathcal{E}'' := \{t^2 = (v^2 + 1)(v^2 + 1 - a^2)\}$$

(where we have set $t := \xi w$), and the elliptic curve

$$\mathcal{E}''' := \{s^2 = (w^2 + a^2)(w^2 + a^2 - 1)\}$$

(where we have set $s := \xi v$).

The first conclusion is that C is isomorphic to \mathcal{E} , the group H is the group of translations by the 2-torsion points of \mathcal{E} , whereas the quotient map $C \rightarrow \mathcal{E} = C/H$ is multiplication by 2 in the elliptic curve $C \cong \mathcal{E}$.

We have another group $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ acting on $C \cong \mathcal{E}$, namely the one with quotient the \mathbb{P}^1 with coordinate ξ . Here, we set

$$g_2 : (\xi, v, w) \mapsto (\xi, -v, w), \quad g_3 := g_1 g_2 = g_2 g_1 : (\xi, v, w) \mapsto (\xi, v, -w).$$

Again, g_1 corresponds to translation by a 2-torsion element η , while we view g_2 as multiplication by -1 . The fixed points of g_2 are the points with $v = 0$, i.e., the 4 points with $v = 0$, $\xi = \pm 1$, $w = \pm\sqrt{1 - a^2}$. Translation by η then acts on them simply by multiplying their w coordinate by -1 .

An important observation is that the covering $C \rightarrow \mathbb{P}^1$, where \mathbb{P}^1 has coordinate u , is a $(\mathbb{Z}/2\mathbb{Z})^3$ -Galois cover of the \mathbb{P}^1 with coordinate u which is the maximal Galois covering of \mathbb{P}^1 branched on the 4 points $0, 1, a^2, \infty$ and with group of the form $(\mathbb{Z}/2\mathbb{Z})^m$.

It would be nice if also for surfaces one could treat such Galois covers with group $(\mathbb{Z}/2\mathbb{Z})^m$ in the same elementary way. This however can be done only in the birational setting, since in dimension ≥ 2 we have different normal models for the same function field. Hence we have to resort to the theory of abelian covers, developed in [Cat84], [Par91], [Cat99].

In this biregular theory, coverings are described through equations holding in certain vector bundles. To compare the surface case with the curve case it is therefore useful first of all to rewrite the above $(\mathbb{Z}/2\mathbb{Z})^3$ -Galois cover in terms of homogeneous coordinates.

And, for later calculations, it will be convenient to replace the points $\xi = \pm 1$ with the points $0, \infty$. We replace then the affine coordinates $(1 : \xi)$ by coordinates $(x' : x)$ with $\frac{x'}{x} = \frac{\xi - 1}{\xi + 1}$.

We can then rewrite the $(\mathbb{Z}/2\mathbb{Z})^2$ cover as the normalization of the curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\{(v' : v), (w' : w), (x' : x) \mid v^2 x' = v'^2 x, \quad w^2 x'^2 = w'^2 (x^2 - x x' (b + \frac{1}{b}) + x'^2)\}.$$

Now the involution exchanging pairs of branch points is simply the involution $(x' : x) \rightarrow (x : x')$. The normalization is obtained simply by considering the curve

of genus 1 which is the subvariety of the vector bundle whose sheaf of sections on \mathbb{P}^1 is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, given by equations

$$V^2 = xx', \quad W^2 = (x^2 - xx'(b + \frac{1}{b}) + x'^2),$$

which is shorthand notation for the following two equations in the local chart outside $x' = 0$, respectively in the local chart outside $x = 0$:

$$\begin{aligned} \left(\frac{v}{v'}\right)^2 &= \left(\frac{x}{x'}\right), \quad \left(\frac{w}{w'}\right)^2 = \left(\frac{x}{x'}\right)^2 - \left(b + \frac{1}{b}\right)\frac{x}{x'} + 1, \\ \left(\frac{v'}{v}\right)^2 &= \left(\frac{x'}{x}\right), \quad \left(\frac{w}{w'}\frac{x'}{x}\right)^2 = \left(\frac{x'}{x}\right)^2 - \left(b + \frac{1}{b}\right)\frac{x'}{x} + 1. \end{aligned}$$

In other words, we have $v^2 = x$, $v'^2 = x'$, hence $V = vv'$. While, setting $W := \left(\frac{w}{w'}\right)x'$, we get $W^2 = (x^2 - xx'(b + \frac{1}{b}) + x'^2)$. We have now the group $(\mathbb{Z}/2\mathbb{Z})^3$ acting on C by the following transformations

$$\begin{aligned} g_1 : ((x' : x), (v' : v), (w' : w)) &\mapsto ((x' : x), (v' : -v), (w' : -w)), \\ f_2 : ((x' : x), (v' : v), (w' : w)) &\mapsto ((x : x'), (v : v'), (w'x : -wx')), \\ f_3 : ((x' : x), (v' : v), (w' : w)) &\mapsto ((x : x'), (-v : v'), (w'x : wx')), \\ g_2 : ((x' : x), (v' : v), (w' : w)) &\mapsto ((x' : x), (v' : -v), (w' : w)), \\ g_3 = g_1g_2 : ((x' : x), (v' : v), (w' : w)) &\mapsto ((x' : x), (v' : v), (w' : -w)). \end{aligned}$$

The sections V and W are clearly eigenvectors for the group action. It is easy to see, in view of the above table, that the image of $V = vv'$ equals $-V, V, -V, -V, V$ respectively, while the image of W equals $-W, -W, W, W, -W$ respectively.

2. Burniat surfaces are Inoue surfaces

The aim of this section is to show that Burniat surfaces are Inoue surfaces. This fact seems to be known to the experts, but, since we did not find any reference, we shall provide a proof of this assertion, which is indeed crucial for our main result.

In [Bu66], P. Burniat constructed a series of families of surfaces of general type with $K^2 = 6, 5, 4, 3, 2$ and $p_g = 0$ (of respective dimensions 4, 3, 2, 1, 0) as singular bidouble covers (Galois covers with group $(\mathbb{Z}/2\mathbb{Z})^2$) of the projective plane branched on 9 lines. We briefly recall the construction.

Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points and denote by $Y := \hat{\mathbb{P}}^2(P_1, P_2, P_3)$ the blow up of \mathbb{P}^2 in P_1, P_2, P_3 .

Y is a Del Pezzo surface of degree 6 and it is the closure of the graph of the rational map

$$\epsilon : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that

$$\epsilon(y_1 : y_2 : y_3) = ((y_2 : y_3), (y_3 : y_1), (y_1 : y_2)).$$

It is immediate to observe that $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the following hypersurface of type $(1, 1, 1)$:

$$Y = \{((x'_1 : x_1), (x'_2 : x_2), (x'_3 : x_3)) \mid x_1 x_2 x_3 = x'_1 x'_2 x'_3\}.$$

Lemma 2.1. *Consider the cartesian diagram*

$$\begin{array}{ccc} p^{-1}(Y) & \xrightarrow{p} & Y \\ \downarrow & & \downarrow i \\ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where $p : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the $(\mathbb{Z}/2\mathbb{Z})^3$ -Galois covering given by $x_i = v_i^2$, $x'_i = (v'_i)^2$. Then $p^{-1}(Y)$ splits as the union $p^{-1}(Y) = Z \cup Z'$ of two degree 6 Del Pezzo surfaces, where

$$Z := \{((v_1 : v'_1), (v_2 : v'_2), (v_3 : v'_3)) : v_1 v_2 v_3 = v'_1 v'_2 v'_3\}$$

and

$$Z' := \{((v_1 : v'_1), (v_2 : v'_2), (v_3 : v'_3)) : v_1 v_2 v_3 = -v'_1 v'_2 v'_3\}.$$

And $p|_Z$ induces on \mathbb{P}^2 the Fermat squaring map

$$(y_0 : y_1 : y_2) \mapsto (y_0^2 : y_1^2 : y_2^2).$$

Moreover, $Z \cap Z' = \{v_1 v_2 v_3 = v'_1 v'_2 v'_3 = 0\}$, which is the union of 6 lines yielding in each Del Pezzo surface the fundamental hexagon of the blow up of \mathbb{P}^2 in three non collinear points (i.e., the pull back of the triangle with vertices the three points).

Proof. The equation of $p^{-1}(Y)$ is $x_1 x_2 x_3 = x'_1 x'_2 x'_3$, i.e.,

$$(v_1 v_2 v_3)^2 = (v'_1 v'_2 v'_3)^2.$$

The surface Z is invariant under the subgroup

$$G^o \subset \{1, -1\}^3 \cong (\mathbb{Z}/2\mathbb{Z})^3, \quad G^o \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

$$G^o = \{(\epsilon_i) \in \{\pm 1\}^3 \mid \prod_i \epsilon_i = 1\}.$$

G^o acts on Z by sending $v_i \mapsto \epsilon_i v_i$, $v'_i \mapsto v'_i$, and this is easily seen to give on \mathbb{P}^2 the Galois group of the Fermat squaring map. \square

We denote by E_i the exceptional curve lying over P_i and by $D_{i,1}$ the unique effective divisor in $|L - E_i - E_{i+1}|$, i.e., the proper transform of the line $y_{i-1} = 0$, side of the triangle joining the points P_i, P_{i+1} .

For the present choice of coordinates E_i is the side $x_{i-1} = x'_{i+1} = 0$ of the hexagon, while $D_{i,1}$ is the side $x_i = x'_{i+1} = 0$ of the hexagon.

Consider on Y the following divisors

$$D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2} \in |3L - 3E_i - E_{i+1} + E_{i+2}|,$$

where $D_{i,j} \in |L - E_i|$, for $j = 2, 3$, $D_{i,j} \neq D_{i,1}$, is the proper transform of another line through P_i and $D_{i,1} \in |L - E_i - E_{i+1}|$ is as above. Assume also that all the corresponding lines in \mathbb{P}^2 are distinct, so that $D := \sum_i D_i$ is a reduced divisor.

Observe that all the indices in $\{1, 2, 3\}$ have here to be understood as residue classes modulo 3.

Note that, if we define the divisor $\mathcal{L}_i := 3L - 2E_{i-1} - E_{i+1}$, then

$$D_{i-1} + D_{i+1} = 6L - 4E_{i-1} - 2E_{i+1} \equiv 2\mathcal{L}_i,$$

and we can consider (cf. [Cat99]) the associated bidouble cover $X \rightarrow Y$ branched on $D := \sum_i D_i$ (but with different ordering of the indices: we take here one which is more apt for our notation).

We recall that this precisely means the following: let $D_i = \text{div}(\delta_i)$, and let u_i be a fibre coordinate of the geometric line bundle \mathbb{L}_i , whose sheaf of holomorphic sections is $\mathcal{O}_Y(\mathcal{L}_i)$. Then $X \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is given by the equations:

$$\begin{aligned} u_1 u_2 &= \delta_1 u_3, & u_1^2 &= \delta_3 \delta_1; \\ u_2 u_3 &= \delta_2 u_1, & u_2^2 &= \delta_1 \delta_2; \\ u_3 u_1 &= \delta_3 u_2, & u_3^2 &= \delta_2 \delta_3. \end{aligned}$$

From the birational point of view, we are simply adjoining to the function field of \mathbb{P}^2 two square roots, namely $\sqrt{\frac{\Delta_1}{\Delta_3}}$ and $\sqrt{\frac{\Delta_2}{\Delta_3}}$, where Δ_i is the cubic polynomial in $\mathbb{C}[x_0, x_1, x_2]$ whose zero set has D_i as strict transform.

This shows clearly that we have a Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$. The equations above give a biregular model X which is nonsingular exactly if the divisor D does not have points of multiplicity 3 (there cannot be points of higher multiplicities). These points give then quotient singularities of type $\frac{1}{4}(1, 1)$, i.e., the quotient of \mathbb{C}^2 by the action of $(\mathbb{Z}/4\mathbb{Z})$ sending $(u, v) \mapsto (iu, iv)$ (or, equivalently, the affine cone over the 4-th Veronese embedding of \mathbb{P}^1).

This (cf. [Cat08] for more details) can be seen by an elementary calculation. Assume in fact that $\delta_1, \delta_2, \delta_3$ are given in local holomorphic coordinates by $x, y, x - y$, and that we define locally w_i as the square root of δ_i . Then:

$$w_1^2 = x, \quad w_2^2 = y, \quad w_3^2 = x - y \quad \Rightarrow \quad w_3^2 = w_1^2 - w_2^2.$$

Therefore the singularity is an A_1 singularity, quotient of \mathbb{C}^2 by the action of $(\mathbb{Z}/2\mathbb{Z})$ sending $(u, v) \mapsto (-u, -v)$ (here, $w_3 = uv$, $u^2 = w_1 + w_2$, $v^2 = w_1 - w_2$). The action of $(\mathbb{Z}/4\mathbb{Z})$ on \mathbb{C}^2 induces the action of $(\mathbb{Z}/2\mathbb{Z})$ on the A_1 singularity, sending $w_i \mapsto -w_i$, $\forall i$. Finally, the functions $u_i = w_{i+1}w_{i+2}$ and $w_i^2 = \delta_i$ generate the $(\mathbb{Z}/4\mathbb{Z})$ -invariants, subject to the linear relation $\delta_1 - \delta_2 = \delta_3$.

The singularity can be resolved by blowing up the point $x = y = 0$, and then the inverse image of the exceptional line is a smooth rational curve with self intersection -4 .

Definition 2.2. A primary Burniat surface is a surface constructed as above, and which is moreover smooth. It is then a minimal surface S with K_S ample, and with $K_S^2 = 6$, $p_g(S) = q(S) = 0$.

A secondary Burniat surface is a surface constructed as above, and which moreover has $1 \leq m \leq 2$ singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, and with $K_S^2 = 6 - m$, $p_g(S) = q(S) = 0$.

A tertiary Burniat surface is a surface constructed as above, and which moreover has $3 \leq m \leq 4$ singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, and with $K_S^2 = 6 - m$, $p_g(S) = q(S) = 0$.

Remark 2.1. 1) We remark that for $K_S^2 = 4$ there are two possible types of configurations. The one where there are three collinear points of multiplicity at least 3 for the plane curve formed by the 9 lines leads to a Burniat surface S which we call of nodal type, and with K_S not ample, since the inverse image of the line joining the 3 collinear points is a (-2) -curve (a smooth rational curve of self intersection -2).

In the other cases with $K_S^2 = 4, 5$, instead, K_S is ample.

2) In the nodal case, if we blow up the two $(1, 1, 1)$ points of D , we obtain a weak Del Pezzo surface, since it contains a (-2) -curve. Its anticanonical model has a node (an A_1 -singularity, corresponding to the contraction of the (-2) -curve). In the non nodal case, we obtain a smooth Del Pezzo of degree 4.

This fact has obviously been overlooked by [Pet77], since he only mentions the nodal case.

In the sequel to this paper we shall show that in the case of secondary Burniat surfaces with $K_S^2 = 4$ these two families indeed give two different connected components of dimension 2 in the moduli space. And also that secondary Burniat surfaces with $K_S^2 = 5$ form a connected component of dimension 3 in the moduli space.

3) We illustrate the possible configurations in the plane in figure 1.

In [In94] Inoue constructed a series of families of surfaces with $K^2 = 6, 5, 4, 3, 2$ and $p_g = 0$ (of respective dimensions 4, 3, 2, 1, 0, exactly as for the Burniat surfaces) as the $(\mathbb{Z}/2\mathbb{Z})^3$ quotient of an invariant hypersurface of type $(2, 2, 2)$ in a product of three elliptic curves. As already mentioned, it seems to be known to the specialists that these Inoue's surfaces are exactly the Burniat's surfaces, but for lack of a reference we show here:

Theorem 2.3. Burniat's surfaces are exactly Inoue's surfaces.

Proof. Consider as in lemma 2.1 the cartesian diagram

$$\begin{array}{ccc}
 p^{-1}(Y) & \xrightarrow{p} & Y \\
 \downarrow & & \downarrow i \\
 \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1
 \end{array}$$

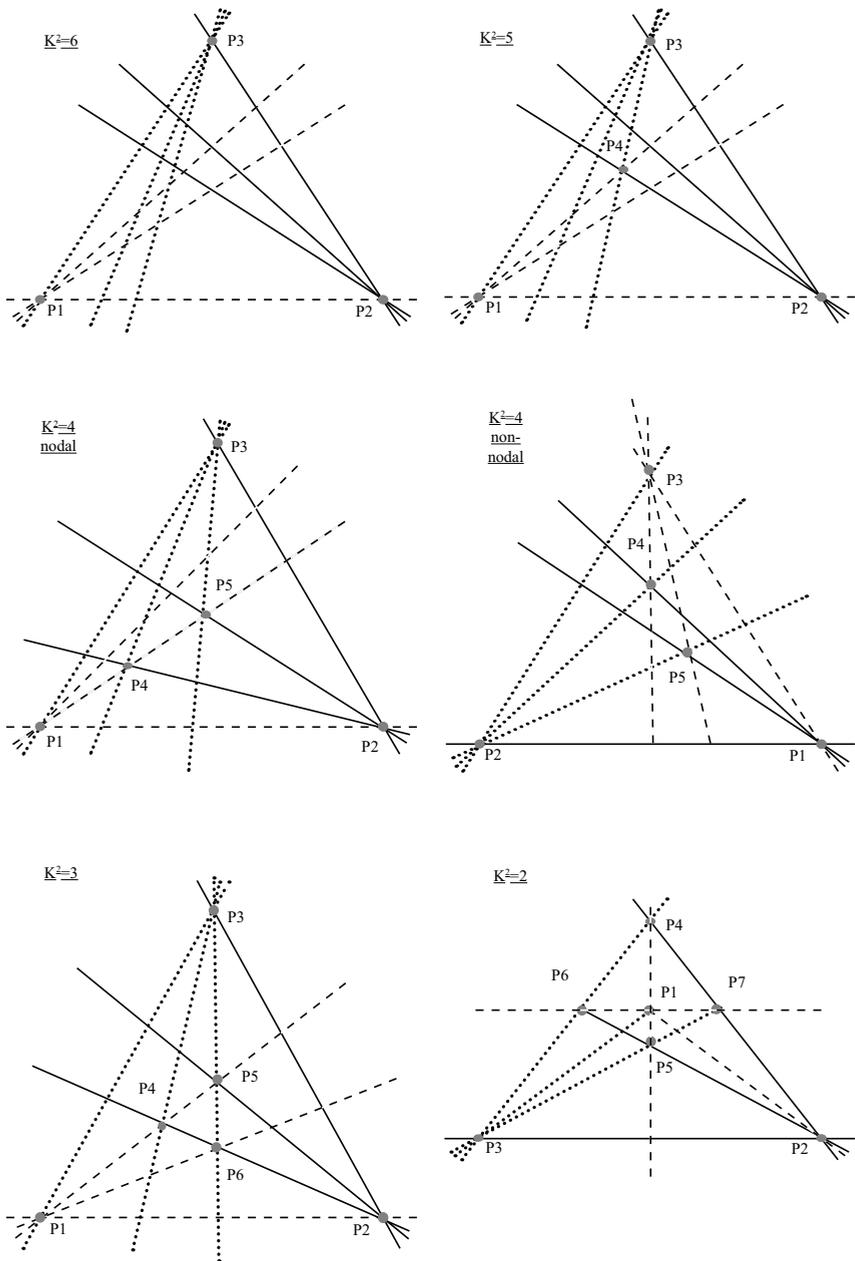


Figure 1. Configurations of lines

where $p : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the $(\mathbb{Z}/2\mathbb{Z})^3$ -Galois covering given by $x_i = v_i^2$, $x'_i = (v'_i)^2$. Then $p^{-1}(Y)$ splits as the union of two degree 6 Del Pezzo surfaces $p^{-1}(Y) = Z \cup Z'$, where

$$Z := \{((v_1 : v'_1), (v_2 : v'_2), (v_3 : v'_3)) : v_1 v_2 v_3 = v'_1 v'_2 v'_3\}$$

and

$$Z' := \{((v_1 : v'_1), (v_2 : v'_2), (v_3 : v'_3)) : v_1 v_2 v_3 = -v'_1 v'_2 v'_3\}.$$

Recall that the subgroup of $(\mathbb{Z}/2\mathbb{Z})^3$ stabilizing Z is $G^o = \{(\epsilon_i) \in \{\pm 1\}^3 \mid \prod_i \epsilon_i = 1\}$.

We can further extend the previous diagram by considering a $(\mathbb{Z}/2\mathbb{Z})^6$ Galois-covering $\hat{p} : \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ obtained by taking, with different choices of the $(x' : x)$ coordinates, the direct product of three $(\mathbb{Z}/2\mathbb{Z})^2$ Galois-coverings $\mathcal{E}_i \rightarrow \mathbb{P}^1$ as in section 1.

What we have now explained is summarized in the bottom two lines of the following commutative diagram, where \hat{X} is defined as the inverse image of the Del Pezzo surface Z .

$$\begin{array}{ccccc} \hat{X} & \xrightarrow{G^2} & X = \hat{X}/G^2 & & \\ \downarrow \hat{i} & & \searrow & & \\ \hat{X} \cup \hat{X}' & \longrightarrow & Z \cup Z' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 & \xrightarrow{(\mathbb{Z}/2\mathbb{Z})^3} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{(\mathbb{Z}/2\mathbb{Z})^3} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

Note that the vertical map $\hat{i} : \hat{X} \cup \hat{X}' \hookrightarrow \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ is the inclusion of $\hat{X} \cup \hat{X}'$ as a divisor of multidegree $(4, 4, 4)$ splitting as a union of two divisors of respective multidegrees $(2, 2, 2)$.

Next we want to show that \hat{X} is a $(\mathbb{Z}/2\mathbb{Z})^3$ Galois covering of X , ramified only in the points of type $\frac{1}{4}(1, 1)$ (and hence étale in the case of a primary Burniat X). In fact, the stabilizer of \hat{X} is

$$G^1 := \{(\epsilon_i, \epsilon'_i) \in \{\pm 1\}^3 \times \{\pm 1\}^3 \mid \prod_i \epsilon_i = 1\} \cong (\mathbb{Z}/2\mathbb{Z})^5.$$

The action of G^1 makes \hat{X} a $(\mathbb{Z}/2\mathbb{Z})^5$ Galois covering of Y , and we claim that we obtain X as an intermediate cover by setting

$$u_i = W_{i-1} W_i v_i v'_i.$$

Let us denote $(D_{i,2} + D_{i,3})$ by D'_i . This is the divisor defined by a section $\delta'_i = 0$ which is the pull back of a homogeneous polynomial of degree 2 on the i th copy of \mathbb{P}^1 (this polynomial is the polynomial $(x_i^2 - x_i x'_i (b_i + \frac{1}{b_i}) + x_i'^2)$ in the notation of section 1). Let us then write

$$D_i = (D_{i,1} + E_{i+2}) + (D_{i,2} + D_{i,3}) = D_{i,1} + E_{i+2} + D'_i.$$

Observe that $\operatorname{div}(x_i) = D_{i,1} + E_{i+1}$, $\operatorname{div}(x'_i) = D_{i-1,1} + E_{i-1}$, whence

$$D_i + D_{i-1} = D'_i + D'_{i-1} + D_{i,1} + E_{i+2} + D_{i-1,1} + E_{i+1} = \operatorname{div}(\delta'_i \delta'_{i-1} x_i x'_i).$$

Now, the $(\mathbb{Z}/2\mathbb{Z})^2$ Galois- covering of the i th copy of \mathbb{P}^1 is given by:

$$(v'_i v_i)^2 = x_i x'_i, \quad W_i^2 = \delta'_i.$$

Since

$$u_i^2 = \delta_i \delta_{i-1},$$

we see that

$$u_i^2 = \delta_i \delta_{i-1} = \delta'_i \delta'_{i-1} x_i x'_i = (W_i W_{i-1} v_i v'_i)^2.$$

Whence we have established our claim that setting

$$u_i = W_{i-1} W_i v_i v'_i,$$

we get a mapping $\hat{X} \rightarrow X$.

We also see that $\hat{X} \rightarrow X$ is Galois with Galois group the subgroup $G^2 < G^1$ leaving each u_i invariant, which, by the above formulae, is given by

$$\{(\epsilon_i, \epsilon'_i) \in G^1 \mid \epsilon'_{i-1} \epsilon'_i \epsilon_i = 1, i \in \{1, 2, 3\}\} \cong (\mathbb{Z}/2\mathbb{Z})^3.$$

The last isomorphism follows since the ϵ'_i 's determine $\epsilon_i = \epsilon'_{i-1} \epsilon'_i$.

A natural basis for $G^2 \leq G^1 \leq (\mathbb{Z}/2\mathbb{Z})^6 \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^3$ is given by

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) =: g_1, \quad \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) =: g_2, \quad \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) =: g_3.$$

Therefore, if z_i is a uniformizing parameter for the elliptic curve \mathcal{E}_i , with $z_i = 0$ corresponding to the origin of \mathcal{E}_i , we see that the action of G^2 on $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ (cf. section 1) is given as follows:

$$g_1 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 + \eta_1 \\ -z_2 \\ z_3 \end{pmatrix}, \quad g_2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 + \eta_2 \\ -z_3 \end{pmatrix}, \quad g_3 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ z_2 \\ z_3 + \eta_3 \end{pmatrix}.$$

Remark 2.2. *If X is a primary Burniat surface, then $\hat{X} \rightarrow X$ is an étale $(\mathbb{Z}/2\mathbb{Z})^3$ -covering.*

Instead, for each $(1, 1, 1)$ -point of $D = D_1 + D_2 + D_3$, X has a singular point of type $\frac{1}{4}(1, 1)$, and $\hat{X} \rightarrow X$ is ramified in exactly these singular points, yielding 4 nodes on \hat{X} for each one of these singular points on X .

Since \hat{X} is a divisor of type $(2, 2, 2)$ invariant by the action of G^2 , we have seen that any Burniat surface X is an Inoue surface.

Conversely, assume that $X = \hat{X}/G^2$ is an Inoue surface: since every such G^2 -invariant surface \hat{X} is the pull back of a Del Pezzo surface

$$Z_c := \{(v'_i, v_i) \mid v_1 v_2 v_3 - c v'_1 v'_2 v'_3 = 0\}, \quad (1)$$

we see that X is a Burniat surface. \square

Remark 2.3. *In the above equation (1) there is a constant c appearing, whereas in the previous description we had normalized this constant to be equal to 1.*

On each \mathbb{P}^1 there are the points $v_i = 0$, $v'_i = 0$, hence these coordinates are determined up to a constant λ_i . In turn, we have two more branch points, forming the locus of zeroes of an equation which we normalized as being $v_i^2 + (b_i + \frac{1}{b_i}) v_i v'_i + v'^2_i = 0$. This normalization now determines the constant λ_i uniquely, and finally with these choice of coordinates we get the equation (1) with $c = \prod_i \lambda_i$, and we see that c is a function of b_1, b_2, b_3 .

3. The fundamental groups of Burniat surfaces

The aim of this section is to combine our and Inoue’s representation of Burniat surfaces in order to calculate the fundamental groups of the Burniat surfaces with $K^2 = 6, 5, 4, 3, 2$.

In [In94] the author gave a table of the respective fundamental groups, but without supplying a proof. As we shall now see, his assertion is right for $K^2 = 6, 5, 4, 3$ but wrong for the case $K^2 = 2$. So we believe it worthwhile to give a detailed proof, especially in order to cast away any doubt on the validity of his assertion for $K^2 = 6, 5, 4, 3$.

Let (\mathcal{E}, o) be any elliptic curve, and consider as in section 1 the $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{0, g_1, g_2, g_3 := g_1 g_2\}$ - action given by

$$g_1(z) := z + \eta, \quad g_2(z) = -z,$$

where $\eta \in \mathcal{E}$ is a 2 - torsion point of \mathcal{E} .

Remark 3.1. *The divisor $[o] + [\eta] \in \text{Div}^2(\mathcal{E})$ is invariant under G , hence the invertible sheaf $\mathcal{O}_{\mathcal{E}}([o] + [\eta])$ carries a natural G -linearization.*

In particular, G acts on the vector space $H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))$ which splits then as a direct sum

$$H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta])) = \bigoplus_{\chi \in G^*} H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))^{\chi}$$

of the eigenspaces corresponding to the characters χ of G .

We shall use a self explanatory notation:

for instance, $H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))^{+-}$ is the eigenspace corresponding to the character χ such that $\chi(g_1) = 1$, $\chi(g_2) = -1$.

We recall the following:

Lemma 3.1 ([BC09a], lemma 2.1). *Let \mathcal{E} be as above. Then*

$$H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta])) = H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))^{++} \oplus H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))^{--}.$$

I.e., $H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))^{+-} = H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}([o] + [\eta]))^{-+} = 0$.

Let now $\mathcal{E}_i := \mathbb{C}/\Lambda_i$, $i = 1, 2, 3$, be three complex elliptic curves, and write $\Lambda_i = \mathbb{Z}e_i \oplus \mathbb{Z}e'_i$. Define now affine transformations $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{A}(3, \mathbb{C})$ as follows:

$$\gamma_1 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 + \frac{e_1}{2} \\ -z_2 \\ z_3 \end{pmatrix}, \quad \gamma_2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 + \frac{e_2}{2} \\ -z_3 \end{pmatrix}, \quad \gamma_3 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ z_2 \\ z_3 + \frac{e_3}{2} \end{pmatrix},$$

and let $\Gamma \leq \mathbb{A}(3, \mathbb{C})$ be the affine group generated by $\gamma_1, \gamma_2, \gamma_3$ and by the translations by the vectors $e_1, e'_1, e_2, e'_2, e_3, e'_3$.

Remark 3.2. Γ contains the lattice $\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$, hence Γ acts on $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ inducing a faithful action of $G^2 := (\mathbb{Z}/2\mathbb{Z})^3$ on $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$.

We prove next the following

Theorem 3.2. *Let S be the minimal model of a Burniat surface.*

- i) $K^2 = 6 \implies \pi_1(S) = \Gamma, H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^6$;
- ii) $K^2 = 5 \implies \pi_1(S) = \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3, H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^5$;
- iii) $K^2 = 4 \implies \pi_1(S) = \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2, H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^4$;
- iv) $K^2 = 3 \implies \pi_1(S) = \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z}), H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^3$;
- v) $K^2 = 2 \implies \pi_1(S) = H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^3$.

Here \mathbb{H} denotes the quaternion group of order 8.

Remark 3.3. *As already said, these results confirm, except for the case $K^2 = 2$, the results of Inoue [In94], stating that for $K^2 \leq 5$, $\pi_1(S) = \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^{K^2-2}$.*

Proof. i) Let S be the minimal model of a Burniat surface with $K_S^2 = 6$. Then, by the previous section 2, $S = X$ has an étale $(\mathbb{Z}/2\mathbb{Z})^3$ Galois covering \hat{X} , which is a hypersurface of multidegree $(2, 2, 2)$ in the product of three elliptic curves $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$. Since \hat{X} is smooth and ample, by Lefschetz's theorem $\pi_1(\hat{X}) = \pi_1(\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3) \cong \mathbb{Z}^6$.

Γ acts on the universal covering of $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \cong \mathbb{C}^3$, and acts freely on the invariant hypersurface $\hat{X} \subset \mathbb{C}^3$, the universal covering of \hat{X} , with quotient $S = X = \hat{X}/\Gamma$. Hence \hat{X} is also the universal covering of $S = X$ and $\pi_1(S) = \Gamma$.

Next we shall prove that $H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^6$. Since $\gamma_i^2 = e_i$, for $i = 1, 2, 3$, it follows that Γ is generated by $g_1, g_2, g_3, e'_1, e'_2, e'_3$. It is clear that

- a) γ_1 commutes with e_1, e'_1, e_3, e'_3 ;
- b) γ_2 commutes with e_2, e'_2, e_1, e'_1 ;
- c) γ_3 commutes with e_2, e'_2, e_3, e'_3 .

Writing $t_{e_i} \in \mathbb{A}(3, \mathbb{C})$ for the translation by the vector e_i , we see that

$$\begin{aligned}\gamma_1 t_{e_2} &= t_{e_2}^{-1} \gamma_1, & \gamma_1 t_{e'_2} &= t_{e'_2}^{-1} \gamma_1; \\ \gamma_2 t_{e_3} &= t_{e_3}^{-1} \gamma_2, & \gamma_2 t_{e'_3} &= t_{e'_3}^{-1} \gamma_2; \\ \gamma_3 t_{e_1} &= t_{e_1}^{-1} \gamma_3, & \gamma_3 t_{e'_1} &= t_{e'_1}^{-1} \gamma_3.\end{aligned}$$

This implies that $2e_1, 2e'_1, 2e_2, 2e'_2, 2e_3, 2e'_3 \in [\Gamma, \Gamma]$. Moreover,

$$\gamma_1 \gamma_2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 + \frac{e_1}{2} \\ -z_2 - \frac{e_2}{2} \\ -z_3 \end{pmatrix} = t_{e_2}^{-1} \gamma_2 \gamma_1,$$

whence $e_2 \in [\Gamma, \Gamma]$. Similarly, we see that (as the respective commutators of γ_1 with γ_3 , γ_2 with γ_3) $e_1, e_3 \in [\Gamma, \Gamma]$.

Therefore $\Gamma' := \Gamma / \langle e_1, e_2, e_3, 2e'_1, 2e'_2, 2e'_3 \rangle$ surjects onto Γ^{ab} . But Γ' is already abelian, since the morphism

$$\Gamma' \rightarrow (\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^3,$$

mapping the residue classes of $\gamma_1, \gamma_2, \gamma_3, e'_1, e'_2, e'_3$ onto the ordered set of coordinate vectors of $(\mathbb{Z}/2\mathbb{Z})^3$ is easily seen to be well defined and an isomorphism. This shows that $H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^6$.

In order to prove the assertions *ii) – v)* observe preliminarily that, if X is the above singular model of S , then, by van Kampen's theorem, $\pi_1(S) \cong \pi_1(X)$. Therefore, for the remaining cases, it suffices to calculate $\pi_1(X)$.

Let X be the above singular model of a Burniat surface with $K^2 \leq 5$. Consider the $G^2 \cong (\mathbb{Z}/2\mathbb{Z})^3$ -Galois cover \hat{X} . Since the singularities of \hat{X} are only nodes, $\pi_1(\hat{X}) \cong \mathbb{Z}^6$ by the theorem of Brieskorn-Tyurina (cf. [Brie68], [Brie71], [Tju70]). By [Arm65], [Arm68] $\pi_1(X) \cong \Gamma / \text{Tors}(\Gamma)$, where $\text{Tors}(\Gamma)$ is the normal subgroup of Γ generated by all elements of Γ having fixed points on the universal covering \tilde{X} of \hat{X} (which is, as we have seen before, a Γ -invariant hypersurface in \mathbb{C}^3).

Note that the elements in $G^2 \cong (\mathbb{Z}/2\mathbb{Z})^3$ induced by the elements

$$\gamma_1, \gamma_2, \gamma_3, \gamma_1 \gamma_2, \gamma_1 \gamma_3, \gamma_2 \gamma_3 \in \Gamma$$

do not have fixed points on $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$. Instead,

$$\gamma_1 \gamma_2 \gamma_3 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 + \frac{e_1}{2} \\ -z_2 - \frac{e_2}{2} \\ -z_3 - \frac{e_3}{2} \end{pmatrix}$$

has as fixed points the 64 points on $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ corresponding to vectors in \mathbb{C}^3 such that

$$2z_i \equiv \frac{e_i}{2} \pmod{\Lambda_i, \forall i.} \quad (2)$$

Equivalently,

$$z_i \equiv \frac{e_i}{4} \pmod{\frac{1}{2}\Lambda_i, \forall i.} \quad (3)$$

ii) Let X be the singular model of a Burniat surface with $K^2 = 5$. Then \hat{X} has 4 nodes (lying over the point $P_4 \in \mathbb{P}^2$, see figure 1).

We observed that if $\gamma \in \Gamma$ has a fixed point on \tilde{X} , then there is a

$$\lambda \in \mathbb{Z}^6 \cong \langle e_1, e_2, e_3, e'_1, e'_2, e'_3 \rangle =: \Lambda$$

such that

$$\gamma = \gamma_1 \gamma_2 \gamma_3 t_\lambda.$$

Let now $z = (z_1, z_2, z_3) \in \tilde{X} \subset \mathbb{C}^3$. Then z yields a fixed point of $\gamma_1 \gamma_2 \gamma_3$ on \hat{X} if and only if there is a $\hat{\lambda} \in \Lambda$ such that

$$2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e_1 \\ -e_2 \\ -e_3 \end{pmatrix} + \hat{\lambda} \iff z = \frac{1}{4}\epsilon + \frac{\hat{\lambda}}{2},$$

where we have set $\epsilon := \begin{pmatrix} e_1 \\ -e_2 \\ -e_3 \end{pmatrix}$.

We show now that z is a fixed point of a γ as above iff $\lambda = -\hat{\lambda}$. In fact:

$$\begin{aligned} \gamma(z) &= \gamma_1 \gamma_2 \gamma_3 t_\lambda(z) = -(z + \lambda) + \frac{1}{2}\epsilon = \\ &= -\frac{1}{4}\epsilon - \frac{\hat{\lambda}}{2} - \lambda + \frac{1}{2}\epsilon = z - \lambda - \hat{\lambda}. \end{aligned}$$

Modifying z modulo Λ , we replace z by $z + \lambda'$, and the corresponding $\hat{\lambda}$ gets replaced by $\hat{\lambda} + 2\lambda'$; hence we see that $\gamma_1 \gamma_2 \gamma_3 t_\lambda$ has a fixed point on \hat{X} for all $\lambda \in -\hat{\lambda} + 2\Lambda$. Therefore 2Λ is contained in $\text{Tors}(\Gamma)$. Since the above arguments apply to all remaining cases (ii) - (v) we summarize what we have seen in the following

Lemma 3.3. *If Γ has a fixed point z on the universal covering of \hat{X} (i.e., we are in one of the cases ii) - v)), then $\pi_1(X)$ is a quotient of $\bar{\Gamma} := \Gamma/2\Lambda$. We have thus an exact sequence*

$$1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^6 \rightarrow \bar{\Gamma} \rightarrow (\mathbb{Z}/2\mathbb{Z})^3 \rightarrow 1.$$

In particular, we already showed that the fundamental group of a Burniat surface with $K^2 \leq 5$ is finite: we are now going to write its structure explicitly.

Remark 3.4. 1) *The images of e_i, e'_j , $i, j \in \{1, 2, 3\}$ in $\bar{\Gamma}$ are contained in the center of $\bar{\Gamma}$, i.e., the above exact sequence yields a central extension.*

2) *Note that over each $(1, 1, 1)$ point of the branch divisor $D \subset \mathbb{P}^2$ there are 4 nodes of \hat{X} , which are a G^2 -orbit of fixed points of $\gamma_1 \gamma_2 \gamma_3$ on \hat{X} . Let $z \in \hat{X}$ induce a fixed point of $\gamma_1 \gamma_2 \gamma_3$ on \hat{X} : then $z = \frac{1}{4}\epsilon + \frac{\hat{\lambda}}{2}$, and the other 3 fixed points in the orbit are exactly the points induced by $\gamma_i(z)$ on \hat{X} , for $i = 1, 2, 3$. We have:*

$$\gamma_1(z) = \gamma_1 \left(\begin{pmatrix} \frac{e_1}{4} + \frac{1}{2}(\hat{\lambda}_z)_1 \\ -\frac{e_2}{4} - \frac{1}{2}(\hat{\lambda}_z)_2 \\ -\frac{e_3}{4} + \frac{1}{2}(\hat{\lambda}_z)_3 \end{pmatrix} \right) = \frac{1}{4}\epsilon + \frac{\hat{\lambda}_{\gamma_1(z)}}{2}.$$

This implies that

$$\hat{\lambda}_{\gamma_1(z)} \equiv \hat{\lambda}_z + \begin{pmatrix} e_1 \\ e_2 \\ 0 \end{pmatrix} \pmod{2\Lambda},$$

and similarly

$$\hat{\lambda}_{\gamma_2(z)} \equiv \hat{\lambda}_z + \begin{pmatrix} 0 \\ e_2 \\ e_3 \end{pmatrix} \pmod{2\Lambda}, \quad \hat{\lambda}_{\gamma_3(z)} \equiv \hat{\lambda}_z + \begin{pmatrix} e_1 \\ 0 \\ e_3 \end{pmatrix} \pmod{2\Lambda}.$$

3) Let X be the singular model of a Burniat surface, and choose w.l.o.g. one of the points in \tilde{X} over the $(1, 1, 1)$ -point P_4 to be $z := \frac{1}{4}\epsilon$. This is equivalent to $\hat{\lambda}_z = 0$. For each $(1, 1, 1)$ point of the branch divisor $D \subset \mathbb{P}^1$ choose one singular point of \tilde{X} lying over it.

Let $\mathcal{S} := \{z(4) = z, \dots, z(9 - K^2)\}$ be a choice of representatives for each G^2 -orbit of points of \tilde{X} lying over the respective $(1, 1, 1)$ -points. Then:

$$\pi_1(X) = \Gamma / \langle \gamma_1 \gamma_2 \gamma_3 t_\lambda : \lambda \in -\hat{\lambda}_z + 2\Lambda, z \in \mathcal{S} \rangle.$$

In particular, we have the relations:

$$\gamma_1 \gamma_2 \gamma_3 = 1,$$

and, by 2) :

$$e_1 = e_2 = e_3.$$

Recall now that $\gamma_i^2 = e_i$. Therefore in $\pi_1(X)$ we have:

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = e_1 + e_2 + e_3.$$

Thus we get an exact sequence (cf. Lemma 3.3):

$$1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z}) \rightarrow \pi_1(X) \rightarrow (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow 1,$$

where the map $\varphi : \pi_1(X) \rightarrow (\mathbb{Z}/2\mathbb{Z})^2$ is given by $\gamma_1 \mapsto (1, 0)$, $\gamma_2 \mapsto (0, 1)$, $e'_i \mapsto 0$. This immediately shows that the kernel of φ is equal to $\langle e'_1, e'_2, e'_3, e_1 + e_2 + e_3 = \gamma_i^2 \rangle$.

Let $\mathbb{H} := \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group, and let $\gamma_1, \gamma_2, \gamma_3$ correspond respectively to $i, j, -k$: then we obtain an isomorphism

$$\pi_1(X) \cong \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3.$$

This proves the assertion on the fundamental group for Burniat surfaces with $K^2 = 5$. That $H_1(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^5$ follows, since $\mathbb{H}^{ab} = (\mathbb{Z}/2\mathbb{Z})^2$.

iii), iv) First observe that, by the above, if X is the singular model of a Burniat surfaces with $K^2 \leq 5$, then $\pi_1(X)$ is the quotient of $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ by the relations $\hat{\lambda}_{z(i)} - \hat{\lambda}_{z(j)} = 0$, where $z(i) \neq z(j) \in \mathcal{S}$.

Note that for $K^2 = 4$ (nodal or non nodal), the projections of $z(4)$ and $z(5)$ to \mathcal{E}_2 , resp. \mathcal{E}_3 , are points whose differences are non trivial 2-torsion elements. Since each of the corresponding $(1, 1, 1)$ points lies on two different lines $D_{2,2}$, $D_{2,3}$, respectively $D_{3,2}$, $D_{3,3}$, the images of $z(4)$ and $z(5)$ under the composition of the projection to \mathcal{E}_2 (resp. to \mathcal{E}_3) with the quotient map $\mathcal{E}_2 \rightarrow \mathbb{P}^1$ (resp. $\mathcal{E}_3 \rightarrow \mathbb{P}^1$) have different x_2 -value (resp. x_3 -value).

Claim. The image of $\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(5)}$ in $\bigoplus_{i=1}^3 e'_i \mathbb{Z}/2\mathbb{Z}$ is non zero.

Proof of the claim. Again, we look at the image of $z(4)$ (resp. $z(5)$) in $\mathcal{E}_2 \rightarrow \mathbb{P}^1$ (with coordinate of \mathbb{P}^1 equal to x_2). We have seen that the corresponding $(1, 1, 1)$ points P_4, P_5 lie on two different lines $D_{2,2}$, $D_{2,3}$, respectively $D_{3,2}$, $D_{3,3}$; hence the respective x_2 coordinates of the projection of $z(4)$ and $z(5)$ to \mathbb{P}^1 are different.

We conclude since the description of the transformations of order 2 of \mathcal{E}_2 given by translation by 2-torsion elements (cf. section 1) shows that translation by $\frac{e_2}{2}$ is the only one which leaves the x_2 coordinate invariant. *QED for the claim.*

Therefore, if X is the singular model of a Burniat surface with $K^2 = 4$, $\pi_1(X)$ is the quotient of $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ by an element having a non trivial component in $(\mathbb{Z}/2\mathbb{Z})^3$, hence $\pi_1(X) \cong \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

Assume now that X is the singular model of a Burniat surface with $K^2 = 3$. Here the branch divisor on \mathbb{P}^2 has three $(1, 1, 1)$ -points. Repeating the above argument, we see that $\pi_1(X)$ is the quotient of $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ by $\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(5)}$ and $\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(6)}$.

As above, we look at the image in $\bigoplus_{i=1}^3 e'_i \mathbb{Z}/2\mathbb{Z}$ and see that they give (up to a permutation of indices) the elements $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. This implies that $\pi_1(X)$ is the quotient of $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ by two linear independent relations in $(\mathbb{Z}/2\mathbb{Z})^3$. Therefore $\pi_1(X) = \mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})$.

v) Let X be the singular model of a Burniat surfaces with $K^2 = 2$.

Remark 3.5. *Observe that, by [Rei], [Rei79], [Miy77], $|\pi_1(X)| \leq 9$. Since $\pi_1(X)$ is a quotient of $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})$ by a relation coming from an element of Λ there are only two possibilities: either $\pi_1(X) = \mathbb{H}$ or $\pi_1(X) = (\mathbb{Z}/2\mathbb{Z})^3$.*

We are going to show that the second alternative holds. Here the branch divisor D on \mathbb{P}^2 has four $(1, 1, 1)$ -points P_4, P_5, P_6, P_7 . As above, $\pi_1(X)$ is the quotient of $\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ by the relations:

$$\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(5)} = 0, \hat{\lambda}_{z(4)} - \hat{\lambda}_{z(6)} = 0, \hat{\lambda}_{z(4)} - \hat{\lambda}_{z(7)} = 0,$$

where $z(j) \in \hat{X}$ is a point lying over P_j .

Looking at figure 1 of the configuration of lines in \mathbb{P}^2 for the Burniat surface with $K^2 = 2$, we see that

$$P_4, P_5 \in D_{1,2}, P_4, P_6 \in D_{2,2}, P_4, P_7 \in D_{3,2},$$

i.e., P_4, P_5 lie in the same dashed line, but in two different dotted and two different black lines, P_4, P_6 lie in the same dotted line, but in two different dashed and two different black lines, and P_4, P_7 lie in the same black line, but in two different dashed and two different dotted lines.

This means that if we look at the image of $\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(5)}, \hat{\lambda}_{z(4)} - \hat{\lambda}_{z(6)}, \hat{\lambda}_{z(4)} - \hat{\lambda}_{z(7)}$ in $\oplus_{i=1}^3 e'_i \mathbb{Z} / 2\mathbb{Z}$ we see that they give (up to a permutation of indices) the elements

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

These three vectors are linearly dependent, hence, taking the quotient by these relations, the rank of $\oplus_{i=1}^3 e'_i \mathbb{Z} / 2\mathbb{Z}$ drops only by two.

In order to determine the component of the image of $\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(5)}, \hat{\lambda}_{z(4)} - \hat{\lambda}_{z(6)}, \hat{\lambda}_{z(4)} - \hat{\lambda}_{z(7)}$ in the center of the quaternion group, we have to write the points $z(j), j \in \{4, 5, 6, 7\}$ more explicitly, using section 1.

Observe that in the case $K^2 = 2$, we have $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 =: \mathcal{E}$. The fixed points of $\gamma_1 \gamma_2 \gamma_3$ are given by $w_i = 0, i = 1, 2, 3$. Setting $x'_i = 1$, and $a := \sqrt{b}$, we can assume w.l.o.g. that

$$z(4) = \begin{pmatrix} (1 : b), (1 : a), (1 : 0) \\ (1 : b), (1 : a), (1 : 0) \\ (1 : b), (1 : a), (1 : 0) \end{pmatrix}.$$

By equation (1), we have $v_1 v_2 v_3 = c v'_1 v'_2 v'_3$, whence $c = a^3$. W.l.o.g., by (2) of remark 3.4, we can assume that $z(5)$ is given by

$$z(5) = \begin{pmatrix} (1 : b), (1 : \zeta), (1 : 0) \\ (b : 1), (a : 1), (1 : 0) \\ (b : 1), (a : 1), (1 : 0) \end{pmatrix} = \begin{pmatrix} (1 : b), (1 : \zeta), (1 : 0) \\ f_2((1 : b), (1 : a), (1 : 0)) \\ f_2((1 : b), (1 : a), (1 : 0)) \end{pmatrix}.$$

We have now to determine ζ in such a way that $v_1 v_2 v_3 = a^3 v'_1 v'_2 v'_3$.

For $z(5)$ we have

$$v_1 v_2 v_3 = \zeta = c v'_1 v'_2 v'_3 = a^3 a^2 = a^5. \tag{4}$$

Since by section 1 the only two translations of order 2 leaving $(x' : x)$ unchanged are the identity and g_1 , we have

$$((1 : b), (1 : \zeta), (1 : 0)) = ((1 : b), (1 : a), (1 : 0))$$

or

$$((1 : b), (1 : \zeta), (1 : 0)) = g_1((1 : b), (1 : a), (1 : 0)) = ((1 : b), (1 : -a), (1 : 0)).$$

Together with equation (4) we get: $a^5 = \zeta = \pm a$, i.e., $a^4 = \pm 1$. $a^4 = 1$ is not possible, because this would imply that $b = a^2 = \pm 1$, a contradiction to $b \neq \frac{1}{b}$.

Hence we have that $a^4 = -1$, i.e., $\zeta = a^5 = -a$, and we see that the relation $\hat{\lambda}_{z(4)} - \hat{\lambda}_{z(5)} = 0$ is given by

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, 1 \right\rangle \in \bigoplus_{i=1}^3 e'_i(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z},$$

where the last summand is the center of the quaternion group.

Using for $z(6)$ and $z(7)$ the same argument as for $z(5)$, we get two further elements which have to be set equal to zero in the quotient:

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, 1 \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, 1 \right\rangle \in \bigoplus_{i=1}^3 e'_i(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

Taking the sum of these three elements in $\bigoplus_{i=1}^3 e'_i(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$ we see that we get

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, 1 \right\rangle = 0,$$

and we have concluded the proof of the theorem. \square

Remark 3.6. *We shall show in the last section how Burniat surfaces with $K^2 = 2$ are classical Campedelli surfaces, i.e., obtained as the tautological $(\mathbb{Z}/2\mathbb{Z})^3$ Galois covering of \mathbb{P}^2 branched on seven lines.*

4. The moduli space of primary Burniat surfaces

In this section we finally devote ourselves to the main result of the paper. First of all, we show

Theorem 4.1. *The subset of the Gieseker moduli space corresponding to primary Burniat surfaces is an irreducible connected component, normal, rational and of dimension equal to 4.*

This result was already proven in [MLP01] using the fact that the bicanonical map of the canonical model X' of a Burniat surface is exactly the bidouble covering $X' \rightarrow Y'$ onto the normal Del Pezzo surface Y' of degree $K_{Y'}^2$, obtained as the anticanonical model of the weak Del Pezzo surface obtained blowing up not only the points P_1, P_2, P_3 , but also all the other triple points of D .

We shall now give an alternative proof of their theorem.

Proof. The singular model X of a primary Burniat surface is smooth, and has ample canonical divisor. Hence it equals the minimal model S (and the canonical model X').

Since $\hat{X} \rightarrow X$ is étale with group G^2 , it suffices to show that the Kuranishi family of \hat{X} is smooth. Then it will also follow that the Kuranishi family of X is smooth, whence the Gieseker moduli space is normal (being locally analytically isomorphic to the quotient of the base of the Kuranishi family by the finite group $\text{Aut}(X)$).

Since $\hat{X} \subset \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ is a smooth hypersurface, setting for convenience $T := \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$, we have the tangent bundle sequence

$$0 \rightarrow \Theta_{\hat{X}} \rightarrow \Theta_T \otimes \mathcal{O}_{\hat{X}} \cong \mathcal{O}_{\hat{X}}^3 \rightarrow \mathcal{O}_{\hat{X}}(\hat{X}) \rightarrow 0$$

with exact cohomology sequence

$$\begin{aligned} 0 \rightarrow \mathbb{C}^3 \rightarrow H^0(\mathcal{O}_{\hat{X}}(\hat{X})) \cong \mathbb{C}^{10} \rightarrow \\ \rightarrow H^1(\Theta_{\hat{X}}) \rightarrow H^1(\Theta_T \otimes \mathcal{O}_{\hat{X}}) \cong \mathbb{C}^9 \rightarrow H^1(\mathcal{O}_{\hat{X}}(\hat{X})) \cong \mathbb{C}^3. \end{aligned}$$

Since \hat{X} moves in a smooth family of dimension $13 = 6 + 7$, a fibre bundle over the family of deformations of the principally polarized Abelian variety T , with fibre the linear system $\mathbb{P}(H^0(T, \mathcal{O}_T(\hat{X})))$, it suffices to show that the map $H^1(\Theta_T \otimes \mathcal{O}_{\hat{X}}) \rightarrow H^1(\mathcal{O}_{\hat{X}}(\hat{X}))$ is surjective.

It suffices to observe that $H^1(\Theta_T \otimes \mathcal{O}_{\hat{X}}) \cong H^1(\Theta_T)$, $H^1(\mathcal{O}_{\hat{X}}(\hat{X})) \cong H^2(\mathcal{O}_T)$, and that, as well known, the above map corresponds via these isomorphisms to the contraction with the first Chern class of \hat{X} , an element of $H^1(\Omega_T^1)$ which represents a non degenerate alternating form. Whence surjectivity follows.

Thus the base of the Kuranishi family of \hat{X} is smooth (moreover the Kodaira-Spencer map of the above family is a bijection, but we omit the verification here), whence the base of the Kuranishi family of X , which is the G^2 -invariant part of the base of the Kuranishi family of \hat{X} , is also smooth.

Moreover the Kuranishi family of X fibres onto the family of G^2 -invariant deformations of T , which coincides with the deformations of the three individual elliptic curves.

The fibres of the corresponding morphism between the bases of the respective families are given by the G^2 -invariant part of the linear system $|\hat{X}|$, which we are going to calculate explicitly as being isomorphic to \mathbb{P}^1 .

We obtain thereby a rational family of dimension 4 parametrizing the primary Burniat surfaces. This proves the unirationality of the 4 dimensional irreducible component.

That the irreducible component of the moduli space is in fact a connected component follows from the more general result below (theorem 4.2).

We calculate now

$$H^0(\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3, p_1^* \mathcal{O}_{\mathcal{E}_1}([o_1] + [\frac{e_1}{2}]) \otimes p_2^* \mathcal{O}_{\mathcal{E}_2}([o_2] + [\frac{e_2}{2}]) \otimes p_3^* \mathcal{O}_{\mathcal{E}_3}([o_3] + [\frac{e_3}{2}]))^{G^2},$$

where $p_i : \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \rightarrow \mathcal{E}_i$ is the i -th projection.

From lemma 3.1 it follows that

$$\begin{aligned} H_1 &:= H^0(\mathcal{E}_1, \mathcal{O}_{\mathcal{E}_1}([o_1] + [\frac{e_1}{2}])) = H_1^{+++} \oplus H_1^{-+-} = \\ &= H^0(\mathcal{E}_1, \mathcal{O}_{\mathcal{E}_1}([o_1] + [\frac{e_1}{2}]))^{+++} \oplus H^0(\mathcal{E}_1, \mathcal{O}_{\mathcal{E}_1}([o_1] + [\frac{e_1}{2}]))^{-+-}, \end{aligned}$$

$$\begin{aligned} H_2 &:= H^0(\mathcal{E}_2, \mathcal{O}_{\mathcal{E}_2}([o_2] + [\frac{e_2}{2}])) = H_2^{+++} \oplus H_2^{-+-} = \\ &= H^0(\mathcal{E}_2, \mathcal{O}_{\mathcal{E}_2}([o_2] + [\frac{e_2}{2}]))^{+++} \oplus H^0(\mathcal{E}_2, \mathcal{O}_{\mathcal{E}_2}([o_2] + [\frac{e_2}{2}]))^{-+-}, \end{aligned}$$

$$\begin{aligned} H_3 &:= H^0(\mathcal{E}_3, \mathcal{O}_{\mathcal{E}_3}([o_3] + [\frac{e_3}{2}])) = H_3^{+++} \oplus H_3^{+--} \\ &= H^0(\mathcal{E}_3, \mathcal{O}_{\mathcal{E}_3}([o_3] + [\frac{e_3}{2}]))^{+++} \oplus H^0(\mathcal{E}_3, \mathcal{O}_{\mathcal{E}_3}([o_3] + [\frac{e_3}{2}]))^{+--}. \end{aligned}$$

As a consequence of this, we get

$$\begin{aligned} H^0(\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3, p_1^* \mathcal{O}_{\mathcal{E}_1}([o_1] + [\frac{e_1}{2}]) \otimes p_2^* \mathcal{O}_{\mathcal{E}_2}([o_2] + [\frac{e_2}{2}]) \otimes p_3^* \mathcal{O}_{\mathcal{E}_3}([o_3] + [\frac{e_3}{2}]))^{G^2} = \\ (H_1^{+++} \otimes H_2^{+++} \otimes H_3^{+++}) \oplus (H_1^{-+-} \otimes H_2^{-+-} \otimes H_3^{+--}) \cong \mathbb{C}^2. \end{aligned}$$

□

We have obtained a 4 - dimensional rational family parametrizing all the primary Burniat surfaces. This can also be seen in a more direct fashion by the fact that, fixing 4 points in \mathbb{P}^2 in general position, we can fix the 3 lines $D_{i,1}, i = 1, 2, 3$ and 2 lines $D_{1,2}, D_{2,2}$. Then the other 4 lines vary each in a pencil, hence we get 4 moduli.

In the remaining part of this section, we will prove the following result:

Theorem 4.2. *Let S be a smooth complex projective surface which is homotopically equivalent to a primary Burniat surface. Then S is a Burniat surface.*

Proof. Let S be a smooth complex projective surface with $\pi_1(S) = \Gamma$.

Recall that $\gamma_i^2 = e_i$ for $i = 1, 2, 3$. Therefore $\Gamma = \langle \gamma_1, e'_1, \gamma_2, e'_2, \gamma_3, e'_3 \rangle$ and we have the exact sequence

$$1 \rightarrow \mathbb{Z}^6 \cong \langle e_1, e'_1, e_2, e'_2, e_3, e'_3 \rangle \rightarrow \Gamma \rightarrow (\mathbb{Z}/2\mathbb{Z})^3 \rightarrow 1,$$

where $e_i \mapsto \gamma_i^2$. If we set $\Lambda_i := \mathbb{Z}e_i \oplus \mathbb{Z}e'_i$, $i = 1, 2, 3$ then

$$\pi_1(\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3) = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3.$$

We also have the three lattices $\Lambda'_i := \mathbb{Z}\frac{e_i}{2} \oplus \mathbb{Z}e'_i$.

Remark 4.1. 1) Γ is a group of affine transformations on $\Lambda'_1 \oplus \Lambda'_2 \oplus \Lambda'_3$.
 2) We have an étale double cover $\mathcal{E}_i = \mathbb{C}/\Lambda_i \rightarrow \mathcal{E}'_i := \mathbb{C}/\Lambda'_i$, which is the quotient by the semiperiod $\frac{e_i}{2}$ of \mathcal{E}_i .

Γ has the following three subgroups of index two:

$$\Gamma_3 := \langle \gamma_1, e'_1, e_2, e'_2, \gamma_3, e'_3 \rangle,$$

$$\Gamma_1 := \langle \gamma_1, e'_1, \gamma_2, e'_2, e_3, e'_3 \rangle,$$

$$\Gamma_2 := \langle e_1, e'_1, \gamma_2, e'_2, \gamma_3, e'_3 \rangle,$$

corresponding to three étale double covers of S : $S_i \rightarrow S$, for $i = 1, 2, 3$.

Lemma 4.3. *The Albanese variety of S_i is \mathcal{E}'_i . In particular, $q(S_1) = q(S_2) = q(S_3) = 1$.*

Proof. Observe once more that

- i) γ_1 commutes with e_1, e'_1, e_3, e'_3 ;
- ii) γ_2 commutes with e_2, e'_2, e_1, e'_1 ;
- iii) γ_3 commutes with e_2, e'_2, e_3, e'_3 .

Denoting by $t_{e_i} \in \mathbb{A}(3, \mathbb{C})$ the translation with vector e_i , we see that

$$\gamma_1 t_{e_2} = t_{e_2}^{-1} \gamma_1 \quad \gamma_1 t_{e'_2} = t_{e'_2}^{-1} \gamma_1;$$

$$\gamma_3 t_{e_1} = t_{e_1}^{-1} \gamma_3 \quad \gamma_3 t_{e'_1} = t_{e'_1}^{-1} \gamma_3.$$

This implies that $2e_2, 2e'_2, 2e_1, 2e'_1 \in [\Gamma_3, \Gamma_3]$.

Moreover, $\gamma_1 \gamma_3 = t_{e_1}^{-1} \gamma_3 \gamma_1$, whence already $e_1 \in [\Gamma_3, \Gamma_3]$. Therefore we have a surjective homomorphism

$$\Gamma'_3 := \Gamma_3 / \langle 2e_2, 2e'_2, e_1, 2e'_1 \rangle = \Gamma_3 / \langle 2\mathbb{Z}^3 \oplus \mathbb{Z} \rangle \rightarrow \Gamma_3^{ab} = \Gamma_3 / [\Gamma_3, \Gamma_3].$$

Since the images of γ_3 and e'_3 are in the centre of Γ'_3 , we get that Γ'_3 is abelian, hence $H_1(S_3, \mathbb{Z}) = \Gamma_3^{ab} = \Gamma'_3$ and

$$\Gamma'_3 = \langle \gamma_3, e'_3 \rangle \oplus (\mathbb{Z}/2\mathbb{Z})^4 = \mathbb{Z} \frac{e_3}{2} \oplus \mathbb{Z} e'_3 \oplus (\mathbb{Z}/2\mathbb{Z})^4 = \Lambda_3 \oplus (\mathbb{Z}/2\mathbb{Z})^4.$$

This implies that $\text{Alb}(S_3) = \mathbb{C}/\Lambda'_3 = \mathcal{E}'_3$.

The same calculation shows that $\Gamma_i^{ab} = H_1(S_i, \mathbb{Z}) = \Lambda'_i \oplus (\mathbb{Z}/2\mathbb{Z})^4$, whence $\text{Alb}(S_i) = \mathbb{C}/\Lambda'_i = \mathcal{E}'_i$, also for $i = 2, 3$. \square

Let now $\hat{S} \rightarrow S$ be the étale $(\mathbb{Z}/2\mathbb{Z})^3$ -covering associated to the subgroup $\mathbb{Z}^6 \cong \langle e_1, e'_1, e_2, e'_2, e_3, e'_3 \rangle \triangleleft \Gamma$. Since $\hat{S} \rightarrow S_i \rightarrow S$, and S_i maps to \mathcal{E}'_i (via the Albanese map), we get a morphism

$$f : \hat{S} \rightarrow \mathcal{E}'_1 \times \mathcal{E}'_2 \times \mathcal{E}'_3 = \mathbb{C}/\Lambda'_1 \times \mathbb{C}/\Lambda'_2 \times \mathbb{C}/\Lambda'_3.$$

Since the covering of $\mathcal{E}'_1 \times \mathcal{E}'_2 \times \mathcal{E}'_3$ associated to $\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 < \Lambda'_1 \oplus \Lambda'_2 \oplus \Lambda'_3$ is $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$, we see that f factors through $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ and the Albanese map of \hat{S} is $\hat{\alpha} : \hat{S} \rightarrow \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$.

Let $Y := \hat{\alpha}(\hat{S}) \subset T = \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ be the Albanese image of \hat{S} .

We consider for $i \neq j \in \{1, 2, 3\}$, the natural projections

$$\pi_{ij} : \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \rightarrow \mathcal{E}_i \times \mathcal{E}_j.$$

Claim 4.4. *If S is homotopically equivalent to a primary Burniat surface, then for $i \neq j \in \{1, 2, 3\}$ we have $\deg(\pi_{ij} \circ \hat{\alpha}) = 2$.*

Moreover the cohomology class of the image Y equals $2F_1 + 2F_2 + 2F_3$, where F_i is the pull back of a point in the elliptic curve \mathcal{E}_i .

Proof. The degree of $\pi_{ij} \circ \hat{\alpha}$ is the index of the image of $H^4(\mathcal{E}_i \times \mathcal{E}_j, \mathbb{Z})$ inside $H^4(\hat{S}, \mathbb{Z})$. But the former equals $\wedge^4(\Lambda_i \oplus \Lambda_j)$, hence we see that this number is an invariant of the cohomology algebra of \hat{S} .

There remains to show that, identifying the first cohomology group $H^1(\hat{S}, \mathbb{Z})$ to the one which we obtain for a homotopically equivalent primary Burniat surface, the class of Y remains the same.

It suffices to show that, for each class $\gamma \in H^4(T, \mathbb{Z}) = \wedge^4(H^1(T, \mathbb{Z}))$ the intersection product $\gamma \cdot Y$ is the same. But this number is the multiple of the fundamental class of \hat{S} yielding $\hat{\alpha}^*(\gamma)$, and, identifying $H^1(T, \mathbb{Z})$ with $H^1(\hat{S}, \mathbb{Z})$, it is an invariant of the cohomology algebra $H^*(\hat{S}, \mathbb{Z})$, which is isomorphic to the one we obtain for a homotopically equivalent primary Burniat surface. \square

The above claim implies that $\hat{S} \rightarrow Y$ is a birational morphism and that $Y \subset Z$ has cohomology class $2F_1 + 2F_2 + 2F_3$. Thus $K_Y = \mathcal{O}_Y(Y)$, and $K_Y^2 = (2F_1 + 2F_2 + 2F_3)^3 = 48$. On the other hand, since S is homotopically equivalent to a primary Burniat surface, we have that $K_S^2 = 6$, whence $K_S^2 = 6 \cdot 2^3 = 48$.

Moreover, we have

$$p_g(\hat{S}) = q(\hat{S}) + \chi(\hat{S}) - 1 = 3 + 8\chi(S) - 1 = 10.$$

The short exact sequence

$$0 \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_T(Y) \rightarrow \omega_Y \rightarrow 0,$$

induces a long exact cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(T, \mathcal{O}_T) \rightarrow H^0(T, \mathcal{O}_T(Y)) \rightarrow H^0(Y, \omega_Y) \rightarrow \\ \rightarrow H^1(T, \mathcal{O}_T) \rightarrow H^1(T, \mathcal{O}_T(Y)) = 0, \end{aligned}$$

where the last equality holds since Y is an ample divisor on T .

Moreover $H^0(T, \mathcal{O}_T) \cong \mathbb{C}$, and $H^1(T, \mathcal{O}_T) \cong \mathbb{C}^3$, and therefore

$$p_g(Y) = h^0(Y, \omega_Y) = 10 = p_g(\hat{S}).$$

Since $|\omega_Y|$ is base point free, and it has the same dimension as $|\omega_{\hat{S}}|$, this implies that Y has at most rational double points as singularities. This concludes the proof that S is a primary Burniat surface. \square

4.1. The rationality of the moduli space of primary Burniat surfaces.

This subsection is devoted to the proof of the following:

Theorem 4.5. *Let \mathcal{C} be the connected component of the moduli space constituted by the primary Burniat surfaces ($K_S^2 = 6$). Then \mathcal{C} is a rational 4-fold.*

Proof. The proof goes along similar lines as the one of the rationality of the moduli space of secondary Burniat surfaces with $K^2 = 5$, explained in [BC09b].

We have to divide a parameter space $\cong \mathbb{C}^6$, parametrizing three pairs of lines of equations $x_{i+2} = a_i x_{i+1}, x_{i+2} = b_i x_{i+1}$ by the action of $(\mathbb{C}^*)^2$, of \mathfrak{S}_3 , of the $(\mathbb{Z}/2\mathbb{Z})^3$ generated by the transformations τ_i such that τ_i exchanges a_i with b_i , and finally of the Cremona transformation (mapping a_i to a_i^{-1} , b_i to b_i^{-1}).

Now, we can replace the action of \mathfrak{S}_3 by the direct sum of two copies of the standard permutation representation (of the a_i 's and of the b_i 's).

Moreover, we have the action of the subgroup $(\mathbb{C}^*)^2 \subset \mathbb{P}GL(3, \mathbb{C})$ of diagonal matrices $(\mathbb{C}^*)^2 := \{\text{diag}(t_1, t_2, t_3) | t_i \in \mathbb{C}^*\}$:

$$a_i \mapsto a_i \frac{t_{i+1}}{t_{i+2}}, \quad b_i \mapsto b_i \frac{t_{i+1}}{t_{i+2}}, \quad i \in \{1, 2, 3\}.$$

We set: $\lambda_i := \frac{t_{i+1}}{t_{i+2}}$, thus $\prod_i \lambda_i = 1$ and our $(\mathbb{C}^*)^2$ is the subgroup of $(\mathbb{C}^*)^3$,

$$(\mathbb{C}^*)^2 = \{(\lambda_1, \lambda_2, \lambda_3) | \prod_i \lambda_i = 1\}.$$

The invariants for the $(\mathbb{Z}/2\mathbb{Z})^3$ -action are: $u_i := a_i b_i$, $v_i := a_i + b_i$ and $(\mathbb{C}^*)^3$ acts by

$$u_i \mapsto \lambda_i^2 u_i, \quad v_i \mapsto \lambda_i v_i.$$

Claim 4.6. *The invariants for the $(\mathbb{C}^*)^2$ -action are*

$$w_i := \frac{u_i}{v_i^2}, \quad i = 1, 2, 3; \quad v := \prod_{i=1}^3 v_i.$$

Proof of the claim. Clearly the field of $(\mathbb{C}^*)^3$ -invariants is generated by the w_i 's, and we can replace the generators u_i, v_i ($i = 1, 2, 3$) by the generators w_i, v_i ($i = 1, 2, 3$). Now the exact sequence of algebraic groups

$$1 \rightarrow (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^3 \rightarrow \mathbb{C}^* \rightarrow 1$$

where $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda := \prod_i \lambda_i$, shows that the projection $(\mathbb{C}^*)^3 \rightarrow \mathbb{C}^*$ is the quotient map by the $(\mathbb{C}^*)^2$ action. Since $\mathbb{C}(v_1, v_2, v_3)$ is the function field of $(\mathbb{C}^*)^3$, the field of invariants for $(\mathbb{C}^*)^2$ acting on $\mathbb{C}(v_1, v_2, v_3)$ is $\mathbb{C}(v)$. \square

Note that \mathfrak{S}_3 acts on $\{w_1, w_2, w_3\}$ by the permutation representation, whereas the Cremona transformation acts by $w_i \mapsto w_i, v \mapsto \prod \frac{v_i}{u_i} =: \frac{v}{u}$. In fact, the Cremona transformation sends u_i to u_i^{-1} and v_i to $\frac{1}{a_i} + \frac{1}{b_i} = \frac{a_i + b_i}{a_i b_i} = \frac{v_i}{u_i}$. Since

$\frac{u}{v^2} = \prod w_i$ it follows that $u = \prod w_i v^2$, thus $v \mapsto (\prod w_i)^{-1} v^{-1}$. The invariants for the Cremona transformations are therefore

$$w_1, w_2, w_3, v + \frac{1}{v \prod w_i}$$

where the last element is \mathfrak{S}_3 -invariant.

Finally, the invariants for the action of \mathfrak{S}_3 are: the elementary symmetric functions $\sigma_1(w_1, w_2, w_3)$, $\sigma_2(w_1, w_2, w_3)$, $\sigma_3(w_1, w_2, w_3)$, and $v + \frac{1}{v \prod w_i}$. Thus the field of invariants is rational. \square

5. The Burniat surface with $K^2 = 2$ is a classical Campedelli surface

The aim of this short last section is to illustrate how the Burniat surface with $K^2 = 2$ can be seen as a classical Campedelli surface (with fundamental group $(\mathbb{Z}/2\mathbb{Z})^3$).

A classical Campedelli surface can be described as the tautological $(\mathbb{Z}/2\mathbb{Z})^3$ Galois-covering of \mathbb{P}^2 branched in seven lines. This means that each line $\{l_\alpha = 0\}$ is set to correspond to a non zero element of the Galois group $(\mathbb{Z}/2\mathbb{Z})^3$, and then, for each character $\chi \in (\mathbb{Z}/2\mathbb{Z})^{3*}$, we consider the covering given (cf. [Par91]) by

$$w_\chi w_{\chi'} = \prod_{\chi(\nu)=\chi'(\nu)=1} l_\nu w_{\chi+\chi'}$$

in the vector bundle whose sheaf of sections is $\bigoplus_{\chi \in (\mathbb{Z}/2\mathbb{Z})^{3*}} \mathcal{O}_{\mathbb{P}^2}(1)$.

As we have seen before, the singular model X of a Burniat surface S with $K_S^2 = 2$ (i.e., $K_X^2 = 6$) is the $(\mathbb{Z}/2\mathbb{Z})^2$ Galois covering branched in 9 lines having 4 points of type $(1, 1, 1)$, whereas the minimal model S of a Burniat surface with $K_S^2 = 2$ is the smooth bidouble cover of a weak Del Pezzo surface Y'' of degree 2. Note that the strict transforms of the lines of $D \subset \mathbb{P}^2$ passing through 2 of the points P_4, P_5, P_6, P_7 yield rational (-2) -curves on Y'' . There are six of them on Y'' , namely $D_{i,j}$ for $1 \leq i \leq 3, j \in \{2, 3\}$.

Contracting these six (-2) curves, we obtain a normal Del Pezzo surface Y' of degree 2 having six nodes, and with $-K_{Y'}$ ample. Then the anticanonical map $\varphi := \varphi_{|-K_{Y'}|} : Y' \rightarrow \mathbb{P}^2$ is a finite double cover branched on a quartic curve, which has 6 nodes (since Y' has six nodes). But a plane quartic having 6 nodes has to be the union of four lines L_1, L_2, L_3, L_4 in general position.

Let $S \rightarrow Y''$ be the bidouble cover branched in the Burniat configuration yielding a minimal model of the Burniat surface with $K_S^2 = 2$. Then the preimages of the (-2) -curves $D_{i,j}$ for $1 \leq i \leq 3, j \in \{2, 3\}$ on Y are rational (-2) curves of S .

Let now X' be the canonical model of S and consider the composition of the bidouble cover $\psi : X' \rightarrow Y'$ with φ . Since ψ branches on the image Δ of $D_{1,1} + D_{2,1} + D_{3,1}$ in Y' (the other 6 lines being contracted), we see that the branch divisor of $\varphi \circ \psi$ consists of $Q := L_1 + L_2 + L_3 + L_4$ and the image of Δ in \mathbb{P}^2 .

Looking at the configuration of the lines (cf. figure 1), we see that $D_{1,1}$ intersects $D_{2,2}$ and $D_{2,3}$, that $D_{2,1}$ intersects $D_{3,2}$ and $D_{3,3}$, and that $D_{3,1}$ intersects $D_{1,2}$ and $D_{1,3}$. Hence the image of $D_{i,1}$ under φ has to intersect two nodes of the plane quartic $Q := L_1 + L_2 + L_3 + L_4$, which implies that, denoting the image of $D_{i,1}$ under φ by L'_i , the branch divisor of the $(\mathbb{Z}/2\mathbb{Z})^3$ Galois-covering $\varphi \circ \psi : X \rightarrow \mathbb{P}^2$ is a configuration of seven lines $L_1 + L_2 + L_3 + L_4 + L'_1 + L'_2 + L'_3$, where L_1, L_2, L_3, L_4 are four lines in general position, i.e., form a complete quadrilateral, and L'_1, L'_2, L'_3 are the three diagonals.

The covering $\varphi \circ \psi : X \rightarrow \mathbb{P}^2$ is a Galois covering with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$. In fact we already have as covering transformations the elements of the Galois group $G'' := (\mathbb{Z}/2\mathbb{Z})^2$ of ψ . Moreover the involution $i : Y' \rightarrow Y'$ can be lifted to X' since i leaves the individual branch curves invariant (as they are inverse image of the diagonals of the quadrilateral), and also the line bundles associated to the covering of Y'' (Y'' is simply connected, whence division by 2 is unique in $\text{Pic}(Y'')$).

To show that the covering is the tautological one it suffices to verify that for each non trivial element of the Galois group its fixed divisor is exactly the inverse image of one of the 7 lines in \mathbb{P}^2 .

We omit further details since they are contained in the article [Ku04] by Kulikov. The idea there is simply to take the tautological cover and observe that it factors as a bidouble cover of Y' branched on the inverse image of the diagonals, each splitting into the divisor corresponding to the line $D_{i,1}$ and the divisor corresponding to E_{i+2} . Whence Kulikov verifies that one gets in this way the Burniat surface with $K^2 = 2$.

Remark 5.1. *There are other interesting $(\mathbb{Z}/2\mathbb{Z})^3$ -Galois covers of the plane branched on the seven lines $L_1, L_2, L_3, L_4, L'_1, L'_2, L'_3$.*

One such is the fibre product Z of the standard bidouble cover $\mathbb{P}^2 \rightarrow \mathbb{P}^2$ branched on the diagonals L'_1, L'_2, L'_3 with the double covering Y' branched on L_1, L_2, L_3, L_4 . This gives Z as a double plane branched on four conics touching in 12 points. Z is a surface with $K_Z^2 = 2$, $p_g(Z) = 3$, whose singularities are precisely 12 points of type A_3 .

References

- [Arm65] Armstrong, M. A., *On the fundamental group of an orbit space*. Proc. Cambridge Philos. Soc. **61** 639–646 (1965).
- [Arm68] Armstrong, M. A., *The fundamental group of the orbit space of a discontinuous group*. Proc. Cambridge Philos. Soc. **64** 299–301 (1968).
- [BCGP09] Bauer, I., Catanese, F., Grunewald, F., Pignatelli, R. *Quotients of a product of curves by a finite group and their fundamental groups*. arXiv:0809.3420.
- [BC09a] Bauer, I., Catanese, F. *The moduli space of Keum - Naie surfaces* arXiv:0909.1733.
- [BC09b] Bauer, I., Catanese, F. *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space*. arXiv:0911.1466.

- [Brie68] Brieskorn, E. *Die Auflösung der rationalen Singularitäten holomorpher Abbildungen*. Math. Ann. 178 (1968) 255–270.
- [Brie71] Brieskorn, E. *Singular elements of semi-simple algebraic groups*. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 279–284. Gauthier-Villars, Paris, 1971.
- [Bu66] Burniat, P. *Sur les surfaces de genre $P_{12} > 1$* . Ann. Mat. Pura Appl. (4) 71 (1966) 1–24.
- [Cat84] Catanese, F. *On the moduli spaces of surfaces of general type*. J. Differential Geom. 19 (1984), no. 2, 483–515.
- [Cat99] Catanese, F. *Singular bidouble covers and the construction of interesting algebraic surfaces*. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97–120, Contemp. Math. 241, Amer. Math. Soc., Providence, RI, 1999.
- [Cat08] Catanese, F. *Differentiable and deformation type of algebraic surfaces, real and symplectic structures*. Symplectic 4-manifolds and algebraic surfaces, 55–167, Lecture Notes in Math., 1938, Springer, Berlin, 2008.
- [In94] Inoue, M. *Some new surfaces of general type*. Tokyo J. Math. 17 (1994), no. 2, 295–319.
- [Ku04] Kulikov, V., S. *Old examples and a new example of surfaces of general type with $p_g = 0$* . (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 5, 123–170; translation in Izv. Math. 68 (2004), no. 5, 965–1008.
- [MLP01] Mendes Lopes, M., Pardini, R. *A connected component of the moduli space of surfaces with $p_g = 0$* . Topology 40 (2001), no. 5, 977–991.
- [Miy77] Miyaoka, Y. *On numerical Campedelli surfaces*. In: Complex analysis and algebraic geometry, pp. 113–118. Iwanami Shoten, Tokyo, 1977.
- [Par91] Pardini, R. *Abelian covers of algebraic varieties*. J. Reine Angew. Math. 417 (1991), 191–213.
- [Pet77] Peters, C. A. M. *On certain examples of surfaces with $p_g = 0$ due to Burniat*. Nagoya Math. J. 66 (1977), 109–119.
- [Rei] Reid, M. *Surfaces with $p_g = 0$, $K^2 = 2$* . Preprint available at <http://www.warwick.ac.uk/~masda/surf/>.
- [Rei79] Reid, M.: π_1 for surfaces with small K^2 . In: Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), 534–544. Lecture Notes in Math. 732, Springer, Berlin (1979).
- [Tju70] Tjurina, G. N. *Resolution of singularities of flat deformations of double rational points*. Funkcional. Anal. i Priložen. 4 (1970) no. 1, 77–83.

I. Bauer, Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth
E-mail: ingrid.bauer@uni-bayreuth.de

F. Catanese, Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth
E-mail: fabrizio.catanese@uni-bayreuth.de