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Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space

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This article is dedicated, with gratitude and admiration, to David Mumford on the occasion of his $2^3 \cdot 3^2$-th birthday

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1 Introduction

The so called Burniat surfaces were constructed by Pol Burniat in 1966 [7], where the method of singular bidouble covers was introduced in order to solve the geography problem for surfaces of general type.

The special construction of surfaces with geometric genus $p_g(S) = 0$, done in [7], was brought to attention by [30], which explained Burniat's calculation of invariants in the modern language of algebraic geometry, and nowadays the name of Burniat surfaces is reserved for these surfaces with $p_g(S) = 0$.

Burniat surfaces are especially interesting examples for the nonbirationality of the bicanonical map (see [19]). For all the Burniat surfaces S with $K_S^2 \geq 3$ the bicanonical map turns out to be a Galois morphism of degree 4.

We refer to our joint paper with Grunewald and Pignatelli [2] for a general introduction on the classification and moduli problem for surfaces with

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$p_g(S) = 0$ and its applications: as an example we mention our final corollary here on the validity of Bloch's conjecture for all deformations of secondary Burniat surfaces.

The main achievement of the present series of three articles is to completely solve the moduli problem for Burniat surfaces, determining the connected components of the moduli space of surfaces of general type containing the Burniat surfaces, and describing their geometry.

The minimal models S of Burniat surfaces have as invariant the positive integer K_S^2, which can take values $K_S^2 = 6, 5, 4, 3, 2$.

We get a rationally parametrized family of dimension $K_S^2 - 2$ for each value of $K_S^2 = 6, 5, 3, 2$, and two such families for $K_S^2 = 4$, one called of nonnodal type, the other of nodal type. We proposed in [4] to call *primary Burniat surfaces* those with $K_S^2 = 6$, *secondary Burniat surfaces* those with $K_S^2 = 5, 4$, and *tertiary Burniat surfaces* those with $K_S^2 = 3$. The reason not to consider the Burniat surface with $K_S^2 = 2$ is that it is just one special element of the family of standard Campedelli surfaces (i.e., with torsion group $(\mathbb{Z}/2\mathbb{Z})^3$) (see [4, 24]), whose geometry is completely understood (see [26, 32]).

An important result was obtained by Mendes Lopes and Pardini in [25] who proved that primary Burniat surfaces form a connected component of the moduli space of surfaces of general type. A stronger result concerning primary Burniat surfaces was proved in part one [4], namely that any surface homotopically equivalent to a primary Burniat surface is a primary Burniat surface. Alexeev and Pardini (cf. [1]) reproved the result of Mendes Lopes and Pardini by studying more generally the component of the moduli space of stable surfaces of general type containing primary Burniat surfaces.

Here, we shall prove in one go that each of the 4 families of Burniat surfaces with $K_S^2 \geq 4$, i.e., of primary and secondary Burniat surfaces, is a connected component of the moduli space of surfaces of general type.

The case of tertiary Burniat surfaces will be treated in the third one of this series of papers, and we limit ourselves here to say that the general deformation of a Burniat surface with $K_S^2 = 3$ is not a Burniat surface, but it is always a bidouble cover (through the bicanonical map) of a cubic surface with three nodes.

At the moment when we started the redactional work for the present paper we became aware of the fact that a weaker result was stated in [24], namely that each family of Burniat surfaces of secondary type yields a dense set in an irreducible component of the moduli space. The result is derived by Kulikov from the assertion that the base of the Kuranishi family of deformations is smooth. This result is definitely false for the Burniat surfaces with $K_S^2 = 4$ of nodal type (Proposition 4.12 and Corollary 4.23(iii) of [24]), as we shall now see.

Indeed one of the main technical contributions of this paper is the study of the deformations of secondary Burniat surfaces, through diverse techniques.

A very surprising and new phenomenon occurs for nodal surfaces, confirming Vakil's 'Murphy's law' philosophy [35].

To explain it, recall that indeed there are two different structures as complex analytic spaces for the moduli spaces of surfaces of general type.

One is the moduli space $\mathfrak{M}^{min}_{\chi,K^2}$ for minimal models S having $\chi(\mathcal{O}_S) = \chi$, $K_S^2 = K^2$, the other is the Gieseker moduli space $\mathfrak{M}^{can}_{\chi,K^2}$ for canonical models X having $\chi(\mathcal{O}_X) = \chi$, $K_X^2 = K^2$ (cf. [20]). The Gieseker moduli space is a quasi projective scheme and there is a natural holomorphic map $\mathfrak{M}^{min}_{\chi,K^2} \to \mathfrak{M}^{can}_{\chi,K^2}$ which is a homeomorphism in the Hausdorff topology. Their local structure as complex analytic spaces is the respective quotient of the base of the Kuranishi family by the action of the finite group $\mathrm{Aut}(S) = \mathrm{Aut}(X)$.

In [14] series of examples were exhibited where $\mathfrak{M}^{can}_{\chi,K^2}$ was smooth, but $\mathfrak{M}^{min}_{\chi,K^2}$ was everywhere nonreduced.

For nodal Burniat surfaces with $K_S^2 = 4$ both spaces are everywhere nonreduced, but the nilpotence order is higher for $\mathfrak{M}^{min}_{\chi,K^2}$; this is a further pathology, which adds to the ones presented in [14] and in [35].

More precisely, this is one of our two main results:

Theorem 1.1 *The subset of the Gieseker moduli space $\mathfrak{M}^{can}_{1,4}$ of canonical surfaces of general type X corresponding to Burniat surfaces S with $K_S^2 = 4$ and of nodal type is an irreducible connected component of dimension* 2, *rational and everywhere nonreduced.*

More precisely, there exists an integer $m \geq 2$ such that the base $\mathrm{Def}(X)$ *of the Kuranishi family of X is locally analytically isomorphic to $\mathbb{C}^2 \times$ $Spec(\mathbb{C}[t]/(t^m))$.*

The corresponding subset of the moduli space $\mathfrak{M}^{min}_{1,4}$ of minimal surfaces S of general type is also everywhere nonreduced and the base $\mathrm{Def}(S)$ *of the Kuranishi family of S is locally analytically isomorphic to $\mathbb{C}^2 \times$ $Spec(\mathbb{C}[t]/(t^{2m}))$.*

For the general such surface $\mathrm{Aut}(S) = \mathrm{Aut}(X) = (\mathbb{Z}/2\mathbb{Z})^2$, *and this group acts trivially on the bases* $\mathrm{Def}(S), \mathrm{Def}(X)$ *of the respective Kuranishi families.*

Whereas for the nonnodal case we get the following second main result:

Theorem 1.2 *The three respective subsets of the moduli spaces of minimal surfaces of general type $\mathfrak{M}^{min}_{1,K^2}$ corresponding to Burniat surfaces with $K^2 = 6$, resp. with $K^2 = 5$, resp. Burniat surfaces with $K^2 = 4$ of nonnodal* *type, are irreducible connected components, normal, rational of respective dimensions* 4, 3, 2.

Moreover, the base of the Kuranishi family of such surfaces S is smooth.

Theorem 1.1 poses the challenging deformation theoretic question to calculate the number m giving the order of nilpotence of the local moduli space (and also of the moduli space at the general point).

2 The local moduli spaces of Burniat surfaces

2.1 Definition of the Burniat surfaces

Burniat surfaces are minimal surfaces of general type with $K^2 = 6, 5, 4, 3, 2$ and $p_g = 0$, which were constructed in [7] as minimal resolutions of singular bidouble covers (Galois covers with group $(\mathbb{Z}/2\mathbb{Z})^2$) of the projective plane branched on 9 lines.

We briefly recall their construction: this will also be useful to fix our notation. For more details, and for the proof that Burniat surfaces are exactly certain Inoue surfaces we refer to [4].

Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three noncollinear points (which we assume to be the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$) and let's denote by $Y := \hat{\mathbb{P}}^2(P_1, P_2, P_3)$ the Del Pezzo surface of degree 6, blow up of $\mathbb{P}^2$ in P_1, P_2, P_3.

Y is 'the' smooth Del Pezzo surface of degree 6, and it is the closure of the graph of the rational map

$$\epsilon : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that

$$\epsilon(y_1 : y_2 : y_3) = ((y_2 : y_3)(y_3 : y_1)(y_1 : y_2)).$$

One sees immediately that $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the hypersurface of type $(1, 1, 1)$:

$$Y = \{((x_1' : x_1), (x_2' : x_2), (x_3' : x_3)) \mid x_1 x_2 x_3 = x_1' x_2' x_3'\}.$$

We denote by L the total transform of a general line in $\mathbb{P}^2$, by E_i the exceptional curve lying over P_i, and by $D_{i,1}$ the unique effective divisor in $|L - E_i - E_{i+1}|$, i.e., the proper transform of the line $y_{i-1} = 0$, side of the triangle joining the points P_i, P_{i+1}.

Consider on Y, for each $i \in \mathbb{Z}/3\mathbb{Z} \cong \{1, 2, 3\}$, the following divisors

$$D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2} \in |3L - 3E_i - E_{i+1} + E_{i+2}|,$$

where $D_{i,j} \in |L - E_i|$, for j = 2, 3, is the proper transform of another line through P_i. Assume further that all the nine corresponding lines in $\mathbb{P}^2$ are distinct, so that $D := \sum_i D_i$ is a reduced divisor.

Note that, if we define the divisor $\mathcal{L}_i := 3L - 2E_{i-1} - E_{i+1}$, then

$$D_{i-1} + D_{i+1} \equiv 6L - 4E_{i-1} - 2E_{i+1} \equiv 2\mathcal{L}_i,$$

and we can consider (cf. [15]) the associated bidouble cover $X' \to Y$ branched on $D := \sum_i D_i$ (but we take a different ordering of the indices of the fibre coordinates u_i, using the same choice as the one made in [4], where however X' was denoted by X).

We recall that this precisely means the following: let δ_i be a section of the line bundle $\mathcal{O}_Y(D_i)$ such that $D_i = \mathrm{div}(\delta_i)$, and let u_i be a fibre coordinate of the geometric line bundle $\mathbb{L}_{i+1}$, whose sheaf of holomorphic sections is $\mathcal{O}_Y(\mathcal{L}_{i+1})$.

Then $X \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is given by the equations:

$$u_1u_2 = \delta_1u_3, \qquad u_1^2 = \delta_3\delta_1;$$
$$u_2u_3 = \delta_2u_1, \qquad u_2^2 = \delta_1\delta_2;$$
$$u_3u_1 = \delta_3u_2, \qquad u_3^2 = \delta_2\delta_3.$$

From the birational point of view, as done by Burniat, we are simply adjoining to the function field of $\mathbb{P}^2$ two square roots, namely $\sqrt{\frac{\Delta_1}{\Delta_3}}$ and $\sqrt{\frac{\Delta_2}{\Delta_3}}$, where Δ_i is the cubic polynomial in $\mathbb{C}[x_0, x_1, x_2]$ whose zero set has $D_i - E_{i+2}$ as strict transform.

This shows clearly that we have a Galois cover $X' \to Y$ with group $(\mathbb{Z}/2\mathbb{Z})^2$.

The equations above give a biregular model X' which is nonsingular exactly when the divisor D does not have points of multiplicity 3 (there cannot be points of higher multiplicities). These points give then quotient singularities of type $\frac{1}{4}(1, 1)$, i.e., isomorphic to the quotient of $\mathbb{C}^2$ by the action of $(\mathbb{Z}/4\mathbb{Z})$ sending $(u, v) \mapsto (iu, iv)$ (or, equivalently , the affine cone over the 4-th Veronese embedding of $\mathbb{P}^1$).

Definition 2.1 A *primary Burniat surface* is a surface constructed as above, and which is moreover smooth. It is then a minimal surface S with K_S ample, and with $K_S^2 = 6$, $p_g(S) = q(S) = 0$.

A *secondary Burniat surface* is the minimal resolution of a surface X' constructed as above, and which moreover has $1 \le m \le 2$ singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, and with $K_S^2 = 6 - m$, $p_g(S) = q(S) = 0$.

A *tertiary (respectively, quaternary) Burniat surface* is the minimal resolution of a surface X' constructed as above, and which moreover has $m=3$ (respectively $m=4$) singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, but not ample, and with $K_S^2 = 6-m$, $p_g(S)=q(S)=0$.

Remark 2.2

(1) We remark that for $K_S^2=4$ there are two possible types of configurations. The one where there are three collinear points of multiplicity at least 3 for the plane curve formed by the 9 lines leads to a Burniat surface S which we call of *nodal type*, and with K_S not ample, since the inverse image of the line joining the 3 collinear points is a (-2)-curve (a smooth rational curve of self intersection -2).

In the other cases with $K_S^2 = 4, 5, 6$, instead, K_S is ample.

(2) In the nodal case, if we blow up the two $(1,1,1)$ points of D, we obtain a weak Del Pezzo surface $\tilde{Y}$, i.e., a surface with nef and big anticanonical divisor $-K_{\tilde{Y}}$. In fact, $-K_{\tilde{Y}}^2 = 4$ but $-K_{\tilde{Y}}$ contains a (-2)-curve, hence $-K_{\tilde{Y}}$, which is nef, is not ample.

Its anticanonical model Y' has a node (an A_1-singularity, corresponding to the contraction of the (-2)-curve). In the nonnodal case, we obtain a smooth Del Pezzo surface $\tilde{Y} = Y'$ of degree 4.

(3) We illustrated the possible configurations of the lines in the plane in Fig. 1.

We will mostly restrict ourselves in the following to *secondary Burniat surfaces*. In this cases the branch divisor D on Y has either one or two singular points of type $(1,1,1)$. In the case $K_S^2=5$ there is one point, denoted by P_4, in the case $K_S^2=4$ there are two such points, denoted by P_4, P_5; in the nodal case we shall assume that P_1, P_4, P_5 are collinear.

We finally let $\tilde{Y} \to Y$ be the blow up of Y in P_4 (in the case $K_S^2=5$), respectively in the points P_4, P_5 (if $K^2=4$). We let E_4 (resp. E_5) be the exceptional curve lying over P_4 (resp. over P_5).

We have summarized in Tables 1, 2, 3 the linear equivalence classes of the divisors $D_{i,j}$, which are the strict transforms of lines $D'_{i,j}$ in $\mathbb{P}^2$.

2.2 Local deformations of the Burniat surfaces

In order to get some grip on the local deformations of a Burniat surface, we show a preliminary result, which, although not used in the sequel, suggests the main idea, namely that for $K_S^2 \geq 4$ all deformations carry along the $(\mathbb{Z}/2\mathbb{Z})^2$-action.

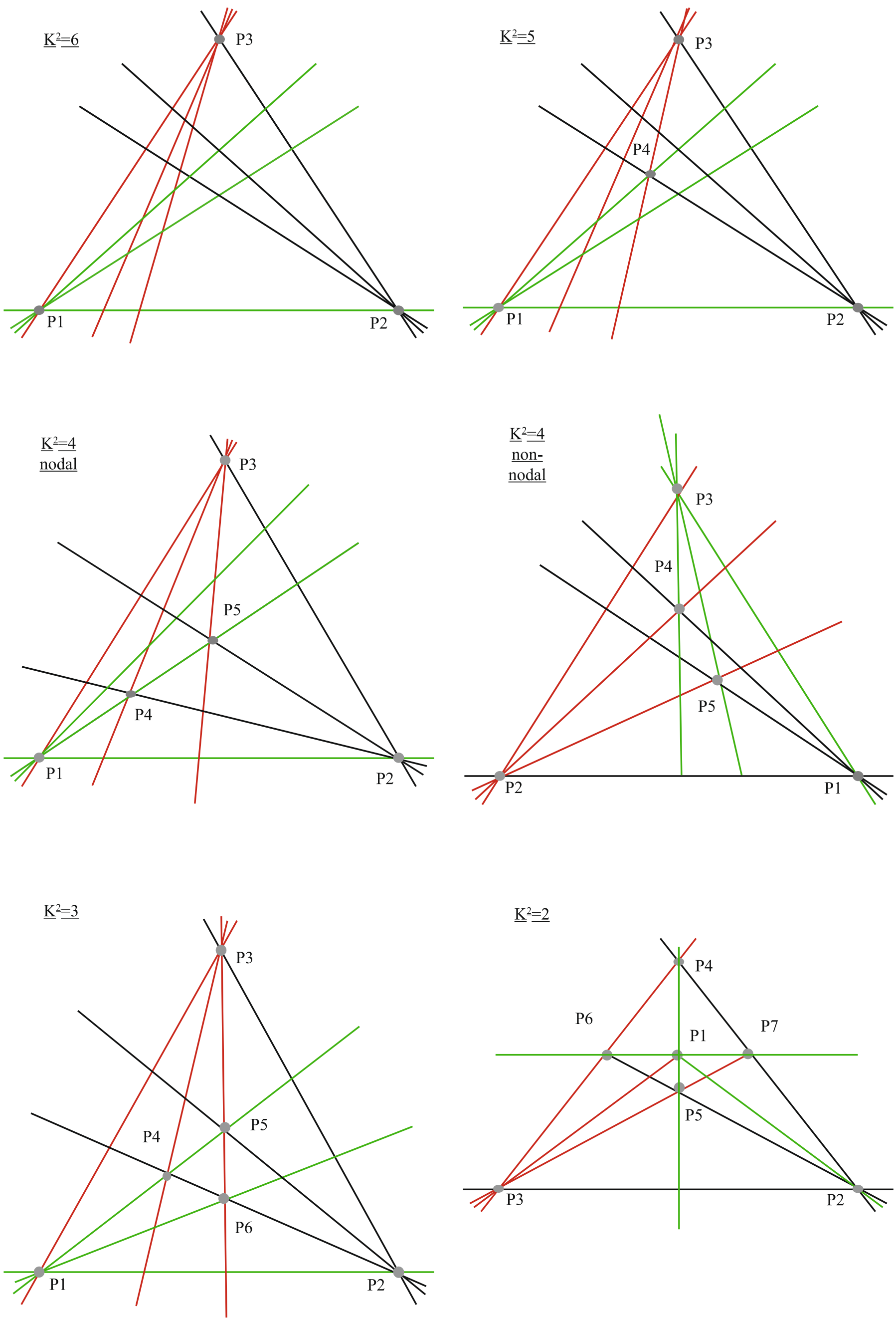

Fig. 1 Configurations of lines

Table 1 $K_S^2 = 5$

(i, j)	$D_{i,j}$
$(i, 1)$	$L - E_i - E_{i+1}$
$(i, 2)$	$L - E_i - E_4$
$(i, 3)$	$L - E_i$

Table 2 $K_S^2 = 4$: nonnodal

(i, j)	$D_{i,j}$
$(i, 1)$	$L - E_i - E_{i+1}$
$(i, 2)$	$L - E_i - E_4$
$(i, 3)$	$L - E_i - E_5$

Table 3 $K_S^2 = 4$: nodal

(i, j)	$D_{i,j}$
$(i, 1)$	$L - E_i - E_{i+1}$
$(1, 2)$	$L - E_1 - E_4 - E_5$
$(1, 3)$	$L - E_1$
$(2, 2)$	$L - E_2 - E_4$
$(2, 3)$	$L - E_2 - E_5$
$(3, 2)$	$L - E_3 - E_4$
$(3, 3)$	$L - E_3 - E_5$

Proposition 2.3 *Let S be the minimal model of a Burniat surface, given as Galois $(\mathbb{Z}/2\mathbb{Z})^2$-cover of the (weak) Del Pezzo surface $\tilde{Y}$. Then all natural deformations of $\pi : S \to \tilde{Y}$ are Galois $(\mathbb{Z}/2\mathbb{Z})^2$-covers of $\tilde{Y}$.*

Proof The natural deformations of a bidouble cover (we refer to [11], Definition 2.8, p. 494, and to [15], p. 106 for the definition of the family of natural deformations of a bidouble cover) are parametrized by the direct sum of the vector spaces $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D_i))$ with the vector spaces $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D_i - L_i))$. The second summand is zero exactly when all the natural deformations are Galois.

As easily read off from Tables 1, 2, 3, in all cases we have

$$D_i - L_i \equiv -3E_i + 3E_{i+2}, \quad \forall i \in \{1, 2, 3\}$$

Assume that there exists an effective divisor $C \in |D_i - L_i|$. Then $C \cdot E_{i+2} = -3$, whence $C \geq 3E_{i+2}$. Therefore we can write $C = C' + 3E_{i+2}$, with $C' \in |-3E_i|$, a contradiction. This implies that $|D_i - L_i| = \emptyset$. $\square$

Remark 2.4 It is easy to see that the respective dimensions of the families of Burniat surfaces are

- 4 for $K^2 = 6$;
- 3 for $K^2 = 5$;
- 2 for $K^2 = 4$, nonnodal;
- 2 for $K^2 = 4$, nodal;
- 1 for $K^2 = 3$.

An important feature of each family of Burniat surfaces is that the canonical models do not get worse singularities for special elements of the family.

The minimal model S of a Burniat surface is a smooth bidouble cover of a smooth weak Del Pezzo surface $\tilde{Y}$, branched over a normal crossing divisor. K_S is ample for $K_S^2 \geq 4$ unless we are in the nodal case with $K_S^2 = 4$.

In this nodal case one has a singular Del Pezzo surface Y' with an A_1-singularity obtained contracting the (-2) curve $D_{1,2}$.

The canonical model X of S is obtained contracting the (-2) curve E which is the inverse image of $D_{1,2}$. X is a finite bidouble cover of Y'.

Our strategy is to preliminarily investigate the tangent space and the obstruction space for the Kuranishi family of S as representations of the group $G := (\mathbb{Z}/2\mathbb{Z})^2$, and to later use this information to describe the Kuranishi family of X, showing in particular that all the deformations preserve the G-action.

First of all, we determine the several character spaces and their dimension. Here, V^i, for $i \in 1, 2, 3$ is the character spaces and their respective dimensions.

Proposition 2.5 *Let S be the minimal model of a Burniat surface.*

Then the dimensions of the eigenspaces of the cohomology groups of the tangent sheaf Θ_S *(for the natural* $(\mathbb{Z}/2\mathbb{Z})^2$*-action) are as follows.*

- $\underline{K^2 = 6}$: $h^1(S, \Theta_S)^{\text{inv}} = 4$, $h^2(S, \Theta_S)^{\text{inv}} = 0$,
  $h^1(S, \Theta_S)^i = 0$, $h^2(S, \Theta_S)^i = 2$, *for* $i \in \{1, 2, 3\}$;
- $\underline{K^2 = 5}$: $h^1(S, \Theta_S)^{\text{inv}} = 3$, $h^2(S, \Theta_S)^{\text{inv}} = 0$,
  $h^1(S, \Theta_S)^i = 0$, $h^2(S, \Theta_S)^i = 1$, *for* $i \in \{1, 2, 3\}$;
- $\underline{K^2 = 4}$ *of nonnodal type*: $h^1(S, \Theta_S)^{\text{inv}} = 2$, $h^2(S, \Theta_S)^{\text{inv}} = 0$,
  $h^1(S, \Theta_S)^i = h^2(S, \Theta_S)^i = 0$, *for* $i \in \{1, 2, 3\}$.
- $\underline{K^2 = 4}$ *of nodal type*: $h^1(S, \Theta_S)^{\text{inv}} = 2$, $h^2(S, \Theta_S)^{\text{inv}} = 0$,
  $h^1(S, \Theta_S)^1 = 1 = h^2(S, \Theta_S)^1$, $h^j(S, \Theta_S)^i = 0$, *for* $i \in \{2, 3\}$.

We shall postpone the proof of the above proposition to the next subsection and state right away the main result of this subsection.

Recall that for surfaces of general type we have two moduli spaces: one is the moduli space $\mathfrak{M}^{min}_{\chi,K^2}$ for minimal models S having $\chi(\mathcal{O}_S) = \chi$, $K_S^2 = K^2$, the other is the moduli space $\mathfrak{M}^{can}_{\chi,K^2}$ for canonical models X having $\chi(\mathcal{O}_X) = \chi$, $K_X^2 = K^2$; the latter is called the Gieseker moduli space. Both are complex analytic spaces and there is a natural morphism $\mathfrak{M}^{min}_{\chi,K^2} \to \mathfrak{M}^{can}_{\chi,K^2}$ which is a bijection. Their local analytic structure is the quotient of the base of the Kuranishi family by the action of the finite group $\mathrm{Aut}(S) = \mathrm{Aut}(X)$. Usually, $\mathfrak{M}^{min}_{\chi,K^2}$ tends to be more singular than the one of $\mathfrak{M}^{can}_{\chi,K^2}$ (see e.g. [14]).

Recall moreover that in the following theorem K_S is always ample, thus the minimal and canonical model coincide. Instead, later on, for surfaces with $K^2 = 4$ of nodal type, S contains exactly one -2 curve E, thus the canonical model X has always exactly one singular point, an A_1-singularity.

Theorem 2.6 *The three respective subsets of the moduli spaces of minimal surfaces of general type* $\mathfrak{M}^{min}_{1,K^2}$ *corresponding to Burniat surfaces with* $K^2 = 6$, *resp. with* $K^2 = 5$, *resp. Burniat surfaces with* $K^2 = 4$ *of nonnodal type*, *are irreducible open sets*, *normal*, *unirational of respective dimensions* 4, 3, 2.

Moreover, the base of the Kuranishi family of S is smooth.

Proof By Proposition 2.5 the tangent space to the Kuranishi family of S, $H^1(\Theta_S)$, consists of invariants for the action of the group $G := (\mathbb{Z}/2\mathbb{Z})^2$.

It follows then (cf. [13] lecture three, page 23) that all the local deformations admit the G-action, hence they are bidouble covers of a deformation of the smooth Del Pezzo surface $\tilde{Y}$.

Moreover, the dimension of $H^1(\Theta_S)$ coincides with the dimension of the image of the Burniat family containing S in the moduli space $\mathfrak{M}^{min}_{1,K_S^2}$, hence the Kuranishi family of S is smooth, and coincides with the Burniat family by the above argument.

Alternatively, one could show directly that the Kodaira Spencer map is bijective, or simply observe that a finite morphism between smooth manifolds of the same dimension is open.

Observe then that the quotient of a smooth variety by the action of a finite group is normal.

Finally, the Burniat family is parametrized by a (smooth) rational variety. □

We shall see in the final section that these irreducible components are not only unirational, but indeed rational.

2.3 The proof of Proposition 2.5

This subsection is dedicated to the technical details of the proof of Proposition 2.5.

We see easily from Tables 1, 2, 3 the following formulae which hold uniformly for all Burniat surfaces:

Remark 2.7

(i) $D_i \equiv -K_{\tilde{Y}} - 2E_i + 2E_{i+2}$

(ii) $\mathcal{L}_i \cong \mathcal{O}_{\tilde{Y}}(-K_{\tilde{Y}} + E_i - E_{i-1})$ since $\mathcal{L}_i \cong \mathcal{O}_{\tilde{Y}}(L_i)$, where $L_i \equiv \frac{1}{2}(D_{i-1} + D_{i+1})$.

This yields $L_i \equiv 3L - E_{i+1} - 2E_{i-1} - E_4$ for $K^2 = 5$, and $L_i = 3L - E_{i+1} - 2E_{i-1} - E_4 - E_5$, for $K^2 = 4$.

(iii) $D_i - L_i \equiv -3E_i + 3E_{i-1}$.

In order to determine $\dim H^1(S, \Theta_S)$, we use the following special case of Theorem 2.16 of [11]:

Proposition 2.8 *Let $\pi : S \to \tilde{Y}$ be a Galois $(\mathbb{Z}/2\mathbb{Z})^2$-cover of smooth projective surfaces with branch divisor $D := D_1 + D_2 + D_3$. Then*

$$\pi_*(\Omega^1_S \otimes \Omega^2_S) = (\Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3) \otimes \Omega^2_{\tilde{Y}}) \oplus \left(\bigoplus_{i=1}^{3} \Omega^1_{\tilde{Y}}(\log D_i) \otimes \Omega^2_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{Y}}(L_i) \right),$$

where $\Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3)$ is the subsheaf of the sheaf of rational 1-forms generated by $\Omega^1_{\tilde{Y}}$ and by $dlog(\delta_1), dlog(\delta_2), dlog(\delta_3)$, and where $D_i = div(\delta_i)$.

Moreover the first summand is the invariant one, and the other three correspond to the three nontrivial characters of $(\mathbb{Z}/2\mathbb{Z})^2$.

We are able to use the above result observing in fact that the sheaf $\Omega^1_S \otimes \Omega^2_S = \Omega^1_S(K_S)$ is the Serre dual of Θ_S, and that for each locally free sheaf $\mathcal{F}$ on S we have (the second formula is duality for a finite map, cf. [21], Exercise 6.10, p. 239):

(1) $H^i(\mathcal{F}) = H^i(\pi_*(\mathcal{F}))$,
(2) $\pi_*(\mathcal{F}^\vee(K_S)) \cong (\pi_*\mathcal{F})^\vee(K_{\tilde{Y}})$,
(3) $K_S = \pi^*(K_{\tilde{Y}} + L_1 + L_2 + L_3)$,
(4) $H^i(\Theta_S)^\vee = H^{2-i}(\pi_*(\Omega^1_S \otimes \Omega^2_S))$.

Moreover, we use the following exact residue sequence

$$0 \to \Omega^1_{\tilde{Y}} \to \Omega^1_{\tilde{Y}}(\log D_1, \dots, \log D_k) \to \bigoplus_{i=1}^{k} \mathcal{O}_{D_i} \to 0$$

holding more generally if the divisors D_i are reduced and $\tilde{Y}$ is a factorial variety (see e.g. Lemma 3, p. 675 of [17]).

We are left with the calculation of the cohomology groups of the sheaves:

$$\Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3)(K_{\tilde{Y}}),$$

respectively

$$\Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + L_i).$$

However, the second cohomology groups vanish since S is of general type, hence $H^0(\Theta_S) = 0$. The Riemann Roch theorem tells us what are the alternating sums of the dimensions, thus in the end it suffices to calculate the H^0 of these sheaves.

Let us look at the invariant part, using the exact sequence

$$0 \to \Omega^1_{\tilde{Y}}(K_{\tilde{Y}}) \to \Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3)(K_{\tilde{Y}}) \to \bigoplus_{i=1}^{3} \mathcal{O}_{D_i}(K_{\tilde{Y}}) \to 0.$$

The space $H^0(\Omega^1_{\tilde{Y}}(K_{\tilde{Y}}))$ vanishes since $H^0(\Omega^1_{\tilde{Y}}) = 0$ and $-K_{\tilde{Y}}$ is effective.

Moreover, if $\tilde{Y}$ is a Del Pezzo surface, then $-K_{\tilde{Y}}$ is ample and also $H^0(\mathcal{O}_{D_i}(K_{\tilde{Y}})) = 0$.

Thus $H^0(\Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3)(K_{\tilde{Y}})) = 0$ unless we are in the nodal case. Here there is the (-2) curve $D_{1,2}$ which is a connected component of D_1, hence in this case $H^0(\mathcal{O}_{D_1}(K_{\tilde{Y}})) \cong \mathbb{C}$.

On the other hand the coboundary in the long exact cohomology sequence is given by cup product with the extension class, which is the direct sum of the Chern classes of the divisors D_i, $c_1(D_i) \in H^1(\Omega^1_{\tilde{Y}})$.

Note, in the nodal case, that $|-K_{\tilde{Y}}| = |3L - \sum_{i=1}^5 E_i|$ is base point free. Therefore there is a morphism $\mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}}) \to \mathcal{O}_{\tilde{Y}}$, which is not identically zero on any component of the D_i's.

We get the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_{\tilde{Y}}(K_{\tilde{Y}}) & \longrightarrow & \Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3)(K_{\tilde{Y}}) & \longrightarrow & \bigoplus_{i=1}^3 \mathcal{O}_{D_i}(K_{\tilde{Y}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{\tilde{Y}} & \longrightarrow & \Omega^1_{\tilde{Y}}(\log D_1, \log D_2, \log D_3) & \longrightarrow & \bigoplus_{i=1}^3 \mathcal{O}_{D_i} & \longrightarrow & 0.
\end{array} \tag{1}$$

From this we get the commutative diagram

$$\begin{array}{ccc}
\mathbb{C} \cong H^0(\tilde{Y}, \bigoplus_{i=1}^3 \mathcal{O}_{D_i}(K_{\tilde{Y}})) & \xrightarrow{\delta} & H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(K_{\tilde{Y}})) \\
\downarrow \psi_2 & \searrow^{\varphi} & \downarrow \\
H^0(\tilde{Y}, \bigoplus_{i=1}^3 \mathcal{O}_{D_i}) & \xrightarrow{\psi_1} & H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}).
\end{array}$$

Note that by a straightforward extension of the argument given in [11], Lemma 3.7, the image of the function identically equal to 1 on $D_{1,2}$ maps under ψ_1 to the first Chern class of $D_{1,2}$. Therefore $\varphi = \psi_1 \circ \psi_2 \neq 0$, hence also δ is nonzero.

We have thus accomplished the proof of

Lemma 2.9 *For a primary or secondary Burniat surface the $G := (\mathbb{Z}/2\mathbb{Z})^2$-invariant part $H^0(\Omega^1_S \otimes \Omega^2_S)^G$ of $H^0(\Omega^1_S \otimes \Omega^2_S)$ vanishes.*

Let us now turn to the other characters.

We have then the other exact sequence

$$0 \to \Omega^1_{\tilde{Y}}(K_{\tilde{Y}} + L_i) \to \Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + L_i) \to \mathcal{O}_{D_i}(K_{\tilde{Y}} + L_i) \to 0$$

and we recall that, by Remark 2.7

$$\Omega^1_{\tilde{Y}}(\log D_i)(K_{\tilde{Y}} + L_i) \cong \Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2}).$$

We shall calculate the dimension of the space

$$H^0(\Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2}))$$

taking its direct image sheaf on $\mathbb{P}^2$.

We need a lemma which we state for simplicity in the case of dimension 2: it shows what the effect is of blowing down a (-1) curve.

Lemma 2.10 *Consider a finite set of distinct linear forms*

$$l_\alpha := y - c_\alpha x, \quad \alpha \in A$$

vanishing at the origin in $\mathbb{C}^2$. Let $p\colon Z \to \mathbb{C}^2$ be the blow up of the origin, let D_α be the strict transform of the line $L_\alpha := \{l_\alpha = 0\}$, and let E be the exceptional divisor.

Let $\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha \in A})$ be the sheaf of rational 1-forms η generated by $\Omega^1_{\mathbb{C}^2}$ and by the differential forms $d\log l_\alpha$ as an $\mathcal{O}_{\mathbb{C}^2}$-module and define similarly $\Omega^1_Z((\log D_\alpha)_{\alpha \in A})$. Then:

(1) $p_*\Omega^1_Z(\log E)(-E) = \Omega^1_{\mathbb{C}^2}$,

(2) $p_*\Omega^1_Z(\log E, (\log D_\alpha)_{\alpha \in A}) = p_*\Omega^1_Z((\log D_\alpha)_{\alpha \in A})(E) = \Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha \in A})$,

(3) $p_*\Omega^1_Z((\log D_\alpha)_{\alpha \in A}) = \{\eta \in \Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha \in A}) | \eta = \sum_\alpha g_\alpha d\log l_\alpha + \omega, \omega \in \Omega^1_{\mathbb{C}^2}, \sum_\alpha g_\alpha(0) = 0\}$.

Proof The sheaf $\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A})$ is locally free outside of the origin, and torsion free in view of the residue sequence, since $\bigoplus_{\alpha\in A}\mathcal{O}_{L_\alpha}$ has no section with a 0-dimensional support.

Likewise, all other direct image sheaves are torsion free, and those in 2. and 3. are equal to $\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A})$ outside of the origin.

(1) $p_*\Omega^1_Z(\log E)(-E)\subset\Omega^1_{\mathbb{C}^2}$ holds since the left hand side is torsion free and coincides with the right hand side outside the origin. But $\Omega^1_{\mathbb{C}^2}$ is locally free, hence it enjoys the Hartogs property, so the desired inclusion holds. It suffices then to show that $p^*(\Omega^1_{\mathbb{C}^2})\subset\Omega^1_Z(\log E)(-E)$. This follows since in the affine chart $(x,t)\mapsto(x,y=xt)$ of the blow up, we have $dx=xd\log x$, $dy=x(dt+td\log x)$ (and similarly on the other chart).

(2) It suffices to show the chain of inclusions (where $m\geq 1$)

$$\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A})\subset p_*\Omega^1_Z(\log E,(\log D_\alpha)_{\alpha\in A})$$
$$\subset p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A})(mE)\subset\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A}).$$

The first inclusion follows, the two sheaves being torsion free and equal outside of the origin, from the assertion that $p^*(\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A}))\subset\Omega^1_Z(\log E,(\log D_\alpha)_{\alpha\in A})$.

This assertion is easily verified in each affine chart, since $d\log l_\alpha = d\log x + d\log(\frac{l_\alpha}{x}) = d\log x + d\log(t-c_\alpha)$.

The second inclusion is obvious, while, for the third,

$$p_*\Omega^1_Z((\log D_\alpha)_{\alpha\in A})(mE)$$

consists of rational differential 1-forms ω which, when restricted to $\mathbb{C}^2\setminus\{0\}$, yield sections of $\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A})$.

Therefore in particular $\omega\prod_{\alpha\in A}l_\alpha$ is a regular holomorphic 1-form on $\mathbb{C}^2$. Thus, modulo holomorphic 1-forms, we can write

$$\omega=\frac{f}{\prod_{\alpha\in A}l_\alpha}dx+\frac{g}{\prod_{\alpha\in A}l_\alpha}dy,$$

where f,g are pseudopolynomials of degree in y less than $r:=\mathrm{card}(A)$. By Hermite interpolation we can write $f=\sum_{\alpha\in A}f_\alpha l_\alpha^{-1}\prod_{\beta\in A}l_\beta$, $g=\sum_{\alpha\in A}g_\alpha l_\alpha^{-1}\prod_{\beta\in A}l_\beta$, so that finally, up to a holomorphic 1-form,

$$\omega=\sum_{\alpha\in A}\frac{f_\alpha dx+g_\alpha dy}{l_\alpha}.$$

The condition that ω restricted to $\mathbb{C}^2\setminus\{0\}$ yields a section of $\Omega^1_{\mathbb{C}^2}((d\log l_\alpha)_{\alpha\in A})$ implies that $f_\alpha=-c_\alpha g_\alpha$.

Whence, finally, modulo holomorphic 1-forms, we can write $\omega = \sum_{\alpha\in A} g_\alpha d\log l_\alpha$.

To prove the last statement, pull back such a 1-form ω: $p^*\omega = \sum_{\alpha\in A} p^*(g_\alpha d\log l_\alpha) = (\sum_{\alpha\in A} g_\alpha)d\log x + \sum_{\alpha\in A} g_\alpha d\log(t-c_\alpha)$.

This form lies in $\Omega^1_Z((\log D_\alpha)_{\alpha\in A})$ if and only if $(\sum_{\alpha\in A} g_\alpha(0)) = 0$. □

Corollary 2.11 *The dimension of the space* $H^0(\Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2}))$ *is equal to*

- 2 *in the case* $K_S^2 = 6$,
- 1 *in the case* $K_S^2 = 5$,
- 0 *in the nonnodal case* $K_S^2 = 4$,
- 0, 1 *in the nodal case* $K_S^2 = 4$, *according to* $i \neq 1, i = 1$.

Proof The previous lemma shows that, since $D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2}$, which by the way consists of four disjoint curves, then $H^0(\Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2}))$ maps onto $H^0(\Omega^1_{\mathbb{P}^2}(\log D'_{i,1}, \log D'_{i,2}, \log D'_{i,3}))$, where $D'_{i,j}$ is the line image of the curve $D_{i,j}$.

By the residue exact sequence

$$H^0(\Omega^1_{\mathbb{P}^2}(\log D'_{i,1}, \log D'_{i,2}, \log D'_{i,3})) = \left\{(c_j) \in \mathbb{C}^3 \,\middle|\, \sum_j c_j = 0\right\} \cong \mathbb{C}^2.$$

By 3. we get the subspace of $\{(c_j) \in \mathbb{C}^3 | \sum_j c_j = 0\}$ such that $c_j = 0$ iff $D'_{i,j}$ contains P_4 or P_5. The rest is a trivial verification. □

We deal now with the first cohomology groups using Riemann Roch, as already announced.

Lemma 2.12

(i) $\chi(\mathcal{O}_{D_i}(E_i - E_{i+2})) = 8$,

(ii) $\chi(\Omega^1_{\tilde{Y}}(E_i - E_{i+2})) = -e(\tilde{Y}) = K^2_{\tilde{Y}} - 12$.

In particular, it follows that $\chi(\Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2})) = 8 - e(\tilde{Y}) = K^2_{\tilde{Y}} - 4$.

Proof The third assertion follows from the first two in view of the exact sequence of locally free sheaves on $\tilde{Y}$:

$$0 \to \Omega^1_{\tilde{Y}}(E_i - E_{i+2}) \to \Omega^1_{\tilde{Y}}(\log D_i)(E_i - E_{i+2}) \to \mathcal{O}_{D_i}(E_i - E_{i+2}) \to 0.$$

(i) Observe that for $1 \leq i, j \leq 3$, we have $(E_i - E_{i+2}) \cdot D_{i,j} = 1 = (E_i - E_{i+2}) \cdot E_{i+2}$, whence $\chi(\mathcal{O}_{D_i}(E_i - E_{i+2})) = 4 \cdot \chi(\mathcal{O}_{\mathbb{P}^1}(1)) = 8$.

(ii) In order to calculate $\chi(\Omega^1_{\tilde{Y}}(E_i - E_{i+2}))$ we use the splitting principle and write formally $\Omega^1_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(A_1) \oplus \mathcal{O}_{\tilde{Y}}(A_2)$, where A_1, A_2 are 'divisors' such that $A_1 + A_2 \equiv K_{\tilde{Y}}$, $A_1 \cdot A_2 = e(\tilde{Y}) = 12 - K^2_{\tilde{Y}}$. Using that $(E_i - E_{i+2})^2 = -2$, $K_{\tilde{Y}} \cdot (E_i - E_{i+2}) = 0$, we obtain

$$\begin{aligned}
&\chi(\Omega^1_{\tilde{Y}}(E_i - E_{i+2})) \\
&\quad = \chi(\mathcal{O}_{\tilde{Y}}(A_1 + E_i - E_{i+2})) + \chi(\mathcal{O}_{\tilde{Y}}(A_2 + E_i - E_{i+2})) \\
&\quad = 2 + \frac{1}{2}((A_1 + E_i - E_{i+2})(E_i - E_{i+2} - A_2) \\
&\qquad + (A_2 + E_i - E_{i+2})(E_i - E_{i+2} - A_1)) \\
&\quad = 2 + \frac{1}{2}(-2 - 2 - 2A_1 \cdot A_2) = -e(\tilde{Y}). \qquad \square
\end{aligned}$$

Now Proposition 2.5 follows immediately:

Proof of Proposition 2.5 It is a straightforward consequence of Corollary 2.11, of Lemma 2.12, and of the Enriques-Kuranishi formula $\chi(\Theta_S) = -10\chi(\mathcal{O}_S) + 2K^2_S$. $\square$

2.4 The component of nodal Burniat surfaces is everywhere nonreduced

This section is dedicated to the proof of one of our main results:

Theorem 2.13
The subset of the Gieseker moduli space $\mathfrak{M}^{can}_{1,4}$ of canonical surfaces of general type X corresponding to Burniat surfaces S with $K^2_S = 4$ and of nodal type is an irreducible open set of dimension 2, *unirational and everywhere nonreduced.*

More precisely, there exists an integer $m \geq 2$ such that the base Def(X) *of the Kuranishi family of X is locally analytically isomorphic to $\mathbb{C}^2 \times Spec(\mathbb{C}[t]/(t^m))$.*

The corresponding subset of the moduli space $\mathfrak{M}^{min}_{1,4}$ of minimal surfaces S of general type is also everywhere non reduced and the base Def(S) *of the Kuranishi family of S is locally analytically isomorphic to $\mathbb{C}^2 \times Spec(\mathbb{C}[t]/(t^{2m}))$.*

For the general such surface $\mathrm{Aut}(S) = \mathrm{Aut}(X) = (\mathbb{Z}/2\mathbb{Z})^2$, *and this group acts trivially on the bases* Def(S), Def(X) *of the respective Kuranishi families.*

Since there are some technicalities in the proof of Theorem 2.13, let us first explain the structure of our proof.

Let S be the minimal model of a Burniat surface with $K_S^2 = 4$ and of nodal type, let X be its canonical model, and denote by $\pi : S \to X$ the blow down of the unique (-2)-curve E of S (lying over $D_{1,2}$).

We have the following G- equivariant diagram.

$$\begin{array}{ccccc} \mathrm{Def}(S) & \longrightarrow & T_S^1 = H^1(S, \Theta_S) & \xrightarrow{\kappa_S} & T_S^2 = H^2(S, \Theta_S) \cong \mathbb{C} \\ \downarrow & & \downarrow \beta & & \downarrow \cong \\ \mathrm{Def}(X) & \longrightarrow & T_X^1 = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \xrightarrow{\kappa_X} & T_X^2 = H^2(X, \Theta_X). \end{array} \qquad (2)$$

Its rows express the fact that the Kuranishi base is the zero set of the Kuranishi function (which has order ≥ 2), the left vertical maps express that each deformation (respectively infinitesimal deformation) of S induces a deformation of X. The diagram is indeed commutative as we shall show using the results of Burns and Wahl and Pinkham [8, 31].

By [8] we know that $\pi_* \Theta_S = \Theta_X$ and that $H^1_E(\Theta_S)$ has dimension 1. It follows then from the Leray spectral sequence for π_*:

– $H^2(S, \Theta_S) = H^2(X, \Theta_X)$,
– $H^1(S, \Theta_S) = H^1(X, \Theta_X) \oplus \mathcal{R}^1 \pi_* \Theta_S = H^1(\Theta_X) \oplus H^1_E(\Theta_S)$.

In particular $h^1(\Theta_X) = 2$.

The maps κ_S, κ_X are the Kuranishi obstruction maps and we shall prove that the derivative $d\kappa_X$ (whence also $d\kappa_S$) vanishes identically on $\mathrm{Def}(X)$ (cf. Corollary 2.15).

We have that $H^1_E(\Theta_S) \cong \mathbb{C}$ is the space of infinitesimal smoothings of the node. Since we shall show that all deformations of X are equisingular (i.e., preserve the node), we obtain that $\kappa_S | H^1_E(\Theta_S)$ vanishes only at the origin. On the other hand $\kappa_S | H^1(\Theta_X) \equiv 0$, and we get that set theoretically $\mathrm{Def}(X) = H^1(\Theta_X)$.

Choosing coordinates (t_1, t_2, t_3) for $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ such that $\{t_3 = 0\}$ is the hyperplane $H^1(\Theta_X)$, we see that the Kuranishi equation is a power of t_3, say t_3^m. Since the Kuranishi equation has differential vanishing at the origin, it follows that $m \geq 2$.

Now, the local map $H^1(\Theta_S) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ (cf. Theorem 2.6 of [8], see also [14]) is given by $(s_1, s_2, s_3) \mapsto (s_1, s_2, s_3^2)$, and $\mathrm{Def}(S)$ is the pull back of $\mathrm{Def}(X)$. Hence $\mathrm{Def}(S)$ is the subscheme $s_3^{2m} = 0$.

In the sequel, we shall provide more details for the above scheme of proof.

By the local to global Ext-spectral sequence, we have the “five term exact sequence”:

$$0 \to H^1(X, \Theta_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$$
$$\to H^2(X, \Theta_X) \to \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to 0. \quad (3)$$

Note that the above exact sequence is a $G = (\mathbb{Z}/2\mathbb{Z})^2$-equivariant sequence of $\mathbb{C}$-vector spaces, since all sheaves have a natural G-linearization. Moreover, observe that the map $H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \Theta_X)$ is just the first infinitesimal obstruction map to deforming the singularity.

We proceed now to calculate the decomposition in character spaces of the single terms appearing in the exact sequence.

Lemma 2.14 *The* 1*-dimensional space* $H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$ *is a space of invariants for the* G*-action.*

Proof Recall that $D_{1,2}$ is a (-2)-curve on $\tilde{Y}$ and X is a bidouble cover of the nodal Del Pezzo surface Y' of degree 4 obtained contracting $D_{1,2}$.

Moreover, the curve $D_{1,2}$ intersects exactly two other irreducible components of the branch locus, namely, $D_{2,1}$ and E_1, which is also a component of D_2.

We want to describe the structure of the morphism $f\colon X \to Y'$ locally around the A_1-singular point P'.

Locally around P' we can assume that $Y' = \{z^2 = xy\}$.

Then, locally around the node P of X, $X = \{w^2 = uv\}$, and the bidouble covering $f\colon X \to Y'$ is given by the equations: $w^2 = z$, $u^2 = x$, $v^2 = y$.

In fact, the intermediate double cover branched only on $D_{1,2}$ corresponds to the double cover branched only on P', and given by $\Phi : \mathbb{C}^2 \to Y'$, such that $\Phi(u, v) = (u^2, v^2, uv) := (x, y, z)$, while X is the double cover $w^2 = uv$ branched on the inverse images of the lines $x = z = 0$ and $y = z = 0$ (observe that for A_1 the two G actions listed in Table 3 of [12], page 93 are conjugate to each other).

The local deformation of the A_1-singularity on X is given by

$$X_t = \{w^2 = uv + t\}.$$

Then $t \in \mathbb{C}$ is a trivial representation of G and therefore X_t yields a family of G-coverings of Y' described by the equations $w^2 = z + t$, $u^2 = x$, $v^2 = y$. This proves the claim. □

Corollary 2.15 *The first infinitesimal obstruction map to deforming the singularity*

$$ob\colon H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \Theta_X)$$

is identically zero.

Proof Recall that *ob* is a G-equivariant homomorphism. Since $H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$ is a trivial G-representation, while $H^2(X, \Theta_X) = H^2(S, \Theta_S)$ is a nontrivial G-representation (cf. Proposition 2.5), it follows that $ob \equiv 0$. □

Corollary 2.16 $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)^1$.

Proof Follows immediately from the exact sequence (3) and the above corollary. □

Lemma 2.17 *$H^0(X, \mathcal{R}^1\pi_*\Theta_S) = H^1_E(\Theta_S)$ is a nontrivial character of G.*

Proof By the theorem of Brieskorn-Tjurina [5, 34], the simultaneous resolution of the node on X is given by $\frac{w-\tau}{u} = \frac{v}{w+\tau}$ where one has made the base change $\tau^2 = t$, using the notation of the proof of Lemma 2.14. The action of G lifts in a unique way to the simultaneous resolution of the family since τ must be an eigenvector with character equal to the same character of w (observe that both $w - \tau$, $w + \tau$ are eigenvectors).

Since $\mathbb{C}\tau \cong H^1_E(\Theta_S)$ as G-representation, we have proven that $H^1_E(\Theta_S)$ is an eigenspace corresponding to a nontrivial character of G. □

Since $H^1(S, \Theta_S) = H^1(\Theta_X) \oplus H^1_E(\Theta_S)$, the above lemma and Proposition 2.5 immediately imply the following

Corollary 2.18 *$H^1(X, \Theta_X)$ is a trivial G-representation, hence also* $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.

Now we are ready to prove the following

Proposition 2.19 *Let X be the canonical model of a Burniat surface with $K_S^2 = 4$ of nodal type. Then all deformations of X are deformations of the pair (X, G).*

Proof Since, by the above considerations, G acts trivially on the base of the Kuranishi family of X, it follows that $\mathrm{Def}(X) = \mathrm{Def}(X)^G = \mathrm{Def}(X, G)$. □

The consequence is then that also all deformations of S are deformations of the pair (S, G).

Now we can conclude the

Proof of Theorem 2.13 Let S be the minimal model of a Burniat surface with $K_S^2 = 4$ of nodal type. Then $S/G = \tilde{Y}$, where $\tilde{Y}$ is a weak Del Pezzo surface.

Now, by Proposition 2.19, any small deformation S_t of S is in fact a deformation of (S, G). It suffices to show that S_t/G is again a weak Del Pezzo surface, i.e., the (-2)-curve remains under a small deformation.

We remark that the (-2)-curve on $\tilde{Y}$ is $D_{1,2}$, which is a connected component of D_2, hence E is a connected component of the fixed point set $\mathrm{Fix}(\sigma_2)$ of an element $\sigma_2 \in G$.

Let now $\mathcal{S} \to T$ be a one parameter family of minimal models, such that G acts on $\mathcal{S} \to T$, with trivial action on T and the given action on the central fibre. Then the component of $Fix(\sigma_2)$ in $\mathcal{S}$ has dimension 2, whence all the deformations S_t of S carry a -2 curve E_t deformation of E. It follows that the quotient of E_t yields a -2 curve on $\tilde{Y}$.

In other words, we have shown that all deformations of X are equisingular, therefore $\mathrm{Def}(X) \subset H^1(\Theta_X)$. The Burniat family shows that $\dim(\mathrm{Def}(X)) \geq 2$, whence set theoretically $\mathrm{Def}(X) = H^1(\Theta_X)$.

Choosing coordinates (t_1, t_2, t_3) for $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ such that $\{t_3 = 0\}$ is the hyperplane $H^1(\Theta_X)$, we see that the Kuranishi equation is a power of t_3, say t_3^m. Since the Kuranishi equation has differential vanishing at the origin, it follows that $m \geq 2$.

Now, the local map $H^1(\Theta_S) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ (cf. Theorem 2.6 of [8], see also [14]) is given by $(s_1, s_2, s_3) \mapsto (s_1, s_2, s_3^2)$, and $\mathrm{Def}(S)$ is the pull back of $\mathrm{Def}(X)$. Hence $\mathrm{Def}(S)$ is the subscheme $s_3^{2m} = 0$.

The last assertion $Aut(S) = G = (\mathbb{Z}/2\mathbb{Z})^2$ will be shown in Sect. 4, cf. Remark 4.2. □

3 One parameter limits of secondary Burniat surfaces

In this section we shall show that Burniat surfaces with $K^2 \geq 4$ form a closed set of the moduli space.

In fact, we shall prove the following:

Theorem 3.1 *Each family of Burniat surfaces with* $K^2 = 4, 5, 6$ *yields a closed subset of the moduli space.*

This will be accomplished through the study of limits of one parameter families of such Burniat surfaces.

Note that for $K^2 = 6$, the above was already shown in [25], in [1] and in part I [4].

Let Y' be a normal $\mathbb{Q}$- Gorenstein surface and denote the dualizing sheaf of Y' by $\omega_{Y'}$.

Then there is a minimal natural number m such that the reflexive power $\omega_{Y'}^{\boxtimes m} := i_*(\omega_U^{\otimes m})$ (where $U := Y' \setminus \mathrm{Sing}(Y')$) is an invertible sheaf and it makes sense to define $\omega_{Y'}$ to be ample, respectively anti-ample; Y' is Gorenstein iff $m = 1$.

We shall need the following

Proposition 3.2 *Let Y' be a normal $\mathbb{Q}$-Gorenstein Del Pezzo surface (i.e., $\omega_{Y'}$ is anti-ample) with $K_{Y'}^2 \geq 4$. Then Y' is in fact Gorenstein.*

Proof Assume that $m \geq 2$. Then (cf. [33], Proposition on p. 362), there is a $\mathbb{Z}/m\mathbb{Z}$-Galois covering $p\colon W \to Y'$ such that W is Gorenstein and such that $K_W = p^* K_{Y'}$, where $\omega_{Y'}$ is the sheaf associated to the Weil divisor $K_{Y'}$. p is only branched on the singular points of Y' which are not Gorenstein.

Since $\omega_{Y'}$ is anti-ample, it follows that K_W is anti-ample, hence W is a normal Gorenstein Del Pezzo surface. As it is well known (cf. e.g. [18], Theorem 4.3) W is smoothable and in particular $K_W^2 \leq 9$, indeed $K_W^2 \leq 8$ if W is singular.

On the other hand: $K_W^2 = m K_{Y'}^2 \geq 4m$ and this implies that $m = 2$, $K_{Y'}^2 = 4$.

Therefore $K_W^2 = 8$, whence either W is the blow up of the plane in one point, or $W = Q$ a quadric in $\mathbb{P}^3$.

If W is smooth then $Y' = W/(\mathbb{Z}/2\mathbb{Z})$ has only A_1-singularities and is Gorenstein.

It remains therefore to exclude the case that W is the quadric cone.

In this case $Y' = Q/i$, where i is an involution on Q: since the quotient is not Gorenstein (see [12],Table 2 and Theorem 2.2, page 90) it acts on the tangent space at the node of Q as $-\mathrm{Id}$.

The involution i on Q acts linearly on the anticanonical model of Q, thus i extends to a linear involution I on $\mathbb{P}^3$.

The vertex $v \in Q$ is an isolated fixed point of I, and I acts as $-\mathrm{Id}$ on the tangent space of v. Therefore $H^0(Q, \mathcal{O}_Q(1))$ splits into two eigenspaces of respective dimensions 3, 1.

In particular there is a pointwise fixed hyperplane $H \subset \mathbb{P}^3$ for I. Since then $C := Q \cap H$ is pointwise fixed by I, we contradict the fact that I has only isolated fixed points on Q.

This implies that Y' is Gorenstein. □

Proposition 3.3 *Let T be a smooth affine curve, $t_0 \in T$, and let $f\colon \mathcal{X} \to T$ be a flat family of canonical surfaces. Suppose that $\mathcal{X}_t$ is the canonical model of a Burniat surface with $4 \leq K_{\mathcal{X}_t}^2$ for $t \neq t_0 \in T$. Then there is an action of $G := (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathcal{X}$ yielding a one parameter family of finite $(\mathbb{Z}/2\mathbb{Z})^2$-covers*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & T & \end{array},$$

(*i.e.*, $\mathcal{X}_t \to \mathcal{Y}_t$ *is a finite* $(\mathbb{Z}/2\mathbb{Z})^2$*-cover*), *such that* $\mathcal{Y}_t$ *is a Gorenstein Del Pezzo surface for each* $t \in T$.

Proof Note that $\mathcal{X}$ is Gorenstein, since T is smooth and the fibres have hypersurface singularities.

Since $\mathcal{X} \setminus f^{-1}(t_0) \to T \setminus \{t_0\}$ is a family of canonical models of Burniat surfaces, we have a $(\mathbb{Z}/2\mathbb{Z})^2$-action on $\mathcal{X} \setminus f^{-1}(t_0)$ (this is the Galois group action for the bicanonical map).

By [9], thm. 1.8, the $(\mathbb{Z}/2\mathbb{Z})^2$-action extends to $\mathcal{X}$.

We set $\mathcal{Y} := \mathcal{X}/(\mathbb{Z}/2\mathbb{Z})^2$ and we denote by Φ the finite morphism $\mathcal{X} \to \mathcal{Y}$.

We have for all $t \in T$:

- $K_{\mathcal{Y}_t} = K_{\mathcal{Y}}|_{\mathcal{Y}_t}$;
- $K_{\mathcal{X}_t} = K_{\mathcal{X}}|_{\mathcal{X}_t}$.

Moreover,

$$2K_{\mathcal{X}} = 2\Phi^*(K_{\mathcal{Y}}) + \Phi^*(\mathcal{B}),$$

where $\mathcal{B}$ is the branch divisor of $\Phi\colon \mathcal{X} \to \mathcal{Y}$.

Since for $t \neq t_0$ we have $2K_{\mathcal{X}_t} = -\Phi^*(K_{\mathcal{Y}_t})$, it follows that

$$2K_{\mathcal{X}} + \Phi^*(K_{\mathcal{Y}}) \equiv 0 \quad \text{on} \quad \mathcal{X} \setminus \mathcal{X}_{t_0}.$$

Since however $\mathcal{X}_{t_0} = f^{-1}(t_0)$ is irreducible, we obtain (after possibly restricting T) that $2K_{\mathcal{X}} + \Phi^*(K_{\mathcal{Y}}) \equiv 0$ on $\mathcal{X}$.

In particular, $2K_{\mathcal{X}_t} = -\Phi^*(K_{\mathcal{Y}_t})$ for all $t \in T$, which implies that $-K_{\mathcal{Y}_t}$ is ample for all $t \in T$.

Moreover, $K^2_{\mathcal{X}_t} = K^2_{\mathcal{Y}_t}$ for all $t \in T$.

By construction, $\mathcal{Y}_t$ is a Gorenstein Del Pezzo surface for $t \neq t_0$, and $\mathcal{Y}_{t_0}$ is a normal $\mathbb{Q}$-Gorenstein Del Pezzo surface, whence it is Gorenstein by Proposition 3.2. □

This implies immediately the following:

Corollary 3.4 *Consider a one parameter family of bidouble covers* $\mathcal{X} \to \mathcal{Y}$ *as in Proposition* 3.3. *Then the branch locus of* $\mathcal{X}_{t_0} \to \mathcal{Y}_{t_0}$ *is the limit of the branch locus of* $\mathcal{X}_t \to \mathcal{Y}_t$, *and it is reduced.*

Note that the limit of a line on the Del Pezzo surfaces $\mathcal{Y}_t$ is a line on the Del Pezzo surface $\mathcal{Y}_{t_0}$, and, as a consequence of the above assertion, two lines in the branch locus in $\mathcal{Y}_t$ cannot tend to the same line in $\mathcal{Y}_{t_0}$.

Remark 3.5 Let X be the canonical model of a Burniat surface with $4 \le K_X^2 \le 6$. Recall once more that X is smooth for $K_X^2 = 6, 5$, and for $K_X^2 = 4$ in the nonnodal case. For $K_X^2 = 4$ and the nodal case, X has one ordinary node.

In all three cases the branch locus consists of the union of 3 hyperplane sections, containing ν lines and $\frac{1}{2}(3K_X^2 - \nu)$ conics, where

(a) $\nu = 6$ for $K_X^2 = 6$,
(b) $\nu = 9$ for $K_X^2 = 5$,
(c) $\nu = 12$ for $K_X^2 = 4$ nonnodal,
(d) $\nu = 10$ for $K_X^2 = 4$ nodal.

In fact, in case (a) the 6 lines contained in the branch locus are: $D_{i,1}$, $1 \leq i \leq 3$, E_1, E_2, E_3. In case (b) the 9 lines contained in the branch locus are: $D_{i,j}$, $1 \leq i \leq 3$, $1 \leq j \leq 2$, E_1, E_2, E_3.

In case (c) the 12 lines in the branch locus of the bidouble cover are: $D_{i,j}$, $1 \leq i, j \leq 3$, E_1, E_2, E_3, and finally in case (d) the 10 lines are: $D_{i,1}$, $1 \leq i \leq 3$, $D_{2,2}$, $D_{2,3}$, $D_{3,2}$, $D_{3,3}$, E_1, E_2, E_3.

We shall use the following:

Proposition 3.6 [10, Propostion 1.7] *A weak Del Pezzo surface W, i.e., a smooth projective surface with nef and big anticanonical divisor $-K_W$, is either*

– $\mathbb{P}^1 \times \mathbb{P}^1$, *or*
– $\mathbb{F}_2$, *or*
– *the blow up* $\hat{\mathbb{P}}^2(P_1, \dots, P_r)$, $r \leq 8$,

at r distinct points $P_1, \dots, P_r$ satisfying the following three conditions:

(i) *no more than* 3 *P_i's are collinear*;
(ii) *no more than* 6 *P_i's lie on a conic*;
(iii) *the set $\{P_1, \dots, P_r\}$ can be partitioned into subsets $\{P_{i_1}, \dots, P_{i_k}\}$ with $P_{i_1} \in \mathbb{P}^2$, $P_{i_{(j+1)}}$ infinitely near to P_{i_j}, but not lying on the proper transform of $P_{i_{(j-1)}}$.*

Since weak Del Pezzo surfaces W are exactly the minimal resolutions of singularities of normal Gorenstein Del Pezzo surfaces Z, we use the above result to show the following technical, possibly well known result:

Proposition 3.7 *Let Z be a normal Gorenstein Del Pezzo surface of degree d.*

Then Z contains no line for $d = 9, 8$ unless $Z = \mathbb{F}_1$, which contains one line.

For $d = 7$ Z contains 2 *or* 3 *lines, and is smooth in the latter case.*

If $d = 6, 5, 4$, Z contains at most 6, *respectively* 10, *respectively* 16 *lines. If Z contains at least* 6, *respectively* 9, *respectively* 13 *lines (i.e., irreducible curves C with $C \cdot K_Z = -1$), then Z is smooth.*

Assume that $d = 4$ and that Z contains at least 10 *lines.*

Then we have the following possibilities:

(i) *Z is smooth and has* 16 *lines*;
(ii) *Z has exactly one singular point, of type* A_1, *Z has* 12 *lines and Z is the anticanonical model of the weak Del Pezzo surface obtained blowing up the plane in* 5 *distinct points such that three of them are collinear.*

We postpone the proof of the above result to the Appendix, and conclude instead the proof of the main result of this section.

Proof of Theorem 3.1 Consider a one parameter family of bidouble covers $\mathcal{X} \to \mathcal{Y}$ as in Proposition 3.3, such that $\mathcal{X}_t \to \mathcal{Y}_t$ is the bicanonical map of a Burniat surface for $t \neq t_0$. Then $\mathcal{X}_{t_0} \to \mathcal{Y}_{t_0}$ is a bidouble cover of a normal Gorenstein Del Pezzo surface of degree $K^2_{\mathcal{X}_t}$ and $\mathcal{X}_{t_0}$ has canonical singularities.

Moreover, the branch locus of $\mathcal{X}_{t_0} \to \mathcal{Y}_{t_0}$ is the limit of the branch locus of $\mathcal{X}_t \to \mathcal{Y}_t$, hence it contains at least $3(8 - K^2_{\mathcal{X}_t})$ lines in the nonnodal case, and 10 in the nodal case.

Then by Proposition 3.7 $\mathcal{Y}_{t_0}$ is smooth for $K^2_{\mathcal{X}_t} \geq 5$, while for $K^2_{\mathcal{X}_t} = 4$ it has at most one node.

Thus, for $K^2_{\mathcal{X}_t} \geq 5$, $\mathcal{X}_{t_0}$ is again a Burniat surface.

Assume that $K^2_{\mathcal{X}_t} = 4$ and that we are in the nonnodal case. We are done unless $\mathcal{Y}_{t_0}$ has a node.

In this case every line of $\mathcal{Y}_{t_0}$ is a component of the branch locus.

Note that through the node of $\mathcal{Y}_{t_0}$ pass 4 lines. By [12], Table 3, p. 93, a bidouble cover of a node branched in 4 lines is no longer a rational double point, and we have reached a contradiction.

Finally, in the nodal case, we have seen that the family $\mathcal{Y}_t$ is equisingular. By Proposition 3.7 the minimal resolution of $\mathcal{Y}_{t_0}$ is the blow up of $\mathbb{P}^2$ in 5 distinct points, none infinitely near, with P_1, P_4, P_5 collinear.

A similar representation holds for the minimal resolution W_t of $\mathcal{Y}_t$; by the above argument two of the lines passing through the node cannot be part of the branch locus. Thus the branch locus for each W_t consists of the -2 curve, of 10 (Del Pezzo) lines and a (Del Pezzo) conic. Thus the configuration of the branch locus remains of the same type and the central fibre $\mathcal{X}_{t_0}$ is again a nodal Burniat surface. □

4 Proof of the main theorems and corollaries

All the statements (except the one concerning rationality) of the two main theorems follow combining the two Theorems 2.6 and 2.13, showing that the Burniat families for $K^2_S \geq 4$ form open sets, with Theorem 3.1, showing that they form closed sets.

There remains to prove the rationality of the four connected components $\mathcal{C}$ of the moduli space constituted by Burniat surfaces with $K_S^2 \geq 4$. This is automatical for $K_S^2 = 4$ since $\mathcal{C}$ has dimension 2, and by Castelnuovo's criterion every unirational surface (over $\mathbb{C}$) is rational.

We deal next with the case $K_S^2 = 5$.

Theorem 4.1 *Let $\mathcal{C}$ be the connected component of the moduli space constituted by Burniat surfaces with $K_S^2 = 5$.*

Then $\mathcal{C}$ is a rational 3-fold.

Proof The bicanonical map of S yields a bidouble cover $\Phi_2 \colon S \to \tilde{Y}$, where $\tilde{Y}$ is the Del Pezzo of degree 5 obtained blowing up the plane in the 4 reference points.

As we saw, the branch locus consists of nine Del Pezzo lines and of 3 Del Pezzo conics. Thus there is exactly one line which is not contained in the branch locus, and we can contract it, obtaining a Del Pezzo surface Y of degree 6. The branch locus contains now the six lines of Y.

Let us fix an identification of the Galois group of Φ_2 with $G = (\mathbb{Z}/2\mathbb{Z})^2$. Then these 6 lines, which form a hexagon, are such that each pair of opposite sides is labelled by an element in $G \setminus \{0\}$.

There are two ways to contract three such lines (one for each pair) and obtain the projective plane $\mathbb{P}^2$, and they are related by the standard Cremona transformation $(x_1 : x_2 : x_3) \mapsto (x_1^{-1} : x_2^{-1} : x_3^{-1})$ associated to the linear system $|2L - E_1 - E_2 - E_3|$.

We chose the points P_1, P_2, P_3, P_4 as the reference points ($P_4 = (1:1:1)$), and we consider now the triples of lines corresponding to $D_{i,3}$, which have necessarily an equation of type $x_{i+2} = a_i x_{i+1}$.

The Cremona transformation acts by $a_i \mapsto a_i^{-1}$, the cyclical permutation of coordinates cyclically permutes the three numbers a_1, a_2, a_3, while the transposition exchanging 1 with 2 sends

$$(a_1, a_2, a_3) \mapsto (a_2^{-1}, a_1^{-1}, a_3^{-1}).$$

Composing the action of such a transposition with the action of the Cremona transformation we get the transposition of a_1 and a_2.

We conclude that there is a subgroup of index two, isomorphic to $\mathfrak{S}_3$, acting on the three numbers a_1, a_2, a_3 via the standard permutation action of the symmetric group $\mathfrak{S}_3$.

The full group by which we want to divide is generated by this subgroup and by the Cremona transformation. The invariants for the permutation representation of $\mathfrak{S}_3$ are the three elementary symmetric functions $\sigma_1, \sigma_2, \sigma_3$. The Cremona transformation acts on the field K of $\mathfrak{S}_3$-invariants by

$$\sigma_3 \mapsto \sigma_3^{-1}, \qquad \sigma_1 \mapsto \sigma_2\sigma_3^{-1}, \qquad \sigma_2 \mapsto \sigma_1\sigma_3^{-1}.$$

Obvious invariants are

$$\sigma_1 + \sigma_2\sigma_3^{-1} := y_1, \qquad \sigma_2 + \sigma_1\sigma_3^{-1} := y_2, \qquad \sigma_3 + \sigma_3^{-1} := y_3.$$

Let F be the field $\mathbb{C}(y_1, y_2, y_3)$: to show that F is the whole field of invariants it will suffice to show that $[K : F] = 2$.

Now, $F(\sigma_3)$ is a quadratic extension of F , and the two linear equations in σ_2, σ_1

$$\sigma_1 + \sigma_2\sigma_3^{-1} = y_1, \qquad \sigma_2 + \sigma_1\sigma_3^{-1} = y_2$$

have determinant $1 - \sigma_3^{-2}$, thus $\sigma_2, \sigma_1 \in F(\sigma_3)$ hence $F(\sigma_3) = K$. □

Remark 4.2 The group $\mathrm{Aut}(S) = \mathrm{Aut}(X)$ operates on the bicanonical model, with kernel G. Hence G is normal and $\mathrm{Aut}(X)/G$ operates on the normal Del Pezzo surface Z of degree 4 through a linear action. This action preserves the set of lines, and also the set of lines contained in the branch locus. Thus E_4, E_5 are left invariant and the quotient group acts on the Del Pezzo surface Y of degree 6 leaving the branch locus D invariant. As in the previous theorem, we have only two ways of representing the pair (Y, D) as a Burniat configuration, and the corresponding involution on the parameter space is nontrivial.

Hence the general surface has $\mathrm{Aut}(X) = G$.

We only recall the following

Theorem 4.3 *Let $\mathcal{C}$ be the connected component of the moduli space constituted by the primary Burniat surfaces ($K_S^2 = 6$).*

Then $\mathcal{C}$ is a rational 4*-fold.*

For a proof we refer to [4].

We derive now some easy consequences of the main theorems:

Corollary 4.4 *All surfaces S which are deformations of Burniat surfaces with $K_S^2 \geq 4$ are again Burniat surfaces, and the bicanonical map of their canonical model is a finite morphism $\Phi_2 \colon X \to Y'$, Galois with group $G = (\mathbb{Z}/2\mathbb{Z})^2$, and with image a Del Pezzo surface Y' of degree K_S^2. Y' is singular exactly for the nodal familys with $K_S^2 = 4$ (it has precisely one A_1 singularity then). In particular, Bloch's conjecture $A_0(S) = \mathbb{Z}$ holds for all the surfaces in these* 4 *connected components of the moduli space.*

Proof The last statement follows from our main theorems and the work of Inose and Mizukami [22].

In fact Inose and Mizukami show that Bloch's conjecture holds for certain classes of Inoue surfaces, which we have shown in part one [4] to coincide with the classes of Burniat surfaces. □

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Appendix: Proof of Proposition 3.7

If W is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_2$ or $\mathbb{P}^2$, then obviously W contains no line.

Thus we may assume that W is the blow up of the plane at $P_1, \dots P_r$, with $r = 9 - d$. For $r = 1$ there is only the line E_1, where we denote as customary by E_i the full transform of the point P_i.

Any line C is in particular an effective divisor such that $C^2 = CK_W = -1$, and in particular it is contained in some anticanonical divisor $H = 3L - \sum_j E_j$, where L is the nef and big divisor pull back of a line of $\mathbb{P}^2$.

Thus $C \equiv aL - \sum_j b_j E_j$ and since $LC \geq 0$, $L(H - C) \geq 0$, one gets $0 \leq a \leq 3$.

As usual $C^2 = CK_W = -1$ implies

$$a^2 + 1 = \sum_j b_j^2,$$

$$\sum_j b_j = 3a - 1 \Rightarrow \sum_j b_j(b_j - 1) = (a-1)(a-2).$$

The right hand side vanishes for $a = 1, 2$ and equals 2 for $a = 0, 3$ while each summand on the left side of the last equality is at least 2 unless $b_j = 0$ or $b_j = 1$.

Not considering the b_j's equal to zero, for $a = 0$ one has one $b_j = -1$, for $a = 1$ one has two $b_j = 1$, for $a = 2$ one has five $b_j = 1$.

While $a = 3$ can only occur for $r \geq 7$, with one b_j equal to 2, and six equal to 1.

This gives the a priori bound that the number of lines is at most

$$N(r) := r + \binom{r}{2} + \binom{r}{5}.$$

This gives the number of lines in the case where $-K_W$ is ample, namely, for $d = 7, 6, 5, 4$ we get $r = 2, 3, 4, 5$ and $N = 3, 6, 10, 16$.

Since, if $-K_W$ is ample, each such divisor is linearly equivalent to an unique effective one which is irreducible.

If $-K_W$ is not ample but nef, then there are -2 curves D, i.e., irreducible divisors D with $D \equiv aL - \sum_j b_j E_j$, $0 \le a \le 3$, and $D^2 = -2$, $DK_W = 0$. These conditions are equivalent to

$$a^2 + 2 = \sum_j b_j^2,$$

$$\sum_j b_j = 3a \Rightarrow \sum_j b_j(b_j - 1) = (a-1)(a-2).$$

By the same token $a = 1, 2$ implies $b_j = 1, 0$ and we get for $a = 1$ three $b_j = 1$, for $a = 2$ six $b_j = 1$. For $a = 0$ we get a divisor of the form $E_i - E_j$, for $a = 3$ must be $r \ge 8$ and one $b_j = 2$, seven $b_j = 1$.

What is left is to show that each -2 curve D makes the number of lines diminish sufficiently.

For $a = 2$, we must have $r \ge 6$ (and then we lose 6 lines); for $a = 1$, $D = L - E_i - E_j - E_k$, we lose 3 lines, since $L - E_i - E_j = D + E_k$. We also lose, if $r \ge 5$, $C(r-3, 2)$ lines of the form $D + (L - E_h - E_l)$.

Since we assume $r \le 7$, let us see what happens if $D = E_i - E_j$ is effective. This means that P_j is infinitely near to P_i, so we have a string of infinitely near points as in (iii) of Proposition 3.6.

Assume that this string is $P_{i_1}, \ldots, P_{i_k}$. Then each E_{i_h} is not irreducible for $h = 1, \ldots, k-1$. Also the effective divisor $L - E_j - E_{i_h}$ is not irreducible for $h = 2, \ldots, k$, and for P_j not infinitely near to P_{i_1}. Moreover $L - E_{i_1} - E_{i_2}$ is effective, and contained in $L - E_{i_h} - E_{i_l}$ whenever $h \le l$ are not equal to $1, 2$.

The loss is therefore at least

$$(k-1) + (k-1)(r-k) + \frac{1}{2}(k+1)(k-2)$$
$$= (k-1)[r-(k-1)] + \frac{1}{2}(k+1)(k-2).$$

For $k = 2$ we get a loss of $r - 1$ lines, otherwise a bigger loss.

We want to finally show that the case $r = 5$ and $k = 2$ yields the same surface which is encountered for $r = 5$, no infinitely near points, but 3 collinear points.

Consider then, as in the nodal case, 5 points such that P_1, P_4, P_5 are collinear, and let $\Psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational standard Cremona transformation based on the points P_1, P_2, P_5. On the Del Pezzo $\tilde{Y}$ obtained blowing up the 5 points Ψ corresponds to the linear system $2L - E_1 - E_2 - E_5$.

This system contracts the -2 curve to a point, as well as the lines E_4, E_3, $L - E_1 - E_2$, $L - E_2 - E_5$.

Since the -2 curve intersects E_4, we get also a representation of $\tilde{Y}$ as the blow up of the plane in five points, of which one infinitely near to the other.

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