A superficial working guide to deformations and moduli

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Dedicated to David Mumford with admiration.

Contents

The present work took place in the realm of the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds". A major part of the article was written, and the article was completed, when the author was a visiting research scholar at KIAS. I owe to David Buchsbaum the joke that an expert on algebraic surfaces is a 'superficial' mathematician.

Introduction

There are several ways to look at moduli theory, indeed the same name can at a first glance disguise completely different approaches to mathematical thinking; yet there is a substantial unity since, although often with different languages and purposes, the problems treated are substantially the same.

The most classical approach and motivation is to consider moduli theory as the fine part of classification theory: the big quest is not just to prove that certain moduli spaces exist, but to use the study of their structure in order to obtain geometrical information about the varieties one wants to classify; and using each time the most convenient incarnation of 'moduli'.

For instance, as a slogan, we might think of moduli theory and deformation theory as analogues of the global study of an algebraic variety versus a local study of its singularities, done using power series methods. On the other hand, the shape of an algebraic variety is easily recognized when it has singularities!

In this article most of our attention will be cast on the case of complex algebraic surfaces, which is already sufficiently intricate to defy many attempts of investigation. But we shall try, as much as possible, to treat the higher dimensional and more general cases as well. We shall also stick to the world of complex manifolds and complex projective varieties, which allows us to find so many beautiful connections to related fields of mathematics, such as topology, differential geometry and symplectic geometry.

David Mumford clarified the concept of biregular moduli through a functorial definition, which is extremely useful when we want a precise answer to questions concerning a certain class of algebraic varieties.

The underlying elementary concepts are the concept of normal forms, and of quotients of parameter spaces by a suitable equivalence relation, often given by the action of an appropriate group. To give an idea through an elementary geometric problem: how many are the projective equivalence classes of smooth plane curves of degree 4 admitting 4 distinct collinear hyperflexes?

A birational approach to moduli existed before, since, by the work of Cayley, Bertini, Chow and van der Waerden, varieties $X^n_d \subset \mathbb{P}^N$ in a fixed projective space, having a fixed dimension n and a fixed degree d are parametrized by the so called Chow variety $\mathcal{C}h(n; d; N)$, over which the projective group $G := \mathbb{P}GL(N+1, \mathbb{C})$ acts. And, if Z is an irreducible component of $\mathcal{C}h(n; d; N)$, the transcendence degree of the field of invariant rational functions $\mathbb{C}(Z)^G$ was classically called the number of polarized moduli for the class of varieties parametrized by Z. This topic: 'embedded varieties' is treated in the article by Joe Harris in this Handbook.

A typical example leading to the concept of stability was: take the fourfold symmetric product Z of $\mathbb{P}^2$, parametrizing 4-tuples of points in the plane. Then Z has dimension 8 and the field of invariants has transcendence degree 0. This is not a surprise, since 4 points in linear general position are a projective basis, hence they are projectively equivalent; but, if one takes 4 point to lie on a line, then there is a modulus, namely, the cross ratio. This example, plus the other basic example given by the theory of Jordan normal forms of square matrices (explained in [111] in detail) guide our understanding of the basic problem of Geometric Invariant Theory: in which sense may we consider the quotient of a variety by the action of an algebraic group. In my opinion geometric invariant theory, in spite of its beauty and its conceptual simplicity, but in view of its difficulty, is a foundational but not a fundamental tool in classification theory. Indeed one of the most difficult results, due to Gieseker, is the asymptotic stability of pluricanonical images of surfaces of general type; it has as an important corollary the existence of a moduli space for the canonical models of surfaces of general type, but the methods of proof do not shed light on the classification of such surfaces (indeed boundedness for the families of surfaces with given invariants had followed earlier by the results of Moishezon, Kodaira and Bombieri).

We use in our title the name ‘working’: this may mean many things, but in particular here our goal is to show how to use the methods of deformation theory in order to classify surfaces with given invariants.

The order in our exposition is more guided by historical development and by our education than by a stringent logical nesting.

The first guiding concepts are the concepts of Teichmüller space and moduli space associated to an oriented compact differentiable manifold M of even dimension. These however are only defined as topological spaces, and one needs the Kodaira-Spencer-Kuranishi theory in order to try to give the structure of a complex space to them.

A first question which we investigate, and about which we give some new results (proposition 1.15 and theorem 4.11), is: when is Teichmüller space locally homeomorphic to Kuranishi space?

This equality has been often taken for granted, of course under the assumption of the validity of the so called Wavrik condition (see theorem 1.5), which requires the dimension of the space of holomorphic vector fields to be locally constant under deformation .

An important role plays the example of Atiyah about surfaces acquiring a node: we interpret it here as showing that Teichmüller space is non separated (theorem 2.4). In section 4 we see that it also underlies some recent pathological behaviour of automorphisms of surfaces, recently discovered together with Ingrid Bauer: even if deformations of canonical and minimal models are essentially the same, up to finite base change, the same does not occur for deformations of automorphisms (theorems 4.6 and 4.7). The connected components for deformation of automorphisms of canonical models (X, G, α) are bigger than the connected components for deformation of automorphisms of minimal models (S, G, α'), the latter yielding locally closed sets of the moduli spaces which are locally closed but not closed.

To describe these results we explain first the Gieseker coarse moduli space for canonical models of surfaces of general type, which has the same underlying reduced space as the coarse moduli stack for minimal models of surfaces of general type. We do not essentially talk about stacks (for which an elementary presentation can be found in [67]), but we clarify how moduli spaces are obtained by gluing together Kuranishi spaces, and we show the fundamental difference for the étale equivalence relation in the two respective cases of canonical and minimal models: we exhibit examples showing that the relation is not finite (proper) in the case of minimal models (a fact which underlies the definition of Artin stacks given in [5]).

We cannot completely settle here the question whether Teichmüller space is locally homeomorphic to Kuranishi space for all surfaces of general type, as this question is related to a fundamental question about the non existence of complex automorphisms which are isotopic to the identity, but different from the identity (see however the already mentioned theorem 4.11).

Chapter five is dedicated to the connected components of moduli spaces, and to the action of the absolute Galois group on the set of irreducible components of the moduli space, and surveys many recent results.

We end by discussing concrete issues showing how one can determine a connected component of the moduli space by resorting to topological or differential arguments; we overview several results, without proofs but citing the references, and finally we prove a new result, theorem 5.6, obtained in collaboration with Ingrid Bauer.

There would have been many other interesting topics to treat, but these should probably better belong to a 'part 2' of the working guide.

1. Analytic moduli spaces and local moduli spaces: Teichmüller and Kuranishi space

1.1. Teichmüller space

Consider, throughout this subsection, an oriented real differentiable manifold M of real dimension $2n$ (without loss of generality we may a posteriori assume M and all the rest to be $\mathcal{C}^\infty$ or even $\mathcal{C}^\omega$, i.e., real-analytic).

At a later point it will be convenient to assume that M is compact.

Ehresmann ([63]) defined an **almost complex structure** on M as the structure of a complex vector bundle on the real tangent bundle $TM_{\mathbb{R}}$: namely, the action of $\sqrt{-1}$ on $TM_{\mathbb{R}}$ is provided by an endomorphism

$$J : TM_{\mathbb{R}} \to TM_{\mathbb{R}}, \text{ with } J^2 = -\mathrm{Id}.$$

It is completely equivalent to give the decomposition of the complexified tangent bundle $TM_{\mathbb{C}} := TM_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ as the direct sum of the i, respectively $-i$ eigenbundles:

$$TM_{\mathbb{C}} = TM^{1,0} \oplus TM^{0,1} \text{ where } TM^{0,1} = \overline{TM^{1,0}}.$$

In view of the second condition, it suffices to give the subbundle $TM^{1,0}$, or, equivalently, a section of the associated Grassmannian bundle $\mathcal{G}(n, TM_{\mathbb{C}})$ whose fibre at a point $x \in M$ is the variety of n-dimensional vector subspaces of the complex tangent space at x, $TM_{\mathbb{C},x}$ (note that the section must take values in the open set $\mathcal{T}_n$ of subspaces V such that V and $\bar{V}$ generate).

The space $\mathcal{AC}(M)$ of almost complex structures, once $TM_{\mathbb{R}}$ (hence all associated bundles) is endowed with a Riemannian metric, has a countable number of seminorms (locally, the sup norm on a compact K of all the derivatives of the endomorphism J), and is therefore a Fréchet space. One may for instance assume that M is embedded in some $\mathbb{R}^N$.

Assuming that M is compact, one can also consider the Sobolev k-norms (i.e., for derivatives up order k).

A closed subspace of $\mathcal{AC}(M)$ consists of the set $\mathcal{C}(M)$ of complex structures: these are the almost complex structures for which there are at each point x local holomorphic coordinates, i.e., functions $z_1, \dots, z_n$ whose differentials span the dual $(TM_y^{1,0})^\vee$ of $TM_y^{1,0}$ for each point y in a neighbourhood of x.

In general, the splitting

$$TM_{\mathbb{C}}^\vee = (TM^{1,0})^\vee \oplus (TM^{0,1})^\vee$$

yields a decomposition of exterior differentiation of functions as $df = \partial f + \bar{\partial} f$, and a function is said to be holomorphic if its differential is complex linear, i.e., $\bar{\partial} f = 0$.

This decomposition $d = \partial + \bar{\partial}$ extends to higher degree differential forms.

The theorem of Newlander-Nirenberg ([112]), first proven by Eckmann and Frölicher in the real analytic case ([61], see also [34] for a simple proof) characterizes the complex structures through an explicit equation:

Theorem 1.1. **(Newlander-Nirenberg)** *An almost complex structure J yields the structure of a complex manifold if and only if it is integrable, which means $\bar{\partial}^2 = 0$.*

Obviously the group of oriented diffeomorphisms of M acts on the space of complex structures, hence one can define in few words some basic concepts.

Definition 1.2. *Let $\mathcal{D}\mathrm{iff}^+(M)$ be the group of orientation preserving diffeomorphisms of M, and let $\mathcal{C}(M)$ the space of complex structures on M. Let $\mathcal{D}\mathrm{iff}^0(M) \subset \mathcal{D}\mathrm{iff}^+(M)$ be the connected component of the identity, the so called subgroup of diffeomorphisms which are isotopic to the identity.*

Then Dehn ([54]) defined the mapping class group of M as

$$\mathcal{M}\mathrm{ap}(M) := \mathcal{D}\mathrm{iff}^+(M)/\mathcal{D}\mathrm{iff}^0(M),$$

while the Teichmüller space of M, respectively the moduli space of complex structures on M are defined as

$$\mathcal{T}(M) := \mathcal{C}(M)/\mathcal{D}\mathrm{iff}^0(M), \ \mathfrak{M}(M) := \mathcal{C}(M)/\mathcal{D}\mathrm{iff}^+(M).$$

These definitions are very clear, however, they only show that these objects are topological spaces, and that

$$(*)\ \mathfrak{M}(M) = \mathcal{T}(M)/\mathcal{M}ap(M).$$

The simplest examples here are two: complex tori and compact complex curves.

The example of complex tori sheds light on the important question concerning the determination of the connected components of $\mathcal{C}(M)$, which are called the deformation classes in the large of the complex structures on M (cf. [39], [41]).

Complex tori are parametrized by an open set $\mathcal{T}_n$ of the complex Grassmann Manifold $Gr(n,2n)$, image of the open set of matrices

$$\{\Omega \in Mat(2n, n; \mathbb{C}) \mid (i)^n \det(\Omega\overline{\Omega}) > 0\}.$$

This parametrization is very explicit: if we consider a fixed lattice $\Gamma \cong \mathbb{Z}^{2n}$, to each matrix Ω as above we associate the subspace

$$V = (\Omega)(\mathbb{C}^n),$$

so that $V \in Gr(n,2n)$ and $\Gamma \otimes \mathbb{C} \cong V \oplus \bar{V}$.

Finally, to Ω we associate the torus $Y_V := V/p_V(\Gamma)$, $p_V : V \oplus \bar{V} \to V$ being the projection onto the first addendum.

Not only we obtain in this way a connected open set inducing all the small deformations (cf. [87]), but indeed, as it was shown in [39] (cf. also [41]) $\mathcal{T}_n$ is a connected component of Teichmüller space (as the letter $\mathcal{T}$ suggests).

It was observed however by Kodaira and Spencer already in their first article ([88], and volume II of Kodaira's collected works) that for $n \geqslant 2$ the mapping class group $SL(2n, \mathbb{Z})$ does not act properly discontinuously on $\mathcal{T}_n$. More precisely, they show that for every non empty open set $U \subset \mathcal{T}_n$ there is a point t such that the orbit $SL(2n, \mathbb{Z}) \cdot t$ intersects U in an infinite set.

This shows that the quotient is not Hausdorff at each point, probably it is not even a non separated complex space.

Hence the moral is that for compact complex manifolds it is better to consider, rather than the Moduli space, the Teichmüller space.

Moreover, after some initial constructions by Blanchard and Calabi (cf. [16], [17], , [18], [28]) of non Kähler complex structures X on manifolds diffeomorphic to a product $C \times T$, where C is a compact complex curve and T is a 2-dimensional complex torus, Sommese generalized their constructions, obtaining ([132]) that the space of complex structures on a six dimensional real torus is not connected.

These examples were then generalized in [39] [41] under the name of **Blanchard-Calabi manifolds** showing (corollary 7.8 of [41]) that also the space of complex structures on the product of a curve C of genus $g \geqslant 2$ with a four dimensional real torus is not connected, and that there is no upper bound for the dimension of Teichmüller space (even when M is fixed).

The case of compact complex curves C is instead the one which was originally considered by Teichmüller.

In this case, if the genus g is at least 2, the Teichmüller space $\mathcal{T}_g$ is a bounded domain, diffeomorphic to a ball, contained in the vector space of quadratic differentials $H^0(C, \mathcal{O}_C(2K_C))$ on a fixed such curve C.

In fact, for each other complex structure on the oriented 2-manifold M underlying C we obtain a complex curve C', and there is a unique extremal quasi-conformal map $f : C \to C'$, i.e., a map such that the Beltrami distortion $\mu_f := \bar{\partial} f / \partial f$ has minimal norm (see for instance [79] or [2]).

The fact that the Teichmüller space $\mathcal{T}_g$ is homeomorphic to a ball (see [138] for a simple proof) is responsible for the fact that the moduli space of curves $\mathfrak{M}_g$ is close to be a classifying space for the mapping class group (see [110] and the articles by Edidin and Wahl in this Handbook).

1.2. Kuranishi space

Interpreting the Beltrami distortion as a closed $(0,1)$- form with values in the dual $(TC^{1,0})$ of the cotangent bundle $(TC^{1,0})^\vee$, we obtain a particular case of the Kodaira-Spencer-Kuranishi theory of local deformations.

In fact, by Dolbeault 's theorem, such a closed form determines a cohomology class in $H^1(\Theta_C)$, where Θ_C is the sheaf of holomorphic sections of the holomorphic tangent bundle $(TC^{1,0})$: these cohomology classes are interpreted, in the Kodaira-Spencer-Kuranishi theory, as infinitesimal deformations (or derivatives of a family of deformations) of a complex structure: let us review briefly how.

Local deformation theory addresses precisely the study of the **small deformations** of a complex manifold $Y = (M, J_0)$.

We shall use here unambiguously the double notation $TM^{0,1} = TY^{0,1}$, $TM^{1,0} = TY^{1,0}$ to refer to the splitting determined by the complex structure J_0.

J_0 is a point in $\mathcal{C}(M)$, and a neighbourhood in the space of almost complex structures corresponds to a distribution of subspaces which are globally defined as graphs of an endomorphism

$$\phi : TM^{0,1} \to TM^{1,0},$$

called a **small variation of complex structure**, since one then defines

$$TM^{0,1}_\phi := \{(u, \phi(u)) | \, u \in TM^{0,1}\} \subset TM^{0,1} \oplus TM^{1,0}.$$

In terms of the new $\bar{\partial}$ operator, the new one is simply obtained by considering

$$\bar{\partial}_\phi := \bar{\partial} + \phi,$$

and the integrability condition is given by the Maurer-Cartan equation

$$(MC) \;\; \bar{\partial}(\phi) + \frac{1}{2}[\phi, \phi] = 0,$$

where $[\phi, \phi]$ denotes the Schouten bracket, which is the composition of exterior product of forms followed by Lie bracket of vector fields, and which is graded commutative.

Observe for later use that the form $F(\phi) := (\bar\partial(\phi) + \frac{1}{2}[\phi, \phi])$ is $\bar\partial$ closed, if $\bar\partial(\phi) = 0$, since then

$$\bar\partial F(\phi) = \frac{1}{2}\bar\partial[\phi, \phi] = \frac{1}{2}([\bar\partial\phi, \phi] + [\phi, \bar\partial\phi]) = 0.$$

Recall also the theorem of Dolbeault: if Θ_Y is the sheaf of holomorphic sections of $TM^{1,0}$, then $H^j(\Theta_Y)$ is isomorphic to the quotient space $\frac{Ker(\bar\partial)}{Im(\bar\partial)}$ of the space of $\bar\partial$ closed $(0, j)$-forms with values in $TM^{1,0}$ modulo the space of $\bar\partial$-exact $(0, j)$-forms with values in $TM^{1,0}$.

Our F is a map of degree 2 between two infinite dimensional spaces, the space of (0,1)-forms with values in the bundle $TM^{1,0}$, and the space of (0,2)-forms with values in $TM^{1,0}$.

Observe that, since our original complex structure J_0 corresponds to $\phi = 0$, the derivative DF of the above equation F at $\phi = 0$ is simply

$$\bar\partial(\phi) = 0,$$

hence the tangent space to the space of complex structures consists of the space of $\bar\partial$-closed forms of type (0,1) and with values in the bundle $TM^{1,0}$. Moreover the derivative of F surjects onto the space of $\bar\partial$-exact (0,2)-forms with values in $TM^{1,0}$.

We are now going to show why we can restrict our consideration only to the class of such forms ϕ in the Dolbeault cohomology group

$$H^1(\Theta_Y) := ker(\bar\partial)/Im(\bar\partial).$$

This is done by answering the question: how does the group of diffeomorphisms act on an almost complex structure J?

This is in general difficult to specify, but we can consider the infinitesimal action of a 1-parameter group of diffeomorphisms

$$\{\psi_t := exp(t(\theta + \bar\theta)|t \in \mathbb{R}\},$$

corresponding to a differentiable vector field θ with values in $TM^{1,0}$; from now on, we shall assume that M is compact, hence the diffeomorphism ψ_t is defined $\forall t \in \mathbb{R}$.

We refer to [92] and [81], lemma 6.1.4 , page 260, for the following calculation of the Lie derivative:

Lemma 1.3. *Given a 1-parameter group of diffeomorphisms*
$\{\psi_t := exp(t(\theta + \bar\theta)|t \in \mathbb{R}\}$, $(\frac{d}{dt})_{t=0}(\psi_t^*(J_0))$ *corresponds to the small variation* $\bar\partial(\theta)$.

The lemma says, roughly speaking, that locally at each point J the orbit for the group of diffeomorphisms in $\mathcal{D}iff^0(M)$ contains a submanifold, having as tangent space the forms in the same Dolbeault cohomology class of 0, which has finite codimension inside another submanifold with tangent space the space of $\bar\partial$-closed forms ϕ. Hence the tangent space to the orbit space is the space of such Dolbeault cohomology classes.

Even if we 'heuristically' assume $\bar\partial(\phi) = 0$, it looks like we are still left with another equation with values in an infinite dimensional space. However, the derivative DF surjects onto the space of exact forms, while the restriction of F to the subspace of $\bar\partial$-closed forms ($\{\bar\partial(\phi) = 0\}$ takes values in the space of $\bar\partial$-closed forms: this is the moral reason why indeed one can reduce the above equation $F = 0$, associated to a map between infinite dimensional spaces, to an equation $k = 0$ for a map $k : H^1(\Theta_Y) \to H^2(\Theta_Y)$, called the Kuranishi map.

This is done explicitly via a miraculous equation (see [87], [86],[93] and [34] for details) set up by Kuranishi in order to reduce the problem to a finite dimensional one (here Kuranishi, see [92], uses the Sobolev r- norm in order to be able to use the implicit function theorem for Banach spaces).

Here is how the Kuranishi equation is set up.

Let $\eta_1, \dots, \eta_m \in H^1(\Theta_Y)$ be a basis for the space of harmonic (0,1)-forms with values in $TM^{1,0}$, and set $t := (t_1, \dots, t_m) \in \mathbb{C}^m$, so that $t \mapsto \sum_i t_i\eta_i$ establishes an isomorphism $\mathbb{C}^m \cong H^1(\Theta_Y)$.

Then the *Kuranishi slice* (see [114] for a general theory of slices) is obtained by associating to t the unique power series solution of the following equation:

$$\phi(t) = \sum_i t_i\eta_i + \frac{1}{2}\bar\partial^* G[\phi(t), \phi(t)],$$

satisfying moreover $\phi(t) = \sum_i t_i\eta_i +$ higher order terms (G denotes here the Green operator).

The upshot is that for these forms the integrability equation simplifies drastically; the result is summarized in the following definition.

Definition 1.4. *The Kuranishi space $\mathfrak{B}(Y)$ is defined as the germ of complex subspace of $H^1(\Theta_Y)$ defined by $\{t \in \mathbb{C}^m | \ H[\phi(t), \phi(t)] = 0\}$, where H is the harmonic projector onto the space $H^2(\Theta_Y)$ of harmonic forms of type (0,2) and with values in $TM^{1,0}$.*

Kuranishi space $\mathfrak{B}(Y)$ parametrizes exactly the set of small variations of complex structure $\phi(t)$ which are integrable. Hence over $\mathfrak{B}(Y)$ we have a family of complex structures which deform the complex structure of Y.

It follows from the above arguments that the Kuranishi space $\mathfrak{B}(Y)$ surjects onto the germ of the Teichmüller space at the point corresponding to the given complex structure $Y = (M, J_0)$.

It fails badly to be a homeomorphism, and my favorite example for this is (see [30]) the one of the Segre ruled surfaces $\mathbb{F}_n$, obtained as the blow up at the origin of the projective cone over a rational normal curve of degree n, and described by Hirzebruch biregularly as $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), n \geqslant 0$.

Kuranishi space is here the vector space

$$H^1(\Theta_{\mathbb{F}_n}) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1})$$

parametrizing projectivizations $\mathbb{P}(E)$, where the rank 2 bundle E occurs as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to E \to \mathcal{O}_{\mathbb{P}^1}(n) \to 0.$$

By Grothendieck's theorem, however, E is a direct sum of two line bundles, hence we get as a possible surface only a surface $\mathbb{F}_{n-2k}$, for each $k \leqslant \frac{n}{2}$. Indeed Teichmüller space, in a neighbourhood of the point corresponding to $\mathbb{F}_n$ consists just of a finite number of points corresponding to each $\mathbb{F}_{n-2k}$, and where $\mathbb{F}_{n-2k}$ is in the closure of $\mathbb{F}_{n-2h}$ if and only if $k \leqslant h$.

The reason for this phenomenon is the following. Recall that the form ϕ can be infinitesimally changed by adding $\bar{\partial}(\theta)$; now, for $\phi = 0$, nothing is changed if $\bar{\partial}(\theta) = 0$. i.e., if $\theta \in H^0(\Theta_Y)$ is a holomorphic vector field. But the exponentials of these vector fields, which are holomorphic on $Y = \mathbb{F}_n$, but not necessarily for $\mathbb{F}_{n-2k}$, act transitively on each stratum of the stratification of $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1})$ given by isomorphism type (each stratum is thus the set of surfaces isomorphic to $\mathbb{F}_{n-2k}$).

In other words, the jumping of the dimension of $H^0(\Theta_{Y_t})$ for $t \in \mathfrak{B}(Y)$ is responsible for the phenomenon.

Indeed Kuranishi, improving on a result of Wavrik ([145]) obtained in [92] the following result.

Theorem 1.5. (Kuranishi's third theorem) *Assume that the dimension of* $H^0(\Theta_{Y_t})$ *for* $t \in \mathfrak{B}(Y)$ *is a constant function in a neighbourhood of* 0.

Then there is $k >> 0$ *and a neighbourhood* $\mathfrak{U}$ *of the identity map in the group* $\mathcal{D}\mathrm{iff}(M)$, *with respect to the* k*-th Sobolev norm, and a neighbourhood* U *of* 0 *in* $\mathfrak{B}(Y)$ *such that, for each* $f \in \mathfrak{U}$, *and* $t \neq t' \in U$, f *cannot yield a holomorphic map between* Y_t *and* $Y_{t'}$.

Kuranishi's theorem ([90],[91]) shows that Teichmüller space can be viewed as being locally dominated by a complex space of locally finite dimension (its dimension, as we already observed, may however be unbounded, cf. cor. 7.7 of [41]).

A first consequence is that Teichmüller space is locally connected by holomorphic arcs, hence the determination of the connected components of $\mathcal{C}(M)$, respectively of $\mathcal{T}(M)$, can be done using the original definition of deformation equivalence, given by Kodaira and Spencer in [88].

Corollary 1.6. *Let* $Y = (M, J)$, $Y' = (M, J')$ *be two different complex structures on* M.

Define deformation equivalence as the equivalence relation generated by direct deformation equivalence, where Y, Y' *are said to be* **direct disk deformation equivalent** *if and only if there is a proper holomorphic submersion with connected fibres* $f\colon \mathcal{Y} \to \Delta$, *where $\mathcal{Y}$ is a complex manifold, $\Delta \subset \mathbb{C}$ is the unit disk, and moreover there are two fibres of* f *biholomorphic to* Y, *respectively* Y'.

Then two complex structures on M *yield points in the same connected component of* $\mathcal{T}(M)$ *if and only if they are in the same deformation equivalence class.*

In the next subsections we shall illustrate the meaning of the condition that the vector spaces $H^0(\Theta_{Y_t})$ have locally constant dimension, in terms of deformation theory. Moreover, we shall give criteria implying that Kuranishi and Teichmüller space do locally coincide.

1.3. Deformation theory and how it is used

One can define deformations not only for complex manifolds, but also for complex spaces. The technical assumption of flatness replaces then the condition that π be a submersion.

Definition 1.7. *1) A* **deformation** *of a compact complex space* X *is a pair consisting of*

1.1) a flat proper morphism $\pi : \mathcal{X} \to T$ *between connected complex spaces (i.e.,* $\pi^* : \mathcal{O}_{T,t} \to \mathcal{O}_{\mathcal{X},x}$ *is a flat ring extension for each* x *with* $\pi(x) = t$*)*

1.2) an isomorphism $\psi : X \cong \pi^{-1}(t_0) := X_0$ *of* X *with a fibre* X_0 *of* π.

2.1) A **small deformation** *is the germ* $\pi : (\mathcal{X}, X_0) \to (T, t_0)$ *of a deformation.*

2.2) Given a deformation $\pi : \mathcal{X} \to T$ *and a morphism* $f : T' \to T$ *with* $f(t'_0) = t_0$, *the* **pull-back** $f^*(\mathcal{X})$ *is the fibre product* $\mathcal{X}' := \mathcal{X} \times_T T'$ *endowed with the projection onto the second factor* T' *(then* $X \cong X'_0$*).*

3.1) A small deformation $\pi : \mathcal{X} \to T$ *is said to be* **versal** *or* **complete** *if every other small deformation* $\pi : \mathcal{X}' \to T'$ *is obtained from it via pull back; it is said to be* **semiuniversal** *if the differential of* $f : T' \to T$ *at* t'_0 *is uniquely determined, and* **universal** *if the morphism* f *is uniquely determined.*

4) Two compact complex manifolds X, Y *are said to be* **direct deformation equivalent** *if there are a deformation* $\pi : \mathcal{X} \to T$ *of* X *with* T *irreducible and where all the fibres are smooth, and an isomorphism* $\psi' : Y \cong \pi^{-1}(t_1) := X_1$ *of* Y *with a fibre* X_1 *of* π.

Let's however come back to the case of complex manifolds, observing that in a small deformation of a compact complex manifold one can shrink the base T and assume that all the fibres are smooth.

We can now state the results of Kuranishi and Wavrik (([90], [91], [145]) in the language of deformation theory.

Theorem 1.8. (Kuranishi). *Let* Y *be a compact complex manifold: then*

I) the Kuranishi family $\pi : (\mathcal{Y}, Y_0) \to (\mathfrak{B}(Y), 0)$ *of* Y *is semiuniversal.*

II) $(\mathfrak{B}(Y), 0)$ *is unique up to (non canonical) isomorphism, and is a germ of analytic subspace of the vector space* $H^1(Y, \Theta_Y)$, *inverse image of the origin under a local holomorphic map (called Kuranishi map and denoted by* k*)* $k : H^1(Y, \Theta_Y) \to H^2(Y, \Theta_Y)$ *whose differential vanishes at the origin.*

Moreover the quadratic term in the Taylor development of the Kuranishi map k *is given by the bilinear map* $H^1(Y, \Theta_Y) \times H^1(Y, \Theta_Y) \to H^2(Y, \Theta_Y)$, *called Schouten bracket, which is the composition of cup product followed by Lie bracket of vector fields.*

III) The Kuranishi family is a versal deformation of Y_t *for* $t \in \mathfrak{B}(Y)$.

IV) The Kuranishi family is universal if $H^0(Y, \Theta_Y) = 0$.

V) **(Wavrik)** *The Kuranishi family is universal if* $\mathfrak{B}(Y)$ *is reduced and* $h^0(Y_t, \Theta_{Y_t}) :=$ dim $H^0(Y_t, \Theta_{Y_t})$ *is constant for* $t \in \mathfrak{B}(Y)$ *in a suitable neighbourhood of* 0.

In fact Wavrik in his article ([145]) gives a more general result than V); as pointed out by a referee, the same criterion has also been proven by Schlessinger (prop. 3.10 of [127]).

Wavrik says that the Kuranishi space is a local moduli space under the assumption that $h^0(Y_t, \Theta_{Y_t})$ is locally constant. This terminology can however be confusing, as we shall show, since in no way the Kuranishi space is like the moduli space locally, even if one divides out by the action of the group $\mathrm{Aut}(Y)$ of biholomorphisms of Y.

The first most concrete question is how one can calculate the Kuranishi space and the Kuranishi family. In this regard, the first resource is to try to use the implicit functions theorem.

For this purpose one needs to calculate the Kodaira Spencer map of a family $\pi : (\mathcal{Y}, Y_0) \to (T, t_0)$ of complex manifolds having a smooth base T. This is defined as follows: consider the cotangent bundle sequence of the fibration

$$0 \to \pi^*(\Omega^1_T) \to \Omega^1_{\mathcal{Y}} \to \Omega^1_{\mathcal{Y}|T} \to 0,$$

and the direct image sequence of the dual sequence of bundles,

$$0 \to \pi_*(\Theta_{\mathcal{Y}|T}) \to \pi_*(\Theta_{\mathcal{Y}}) \to \Theta_T \to \mathcal{R}^1\pi_*(\Theta_{\mathcal{Y}|T}).$$

Evaluation at the point t_0 yields a map ρ of the tangent space to T at t_0 into $H^1(Y_0, \Theta_{Y_0})$, which is the derivative of the variation of complex structure (see [87] for a more concrete description, but beware that the definition given above is the most effective for calculations).

Corollary 1.9. *Let* Y *be a compact complex manifold and assume that we have a family* $\pi : (\mathcal{Y}, Y_0) \to (T, t_0)$ *with smooth base* T, *such that* $Y \cong Y_0$, *and such that the Kodaira Spencer map* ρ_{t_0} *surjects onto* $H^1(Y, \Theta_Y)$.

Then the Kuranishi space $\mathfrak{B}(Y)$ *is smooth and there is a submanifold* $T' \subset T$ *which maps isomorphically to* $\mathfrak{B}(Y)$; *hence the Kuranishi family is the restriction of* π *to* T'.

The key point is that, by versality of the Kuranishi family, there is a morphism $f : T \to \mathfrak{B}(Y)$ inducing π as a pull back, and ρ is the derivative of f.

This approach clearly works only if Y is **unobstructed**, which simply means that $\mathfrak{B}(Y)$ is smooth. In general it is difficult to describe the Kuranishi map, and even calculating the quadratic term is nontrivial (see [78] for an interesting example).

In general, even if it is difficult to calculate the Kuranishi map, Kuranishi theory gives a lower bound for the 'number of moduli' of Y, since it shows that $\mathfrak{B}(Y)$ has dimension $\geqslant h^1(Y,\Theta_Y) - h^2(Y,\Theta_Y)$. In the case of curves $H^2(Y,\Theta_Y) = 0$, hence curves are unobstructed; in the case of a surface S

$$\dim\mathfrak{B}(S) \geqslant h^1(\Theta_S) - h^2(\Theta_S) = -\chi(\Theta_S) + h^0(\Theta_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 + h^0(\Theta_S).$$

The above is the Enriques' inequality ([66], observe that Max Noether postulated equality), proved by Kuranishi in all cases and also for non algebraic surfaces.

There have been recently two examples where resorting to the Kuranishi theorem in the obstructed case has been useful.

The first one appeared in a preprint by Clemens ([52]), who then published the proof in [53]; it shows that if a manifold is Kählerian, then there are fewer obstructions than foreseen, since a small deformation Y_t of a Kähler manifold is again Kähler, hence the Hodge decomposition still holds for Y_t.

Another independent proof was given by Manetti in [98].

Theorem 1.10. (Clemens-Manetti) *Let* Y *be a compact complex Kähler manifold.*

Then there exists an analytic automorphism of $H^2(Y,\Theta_Y)$ *with linear part equal to the identity, such that the Kuranishi map* $k : H^1(Y,\Theta_Y) \to H^2(Y,\Theta_Y)$ *takes indeed values in the intersection of the subspaces*

$$\mathrm{Ker}(H^2(Y,\Theta_Y) \to \mathrm{Hom}(H^q(\Omega_Y^p), H^{q+2}(\Omega_Y^{p-1}))$$

(the linear map is induced by cohomology cup product and tensor contraction).

Clemens' proof uses directly the Kuranishi equation, and a similar method was used by Sönke Rollenske in [123], [124] in order to consider the deformation theory of complex manifolds yielding left invariant complex structures on nilmanifolds. Rollenske proved, among other results, the following

Theorem 1.11. (Rollenske) *Let* Y *be a compact complex manifold corresponding to a left invariant complex structure on a real nilmanifold. Assume that the following condition is verified:*

(*) the inclusion of the complex of left invariant forms of pure antiholomorphic type in the Dolbeault complex

$$(\bigoplus_p H^0(\mathcal{A}^{(0,p)}(Y)), \bar\partial)$$

yields an isomorphism of cohomology groups.

Then every small deformation of the complex structure of Y *consists of left invariant complex structures.*

The main idea, in spite of the technical complications, is to look at Kuranishi's equation, and to see that everything is then left invariant.

Rollenske went over in [126] and showed that for the complex structures on nilmanifolds which are complex parallelizable Kuranishi space is defined by explicit polynomial equations, and most of the time singular.

There have been several attempts to have a more direct approach to the understanding of the Kuranishi map, namely to do things more algebraically and giving up to consider the Kuranishi slice. This approach has been pursued for instance in [128] and effectively applied by Manetti. For instance, as already mentioned, Manetti ([98]) gave a nice elegant proof of the above theorem 1.10 using the notion of differential graded Lie algebras, abbreviated by the acronym DGLA 's.

The typical example of such a DGLA is provided by the Dolbeault complex

$$(\bigoplus_p H^0(\mathcal{A}^{(0,p)}(TM_Y^{1,0})), \overline{\partial})$$

further endowed with the operation of Schouten bracket (here: the composition of exterior product followed by Lie bracket of vector fields), which is graded commutative.

The main thrust is to look at solutions of the Maurer Cartan equation $\bar{\partial}(\phi) + \frac{1}{2}[\phi, \phi] = 0$ modulo gauge transformations, that is, exponentials of sections in $H^0(\mathcal{A}^{(0,0)}(TM_Y^{1,0}))$.

The deformation theory concepts generalize from the case of deformations of compact complex manifolds to the more general setting of DGLA's, which seem to govern almost all of the deformation type problems (see for instance [99]).

1.4. Kuranishi and Teichmüller

Returning to our setting where we considered the closed subspace $\mathcal{C}(M)$ of $\mathcal{AC}(M)$ consisting of the set of complex structures on M, it is clear that there is a universal tautological family of complex structures parametrized by $\mathcal{C}(M)$, and with total space

$$\mathfrak{U}_{\mathcal{C}(M)} := M \times \mathcal{C}(M),$$

on which the group $\mathcal{D}\mathrm{iff}^+(M)$ naturally acts, in particular $\mathcal{D}\mathrm{iff}^0(M)$.

A rather simple observation is that $\mathcal{D}\mathrm{iff}^0(M)$ acts freely on $\mathcal{C}(M)$ if and only if for each complex structure Y on M the group of biholomorphisms $\mathrm{Aut}(Y)$ contains no automorphism which is differentiably isotopic to the identity (other than the identity).

Definition 1.12. *A compact complex manifold Y is said to be* **rigidified** *if* $\mathrm{Aut}(Y) \cap \mathcal{D}\mathrm{iff}^0(Y) = \{\mathrm{Id}_Y\}$. *A compact complex manifold Y is said to be cohomologically rigidified if* $\mathrm{Aut}(Y) \to \mathrm{Aut}(H^*(Y, \mathbb{Z}))$ *is injective, and rationally cohomologically rigidified if* $\mathrm{Aut}(Y) \to \mathrm{Aut}(H^*(Y, \mathbb{Q}))$ *is injective.*

The condition of being rigidified is obviously stronger than the condition $H^0(\Theta_Y) = 0$, which is necessary, else there is a positive dimensional Lie group of biholomorphic self maps, and is weaker than the condition of being cohomologically rigidified.

Compact curves of genus $g \geqslant 2$ are rationally cohomologically rigidified since if $\tau : C \to C$ is an automorphism acting trivially on cohomology, then in the product

$C \times C$ the intersection number of the diagonal Δ_C with the graph Γ_τ equals the self intersection of the diagonal, which is the Euler number $e(C) = 2 - 2g < 0$. But, if τ is not the identity, Γ_τ and Δ_C are irreducible and distinct, and their intersection number is a non negative number, equal to the number of fixed points of τ, counted with multiplicity: a contradiction.

It is an interesting question whether compact complex manifolds of general type are rigidified. It is known that already for surfaces of general type there are examples which are not rationally cohomologically rigidified (see a partial classification done by Jin Xing Cai in [27]), while examples which are not cohomologically rigidified might exist among surfaces isogenous to a product (potential candidates have been proposed by Wenfei Liu).

Jin Xing Cai pointed out to us that, for simply connected (compact) surfaces, by a result of Quinn ([118]), every automorphism acting trivially in rational cohomology is isotopic to the identity, and that he conjectures that simply connected surfaces of general type are rigidified (equivalently, rationally cohomologically rigidified).

Remark 1.13. Assume that the complex manifold Y has $H^0(\Theta_Y) = 0$, or satisfies Wavrik's condition, but is not rigidified: then by Kuranishi' s third theorem, there is an automorphism $f \in \mathrm{Aut}(Y) \cap \mathcal{D}iff^0(Y)$ which lies outside of a fixed neighbourhood of the identity. f acts therefore on the Kuranishi space, hence, in order that the natural map from Kuranishi space to Teichmüller space be injective, f must act trivially on $\mathfrak{B}(Y)$, which means that f remains biholomorphic for all small deformations of Y.

At any case, the condition of being rigidified implies that the tautological family of complex structures descends to a universal family of complex structures on Teichmüller space:

$$\mathfrak{U}_{\mathcal{T}(M)} := (M \times \mathcal{C}(M))/\mathcal{D}iff^0(M) \to \mathcal{C}(M))/\mathcal{D}iff^0(M) = \mathcal{T}(M).$$

on which the mapping class group acts.

Fix now a complex structure yielding a compact complex manifold Y, and compare with the Kuranishi family

$$\mathcal{Y} \to \mathfrak{B}(Y).$$

Now, we already remarked that there is a locally surjective continuous map of $\mathfrak{B}(Y)$ to the germ $\mathcal{T}(M)_Y$ of $\mathcal{T}(M)$ at the point corresponding to the complex structure yielding Y. For curves this map is a local homeomorphism, and this fact provides a complex structure on Teichmüller space.

Remark 1.14. Indeed we observe that more generally, if

1) the Kuranishi family is universal at any point

2) $\mathfrak{B}(Y) \to \mathcal{T}(M)_Y$ is a local homeomorphism at every point, then

Teichmüller space has a natural structure of complex space.

Moreover

3) since $\mathfrak{B}(Y) \to \mathcal{T}(M)_Y$ is surjective, it is a local homeomorphism iff it is injective; in fact, since $\mathcal{T}(M)$ has the quotient topology and it is the quotient by a group action, and $\mathfrak{B}(Y)$ is a local slice for a subgroup of $\mathcal{D}\mathrm{iff}^0(M)$, the projection $\mathfrak{B}(Y) \to \mathcal{T}(M)_Y$ is open.

The simple idea used by Arbarello and Cornalba ([2]) to reprove the result for curves is to establish the universality of the Kuranishi family for continuous families of complex structures.

In fact, if any family is locally induced by the Kuranishi family, and we have rigidified manifolds only, then there is a continuous inverse to the map $\mathfrak{B}(Y) \to \mathcal{T}(M)_Y$, and we have the desired local homeomorphism between Kuranishi space and Teichmüller space.

Since there are many cases (for instance, complex tori) where Kuranishi and Teichmüller space coincide, yet the manifolds are not rigidified, we give a simple criterion.

Proposition 1.15. *1) The continuous map $\pi\colon \mathfrak{B}(Y) \to \mathcal{T}(M)_Y$ is a local homeomorphism between Kuranishi space and Teichmüller space if there is an injective continuous map $f\colon \mathfrak{B}(Y) \to Z$, where Z is Hausdorff, which factors through π.*

2) Assume that Y is a compact Kähler manifold and that the local period map f is injective: then $\pi\colon \mathfrak{B}(Y) \to \mathcal{T}(M)_Y$ is a local homeomorphism.

3) In particular, this holds if Y is Kähler with trivial canonical divisor [1].

Proof. 1) : observe that, since $\mathfrak{B}(Y)$ is locally compact and Z is Hausdorff, it follows that f is a homeomorphism with its image $Z' := \mathrm{Im} f \subset Z$. Given the factorization $f = F \circ \pi$, then the inverse of π is the composition $f^{-1} \circ F$, hence π is a homeomorphism.

2) : if Y is Kähler, then every small deformation Y_t of Y is still Kähler, as it is well known (see [87]).

Therefore one has the Hodge decomposition

$$H^*(M, \mathbb{C}) = H^*(Y_t, \mathbb{C}) = \bigoplus_{p,q} H^{p,q}(Y_t)$$

and the corresponding period map $f\colon \mathfrak{B}(Y) \to \mathfrak{D}$, where $\mathfrak{D}$ is the period domain classifying Hodge structures of type $\{(h_{p,q}) | 0 \leqslant p, q, p+q \leqslant 2n\}$.

As shown by Griffiths in [75], see also [76] and [144], the period map is indeed holomorphic, in particular continuous, and $\mathfrak{D}$ is a separated complex manifold, hence 1) applies.

3) the previous criterion applies in several situations, for instance, when Y is a compact Kähler manifold with trivial canonical bundle.

[1] As observed by a referee, the same proof works when Y is Kähler with torsion canonical divisor, since one can consider the local period map of the canonical cover of Y

In this case the Kuranishi space is smooth (this is the so called Bogomolov-Tian-Todorov theorem, compare [19], [134], [137], and see also [119] and [83] for more general results) and the local period map for the period of holomorphic n-forms is an embedding, since the derivative of the period map, according to [75] is given by cup product

$$\mu\colon H^1(Y,\Theta_Y) \to \oplus_{p,q}\mathrm{Hom}(H^q(\Omega^p_Y), H^{q+1}(\Omega^{p-1}_Y))$$

$$= \oplus_{p,q}\mathrm{Hom}(H^{p,q}(Y), H^{p-1,q+1}(Y)).$$

If we apply it for $q = 0, p = n$, we get that μ is injective, since by Serre duality $H^1(Y,\Theta_Y) = H^{n-1}(Y,\Omega^1_Y \otimes \Omega^n_Y)^\vee$ and cup product with $H^0(\Omega^n_Y)$ yields an isomorphism with $H^{n-1}(Y,\Omega^1_Y)^\vee$ which is by Serre duality exactly isomorphic to $H^1(\Omega^{n-1}_Y)$.

□

As we shall see later, a similar criterion applies to show 'Kuranishi= Teichmüller' for most minimal models of surfaces of general type.

For more general complex manifolds, such that the Wavrik condition holds, then the Kuranishi family is universal at any point, so a program which has been in the air for a quite long time has been the one to glue together these Kuranishi families, by a sort of analytic continuation giving another variant of Teichmüller space.

We hope to be able to return on this point in the future.

2. The role of singularities

2.1. Deformation of singularities and singular spaces

The basic analytic result is the generalization due to Grauert of Kuranishi's theorem ([73], see also [130] for the algebraic analogue)

Theorem 2.1. Grauert's Kuranishi type theorem for complex spaces. *Let* X *be a compact complex space: then*

I) there is a semiuniversal deformation $\pi : (\mathcal{X}, X_0) \to (T, t_0)$ *of* X*, i.e., a deformation such that every other small deformation* $\pi' : (\mathcal{X}', X'_0) \to (T', t'_0)$ *is the pull-back of* π *for an appropriate morphism* $f : (T', t'_0) \to (T, t_0)$ *whose differential at* t'_0 *is uniquely determined.*

II) (T, t_0) *is unique up to isomorphism, and is a germ of analytic subspace of the vector space* $\mathbb{T}^1$ *of first order deformations.*

(T, t_0) *is the inverse image of the origin under a local holomorphic map (called Kuranishi map and denoted by* k*)*

$$k : \mathbb{T}^1 \to \mathbb{T}^2$$

to the finite dimensional vector space $\mathbb{T}^2$ *(called* **obstruction space***), and whose differential vanishes at the origin (the point corresponding to the point* t_0*).*

If X *is reduced, or if the singularities of* X *are local complete intersection singularities, then* $\mathbb{T}^1 = \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$.

If the singularities of X *are local complete intersection singularities, then* $\mathbb{T}^2 = \mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$.

Recall once more that this result reproves the theorem of Kuranishi ([90], [91]), which dealt with the case of compact complex manifolds, where $\mathbb{T}^j = \mathrm{Ext}^j(\Omega^1_X, \mathcal{O}_X) \cong H^j(X, \Theta_X)$, $\Theta_X := \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)$ being the sheaf of holomorphic vector fields.

There is also the local variant, concerning isolated singularities, which was obtained by Grauert in [72] extending the earlier result by Tyurina in the unobstructed case where $\mathcal{E}xt^2(\Omega^1_X, \mathcal{O}_X)_{x_0} = 0$ ([135]).

Theorem 2.2. Grauert's theorem for deformations of isolated singularities.. *Let* (X, x_0) *be the germ of an isolated singularity of a reduced complex space: then*

I) there is a semiuniversal deformation $\pi : (\mathcal{X}, X_0, x_0) \to (\mathbb{C}^n, 0) \times (T, t_0)$ *of* (X, x_0)*, i.e., a deformation such that every other small deformation* $\pi' : (\mathcal{X}', X'_0, x'_0) \to (\mathbb{C}^n, 0) \times (T', t'_0)$ *is the pull-back of* π *for an appropriate morphism* $f : (T', t'_0) \to (T, t_0)$ *whose differential at* t'_0 *is uniquely determined.*

II) (T, t_0) *is unique up to isomorphism, and is a germ of analytic subspace of the vector space* $\mathbb{T}^1_{x_0} := \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)_{x_0}$*, inverse image of the origin under a local holomorphic map (called Kuranishi map and denoted by* k*)*

$$k : \mathbb{T}^1_{x_0} = \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)_{x_0} \to \mathbb{T}^2_{x_0}$$

to the finite dimensional vector space $\mathbb{T}^2_{x_0}$ *(called* **obstruction space***), and whose differential vanishes at the origin (the point corresponding to the point* t_0*).*

The obstruction space $\mathbb{T}^2_{x_0}$ *equals* $\mathcal{E}xt^2(\Omega^1_X, \mathcal{O}_X)_{x_0}$ *if the singularity of* X *is normal.*

For the last assertion, see [130], prop. 3.1.14, page 114.

The case of complete intersection singularities was shown quite generally to be unobstructed by Tyurina in the hypersurface case ([135]), and then by Kas-Schlessinger in [82].

This case lends itself to a very explicit description.

Let $(X,0) \subset \mathbb{C}^n$ be the complete intersection $f^{-1}(0)$, where

$$f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0).$$

Then the ideal sheaf $\mathcal{I}_X$ of X is generated by $(f_1, \dots, f_p)$ and the conormal sheaf $\mathcal{N}^\vee_X := \mathcal{I}_X/\mathcal{I}^2_X$ is locally free of rank p on X.

Dualizing the exact sequence

$$0 \to \mathcal{N}^\vee_X \cong \mathcal{O}^p_X \to \Omega^1_{\mathbb{C}^n} \otimes \mathcal{O}_X \cong \mathcal{O}^n_X \to \Omega^1_X \to 0$$

we obtain (as $\Theta_X := \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)$)

$$0 \to \Theta_X \to \Theta_{\mathbb{C}^n} \otimes \mathcal{O}_X \cong \mathcal{O}^n_X \to \mathcal{N}_X \cong \mathcal{O}^p_X \to \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X) \to 0$$

which represents $\mathbb{T}^1_0 := \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)_0$ as a quotient of $\mathcal{O}^p_{X,0}$, and as a finite dimensional vector space (whose dimension will be denoted as usual by τ, which is the so called Tyurina number).

Let $(g^1, \dots, g^\tau) \in \mathcal{O}^p_{X,0}$, $g^i = (g^i_1, \dots, g^i_p)$ represent a basis of $\mathbb{T}^1_0$.

Consider now the complete intersection

$$(\mathfrak{X}, 0) := V(F_1, \dots, F_p) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, 0)$$

where

$$F_j(x,t) := f_j(x) + \sum_{i=1}^{\tau} t_i g^i_j(x).$$

Then

$$(X,0) \xhookrightarrow{i} (\mathfrak{X}, 0) \xrightarrow{\phi} (\mathbb{C}^\tau, 0)$$

where i is the inclusion and ϕ is the projection, yields the semiuniversal deformation of $(X,0)$.

In the case $p = 1$ of hypersurfaces, this representation of $\mathbb{T}^1_0 := \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)_0$ as a quotient of $\mathcal{O}_{X,0}$ yields the well known formula:

$$\mathbb{T}^1_0 = \mathcal{O}_{\mathbb{C}^n,0}/(f, f_{x_1}, \dots, f_{x_n}),$$

where $f_{x_i} := \frac{\partial f}{\partial x_i}$.

The easiest example is then the one of an ordinary quadratic singularity, or node, where we have $p = 1$, and $f = \sum_{i=1,\dots n} x_i^2$.

Then our module $\mathbb{T}^1_0 = \mathcal{O}_{\mathbb{C}^n,0}/(x_i)$ and the deformation is

$$f + t = \sum_{i=1}^{n} x_i^2 + t = 0.$$

2.2. Atiyah's example and three of its implications

Around 1958 Atiyah ([6]) made a very important discovery concerning families of surfaces acquiring ordinary double points. His result was later extended by Brieskorn and Tyurina ([136], [23], [24]) to the more general case of rational double points, which are the rational hypersurface singularities, and which are referred to as RDP's or as Du Val singularities (Patrick Du Val classified them as the surface singularities which do not impose adjunction conditions, see[59], [3], [120], [121])) or as Kleinian singularities (they are analytically isomorphic to a quotient $\mathbb{C}^2/G$, with $G \subset SL(2, \mathbb{C})$).

The crucial part of the story takes place at the local level, i.e., when one deforms the ordinary double point singularity

$$X = \{(u, v, w) \in \mathbb{C}^3 | w^2 = uv\}.$$

In this case the semiuniversal deformation is, as we saw, the family

$$\mathfrak{X} = \{(u, v, w, t) \in \mathbb{C}^4 | w^2 - t = uv\}$$

mapping to $\mathbb{C}$ via the projection over the variable t; and one observes here that $\mathfrak{X} \cong \mathbb{C}^3$.

The minimal resolution of X is obtained blowing up the origin, but we cannot put the minimal resolutions of the X_t together.

One can give two reasons for this fact. The first is algebro geometrical, in that any normal modification of $\mathfrak{X}$ which is an isomorphism outside the origin, and is such that the fibre over the origin has dimension at most 1, must be necessarily an isomorphism.

The second reason is that the restriction of the family (of manifolds with boundary) to the punctured disk $\{t \neq 0\}$ is not topologically trivial, its monodromy being given by a Dehn twist around the vanishing two dimensional sphere (see[102]).

As a matter of fact the square of the Dehn twist is differentiably isotopic to the identity, as it is shown by the fact that the family X_t admits a simultaneous resolution after that we perform a base change

$$t = \tau^2 \Rightarrow w^2 - \tau^2 = uv.$$

Definition 2.3. *Let $\mathfrak{X} \to T'$ be the family where*

$$\mathfrak{X} = \{(u, v, w, \tau) | w^2 - \tau^2 = uv\}$$

and T' is the affine line with coordinate τ.

$\mathfrak{X}$ has an isolated ordinary quadratic singularity which can be resolved either by blowing up the origin (in this way we get an exceptional divisor $\cong \mathbb{P}^1 \times \mathbb{P}^1$) or by taking the closure of one of two distinct rational maps to $\mathbb{P}^1$. The two latter resolutions are called the **small resolutions**.

One defines $\mathcal{S} \subset \mathfrak{X} \times \mathbb{P}^1$ to be one of the small resolutions of $\mathfrak{X}$, and $\mathcal{S}'$ to be the other one, namely:

$$\mathcal{S} : \{(u, v, w, \tau)(\xi) \in \mathfrak{X} \times \mathbb{P}^1 | \ \frac{w - \tau}{u} = \frac{v}{w + \tau} = \xi\}$$
$$\mathcal{S}' : \{(u, v, w, \tau)(\eta) \in \mathfrak{X} \times \mathbb{P}^1 | \ \frac{w + \tau}{u} = \frac{v}{w - \tau} = \eta\}.$$

Now, the two families on the disk $\{\tau \in \mathbb{C} || \tau| < \epsilon\}$ are clearly isomorphic by the automorphism σ_4 such that $\sigma_4(u, v, w, \tau) = (u, v, w, -\tau)$,

On the other hand, the restrictions of the two families to the punctured disk $\{\tau \neq 0\}$ are clearly isomorphic by the automorphism acting as the identity on the variables (u, v, w, τ), since over the punctured disk these two families coincide with the family $\mathfrak{X}$.

This automorphism yields a birational map $\iota : \mathcal{S} \dashrightarrow \mathcal{S}'$ which however does not extend biregularly, since $\xi u = v\eta^{-1}$.

The automorphism $\sigma := \sigma_4 \circ \iota$ acts on the restriction $\mathcal{S}^*$ of the family $\mathcal{S}$ to the punctured disk, and it acts on the given differentiably trivialized family $\mathcal{S}^*$ of manifolds with boundary via the Dehn twist on the vanishing 2-sphere.

For $\tau = 0$ the Dehn twist cannot yield a holomorphic map $\phi\colon S_0 \to S_0$, since every biholomorphism ϕ sends the (-2)-curve E to itself (E is the only holomorphic curve in its homology class), hence it acts on the normal bundle of E by scalar multiplication, therefore by an action which is homotopic to the identity in a neighbourhood of E: a contradiction.

From the above observations, one can derive several 'moral' consequences, when one globalizes the procedure.

Assume now that we have a family of compact algebraic surfaces X_t such that X_t is smooth for $t \neq 0$, and, for $t = 0$, it acquires a node.

We can then take the corresponding families S_τ and S'_τ of smooth surfaces.

We can view the family S_τ as the image of a 1 dimensional complex disk in the Teichmüller space $\mathcal{T}(S_0)$ of S_0, and then the Dehn twist σ yields a self map

$$\sigma^*\colon \mathcal{T}(S_0) \to \mathcal{T}(S_0).$$

It has the property that $\sigma^*(S_\tau) = S_{-\tau}$ for $\tau \neq 0$, but for $\tau = 0$, we have that $\sigma^*(S_0) \neq S_0$, since a map homotopically equivalent to the Dehn twist cannot yield a biholomorphic map.

Hence we get two different points of $\mathcal{T}(S_0)$, namely, $\sigma^*(S_0) \neq S_0$, which are both limits $\lim_{\tau\to 0}\sigma^*(S_\tau) = \lim_{\tau\to 0}S_{-\tau}$ and the conclusion is the following theorem, which is a slightly different version of a result of Burns and Rapoport ([25]).

Theorem 2.4. *Let S_0 be a compact complex surface which contains a (-2)-curve E, i.e., a smooth rational curve with self intersection equal to -2, obtained from the resolution of a normal surface X_0 with exactly one singular point , which is an ordinary quadratic singularity.*

Assume further that X_0 admits a smoothing deformation.

Then the Teichmüller space $\mathcal{T}(S_0)$ is not separated.

That such a surface exists is obvious: it suffices, for each degree $d \geqslant 2$, to consider a surface X_0 in $\mathbb{P}^3$, with equation $f_0(x_0, x_1, x_2, x_3) = 0$, and such that there is no monomial divisible by x_0^{d-1} appearing in f_0 with non zero coefficient. The required smoothing is gotten by setting $X_t := \{f_t := f_0 + tx_0^d = 0\}$.

This example can of course be interpreted in a second way, and with a completely different wording (non separatedness of some Artin moduli stack), which I will try to briefly explain in a concrete way.

It is clear that $\sigma^*(S_0) \neq S_0$ in Teichmüller space, but $\sigma^*(S_0)$ and S_0 yield the same point in the moduli space.

Think of the family S_τ as a 1 dimensional complex disk in the Kuranishi space of S_0: then when we map this disk to the moduli space we have two isomorphic surfaces, namely, since $\sigma^*(S_\tau) = S_{-\tau}$ for $\tau \neq 0$, we identify the point τ with the point $-\tau$.

If we consider a disk Δ, then we get an equivalence relation in $\Delta \times \Delta$ which identifies τ with the point $-\tau$. We do not need to say that $\tau = 0$ is equivalent to itself, because this is self evident. However, we have seen that we cannot extend the self map σ of the family $\mathcal{S}^*$ to the full family $\mathcal{S}$. Therefore, if we require that equivalences come from families, or, in other words, when we glue Kuranishi families, we obtain the following.

The equivalence relation in $\Delta \times \Delta$ is the image of two complex curves, one being the disk Δ, the other being the punctured disk Δ^*.

Δ maps to the diagonal $\Delta \times \Delta$, i.e., $\tau \mapsto (\tau, \tau)$, while the punctured disk Δ^* maps to the antidiagonal, deprived of the origin, that is,$\tau \neq 0, \tau \mapsto (\tau, -\tau)$.

The quotient in the category of complex spaces is indifferent to the fact that we cannot have a family extending the isomorphism ι given previously across $\tau = 0$, and the quotient is the disk Δ_t with coordinate $t := \tau^2$.

But over the disk Δ_t there will not be, as already remarked, a family of smooth surfaces.

This example by Atiyah motivated Artin in [5] to introduce his theory of Artin stacks, where one takes quotients by maps which are étale on both factors, but not proper (as the map of Δ^* into $\Delta \times \Delta$).

A third implication of Atiyah's example will show up in the section on automorphisms.

3. Moduli spaces for surfaces of general type

3.1. Canonical models of surfaces of general type

In the birational class of a non ruled surface there is, by the theorem of Castelnuovo (see e.g. [21]), a unique (up to isomorphism) minimal model S.

We shall assume from now on that S is a smooth minimal (projective) surface of general type: this is equivalent (see [21]) to the two conditions:

(*) $K_S^2 > 0$ and K_S is nef

(we recall that a divisor D is said to be **nef** if, for each irreducible curve C, we have $D \cdot C \geqslant 0$).

It is very important that, as shown by Kodaira in [85], the class of non minimal surfaces is stable by small deformation; on the other hand, a small deformation of a minimal algebraic surface of general type is again minimal (see prop. 5.5 of [8]). Therefore, the class of minimal algebraic surfaces of general type is stable by deformation in the large.

Even if the canonical divisor K_S is nef, it does not however need to be an ample divisor, indeed

The canonical divisor K_S of a minimal surface of general type S is ample iff there does not exist an irreducible curve C ($\neq 0$) on S with $K \cdot C = 0 \Leftrightarrow$ there is no (-2)-curve C on S, i.e., a curve such that $C \cong \mathbb{P}^1$, and $C^2 = -2$.

The number of (-2)-curves is bounded by the rank of the Neron Severi lattice $NS(S)$ of S, and these curves can be contracted by a contraction $\pi\colon S \to X$, where X is a normal surface which is called the **canonical model** of S.

The singularities of X are exactly Rational Double Points (in the terminology of [3]), also called Du Val or Kleinian singularities, and X is Gorenstein with canonical divisor K_X such that $\pi^*(K_X) = K_S$.

The canonical model is directly obtained from the 5-th pluricanonical map of S, but it is abstractly defined as the Projective Spectrum (set of homogeneous prime ideals) of the canonical ring

$$\mathcal{R}(S) := (\mathcal{R}(S, K_S)) := \bigoplus_{m \geqslant 0} H^0(\mathcal{O}_S(mK_S).$$

In fact if S is a surface of general type the canonical ring $\mathcal{R}(S)$ is a graded $\mathbb{C}$-algebra of finite type (as first proven by Mumford in [106]), and then the canonical model is $X = \mathrm{Proj}(\mathcal{R}(S, K_S)) = \mathrm{Proj}(\mathcal{R}(X, K_X))$.

By choosing a minimal homogeneous set of generators of $\mathcal{R}(S)$ of degrees $d_1, \dots, d_r$ one obtains a natural embedding of the canonical model X into a weighted projective space (see[56]). This is however not convenient in order to apply Geometric Invariant Theory, since one has then to divide by non reductive groups, unlike the case of pluricanonical maps, which we now discuss.

In this context the following is the content of the theorem of Bombieri ([20]), which shows with a very effective estimate the boundedness of the family of surfaces of general type with fixed invariants K_S^2 and $\chi(S) := \chi(\mathcal{O}_S)$.

Theorem 3.1. (Bombieri) *Let S be a minimal surface of general type, and consider the linear system $|mK_S|$ for $m \geqslant 5$, or for $m = 4$ when $K_S^2 \geqslant 2$.*

Then $|mK_S|$ yields a birational morphism φ_m onto its image, called the m-th pluricanonical map of S, which factors through the canonical model X as $\varphi_m = \psi_m \circ \pi$, and where ψ_m is the m-th pluricanonical map of X, associated to the linear system $|mK_X|$, and gives an embedding of the canonical model

$$\psi_m \colon X \to \cong X_m \subset \mathbb{P}H^0(\mathcal{O}_X(mK_X))^\vee = \mathbb{P}H^0(\mathcal{O}_S(mK_S))^\vee.$$

3.2. The Gieseker moduli space

The theory of deformations of complex spaces is conceptually simple but technically involved because Kodaira, Spencer, Kuranishi, Grauert et al. had to prove the convergence of the power series solutions which they produced.

It is a matter of life that tori and algebraic K3 surfaces have small deformations which are not algebraic. But there are cases, like the case of curves and of surfaces of general type, where all small deformations are still projective, and then life simplifies incredibly, since one can deal only with projective varieties or projective subschemes.

For these, the most natural parametrization, from the point of view of deformation theory, is given by the Hilbert scheme, introduced by Grothendieck ([77]). Let us illustrate this concept through the case of surfaces of general type.

For these, as we already wrote, the first important consequence of the theorem on pluricanonical embeddings is the finiteness, up to deformation, of the minimal surfaces S of general type with fixed invariants $\chi(S) = a$ and $K_S^2 = b$.

In fact, their 5-canonical models X_5 are surfaces with Rational Double Points as singularities and of degree $25b$ in a fixed projective space $\mathbb{P}^N$, where $N + 1 = P_5 := h^0(5K_S) = \chi(S) + 10K_S^2 = a + 10b$.

The Hilbert polynomial of X_5 equals

$$P(m) := h^0(5mK_S) = a + \frac{1}{2}(5m - 1)5mb.$$

Grothendieck ([77]) showed that there is

i) an integer d and

ii) a subscheme $\mathcal{H} = \mathcal{H}_P$ of the Grassmannian of codimension $P(d)$- subspaces of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$, called Hilbert scheme, such that

iii) $\mathcal{H}$ parametrizes the degree d graded pieces $H^0(\mathcal{I}_\Sigma(d))$ of the homogeneous ideals of all the subschemes $\Sigma \subset \mathbb{P}^N$ having the given Hilbert polynomial P.

We can then talk about the Hilbert point of Σ as the Plücker point

$$\Lambda^{P(d)}(r_\Sigma^\vee)$$

$$r_\Sigma : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(\Sigma, \mathcal{O}_\Sigma(d))$$

being the restriction homomorphism (surjective for d large).

Inside $\mathcal{H}$ one has the open set

$$\mathcal{H}^0 := \{\Sigma | \Sigma \text{ is reduced with only R.D.P.'s as singularities}\}.$$

This is plausible, since rational double points are hypersurface singularities, and first of all the dimension of the Zariski tangent space is upper semicontinuous, as well as the multiplicity: some more work is needed to show that the further property of being a 'rational' double point is open. The result has been extended in greater generality by Elkik in [65].

One can use the following terminology (based on results of Tankeev in [133]).

Definition 3.2. *The 5-pseudo moduli space of surfaces of general type with given invariants* K^2, χ *is the closed subscheme* $\mathcal{H}_0 \subset \mathcal{H}^0$ *(defined by fitting ideals of the direct image of* $\omega_\Sigma^{\otimes 5} \otimes \mathcal{O}_\Sigma(-1)$*),*

$$\mathcal{H}_0(\chi, K^2) := \{\Sigma \in \mathcal{H}^0 | \omega_\Sigma^{\otimes 5} \cong \mathcal{O}_\Sigma(1)\}$$

Since $\mathcal{H}_0$ is a quasi-projective scheme, it has a finite number of irreducible components, called the **deformation types** of the surfaces of general type with given invariants K^2, χ.

As we shall see, the above deformation types of canonical models coincide with the equivalence classes for the relation of deformation equivalence between minimal surfaces of general type.

Remark 3.3. The group $\mathbb{P}GL(N+1,\mathbb{C})$ acts on $\mathcal{H}_0$ with finite stabilizers (corresponding to the groups of automorphisms of each surface) and the orbits correspond to the isomorphism classes of minimal surfaces of general type with invariants K^2, χ.

Tankeev in [133] showed that a quotient by this action exists not only as a complex analytic space, but also as a Deligne Mumford stack ([55]).

Saying that the quotient is a stack is a way to remedy the fact that, over the locus of surfaces with automorphisms, there does not exist a universal family, so we have only, in Mumford's original terminology, a coarse and not a fine moduli space.

In a technically very involved paper ([71]) Gieseker showed that, if one replaces the 5-canonical embedding by an m-canonical embedding with much higher m, then the Hilbert point $\Lambda^{P(d)}(r_\Sigma^\vee)$ is a stable point; this means that, beyond the already mentioned property that the stabilizer is finite, that there are polynomial functions which are invariant for the action of $SL(N+1,\mathbb{C})$ and which do not vanish at the point, so that the Hilbert point maps to a point of the Projective spectrum of the ring of $SL(N+1,\mathbb{C})$-invariants.

The result of Gieseker leads then to the following

Theorem 3.4. (Gieseker) *For m very large, the quotient*

$$\mathfrak{M}^{can}_{\chi,K^2} := \mathcal{H}_0(\chi, K^2)/SL(N+1,\mathbb{C})$$

exists as a quasi-projective scheme. It is independent of m *and called the* **Gieseker moduli space** *of canonical models of surfaces of general type with invariants* χ, K^2.

It should be noted that at that time Gieseker only established the result for a field of characteristic zero; as he remarks in the paper, the only thing which was missing then in characteristic p was the boundedness of the surfaces of general type with given invariants χ, K^2. This result was provided by Ekedahl's extension of Bombieri's theorem to characteristic p ([64], see also [45] and [46] for a simpler proof).

3.3. Minimal models versus canonical models

Let us go back to the assertion that deformation equivalence classes of minimal surfaces of general type are the same thing as deformation types of canonical models (a fact which is no longer true in higher dimension).

We have more precisely the following theorem.

Theorem 3.5. *Given two minimal surfaces of general type* S, S' *and their respective canonical models* X, X', *then*

S *and* S' *are deformation equivalent* $\Leftrightarrow$ X *and* X' *are deformation equivalent.*

The idea of the proof can be simplified by the elementary observation that, in order to analyse deformation equivalence, one may restrict oneself to the case of families parametrized by a base T with $\dim(T) = 1$: since two points in a complex space $T \subset \mathbb{C}^n$ (or in an algebraic variety) belong to the same irreducible component of T if and only if they belong to an irreducible curve $T' \subset T$. And one may further reduce to the case where T is smooth simply by taking the normalization $T^0 \to T_{red} \to T$ of the reduction T_{red} of T, and taking the pull-back of the family to T^0.

But the crucial point underlying the result is the theorem on the so-called simultaneous resolution of singularities (cf. [136],[22], [23], [24])

Theorem 3.6. (Simultaneous resolution according to Brieskorn and Tjurina). *Let* $T := \mathbb{C}^\tau$ *be the basis of the semiuniversal deformation of a Rational Double Point* $(X,0)$*. Then there exists a ramified Galois cover* $T' \to T$*, with* T' *smooth* $T' \cong \mathbb{C}^\tau$ *such that the pull-back* $\mathcal{X}' := \mathcal{X} \times_T T'$ *admits a simultaneous resolution of singularities* $p : \mathcal{S}' \to \mathcal{X}'$ *(i.e.,* p *is bimeromorphic and all the fibres of the composition* $\mathcal{S}' \to \mathcal{X}' \to T'$ *are smooth and equal, for* t_0'*, to the minimal resolution of singularities of* $(X,0)$.

We reproduce Tjurina' s proof for the case of A_n-singularities, observing that the case of the node was already described in the previous section.

Proof. Assume that we have the A_n-singularity

$$\{(x, y, z) \in \mathbb{C}^3 | xy = z^{n+1}\}.$$

Then the semiuniversal deformation is given by

$$\mathcal{X} := \{((x, y, z), (a_2, \dots a_{n+1})) \in \mathbb{C}^3 \times \mathbb{C}^n | xy = z^{n+1} + a_2 z^{n-1} + \dots a_{n+1}\},$$

the family corresponding to the natural deformations of the simple cyclic covering.

We take a ramified Galois covering with group $\mathcal{S}_{n+1}$ corresponding to the splitting polynomial of the deformed degree $n + 1$ polynomial

$$\mathcal{X}' := \{((x, y, z), (\alpha_1, \dots \alpha_{n+1})) \in \mathbb{C}^3 \times \mathbb{C}^{n+1} | \sum_j \alpha_j = 0, \ xy = \prod_j (z - \alpha_j)\}.$$

One resolves the new family $\mathcal{X}'$ by defining $\phi_i : \mathcal{X}' \dashrightarrow \mathbb{P}^1$ as

$$\phi_i := (x, \prod_{j=1}^{i} (z - \alpha_j))$$

and then taking the closure of the graph of $\Phi := (\phi_1, \dots \phi_n) : \mathcal{X}' \dashrightarrow (\mathbb{P}^1)^n$.

□

Here the Galois group G of the covering $T' \to T$ in the above theorem is the Weyl group corresponding to the Dynkin diagram of the singularity (whose vertices are the (-2) curves in the minimal resolution, and whose edges correspond to the intersection points).

I.e., if $\mathcal{G}$ is the simple algebraic group corresponding to the Dynkin diagram (see [80]), and H is a Cartan subgroup, N_H its normalizer, then the Weyl group is the factor group $W := N_H/H$. For example, A_n corresponds to the group $SL(n+1,\mathbb{C})$, its Cartan subgroup is the subgroup of diagonal matrices, which is normalized by the symmetric group $\mathcal{S}_{n+1}$, and N_H is here a semidirect product of H with $\mathcal{S}_{n+1}$. E. Brieskorn ([24]) found later a direct explanation of this phenomenon.

The Weyl group W and the quotient $T = T'/W$ play a crucial role in the understanding of the relations between the deformations of the minimal model S and the canonical model X, which is a nice discovery by Burns and Wahl ([26]).

But, before we do that, let us make the following important observation, saying that the local analytic structure of the Gieseker moduli space is determined by the action of the group of automorphisms of X on the Kuranishi space of X.

Remark 3.7. Let X be the canonical model of a minimal surface of general type S with invariants χ, K^2. The isomorphism class of X defines a point $[X] \in \mathfrak{M}^{can}_{\chi,K^2}$.

Then the germ of complex space $(\mathfrak{M}^{can}_{\chi,K^2}, [X])$ is analytically isomorphic to the quotient $\mathfrak{B}(X)/\mathrm{Aut}(X)$ of the Kuranishi space of X by the finite group $\mathrm{Aut}(X) = \mathrm{Aut}(S)$.

Forgetting for the time being about automorphisms, and concentrating on families, we want to explain the 'local contributions to global deformations of surfaces', in the words of Burns and Wahl ([26]).

Let S be a minimal surface of general type and let X be its canonical model. To avoid confusion between the corresponding Kuranishi spaces, denote by $\mathrm{Def}(S)$ the Kuranishi space for S, respectively $\mathrm{Def}(X)$ the Kuranishi space of X.

Their result explains the relation holding between $\mathrm{Def}(S)$ and $\mathrm{Def}(X)$.

Theorem 3.8. **(Burns - Wahl)** *Assume that K_S is not ample and let $\pi : S \to X$ be the canonical morphism.*

Denote by $\mathcal{L}_X$ the space of local deformations of the singularities of X (Cartesian product of the corresponding Kuranishi spaces) and by $\mathcal{L}_S$ the space of deformations of a neighbourhood of the exceptional locus of π. Then $\mathrm{Def}(S)$ is realized as the fibre product associated to the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Def}(S) & \longrightarrow & \mathrm{Def}(S_{\mathrm{Exc}(\pi)}) =: \mathcal{L}_S \cong \mathbb{C}^\nu, \\ \downarrow & & \downarrow \lambda \\ L\colon\ \mathrm{Def}(X) & \longrightarrow & \mathrm{Def}(X_{\mathrm{Sing}X}) =: \mathcal{L}_X \cong \mathbb{C}^\nu, \end{array}$$

where ν is the number of rational (-2)-curves in S, and λ is a Galois covering with Galois group $W := \oplus_{i=1}^r W_i$, the direct sum of the Weyl groups W_i of the singular points of X (these are generated by reflections, hence yield a smooth quotient, see [50]*).*

An immediate consequence is the following

Corollary 3.9. **(Burns - Wahl)** *1) $\psi : \mathrm{Def}(S) \to \mathrm{Def}(X)$ is a finite morphism, in particular, ψ is surjective.*

2) If the derivative of $\mathrm{Def}(X) \to \mathcal{L}_X$ *is not surjective (i.e., the singularities of* X *cannot be independently smoothen by the first order infinitesimal deformations of* X*), then* $\mathrm{Def}(S)$ *is singular.*

Moreover one has a further corollary

Corollary 3.10. [35]
If the morphism L *is constant, then* $\mathrm{Def}(S)$ *is everywhere non reduced,*

$$\mathrm{Def}(S) \cong \mathrm{Def}(X) \times \lambda^{-1}(0).$$

In [35] several examples were exhibited, extending two previous examples by Horikawa and Miranda. In these examples the canonical model X is a hypersurface of degree d in a weighted projective space:

$$X_d \subset \mathbb{P}(1,1,p,q), d > 2+p+q,$$

where

- $X_d \subset \mathbb{P}(1,1,2,3)$, $d = 1+6k$, X has one singularity of type A_1 and one of type A_2, or
- $X_d \subset \mathbb{P}(1,1,p,p+1)$, $d = p(k(p+1)-1)$, X has one singularity of type A_p, or
- $X_d \subset \mathbb{P}(1,1,p,rp-1)$, $d = (kp-1)(rp-1)$, $r > p-2$, X has one singularity of type A_{p-1}.

The philosophy in these examples (some hard calculations are however needed) is that all the deformations of X remain hypersurfaces in the same projective space, and this forces X to preserve, in view of the special arithmetic properties of the weights and of the degree, its singularities.

3.4. Number of moduli done right

The interesting part of the discovery of Burns and Wahl is that they completely clarified the background of an old dispute going on in the late 1940's between Francesco Severi and Beniamino Segre. The (still open) question was: given a degree d, which is the maximum number $\mu(d)$ of nodes that a normal surface $X \subset \mathbb{P}^3$ of degree d can have ?

The answer is known only for small degree $d \leqslant 6$: $\mu(2) = 1$, $\mu(3) = 4$ (Cayley's cubic), $\mu(4) = 16$ (Kummer surfaces), $\mu(5) = 31$ (Togliatti quintics), $\mu(6) = 65$ (Barth's sextic), and Severi made the following bold assertion: an upper bound is clearly given by the 'number of moduli', i.e., the dimension of the moduli space of the surfaces of degree d in $\mathbb{P}^3$; this number equals the difference between the dimension of the underlying projective space $\frac{(d+3)(d+2)(d+1)}{6} - 1$ and the dimension of the group of projectivities, at least for $d \geqslant 4$ when the general surface of degree d has only a finite group of projective automorphisms.

One should then have $\mu(d) \leqslant \nu(d) := \frac{(d+3)(d+2)(d+1)}{6} - 16$, but Segre ([129]) found some easy examples contradicting this inequality, the easiest of which are some surfaces of the form

$$L_1(x) \cdot \dots \cdot L_d(x) - M(x)^2,$$

where d is even, the $L_i(x)$ are linear forms, and $M(x)$ is a homogeneous polynomial of degree $\frac{d}{2}$.

Whence the easiest Segre surfaces have $\frac{1}{4}d^2(d-1)$ nodes, corresponding to the points where $L_i(x) = L_j(x) = M(x) = 0$, and this number grows asymptotically as $\frac{1}{4}d^3$, versus Severi's upper bound, which grows like $\frac{1}{6}d^3$ (in fact we know nowadays, by Chmutov in [51], resp. Miyaoka in [103], that $\frac{5}{12}d^3 \leqslant \mu(d) \leqslant \frac{4}{9}d^3$).

The problem with Severi's claim is simply that the nodes impose independent conditions infinitesimally, but only for the smooth model S: in other words, if X has δ nodes, and S is its desingularization, then $\mathrm{Def}(S)$ has Zariski tangent dimension at least δ, while it is not true that $\mathrm{Def}(S)$ has dimension at least δ. Burns and Wahl, while philosophically rescuing Severi' s intuition, showed in this way that there are a lot of examples of obstructed surfaces S, thereby killing Kodaira and Spencer's dream that their cohomology dimension $h^1(\Theta_S)$ would be the expected number of moduli.

3.5. The moduli space for minimal models of surfaces of general type

In this section we shall derive some further moral consequences from the result of Burns and Wahl.

For simplicity, consider the case where the canonical model X has only one double point, and recall the notation introduced previously, concerning the local deformation of the node, given by the family

$$uv = w^2 - t,$$

the pull back family

$$\mathfrak{X} = \{(u, v, w, \tau) | w^2 - \tau^2 = uv\}$$

and the two families

$$\mathcal{S} : \{(u, v, w, \tau)(\xi) \in \mathfrak{X} \times \mathbb{P}^1 | \ \frac{w-\tau}{u} = \frac{v}{w+\tau} = \xi\}$$

$$\mathcal{S}' : \{(u, v, w, \tau)(\eta) \in \mathfrak{X} \times \mathbb{P}^1 | \ \frac{w+\tau}{u} = \frac{v}{w-\tau} = \eta\}.$$

There are two cases to be considered in the following oversimplified example:

1) $t \in \Delta$ is a coordinate of an effective smoothing of the node, hence we have a family $\mathcal{S}$ parametrized by $\tau \in \Delta$

2) we have no first order smoothing of the node, hence the Spectrum of the ring $\mathbb{C}[\tau]/(\tau^2)$ replaces Δ.

In case 1), we have two families $\mathcal{S}, \mathcal{S}'$ on a disk Δ with coordinate τ, which are isomorphic over the punctured disk. This induces for the punctured disk Δ with coordinate τ, base of the first family, an equivalence relation induced by the isomorphism with itself where τ is exchanged with $-\tau$; in this case the ring of germs of holomorphic functions at the origin which are invariant for the resulting equivalence relation is just the ring of power series $\mathbb{C}\{t\}$, and we have a smooth 'moduli space'.

In case 2), there is no room for identifying τ with $-\tau$, since if we do this on $\mathbb{C}[\tau]/(\tau^2)$, which has only one point, we glue the families without inducing the identity on the base, and this is not allowed. In this latter case we are left with the non reduced scheme $\mathrm{Spec}(\mathbb{C}[\tau]/(\tau^2))$ as a 'moduli space'.

Recall now Mumford's definition of a coarse moduli space ([107], page 99, definition 5.6 , and page 129, definition 7.4) for a functor of type $\mathcal{S}urf$, such as $\mathcal{S}urf^{min}$, associating to a scheme T the set $\mathcal{S}urf^{min}(T)$ of isomorphism classes of families of smooth minimal surfaces of general type over T, or as $\mathcal{S}urf^{can}$, associating to a scheme T the set $\mathcal{S}urf^{can}(T)$ of isomorphism classes of families of canonical models of surfaces of general type over T.

It should be a scheme A, given together with a morphism Φ from the functor $\mathcal{S}urf$ to $h_A := \mathcal{H}om(-, A)$, such that

(1) for all algebraically closed fields k, $\Phi(\mathrm{Spec}\ k)\colon \mathcal{S}urf(\mathrm{Spec}\ k) \to h_A(\mathrm{Spec}\ k)$ is an isomorphism
(2) any other morphism Ψ from the functor $\mathcal{S}urf$ to a functor h_B factors uniquely through $\chi\colon h_A \to h_B$.

Since any family of canonical models $p\colon \mathcal{X} \to T$ induces, once we restrict T and we choose a local frame for the direct image sheaf $p_*(\omega^m_{\mathcal{X}|T})$ a family of pluricanonical models embedded in a fixed $\mathbb{P}^{P_m - 1}$, follows

Theorem 3.11. *The Gieseker moduli space $\mathfrak{M}^{can}_{\chi,K^2}$ is the coarse moduli space for the functor $\mathcal{S}urf^{can}_{\chi,K^2}$, i.e., for canonical models of surfaces S of general type with given invariants χ, K^2. Hence it gives a natural complex structure on the topological space $\mathfrak{M}(S)$, for S as above.*

As for the case of algebraic curves, we do not have a fine moduli space, i.e., the functor is not representable by this scheme. Here, automorphisms are the main obstruction to the existence of a fine moduli space: dividing the universal family over the Hilbert scheme by the linear group we obtain a family over the quotient coarse moduli space such that the fibre over the isomorphism class of a canonical model X, in the case where the group of automorphisms $\mathrm{Aut}(X)$ is non trivial, is the quotient $X/\mathrm{Aut}(X)$. And $X/\mathrm{Aut}(X)$ is then not isomorphic to X.

Instead, in the case of the functor $\mathcal{S}urf^{min}(T)$, there is a further problem: that the equivalence relation (of isomorphism of families) is not proper on the parameter space, as we already mentioned.

While for curves we have a Deligne-Mumford stack, which amounts roughly speaking to take more general functors than functors which are set valued, this no longer holds for surfaces of general type. Therefore Artin in [5] had to change the definition, allowing more general equivalence relations. The result is ([5], Example 5.5 page 182)

Theorem 3.12. (Artin) *There exists a moduli space $\mathfrak{M}^{min}_{\chi,K^2}$ which is an algebraic Artin stack for minimal surfaces of general type with given invariants χ, K^2.*

The beginning step for Artin was to show that there is a finite number of algebraic families parametrizing all the minimal models with given invariants: this step is achieved by Artin in [4] showing that the simultaneous resolution of a family of canonical models can be done not only in the holomorphic category, but also over schemes, provided that one allows a base change process producing locally quasi-separated algebraic spaces.

After that, one can consider the equivalence relation given by isomorphisms of families.

We shall illustrate the local picture by considering the restriction of this equivalence relation to the base $Def(S)$ of the Kuranishi family.

(I) First of all we have an action of the group $Aut(S)$ on $Def(S)$, and we must take the quotient by this action.

In fact, if $g \in Aut(S)$, then g acts on $Def(S)$, and if $\mathcal{S} \to Def(S)$ is the universal family, we can take the pull back family $g^*\mathcal{S}$. By the universality of the Kuranishi family, we have an isomorphism $g^*\mathcal{S} \cong \mathcal{S}$ lying over the identity of $Def(S)$, and by property (2) we must take the quotient of $Def(S)$ by this action of $Aut(S)$.

(II) Let now $w \in W$ be an element of the Weyl group which acts on $Def(S)$ via the Burns-Wahl fibre product. We let U_w be the open set of $\mathcal{L}_S$ where the transformation w acts freely (equivalently, w being a pseudo reflection, U_w is the complement of the hyperplane of fixed points of w), and we let $Def(S)_w$ be equal to the open set inverse image of U_w.

Since the action of w is free on $Def(S)_w$, we obtain that w induces an isomorphism of the family $\mathcal{S}_w \to Def(S)_w$ with its pull back under w, inducing the identity on the base: hence we have to take the graph of w on $Def(S)_w$, and divide $Def(S)_w$ by the action of w.

(III) The 'equivalence relation' on $Def(S)$ is thus generated by (I) and (II), but it is not really a proper equivalence relation.

The complex space underlying $\mathfrak{M}^{min}_{\chi,K^2}$ is obtained taking the subsheaf of $\mathcal{O}_{Def(S)}$ consisting of the functions which are invariant for this equivalence relation (i.e., in case (II) , their restriction to $Def(S)_w$ should be w- invariant).

$\mathfrak{M}^{min}_{\chi,K^2}$ has the same associated reduced complex space as $\mathfrak{M}^{can}_{\chi,K^2}$, but a different ringed space structure, as the examples of [35] mentioned after corollary 3.10 show, see the next subsection.

In fact, the main difference is that $\mathfrak{M}^{can}_{\chi,K^2}$ is locally, by Burns-Wahl's fibre product theorem, the quotient of $Def(S)$ by the group G' which is the semidirect product of the Weyl group W by $\mathrm{Aut}(S) = \mathrm{Aut}(X)$, as a ringed space (the group G' will make its appearance again in the concrete situation of Lemma 4.9).

Whereas, for $\mathfrak{M}^{min}_{\chi,K^2}$ the action on the set is the same, but the action of an element w of the Weyl group on the sheaf of regular functions is only there on an open set ($\mathrm{Def}(S)_w$) and this set $\mathrm{Def}(S)_w$ may be empty if $\mathrm{Def}(X)$ maps to a branch divisor of the quotient map $\mathcal{L}_S \to \mathcal{L}_X$.

A general question for which we have not yet found the time to provide an answer is whether there is a quasi-projective scheme whose underlying complex space is $\mathfrak{M}^{min}_{\chi,K^2}$: we suspect that the answer should be positive.

3.6. Singularities of moduli spaces

In general one can define

Definition 3.13. *The local moduli space* $(\mathfrak{M}^{min,loc}_{\chi,K^2}, [S])$ *of a smooth minimal surface of general type* S *is the quotient* $\mathrm{Def}(S)/\mathrm{Aut}(S)$ *of the Kuranishi space of* S *by the action of the finite group of automorphisms of* S.

Caveat: whereas, for the canonical model X, $\mathrm{Def}(X)/\mathrm{Aut}(X)$ is just the analytic germ of the Gieseker moduli space at the point corresponding to the isomorphism class of X, the local moduli space $(\mathfrak{M}^{min,loc}_{\chi,K^2}, [S]) := \mathrm{Def}(S)/\mathrm{Aut}(S)$ is different in general from the analytic germ of the moduli space $(\mathfrak{M}^{min}_{\chi,K^2}, [S])$, though it surjects onto the latter. But it is certainly equal to it in the special case where the surface S has ample canonical divisor K_S.

The Cartesian diagram by Burns and Wahl was used in [35] to construct everywhere non reduced moduli spaces $(\mathfrak{M}^{min,loc}_{\chi,K^2}, [S]) := \mathrm{Def}(S)/\mathrm{Aut}(S)$ for minimal models of surfaces of general type.

In this case the basic theorem is

Theorem 3.14. **(**[35]**)** *There are (generically smooth) connected components of Gieseker moduli spaces* $\mathfrak{M}^{can}_{\chi,K^2}$ *such that all the canonical models in it are singular.*

Hence the local moduli spaces $(\mathfrak{M}^{min,loc}_{\chi,K^2}, [S])$ *for the corresponding minimal models are everywhere non reduced, and the same occurs for the germs* $(\mathfrak{M}^{min}_{\chi,K^2}, [S])$.

The reason is simple: we already mentioned that if we take the fibre product

$$\begin{array}{ccc} \mathrm{Def}(S) & \longrightarrow & \mathrm{Def}(S_{\mathrm{Exc}(\pi)}) =: \mathcal{L}_S \cong \mathbb{C}^{\nu}, \\ \downarrow & & \downarrow \lambda \\ \mathrm{Def}(X) & \longrightarrow & \mathrm{Def}(X_{\mathrm{Sing}X}) =: \mathcal{L}_X \cong \mathbb{C}^{\nu}, \end{array}$$

the lower horizontal arrow maps to a reduced point, hence $\mathrm{Def}(S)$ is just the product $\mathrm{Def}(X) \times \lambda^{-1}(0)$, and $\lambda^{-1}(0)$ is a non reduced point, spectrum of an Artin local ring of length equal to the cardinality of the Galois group $W := \oplus_{i=1}^{r} W_i$.

Hence $\mathrm{Def}(S)$ is everywhere non reduced. Moreover, one can show that, in the examples considered, the general surface has no automorphisms, i.e., there is an open set for which the analytic germ of the Gieseker moduli space coincides with the Kuranishi family $\mathrm{Def}(X)$, and the family of canonical models just obtained is equisingular.

Hence, once we consider the equivalence relation on $\mathrm{Def}(S)$ induced by isomorphisms, the Weyl group acts trivially (because of equisingularity, we get $\mathrm{Def}(S)_w = \emptyset$ $\forall w \in W$). Moreover, by our choice of a general open set, $\mathrm{Aut}(S)$ is trivial.

The conclusion is that $(\mathfrak{M}^{min}_{\chi,K^2}, [S])$ is locally isomorphic to the Kuranishi family $\mathrm{Def}(S)$, hence everywhere non reduced. $\square$

Using an interesting result of M'nev about line configurations, Vakil ([141]) was able to show that 'any type of singularity' can occur for the Gieseker moduli space (his examples are such that $S = X$ and S has no automorphisms, hence they produce the desired singularities also for the local moduli space, for the Kuranishi families; they also produce singularities for the Hilbert schemes, because his surfaces have $q(S) := h^1(\mathcal{O}_S) = 0$).

Theorem 3.15. (Vakil's 'Murphy's law') *Given any singularity germ of finite type over the integers, there is a Gieseker moduli space $\mathfrak{M}^{can}_{\chi,K^2}$ and a surface S with ample canonical divisor K_S (hence $S = X$) such that $(\mathfrak{M}^{can}_{\chi,K^2}, [X])$ realizes the given singularity germ.*

In the next section we shall see more instances where automorphisms play an important role.

4. Automorphisms and moduli

4.1. Automorphisms and canonical models

The good thing about differential forms is that any group action on a complex manifold leads to a group action on the vector spaces of differential forms.

Assume that G is a group acting on a surface S of general type, or more generally on a Kähler manifold Y: then G acts linearly on the Hodge vector spaces $H^{p,q}(Y) \cong H^q(\Omega^p_Y)$, and also on the vector spaces $H^0((\Omega^n_Y)^{\otimes m}) = H^0(\mathcal{O}_Y(mK_Y))$, hence on the canonical ring

$$\mathcal{R}(Y) := (\mathcal{R}(Y, K_Y)) := \bigoplus_{m \geqslant 0} H^0(\mathcal{O}_Y(mK_Y)).$$

If Y is a variety of general type, then the group G acts linearly on the vector space $H^0(\mathcal{O}_Y(mK_Y))$, hence linearly on the m-th pluricanonical image Y_m, which is an algebraic variety bimeromorphic to Y. Hence G is contained in the algebraic group $\mathrm{Aut}(Y_m)$ and, if G were infinite, as observed by Matsumura ([101]), $\mathrm{Aut}(Y_m)$ would contain a non trivial Cartan subgroup (hence $\mathbb{C}$ or $\mathbb{C}^*$) and Y would be uniruled, a contradiction. This was the main argument of the following

Theorem 4.1. (Matsumura) *The automorphism group of a variety Y of general type is finite.*

Let us specialize now to S a surface of general type, even if most of what we say shall remain valid also in higher dimension.

Take an m-pseudo moduli space $\mathcal{H}_0(\chi, K^2)$ with m so large that the corresponding Hilbert points of the varieties X_m are stable, and let G be a finite group acting on a minimal surface of general type S whose m-th canonical image is in $\mathcal{H}_0(\chi, K^2)$.

Since G acts on the vector space $V_m := H^0(\mathcal{O}_S(mK_S))$, the vector space splits uniquely, up to permutation of the summands, as a direct sum of irreducible representations

$$(**)\ \ V_m = \bigoplus_{\rho \in Irr(G)} W_\rho^{n(\rho)}.$$

We come now to the basic notion of a family of G-automorphisms

Definition 4.2. *A family of* G*-automorphisms is a triple*

$$((p\colon \mathcal{S} \to T), G, \alpha)$$

where:

(1) $(p\colon \mathcal{S} \to T)$ *is a family in a given category (a smooth family for the case of minimal models of general type)*

(2) G *is a (finite) group*

(3) $\alpha\colon G \times \mathcal{S} \to \mathcal{S}$ *yields a biregular action* $G \to \mathrm{Aut}(\mathcal{S})$, *which is compatible with the projection* p *and with the trivial action of* G *on the base* T *(i.e.,* $p(\alpha(g,x)) = p(x),\ \forall g \in G, x \in \mathcal{S}$*).*

As a shorthand notation, one may also write $g(x)$ *instead of* $\alpha(g,x)$*, and by abuse of notation say that the family of automorphisms is a deformation of the pair* (S_t, G) *instead of the triple* (S_t, G, α_t).

Proposition 4.3. *1) A family of automorphisms of surfaces of general type (not necessarily minimal models) induces a family of automorphisms of canonical models.*

2) A family of automorphisms of canonical models induces, if the basis T *is connected, a constant decomposition type* $(**)$ *for* $V_m(t)$.

3) A family of automorphisms of surfaces of general type admits a differentiable trivialization, i.e., in a neighbourhood of $t_0 \in T$*, a diffeomorphism as a family with* $(S_0 \times T, p_T, \alpha_0 \times \mathrm{Id}_T)$*; in other words, with the trivial family for which* $g(y,t) = (g(y), t)$.

Proof.

We sketch only the main ideas.

1) follows since one can take the relative canonical divisor $K := K_{\mathcal{S}|T}$, the sheaf of graded algebras

$$\mathcal{R}(p) := \oplus_m p_*(\mathcal{O}_{\mathcal{S}}(mK))$$

and take the relative Proj, yielding $\mathfrak{X} := \mathrm{Proj}(\mathcal{R}(p))$, whose fibres are the canonical models.

2) follows since for a representation space (V, ρ') the multiplicity with which an irreducible representation W occurs in V is the dimension of $\mathrm{Hom}(W, V)^G$, which

in turn is calculated by the integral on G of the trace of $\rho''(g)$, where $\rho''(g)$ is the representation $\mathrm{Hom}(W, V)$. If we have a family, we get a continuous integer valued function, hence a constant function.

3) Since G acts trivially on the base T, it follows that for each $g \in G$ the fixed locus $\mathrm{Fix}(g)$ is a relative submanifold with a submersion onto T. By the use of stratified spaces (see [100]), and control data, one finds then a differentiable trivialization for the quotient analytic space $\mathcal{S}/G$, hence a trivialization of the action. □

Let us then consider the case of a family of canonical models: by 2) above, and shrinking the base in order to make the addendum $\mathcal{R}(p)_m = p_*(\mathcal{O}_{\mathcal{S}}(mK))$ free, we get an embedding of the family

$$(\mathfrak{X}, G) \hookrightarrow T \times (\mathbb{P}(V_m = \bigoplus_{\rho \in Irr(G)} W_\rho^{n(\rho)}), G).$$

In other words, all the canonical models X_t are contained in a fixed projective space, where also the action of G is fixed.

Now, the canonical model X_t is left invariant by the action of G if and only if its Hilbert point is fixed by G. Hence, we get a closed set

$\mathcal{H}_0(\chi, K^2)^G \subset \mathcal{H}_0(\chi, K^2)$

of the pseudomoduli space, and a corresponding closed subset of the moduli space. Hence we get the following theorem.

Theorem 4.4. *The surfaces of general type which admit an action of a given pluricanonical type* $(**)$ *i.e., with a fixed irreducible G- decomposition of their canonical ring, form a closed subvariety* $(\mathfrak{M}^{can}_{\chi,K^2})^{G,(**)}$ *of the moduli space* $\mathfrak{M}^{can}_{\chi,K^2}$.

We shall see that the situation for the minimal models is different, because then the subset of the moduli space where one has a fixed differentiable type is not closed.

4.2. Kuranishi subspaces for automorphisms of a fixed type

Proposition 4.3 is quite useful when one analyses the deformations of a given G-action.

In the case of the canonical models, we just have to look at the fixed points for the action on a subscheme of the Hilbert scheme; whereas, for the case of the deformations of the minimal model, we have to look at the complex structures for which the given differentiable action is biholomorphic. Hence we derive

Proposition 4.5. *Consider a fixed action of a finite group* G *on a minimal surface of general type* S*, and let* X *be its canonical model. Then we obtain closed subsets of the respective Kuranishi spaces, corresponding to deformations which preserve the given action, and yielding a maximal family of deformations of the* G*-action.*

These subspaces are $\mathfrak{B}(S) \cap H^1(\Theta_S)^G = \mathrm{Def}(S) \cap H^1(\Theta_S)^G$, *respectively* $\mathfrak{B}(X) \cap \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)^G = \mathrm{Def}(X) \cap \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)^G$.

We refer to [34] for a proof of the first fact, while for the second the proof is based again on Cartan's lemma ([29]), that the action of a finite group in an analytic neighbourhood of a fixed point can be linearized.

Just a comment about the contents of the proposition: it says that in each of the two cases, the locus where a group action of a fixed type is preserved is a locally closed set of the moduli space. We shall see more clearly the distinction in the next subsection.

4.3. Deformations of automorphisms differ for canonical and for minimal models

The scope of this subsection is to illustrate the main principles of a rather puzzling phenomenon which we discovered in my joint work with Ingrid Bauer ([10], [11]) on the moduli spaces of Burniat surfaces.

Before dwelling on the geometry of these surfaces, I want to explain clearly what happens, and it suffices to take the example of nodal secondary Burniat surfaces, which I will denote by BUNS in order to abbreviate the name.

For BUNS one has $K_S^2 = 4, p_g(S) := h^0(K_S) = 0$, and the bicanonical map is a Galois cover of the Del Pezzo surface Y of degree 4 with just one node as singularity (the resolution of Y is the blow up Y' of the plane in 5 points, of which exactly 3 are collinear). The Galois group is $G = (\mathbb{Z}/2)^2$, and over the node of Y lies a node of the canonical model X of S, which does not have other singularities.

Then we have BUES, which means extended secondary Burniat surfaces, whose bicanonical map is again a finite $(\mathbb{Z}/2)^2$ - Galois cover of the 1-nodal Del Pezzo surface Y of degree 4 (and for these $S = X$, i.e., the canonical divisor K_S is ample).

All these actions on the canonical models fit together into a single family, but, if we pass to the minimal models, then the topological type of the action changes in a discontinuous way when we pass from the closed set of BUNS to the open set of BUES, and we have precisely two families.

We have , more precisely, the following theorems ([11]):

Theorem 4.6. **(Bauer-Catanese)** *An irreducible connected component, normal, of dimension 3 of the moduliunirational space of surfaces of general type $\mathfrak{M}^{can}_{1,4}$ is given by the subset $\mathcal{NEB}_4$, formed by the disjoint union of the open set corresponding to BUES (*extended secondary Burniat surfaces*), with the irreducible closed set parametrizing BUNS (nodal secondary Burniat surfaces).*

For all surfaces S in $\mathcal{NEB}_4$ the bicanonical map of the canonical model X is a finite cover of degree 4, with Galois group $G = (\mathbb{Z}/2)^2$, of the 1-nodal Del Pezzo surface Y of degree 4 in $\mathbb{P}^4$.

Moreover the Kuranishi space $\mathfrak{B}(S)$ of any such a minimal model S is smooth.

Theorem 4.7. **(Bauer-Catanese)** *The deformations of nodal secondary Burniat surfaces (secondary means that $K_S^2 = 4$) to extended secondary Burniat surfaces yield examples where* $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \to \mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2)$ *is not surjective.*

Indeed the pairs (X, G), *where* $G := (\mathbb{Z}/2\mathbb{Z})^2$ *and* X *is the canonical model of an extended or nodal secondary Burniat surface, where the action of* G *on* X *is induced by the bicanonical map of* X, *belong to only one deformation type.*

If S *is a BUNS, then* $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2) \subsetneq \mathrm{Def}(S)$, *and* $\mathrm{Def}(S, (\mathbb{Z}/2\mathbb{Z})^2)$ *consists exactly of all the BUNS '; while for the canonical model* X *of* S *we have:* $\mathrm{Def}(X, (\mathbb{Z}/2\mathbb{Z})^2) = \mathrm{Def}(X)$.

Indeed for the pairs (S, G), *where* S *is the minimal model of an extended or nodal Burniat surface,* $G := (\mathbb{Z}/2\mathbb{Z})^2$ *and the action is induced by the bicanonical map (it is unique up to automorphisms of* G*), they belong to exactly two distinct deformation types, one given by BUNS, and the other given by BUES.*

The discovery of BUES came later as a byproduct of the investigation of tertiary (3-nodal) Burniat surfaces, where we knew by the Enriques-Kuranishi inequality that tertiary Burniat surfaces cannot form a component of the moduli space: and knowing that there are other deformations helped us to find them eventually.

For BUNS, we first erroneously thought (see [10]) that they form a connected component of the moduli space, because $G = (\mathbb{Z}/2\mathbb{Z})^2 \subset \mathrm{Aut}(S) = \mathrm{Aut}(X)$ for a BUNS, and BUNS are exactly the surfaces S for which the action deforms, while we proved that for all deformations of the canonical model X the action deforms.

The description of BUNS and especially of BUES is complicated, so I refer simply to [11]; but the essence of the pathological behaviour can be understood from the local picture around the node of the Del Pezzo surface Y.

We already described most of this local picture in a previous section.

We make here a first additional observation:

Proposition 4.8. *Let* $t \in \mathbb{C}$, *and consider the action of* $G := (\mathbb{Z}/2\mathbb{Z})^2$ *on* $\mathbb{C}^3$ *generated by* $\sigma_1(u, v, w) = (u, v, -w)$, $\sigma_2(u, v, w) = (-u, -v, w)$. *Then the hypersurfaces* $X_t = \{(u, v, w) | w^2 = uv + t\}$ *are* G*-invariant, and the quotient* X_t/G *is the hypersurface*

$$Y_t = Y_0 = Y := \{(x, y, z) | z^2 = xy\},$$

which has a nodal singularity at the point $x = y = z = 0$.

$X_t \to Y$ *is a family of finite bidouble coverings (Galois coverings with group* $G := (\mathbb{Z}/2\mathbb{Z})^2$*).*

We get in this way a flat family of (non flat) bidouble covers.

Proof. The invariants for the action of G on $\mathbb{C}^3 \times \mathbb{C}$ are:

$$x := u^2, y := v^2, z := uv, s := w^2, t.$$

Hence the family $\mathfrak{X}$ of the hypersurfaces X_t is the inverse image of the family of hypersurfaces $s = z + t$ on the product

$$Y \times \mathbb{C}^2 = \{(x, y, z, s, t) | xy = z^2\}.$$

Hence the quotient of X_t is isomorphic to Y.

□

The following is instead a rephrasing and a generalization of the discovery of Atiyah in the context of automorphisms, which is the main local content of the above theorem. It says that the family of automorphisms of the canonical models X_t, i.e., the automorphism group of the family $\mathfrak{X}$, does not lift, even after base change, to the family $\mathcal{S}$ of minimal surfaces S_τ.

Lemma 4.9. *Let* G *be the group* $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ *acting on* $\mathfrak{X}$ *trivially on the variable* τ, *and else as follows on* $\mathfrak{X}$*: the action of* $G := (\mathbb{Z}/2\mathbb{Z})^2$ *on* $\mathbb{C}^3$ *is generated by* $\sigma_1(u, v, w) = (u, v, -w)$, $\sigma_2(u, v, w) = (-u, -v, w)$ *(we set then* $\sigma_3 := \sigma_1\sigma_2$, *so that* $\sigma_3(u, v, w) = (-u, -v, -w)$*).*

The invariants for the action of G *on* $\mathbb{C}^3 \times \mathbb{C}$ *are:*

$$x := u^2, y := v^2, z := uv, s := w^2, t.$$

Observe that the hypersurfaces $X_t = \{(u, v, w) | w^2 = uv + t\}$ *are* G*-invariant, and the quotient* X_t/G *is the hypersurface*

$$Y_t \cong Y_0 = \{(x, y, z) | z^2 = xy\},$$

which has a nodal singularity at the point $x = y = z = 0$.

Let further σ_4 *act by* $\sigma_4(u, v, w, \tau) = (u, v, w, -\tau)$, *let* $G' \cong (\mathbb{Z}/2\mathbb{Z})^3$ *be the group generated by* G *and* σ_4, *and let* $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ *be the subgroup* $\{\mathrm{Id}, \sigma_2, \sigma_1\sigma_4, \sigma_3\sigma_4\}$.

The biregular action of G' *on* $\mathfrak{X}$ *lifts only to a birational action on* $\mathcal{S}$, *respectively* $\mathcal{S}'$. *The subgroup* H *acts on* $\mathcal{S}$, *respectively* $\mathcal{S}'$, *as a group of biregular automorphisms.*

The elements of $G' \setminus H = \{\sigma_1, \sigma_3, \sigma_4, \sigma_2\sigma_4\}$ *yield isomorphisms between* $\mathcal{S}$ *and* $\mathcal{S}'$.

The group G *acts on the punctured family* $\mathcal{S} \setminus S_0$, *in particular it acts on each fibre* S_τ.

Since σ_4 *acts trivially on* S_0, *the group* G' *acts on* S_0 *through its direct summand* G.

The biregular actions of G *on* $\mathcal{S} \setminus S_0$ *and on* S_0 *do not patch together to a biregular action on* $\mathcal{S}$, *in particular* σ_1 *and* σ_3 *yield birational maps which are not biregular: they are called Atiyah flops (cf.* [6]*).*

Another more geometrical way to see that there is no G-action on the family $\mathcal{S}$ is the following: if G would act on $\mathcal{S}$, and trivially on the base, then the fixed loci for the group elements would be submanifolds with a smooth map onto the parameter space $\mathbb{C}$ with parameter τ. Hence all the quotients S_τ/G would be homeomorphic.

But for BUNS the quotient of S_0 by G is the blow up Y' of Y at the origin, while for $\tau \neq 0$, S_τ/G is just Y! [2] In fact, if one wants to construct the family of smooth models as a family of bidouble covers of a smooth surface, one has to take the blown up surface Y' and its exceptional divisor N (N is called the nodal curve).

Remark 4.10. i) The simplest way to view X_t is to see $\mathbb{C}^2$ as a double cover of Y branched only at the origin, and then X_t as a family of double covers of $\mathbb{C}^2$ branched on the curve $uv + t = 0$, which acquires a double point for $t = 0$.

[2] In the case of BUNS, Y is a nodal Del Pezzo surface of degree 4, whereas in the local analysis we use the same notation Y for the quadric cone, which is the germ of the nodal Del Pezzo surface at the nodal singular point.

ii) If we pull back the bidouble cover X_t to Y', and we normalize it, we can see that the three branch divisors, corresponding to the fixed points for the three non trivial elements of the group G, are as follows:

- D_3 is, for $t = 0$, the nodal curve N, and is the empty divisor for $t \neq 0$;
- D_1 is, for $t \neq 0$, the inverse image of the curve $z + t = 0$; while, for $t = 0$, it is only its strict transform, i.e. a divisor made up of F_1, F_2, the proper transforms of the two branch lines ($\{x{=}z{=}0\}$, resp. $\{y{=}z{=}0\}$) on the quadric cone Y
- D_2 is an empty divisor for $t = 0$, and the nodal curve N for $t \neq 0$.

The above remark shows then that in order to construct the smooth models, one has first of all to take a discontinuous family of branch divisors; and, moreover, for $t \neq 0$, we obtain then a non minimal surface which contains two (-1)-curves ($S_t = X_t$ is then gotten by contracting these two (-1)-curves).

4.4. Teichmüller space for surfaces of general type

Recall the fibre product considered by Burns and Wahl:

$$\begin{tikzcd} \mathrm{Def}(S) \arrow[r] \arrow[d] & \mathrm{Def}(S_{\mathrm{Exc}(\pi)}) =: \mathcal{L}_S \cong \mathbb{C}^\nu, \arrow[d, "\lambda"] \\ \mathrm{Def}(X) \arrow[r] & \mathrm{Def}(X_{\mathrm{Sing}X}) =: \mathcal{L}_X \cong \mathbb{C}^\nu, \end{tikzcd}$$

This gives a map $f\colon \mathrm{Def}(S) \to \mathrm{Def}(X)/\mathrm{Aut}(X)$ of the Kuranishi space of S into an open set of a quasiprojective variety, which factors through Teichmüller space.

Theorem 4.11. *Let S be the minimal model of a surface of general type.*

Then the continuous map $\pi\colon \mathrm{Def}(S) \to \mathcal{T}(M)_S$ is a local homeomorphism between Kuranishi space and Teichmüller space if

1) $\mathrm{Aut}(S)$ is a trivial group, or

2) K_S is ample and S is rigidified.

Proof. We need only to show that π is injective. Assume the contrary: then there are two points $t_1, t_2 \in \mathrm{Def}(S)$ yielding surfaces S_1 and S_2 isomorphic through a diffeomorphism Ψ isotopic to the identity.

By the previous remark, the images of t_1, t_2 inside $\mathrm{Def}(X)/\mathrm{Aut}(X)$ must be the same.

Case 1): there exists then an element w of the Weyl group of λ carrying t_1 to t_2, hence the composition of w and Ψ yields an automorphism of S_1. Since $\mathrm{Aut}(S_1) = \mathrm{Aut}(X_1)$ and the locus of canonical models with non trivial automorphisms is closed, we conclude that, taking $\mathrm{Def}(S)$ as a suitably small germ, then this automorphism is the identity. This is however a contradiction, since w acts non trivially on the cohomology of the exceptional divisor, while Ψ acts trivially.

Case 2) : In this case there is an automorphism g of S carrying t_1 to t_2, and again the composition of g and Ψ yields an automorphism of S_1. We apply the same argument, since g is not isotopic to the identity by our assumption.

$\square$

Remark 4.12. With more work one should be able to treat the more general case where we assume that $\mathrm{Aut}(S)$ is non trivial, but S is rigidified. In this case one should show that a composition $g \circ w$ as above is not isotopic to the identity.

The most interesting question is however whether every surface of general type is rigidified.

5. Connected components and arithmetic of moduli spaces for surfaces

5.1. Gieseker's moduli space and the analytic moduli spaces

As we saw, all 5-canonical models of surfaces of general type with invariants K^2, χ occur in a big family parametrized by an open set of the Hilbert scheme $\mathcal{H}^0$ parametrizing subschemes with Hilbert polynomial $P(m) = \chi + \frac{1}{2}(5m-1)5mK^2$, namely the open set

$$\mathcal{H}^0(\chi, K^2) := \{\Sigma | \Sigma \text{ is reduced with only R.D.P.'s as singularities }\}.$$

Indeed, it is not necessary to consider the 5-pseudo moduli space of surfaces of general type with given invariants K^2, χ, which was defined as the closed subset $\mathcal{H}_0 \subset \mathcal{H}^0$,

$$\mathcal{H}_0(\chi, K^2) := \{\Sigma \in \mathcal{H}^0 | \omega_\Sigma^{\otimes 5} \cong \mathcal{O}_\Sigma(1)\}.$$

At least, if we are only interested about having a family which contains all surfaces of general type, and are not interested about taking the quotient by the projective group.

Observe however that if $\Sigma \in \mathcal{H}^0(\chi, K^2)$, then Σ is the canonical model X of a surface of general type , embedded by a linear system $|D|$, where D is numerically equivalent to $5K_S$, i.e., $D = 5K_S + \eta$, where η is numerically equivalent to 0.

Therefore the connected components $\mathcal{N}$, respectively the irreducible components $\mathcal{Z}$ of the Gieseker moduli space correspond to the connected , resp. irreducible, components of $\mathcal{H}_0(\chi, K^2)$, and in turn to the connected , resp. irreducible, components of $\mathcal{H}^0(\chi, K^2)$ which intersect $\mathcal{H}_0(\chi, K^2)$.

We shall however, for the sake of brevity, talk about connected components $\mathcal{N}$ of the Gieseker moduli space $\mathfrak{M}^{can}_{a,b}$ even if these do not really parametrize families of canonical models.

We refer to [44] for a more ample discussion of the basic ideas which we are going to sketch here.

$\mathfrak{M}^{can}_{a,b}$ has a finite number of connected components, and these parametrize the deformation classes of surfaces of general type. By the classical theorem of Ehresmann

([62]), deformation equivalent varieties are diffeomorphic, and moreover, by a diffeomorphism carrying the canonical class to the canonical class.

Hence, fixed the two numerical invariants $\chi(S) = a$, $K_S^2 = b$, which are determined by the topology of S (indeed, by the Betti numbers of S), we have a finite number of differentiable types.

It is clear that the analytic moduli space $\mathfrak{M}(S)$ that we defined at the onset is then the union of a finite number of connected components of $\mathfrak{M}^{can}_{a,b}$. But how many, and how?

A very optimistic guess was: one.

A basic question was really whether a moduli space $\mathfrak{M}(S)$ would correspond to a unique connected component of the Gieseker moduli space, and this question was abbreviated as the DEF = DIFF question.

I.e., the question whether differentiable equivalence and deformation equivalence would coincide for surfaces.

I conjectured (in [140]) that the answer should be negative, on the basis of some families of simply connected surfaces of general type constructed in [31]: these were then homeomorphic by the results of Freedman (see [68], and [69]), and it was then relatively easy to show then ([32]) that there were many connected components of the moduli space corresponding to homeomorphic but non diffeomorphic surfaces. It looked like the situation should be similar even if one would fix the diffeomorphism type.

Friedman and Morgan instead made the 'speculation' that the answer to the DEF= DIFF question should be positive (1987) (see [70]), motivated by the new examples of homeomorphic but not diffeomorphic surfaces discovered by Donaldson (see [57] for a survey on this topic).

The question was finally answered in the negative, and in every possible way ([97],[84],[40],[48],[13].

Theorem 5.1. *(Manetti '98, Kharlamov -Kulikov 2001, C. 2001, C. - Wajnryb 2004, Bauer- C. - Grunewald 2005)*

The Friedman- Morgan speculation does not hold true and the DEF= DIFF question has a negative answer.

In my joint work with Bronek Wajnryb ([48]) the question was also shown to have a negative answer even for simply connected surfaces.

I showed later ([39]) that each minimal surface of general type S has a natural symplectic structure with class of the symplectic form equal to $c_1(K_S)$, and in such a way that to each connected component $\mathcal{N}$ of the moduli space one can associate the pair of a differentiable manifold with a symplectic structure, unique up to symplectomorphism.

Would this further datum determine a unique connected component, so that DEF = SIMPL ?

This also turned out to have a negative answer ([43]).

Theorem 5.2. *Manetti surfaces provide counterexamples to the DEF = SIMPL question.*

I refer to [42] for a rather comprehensive treatment of the above questions.

Let me just observe that the Manetti surfaces are not simply connected, so that the DEF=SYMPL question is still open for the case of simply connected surfaces. Concerning the question of canonical symplectomorphism of algebraic surfaces, Auroux and Katzarkov ([7]) defined asymptotic braid monodromy invariants of a symplectic manifold, extending old ideas of Moishezon (see [104]).

Quite recent work, not covered in [42], is my joint work with Lönne and Wajnryb ([47]), which investigates in this direction the braid monodromy invariants (especially the 'stable' ones) for the surfaces introduced in [31].

5.2. Arithmetic of moduli spaces

A basic remark is that all these schemes are defined by equations involving only $\mathbb{Z}$ coefficients, since the defining equation of the Hilbert scheme is a rank condition on a multiplication map (see for instance [74]), and similarly the condition $\omega_\Sigma^{\otimes 5} \cong \mathcal{O}_\Sigma(1)$ is also closed (see [108]) and defined over $\mathbb{Z}$..

It follows that the absolute Galois group $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ acts on the Gieseker moduli space $\mathfrak{M}^{can}_{a,b}$.

To explain how it concretely acts, it suffices to recall the notion of a conjugate variety.

Remark 5.3. 1) $\phi \in Aut(\mathbb{C})$ acts on $\mathbb{C}[z_0, \dots z_n]$, by sending $P(z) = \sum_{i=0}^n a_i z^i \mapsto \phi(P)(z) := \sum_{i=0}^n \phi(a_i) z^i$.

2) Let X be as above a projective variety

$$X \subset \mathbb{P}^n_{\mathbb{C}}, X := \{z | f_i(z) = 0 \ \forall i\}.$$

The action of ϕ extends coordinatewise to $\mathbb{P}^n_{\mathbb{C}}$, and carries X to another variety, denoted X^ϕ, and called the **conjugate variety**. Since $f_i(z) = 0$ implies $\phi(f_i)(\phi(z)) = 0$, we see that

$$X^\phi = \{w | \phi(f_i)(w) = 0 \ \forall i\}.$$

If ϕ is complex conjugation, then it is clear that the variety X^ϕ that we obtain is diffeomorphic to X, but in general, what happens when ϕ is not continuous ?

Observe that, by the theorem of Steiniz, one has a surjection $Aut(\mathbb{C}) \to Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and by specialization the heart of the question concerns the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on varieties X defined over $\bar{\mathbb{Q}}$.

For curves, since in general the dimensions of spaces of differential forms of a fixed degree and without poles are the same for X^ϕ and X, we shall obtain a curve of the same genus, hence X^ϕ and X are diffeomorphic.

But for higher dimensional varieties this breaks down, as discovered by Jean Pierre Serre in the 60's ([131]), who proved the existence of a field automorphism $\phi \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over $\bar{\mathbb{Q}}$ such that X and the Galois conjugate variety X^{ϕ} have non isomorphic fundamental groups.

In work in collaboration with Ingrid Bauer and Fritz Grunewald ([14], [15]) we discovered wide classes of algebraic surfaces for which the same phenomenon holds.

A striking result in a similar direction was obtained by Easton and Vakil ([60])

Theorem 5.4. *The absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of irreducible components of the (coarse) moduli space of canonical surfaces of general type,*

$$\mathfrak{M}^{can} := \cup_{a,b \geqslant 1} \mathfrak{M}^{can}_{a,b}.$$

5.3. Topology sometimes determines connected components

There are cases where the presence of a big fundamental group implies that a connected component of the moduli space is determined by some topological invariants.

A typical case is the one of surfaces isogenous to a product ([38]), where a surface is said to be isogenous to a (higher) product if and only if it is a quotient

$$(C_1 \times C_2)/G,$$

where C_1, C_2 are curves of genera $g_1, g_2 \geqslant 2$, and G is a finite group acting freely on $(C_1 \times C_2)$.

Theorem 5.5. *(see* [38]*).*

a) A projective smooth surface is isogenous to a higher product if and only if the following two conditions are satisfied:

1) there is an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \pi = \pi_1(S) \to G \to 1,$$

where G is a finite group and where Π_{g_i} denotes the fundamental group of a compact curve of genus $g_i \geqslant 2$;

2) $e(S)(= c_2(S)) = \frac{4}{|G|}(g_1 - 1)(g_2 - 1)$.

b) Any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S. The corresponding subset of the moduli space, $\mathfrak{M}^{top}_S = \mathfrak{M}^{diff}_S$, corresponding to surfaces orientedly homeomorphic, resp. orientedly diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.

In particular, if S' is orientedly diffeomorphic to S, then S' is deformation equivalent to S or to $\bar{S}$.

Other non trivial examples are the cases of Keum-Naie surfaces, Burniat surfaces and Kulikov surfaces ([9] , [12], [49]): for these classes of surfaces the main result is that any surface homotopically equivalent to a surface in the given class belongs to a unique irreducible connected component of the moduli space.

Just to give a flavour of some of the arguments used, let us consider a simple example which I worked out together with Ingrid Bauer.

Let S be a minimal surface of general type with $q(S) \geqslant 2$. Then we have the Albanese map

$$\alpha\colon S \to A := \mathrm{Alb}(S),$$

and S is said to be of Albanese general type if $\alpha(S) := Z$ is a surface. This property is a topological property (see [36]), since α induces a homomorphism of cohomology algebras

$$\alpha^*\colon H^*(A, \mathbb{Z}) \to H^*(S, \mathbb{Z})$$

and $H^*(A, \mathbb{Z})$ is the full exterior algebra $\Lambda^*(H^1(A, \mathbb{Z})) \cong \Lambda^*(H^1(S, \mathbb{Z}))$ over $H^1(S, \mathbb{Z})$.

In particular, in the case where $q(S) = 2$, the degree d of the Albanese map equals the index of the image of $\Lambda^4 H^1(S, \mathbb{Z})$ inside $H^4(S, \mathbb{Z}) = \mathbb{Z}[S]$.

The easiest case is the one when $d = 2$, because then $K_S = R$, R being the ramification divisor. Observe that the Albanese morphism factors through the canonical model X of S, and a morphism $a\colon X \to A$.

Assume now that a is a finite morphism, so that $2K_X = a^*(a_*(K_X))$. In particular, if we set $D := a_*(K_X)$, then $D^2 = 2K_X^2 = 2K_S^2$, and this number is also a topological invariant.

By the standard formula for double covers we have that $p_g(S) = h^0(L) + 1$, where D is linearly equivalent to $2L$; hence, if L is a polarization of type (d_1, d_2), then $p_g(S) = d_1 d_2 + 1$, D is a polarization of type $(2d_1, 2d_2)$, and $4d_1 d_2 = 2L^2 = K_S^2$, hence in particular we have

$$K_S^2 = 4(p_g - 1) = 4\chi(S),$$

since $q(S) = 2$.

I can moreover recover the polarization type (d_1, d_2) (where d_1 divides d_2) using the fact that $2d_1$ is exactly the divisibility index of D. This is in turn the divisibility of K_S, since K_S gives a linear form on $H^2(A, \mathbb{Z})$ simply by the composition of pushforward and cup product, and this linear form is represented by the class of D. Finally, the canonical class K_S is a differentiable invariant of S (see [58] or [105]).

The final argument is that, by formulae due to Horikawa ([78]), necessarily if $K_S^2 = 4\chi(S)$ the branch locus has only negligible singularities (see [9]), which means that the normal finite cover branched over D has rational double points as singularities.

Theorem 5.6. (Bauer-Catanese) *Let S be a minimal surface of general type whose canonical model X is a finite double cover of an Abelian surface A, branched on a divisor D of type $(2d_1, 2d_2)$. Then S belongs to an irreducible connected component $\mathcal{N}$ of the moduli space of dimension $4d_1 d_2 + 2 = 4\chi(S) + 2$.*

Moreover,

1) any other surface which is diffeomorphic to such a surface S *belongs to the component* $\mathcal{N}$.

2) The Kuranishi space $\mathrm{Def}(X)$ *is always smooth.*

The assumption that X is a finite double cover is a necessary one.

For instance, Penegini and Polizzi ([117]) construct surfaces with $p_g(S) = q(S) = 2$ and $K_S^2 = 6$ such that for the general surface the canonical divisor is ample (whence $S = X$), while the Albanese map, which is generically finite of degree 2, contracts an elliptic curve Z with $Z^2 = -2$ to a point. The authors show then that the corresponding subset of the moduli space consists of three irreducible connected components.

Other very interesting examples with degree $d = 3$ have been considered by Penegini and Polizzi in [116].

6. Smoothings and surgeries

Lack of time prevents me to develop this section.

I refer the reader to [42] for a general discussion, and to the articles [97] and [94] for interesting applications of the $\mathbb{Q}$-Gorenstein smoothings technique (for the study of components of moduli spaces, respectively for the construction of new interesting surfaces).

There is here a relation with the topic of compactifications of moduli spaces. Arguments to show that certain subsets of the moduli spaces are closed involved taking limits of canonical models and studying certain singularities (see [33], [96], see also [122] for the relevant results on deformations of singularities); in [89] a more general study was pursued of the singularities allowed on the boundary of the moduli space of surfaces. I refer to the article by Kollár in this Handbook for the general problem of compactifying moduli spaces (also Viehweg devoted a big effort into this enterprise, see [142], [143], another important reference is [139]).

An explicit study of compactifications of the moduli spaces of surfaces of general type was pursued in [113], [1], [95], [125].

There is here another relation, namely to the article by Abramovich and others in this Handbook, since the deformation of pairs (Y, D) where Y is a smooth complex manifold and $D = \cup_{i=1,\dots,h} D_i$ is a normal crossing divisor, are governed by the cohomology groups

$$H^i(\Theta_Y(-\log D_1, \dots, -\log D_h)),$$

for $i = 1,2$, and where the sheaf $\Theta_Y(-\log D_1, \dots, -\log D_h)$ is the Serre dual of the sheaf $\Omega^1_Y(\log D_1, \dots, \log D_h)(K_Y)$, with its residue sequence

$$0 \to \Omega^1_Y(K_Y) \to \Omega^1_Y(\log D_1, \dots, \log D_h)(K_Y) \to \bigoplus_{i=1}^{3} \mathcal{O}_{D_i}(K_Y) \to 0.$$

These sheaves are the appropriate ones to calculate the deformations of ramified coverings, see for instance [31]),[115], [37], [10], and especially [11]).

I was taught about these by David Mumford back in 1977, when he had just been working on the Hirzebruch proportionality principle in the non compact case ([109]).

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