

Inoue type manifolds and Inoue surfaces: a connected component of the moduli space of surfaces with $K^2 = 7$, $p_g = 0$

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Dedicated to Gerard van der Geer on the occasion of his 60-th birthday.

Dedicato a Gherardo in occasione del 60-esimo genetliaco.

Gerard zu seinem 60. Geburtstag gewidmet.

Abstract. We show that a family of minimal surfaces of general type with $p_g = 0$, $K^2 = 7$, constructed by Inoue in 1994, is indeed a connected component of the moduli space: indeed that any surface which is homotopically equivalent to an Inoue surface belongs to the Inoue family.

The ideas used in order to show this result motivate us to give a new definition of varieties, which we propose to call Inoue type manifolds: these are obtained as quotients \hat{X}/G , where \hat{X} is an ample divisor in a $K(\Gamma, 1)$ projective manifold Z , and G is a finite group acting freely on \hat{X} . For these types of manifolds we prove a similar theorem to the above, even if weaker, that manifolds homotopically equivalent to Inoue type manifolds are again Inoue type manifolds.

Sunto. Lo scopo di questo lavoro e' duplice: da una parte vogliamo qui mostrare che una famiglia di superficie minimali di tipo generale con genere geometrico nullo, e genere lineare $p_1 = 8$, costruite dal signor Inoue nel 1994, formano una componente connessa dello spazio dei moduli. Anzi, piu' precisamente, mostriamo che ogni superficie omotopicamente equivalente ad una superficie di Inoue appartiene alla suddetta famiglia.

Le idee su cui si appoggiano le tecniche dimostrative sono di carattere assai generale e ci inducono a proporre come oggetto di studio una classe di varietá proiettive che vogliamo qui chiamare varietá di tipo Inoue.

Queste varietá vengono definite come quozienti \hat{X}/G (per la azione di un gruppo finito G che agisca liberamente su \hat{X}), dove \hat{X} é un divisore ampio in una varietá proiettiva Z che sia uno spazio di Eilenberg MacLane $K(\Gamma, 1)$. Per queste varietá siamo in grado di mostrare un teorema analogo al precedente, anche se piu' debole, che stabilisce che varietá omotopicamente equivalenti a varietá di tipo Inoue sono anche esse varietá di tipo Inoue.

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Introduction

Minimal surfaces of general type with $p_g(S) = 0$ have invariants $p_g(S) = q(S) = 0$, $1 \leq K_S^2 \leq 9$, and, for each value of $y \in \{1, 2, \dots, 9\}$, such surfaces with $K_S^2 = y$ yield a finite number of irreducible components of the Gieseker moduli space of surfaces of general type $\mathfrak{M}_{1,y}^{can}$.

After the first surfaces of general type with $p_g = q = 0$ were constructed in the 1930's by Luigi Campedelli and by Lucien Godeaux (cf. [Cam32], [God35]) there was in the 1970's a big revival of interest in the construction of these surfaces, as documented by a long and influential survey paper written by Dolgachev ([Dolga81]).

The Bloch conjecture and differential topological questions raised by Donaldson Theory were a further reason for the remarkable and ongoing interest about surfaces of general type with $p_g = 0$, and we refer to [BCP11] for an update about recent important progress on the topic, and about the state of the art.

Looking at the tables 1–3 of [BCP11] one finds it striking that for the value $K_S^2 = 7$ there is only one known family of such surfaces of general type. This family was constructed by Inoue (cf. [In94]). Further interest concerning this family comes from the question of when the fundamental group of surfaces with $p_g = 0$ must be finite, respectively infinite, a problem which was raised in [BCP11]. Indeed this paper was motivated by the observation that Inoue's surfaces have a “big” fundamental group. In fact, the fundamental group of an Inoue surface with $p_g = 0$ and $K_S^2 = 7$ sits in an extension (π_g denotes the fundamental group of a compact curve of genus g):

$$1 \rightarrow \pi_5 \times \mathbb{Z}^4 \rightarrow \pi_1(S) \rightarrow (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow 1.$$

This extension is given geometrically, i.e., stems from our observation that an Inoue surface S admits an unramified $(\mathbb{Z}/2\mathbb{Z})^5$ -Galois covering \hat{S} which is an ample divisor in $E_1 \times E_2 \times D$, where E_1, E_2 are elliptic curves and D is a compact curve of genus 5; this convinced us that the topological type of an Inoue surface determines an irreducible connected component of the moduli space (a phenomenon similar to the one which was already observed in [BC09a], [BC09b], [ChCou10]).

The following is one of the main results of this paper:

Theorem 0.1.

- (1) *Let S' be a smooth complex projective surface which is homotopically equivalent to an Inoue surface (with $K^2 = 7$ and $p_g = 0$). Then S' is an Inoue surface.*
- (2) *The connected component of the Gieseker moduli space $\mathfrak{M}_{1,7}^{can}$ corresponding to Inoue surfaces is irreducible, generically smooth, normal and unirational of dimension 4. Moreover, each Inoue surface S has ample canonical divisor¹ and the base $\text{Def}(S)$ of the Kuranishi family of S is smooth.*

¹This is proven by Inoue, see page 318 of [In94].

(3) *Finally, the first homology group of an Inoue surface equals*

$$\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^4.$$

Indeed (see theorem 4.1), we can considerably relax the above assumption of homotopy equivalence to a weaker assumption concerning homology groups; there remains the interesting question whether the conditions that S' has $K_{S'}$ ample, $K_{S'}^2 = 7$ and the same fundamental group as an Inoue surface S do indeed suffice.

Since this theorem is similar in flavour to other results that we mentioned above, the main purpose of this paper is not only to give a more general proof, but also to set the stage for the investigation and search for a new class of varieties, which we propose to call Inoue type varieties.

Definition 0.2. We define a complex projective manifold X to be an **Inoue type manifold** if

- (1) $\dim(X) \geq 2$;
- (2) there is a finite group G and a Galois unramified covering $\hat{X} \rightarrow X$ with group G , (hence $X = \hat{X}/G$) so that
- (3) \hat{X} is an ample divisor inside a $K(\Gamma, 1)$ -projective manifold Z (this means that $\pi_1(Z) \cong \Gamma$, $\pi_i(Z) = 0$, $\forall i \geq 2$): hence by Lefschetz $\pi_1(\hat{X}) \cong \pi_1(Z) \cong \Gamma$ and moreover
- (4) the action of G on \hat{X} yields a faithful action on $\pi_1(\hat{X}) \cong \Gamma$: in other words the exact sequence

$$1 \rightarrow \Gamma \cong \pi_1(\hat{X}) \rightarrow \pi_1(X) \rightarrow G \rightarrow 1$$

gives an injection $G \rightarrow \text{Out}(\Gamma)$, defined by conjugation;

- (5) the action of G on \hat{X} is induced by an action on Z .

Similarly one defines the notion of an **Inoue type variety**, by requiring the same properties for a variety X with canonical singularities.

We should warn the reader that our approach was inspired by, but is different from the original construction of Inoue, who considers hypersurfaces and complete intersections (of not necessarily ample divisors) in a product of elliptic curves. In fact, this change in perspective in the particular case of an Inoue surface with $K^2 = 7, p_g = 0$ produces a different realization: instead of Inoue's original realization as a complete intersection of two hypersurfaces of multidegrees $(2, 2, 2, 0)$ and $(0, 0, 2, 2)$ in a product of 4 elliptic curves, we view the same surface as a hypersurface of multidegrees $(2, 2, 4)$ in the product $E_1 \times E_2 \times D$ of two elliptic curves with a curve D of genus 5.

One can see that our definition, although imposing a strong restriction on X , is not yet satisfactory in order to obtain some weak rigidity result (of the type of theorems 4.13 and 4.14 of [Cat00], amended in [Cat03], theorem 1.3). Some

hypotheses must be made on the fundamental group Γ of Z , for instance the most interesting case is the one where Z is a product of Abelian varieties, curves, and other locally symmetric varieties with ample canonical bundle.

Definition 0.3. We shall say that an Inoue type manifold X is

- (1) a **SIT := special Inoue type manifold** if moreover

$$Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h) \times (M_1 \times \cdots \times M_s)$$

where each A_i is an Abelian variety, each C_j is a curve of genus $g_j \geq 2$, and M_i is a compact quotient of an irreducible bounded symmetric domain of dimension at least 2 by a torsion free subgroup.

- (2) a **CIT := classical Inoue type manifold** if moreover

$Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h)$ where each A_i is an Abelian variety, each C_j is a curve of genus $g_j \geq 2$.

- (3) a special Inoue type manifold is said to be a

diagonal SIT manifold := diagonal special Inoue type manifold if moreover:

- (I) the action of G on \hat{X} is induced by a diagonal action on Z , i.e.,

$$G \subset \prod_{i=1}^r \text{Aut}(A_i) \times \prod_{j=1}^h \text{Aut}(C_j) \times \prod_{l=1}^s \text{Aut}(M_l) \quad (0.1)$$

and furthermore:

- (II) the faithful action on $\pi_1(\hat{X}) \cong \Gamma$, induced by conjugation by lifts of elements of G in the exact sequence

$$1 \rightarrow \Gamma = \prod_{i=1}^r (\Lambda_i) \times \prod_{j=1}^h (\pi_{g_j}) \times \prod_{l=1}^s (\pi_1(M_l)) \rightarrow \pi_1(X) \rightarrow G \rightarrow 1 \quad (0.2)$$

(observe that each factor Λ_i , resp. $\pi_{g_j}, \pi_1(M_l)$ is normal), has the Schur property

$$(SP) \quad \text{Hom}(V_i, V_j)^G = 0, \quad \forall i \neq j,$$

where $V_j := \Lambda_j \otimes \mathbb{Q}$ (it suffices then to verify that for each Λ_i there is a subgroup H_i of G for which $\text{Hom}(V_i, V_j)^{H_i} = 0, \forall j \neq i$).

- (4) similarly we define a **diagonal CIT manifold := diagonal classical Inoue type manifold**.

We can define analogous notions for Inoue type varieties X with canonical singularities.

Property (SP) plays an important role in order to show that an Abelian variety with such a G -action on its fundamental group must split as a product.

There is however a big difference between the curve and locally symmetric factors on one side and the Abelian variety factors on the other. Namely: for curves we have weak rigidity, i.e., the action of G on π_{g_j} determines a connected family of curves; for compact free quotients of bounded symmetric domains of dimension ≥ 2 we have strong rigidity, i.e., the action on the fundamental group determines uniquely the holomorphic action; for Abelian varieties it is not necessarily so.

Hence, in order to hope for weak rigidity results, one has to introduce a further invariant, called Hodge type (see section 1).

Before stating our main general result we need the following

Definition 0.4. Let Y, Y' be two projective manifolds with isomorphic fundamental groups. We identify the respective fundamental groups $\pi_1(Y) = \pi_1(Y') = \Gamma$. Then we say that the condition (**SAME HOMOLOGY**) is satisfied for Y and Y' if there is an isomorphism $\Psi : H_*(Y', \mathbb{Z}) \cong H_*(Y, \mathbb{Z})$ of homology groups which is compatible with the homomorphisms

$$u : H_*(Y, \mathbb{Z}) \rightarrow H_*(\Gamma, \mathbb{Z}), \quad u' : H_*(Y', \mathbb{Z}) \rightarrow H_*(\Gamma, \mathbb{Z}),$$

i.e., $u \circ \Psi = u'$.

We can now state the following

Theorem 0.5. *Let X be a diagonal SIT (special Inoue type) manifold, and let X' be a projective manifold with the same fundamental group as X , which moreover either*

(A) *is homotopically equivalent to X ;*

or satisfies the following weaker property:

(B) *let \hat{X}' be the corresponding unramified covering of X' . Then \hat{X} and \hat{X}' satisfy the condition (**SAME HOMOLOGY**).*

Setting $W := \hat{X}'$, we have that

- (1) *$X' = W/G$ where W admits a generically finite morphism $f : W \rightarrow Z'$, and where Z' is also a $K(\Gamma, 1)$ projective manifold, of the form $Z' = (A'_1 \times \cdots \times A'_r) \times (C'_1 \times \cdots \times C'_h) \times (M'_1 \times \cdots \times M'_s)$.*

Moreover here M'_i is either M_i or its complex conjugate, and the product decomposition corresponds to the product decomposition (0.2) of the fundamental group of Z .

The image cohomology class $f_([W])$ corresponds, up to sign, to the cohomology class of \hat{X} .*

- (2) *The morphism f is finite if $n = \dim X$ is odd, and it is generically injective if*

*(**) the cohomology class of \hat{X} is indivisible, or if every strictly submultiple cohomology class cannot be represented by an effective G -invariant divisor on any pair (Z', G) homotopically equivalent to (Z, G) .*

(3) f is an embedding if moreover $K_{X'}$ is ample and

$$(***) \quad K_{X'}^n = K_X^n.^2$$

In particular, if $K_{X'}$ is ample and $(**)$ and $(***)$ hold, also X' is a diagonal SIT (special Inoue type) manifold.

A similar conclusion holds under the alternative assumption that the homotopy equivalence sends the canonical class of W to that of \hat{X} : then X' is a diagonal SIT (special Inoue type) variety.

Remark 0.6. If two projective manifolds Y and Y' are homotopically equivalent, they obviously satisfy the condition (**SAME HOMOMOLOGY**).

Hypothesis (A) in theorem 0.5 allows to derive the conclusion that also $W := \hat{X}'$ admits a holomorphic map f' to a complex manifold Z' with the same structure as Z , while hypotheses (B) and following ensure that the morphism is birational to its image, and the class of the image divisor $f'(\hat{X}')$ corresponds to \pm that of \hat{X} under the identification

$$H_*(Z', \mathbb{Z}) \cong H_*(\Gamma, \mathbb{Z}) \cong H_*(Z, \mathbb{Z}).$$

Since $K_{X'}$ is ample, one uses $(***)$ to conclude that f' is an isomorphism with its image.

The next question is weak rigidity, which amounts to the existence of a connected complex manifold parametrizing all such maps (or the complex conjugate). Here several ingredients come into play, namely, firstly the Hodge type, secondly a fine analysis of the structure of the action of G on Z , in particular concerning the existence of hypersurfaces on which G acts freely. Finally, one would have to see whether the family of the invariant effective divisors thus obtained is parametrized by a connected family: this also requires further work which we do not undertake here except that for the case of Inoue surfaces.

It would take too long to analyse here the most general situation, yet there is an even more general situation which is worthy of investigation. This is the case of **orbifold Inoue type varieties**, where the action of G is no longer free: this situation is especially appealing from the point of view of the construction of new interesting examples.

The paper is organized as follows: in the first section we deal with general Inoue type manifolds, establish the first general properties of Inoue type manifolds, and prove our main theorem 0.5. Further, more complete, results dealing with weak rigidity will be given elsewhere.

Section two is devoted to preliminaries, for instance on curves of genus 5 admitting symmetries by $(\mathbb{Z}/2\mathbb{Z})^4$. This is important background for the construction of Inoue surfaces with $K_S^2 = 7$ and $p_g(S) = 0$, which is explained in detail in section three. The end of section three is then devoted to a new result, namely, the calculation of the first homology group of an Inoue surface, which is shown to be equal to the group $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^4$.

²This last property for algebraic surfaces follows automatically from homotopy invariance.

Section four proves the main result on Inoue surfaces with $K_S^2 = 7$ and $p_g(S) = 0$.

Finally, section five is devoted to showing that the moduli space of Inoue surfaces is generically smooth: this is achieved by looking at another realization as bidouble covers of a four-nodal cubic.

1. Inoue type manifolds and varieties

Assume that $X = \hat{X}/G$ is an Inoue type manifold, so that there is an isomorphism $\pi_1(\hat{X}) \cong \pi_1(Z) =: \Gamma$, by virtue of Lefschetz' theorem.

In general, if W is another Kähler manifold with $\pi_1(W) \cong \pi_1(Z) =: \Gamma$, we would like to be able to assert that there exists a holomorphic map $f : W \rightarrow Z'$ where Z' is another $K(\Gamma, 1)$ manifold and where $f_* : \pi_1(W) \rightarrow \pi_1(Z') \cong \Gamma$ realizes the above isomorphism.

This is for instance the case if Z is a compact quotient of an irreducible bounded symmetric domain of dimension at least 2 by a torsion free subgroup; this follows by combining the results of Eells and Sampson ([EeSam64]) proving the existence of a harmonic map in each homotopy class of maps $f : W \rightarrow Z$, since Z has negative curvature, with the results of Siu ([Siu80] and [Siu81]), showing the complex dianalyticity of the resulting harmonic map (i.e., the map f is holomorphic or antiholomorphic) in the case where $f_* : \pi_1(W) \rightarrow \pi_1(Z') \cong \Gamma$ is an isomorphism, since then the differential of f has rank ≥ 4 (as a linear map of real vector spaces). Observe that in this case Z' is Z , or the complex conjugate \bar{Z} .

Another such situation is the case where Z is a compact curve of genus $g \geq 2$. Here, after several results by Siu, Beauville and others (see [Cat91]), a simple criterion was shown to be the existence of a surjection $\pi_1(W) \rightarrow \pi_1(Z)$ with finitely generated kernel (see [Cat03b] and [Cat08], Theorem 5.14), an assumption which holds true in our situation.

It is on the above grounds that we restricted ourselves to special Inoue type manifolds (the diagonality assumption is only a simplifying assumption).

Let us prove the first general result, namely, theorem 0.5.

Proof of Theorem 0.5.

Step 1

The first step consists in showing that $W := \hat{X}'$ admits a holomorphic mapping to a manifold Z' of the above type $Z' = (A'_1 \times \cdots \times A'_r) \times (C'_1 \times \cdots \times C'_h) \times (M'_1 \times \cdots \times M'_s)$, where M'_i is either M_i or its complex conjugate.

First of all, by the cited results of Siu and others ([Siu80], [Siu81], [Cat03b], [Cat08], Theorem 5.14), W admits a holomorphic map to a product manifold

$$Z'_2 \times Z'_3 = (C'_1 \times \cdots \times C'_h) \times (M'_1 \times \cdots \times M'_s).$$

Look now at the Albanese variety $\text{Alb}(W)$ of the Kähler manifold W , whose fundamental group is the quotient of the Abelianization of $\Gamma = \pi_1(Z)$ by its torsion subgroup. Write the fundamental group of $\text{Alb}(W)$ as the first homology group of

$A \times Z_2 \times Z_3$, i.e., as

$$H_1(\text{Alb}(W)) = \Lambda \oplus H_1(Z_2, \mathbb{Z}) \oplus (H_1(Z_3, \mathbb{Z})/\text{Torsion})$$

($\text{Alb}(Z_2)$ is the product of Jacobians $J := (\text{Jac}(C_1) \times \cdots \times \text{Jac}(C_h))$).

Since however, by the universal property, $\text{Alb}(W)$ has a holomorphic map to

$$B' := \text{Alb}(Z_2) \times \text{Alb}(Z_3),$$

inducing a splitting of the lattice $H_1(\text{Alb}(W), \mathbb{Z}) = \Lambda \oplus H_1(B', \mathbb{Z})$, it follows that $\text{Alb}(W)$ splits as $A' \times B'$.

Now, we want to show that the Abelian variety A' (W is assumed to be a projective manifold) splits as desired. This is in turn a consequence of assumption (3) in definition 0.3. In fact, the group G acts on the Abelian variety A' as a group of biholomorphisms, hence it acts on $\Lambda \otimes \mathbb{R}$ commuting with multiplication by $\sqrt{-1}$. Hence multiplication by $\sqrt{-1}$ is an isomorphism of G representations, and then (3) implies that $\Lambda_i \otimes \mathbb{R}$ is stable by multiplication by $\sqrt{-1}$; whence $\Lambda_i \otimes \mathbb{R}$ generates a subtorus A'_i . Finally, A' splits because Λ is the direct sum of the sublattices Λ_i . We are through with the proof of step 1.

Step 2

Consider now the holomorphic map $f: W \rightarrow Z'$. We shall show that the image $W' := f(W)$ is indeed a divisor in Z' . For this we use the assumption (SAME HOMOMOLOGY) and, in fact, the claim is an immediate consequence of the following lemma.

Lemma 1.1. *Assume that W is a Kähler manifold, such that*

- i) *there is an isomorphism of fundamental groups $\pi_1(W) = \pi_1(\hat{X}) = \Gamma$, where \hat{X} is a smooth ample divisor in a $K(\Gamma, 1)$ complex projective manifold Z ;*
- ii) *there exists a holomorphic map $f: W \rightarrow Z'$, where Z' is another $K(\Gamma, 1)$ complex manifold, such that $f_*: \pi_1(W) \rightarrow \pi_1(Z') = \Gamma$ is an isomorphism, and moreover*
- iii) **(SAME HOMOMOLOGY)** *there is an isomorphism $\Psi: H_*(W, \mathbb{Z}) \cong H_*(\hat{X}, \mathbb{Z})$ of homology groups which is compatible with the homomorphisms*

$$u: H_*(\hat{X}, \mathbb{Z}) \rightarrow H_*(\Gamma, \mathbb{Z}), \quad u': H_*(W, \mathbb{Z}) \rightarrow H_*(\Gamma, \mathbb{Z}),$$

$$\text{i.e., } u \circ \Psi = u'.$$

Then f is a generically finite morphism of W into Z' , and the cohomology class $f_([W])$ in*

$$H^*(Z', \mathbb{Z}) = H^*(Z, \mathbb{Z}) = H^*(\Gamma, \mathbb{Z})$$

corresponds to ± 1 times the one of \hat{X} .

Proof of the Lemma. We can identify the homology groups of W and \hat{X} under $\Psi: H_*(W, \mathbb{Z}) \cong H_*(\hat{X}, \mathbb{Z})$, and then the image in the homology groups of $H_*(Z', \mathbb{Z}) = H_*(Z, \mathbb{Z}) = H_*(\Gamma, \mathbb{Z})$ is the same.

We apply the above consideration to the fundamental classes of the oriented manifolds W and \hat{X} , which are generators of the infinite cyclic top degree homology groups $H_{2n}(W, \mathbb{Z})$, respectively $H_{2n}(\hat{X}, \mathbb{Z})$.

This implies a fortiori that $f: W \rightarrow Z'$ is generically finite: since then the homology class $f_*([W])$ (which we identify to a cohomology class by virtue of Poincaré duality) equals the class of \hat{X} , up to sign. \square

Step 3

We claim that $f: W \rightarrow Z'$ is generically 1-1 onto its image W' .

Let d be the degree of $f: W \rightarrow W'$. Then $f_*([W]) = d[W']$, hence if the class of \hat{X} is indivisible, then obviously $d = 1$. Otherwise, observe that the divisor W' is an effective G -invariant divisor and use our assumption (**).

Step 4

Here we are going to prove that f is an embedding under the additional hypotheses that $K_X^n = K_{X'}^n$, and that $K_{X'}$ is ample. We established that f is birational onto its image W' , hence it is a desingularization of W' .

We now use adjunction. We claim that, since K_W is nef, there exists an effective divisor \mathfrak{A} , called the adjunction divisor, such that

$$K_W = f^*(K_{Z'} + W') - \mathfrak{A}.$$

This can be shown by taking the Stein factorization

$$p \circ h: W \rightarrow W^N \rightarrow W',$$

where W^N is the normalization of W' .

Let \mathcal{C} be the conductor ideal $\mathcal{H}om(p_*\mathcal{O}_{W^N}, \mathcal{O}_{W'})$ viewed as an ideal $\mathcal{C} \subset \mathcal{O}_{W^N}$; then the Zariski canonical divisor of W^N satisfies

$$K_{W^N} = p^*(K_{W'}) - C = p^*(K_{Z'} + W') - C,$$

where C is the Weil divisor associated to the conductor ideal (the equality on the Gorenstein locus of W^N is shown for instance in [Cat84b], then it suffices to take the direct image from the open set to the whole of W^N).

In turn, we would have in general $K_W = h^*(K_{W^N}) - \mathfrak{B}$, with \mathfrak{B} not necessarily effective; but, by Lemma 2.5 of [Cor95], see also Lemma 3.39 of [K-M], and since $-\mathfrak{B}$ is h -nef, we conclude that \mathfrak{B} is effective. We establish the claim by setting $\mathfrak{A} := \mathfrak{B} + h^*C$.

Observe that, under the isomorphism of homology groups, $f^*(K_{Z'} + W')$ corresponds to $(K_Z + \hat{X})|_{\hat{X}} = K_{\hat{X}}$, in particular we have

$$K_{\hat{X}}^n = f^*(K_{Z'} + W')^n = (K_W + \mathfrak{A})^n.$$

If we assume that K_W is ample, then $(K_W + \mathfrak{A})^n \geq (K_W)^n$, equality holding if and only if $\mathfrak{A} = 0$.

Under assumption (**), it follows that

$$K_{\hat{X}}^n = |G|K_X^n = |G|K_{X'}^n = K_W^n,$$

hence $\mathfrak{A} = 0$. Since however K_W is ample, it follows that f is an embedding.

If instead we assume that K_W has the same class as $f^*(K_{Z'} + W')$, we conclude first that necessarily $\mathfrak{B} = 0$, and then we get that $C = 0$.

Hence W' is normal and W has canonical singularities.

Step 5

Finally, the group G acts on W , preserving the direct summands of its fundamental group Γ . Hence, G acts on the curve-factors, and the locally symmetric factors.

By assumption, moreover, it sends the summand Λ_i to itself, hence we get a well defined linear action on each Abelian variety A'_i , so that we have a diagonal linear action of G on A' .

Since however the image of W generates A' , we can extend the action of G on W to a compatible affine action on A' . It remains to show that the *real affine type* of the action on A' is uniquely determined. This will be taken care of by the following lemma.

Lemma 1.2. *Given a diagonal special Inoue type manifold, the real affine type of the action of G on the Abelian variety $A = (A_1 \times \cdots \times A_r)$ is determined by the fundamental group exact sequence*

$$1 \rightarrow \Gamma = \prod_{i=1}^r (\Lambda_i) \times \prod_{j=1}^h (\pi_{g_j}) \times \prod_{l=1}^s (\pi_1(M_l)) \rightarrow \pi_1(X) \rightarrow G \rightarrow 1.$$

Proof. Define as before $\Lambda := \prod_{i=1}^r (\Lambda_i) = \pi_1(A)$; moreover, since all the summands in the left hand side are normal in $\pi_1(X)$, set

$$\overline{G} := \pi_1(X) / (\prod_{j=1}^h (\pi_{g_j}) \times \prod_{l=1}^s (\pi_1(M_l))).$$

Observe now that X is the quotient of its universal covering

$$\tilde{X} = \mathbb{C}^m \times \prod_{j=1}^h \mathbb{H}_j \times \prod_{l=1}^s \mathcal{D}_l$$

by its fundamental group, acting diagonally (here \mathbb{H}_j is a copy of Poincaré's upper half plane, while \mathcal{D}_l is an irreducible bounded symmetric domain of dimension at least two), hence we obtain that \overline{G} acts on \mathbb{C}^m as a group of affine transformations.

Let \mathcal{K} be the kernel of the associated homomorphism

$$\alpha : \overline{G} \rightarrow \text{Im}(\alpha) =: \hat{G} \subset \text{Aff}(m, \mathbb{C}),$$

and let

$$\overline{G}_1 := \ker(\alpha_L : \overline{G} \rightarrow \text{GL}(m, \mathbb{C})).$$

\overline{G}_1 is obviously Abelian, and contains Λ , and maps onto a lattice $\Lambda' \subset \hat{G}$.

Since Λ injects into Λ' , $\Lambda \cap \mathcal{K} = 0$, whence \mathcal{K} injects into G , therefore \mathcal{K} is a torsion subgroup; since Λ' is free, we obtain

$$\overline{G}_1 = \Lambda' \oplus \mathcal{K},$$

and we finally get

$$\mathcal{K} = \text{Tors}(\overline{G}_1), \quad \hat{G} = \overline{G}/\text{Tors}(\overline{G}_1).$$

Since our action is diagonal, we can write $\Lambda' = \bigoplus_{i=1}^r (\Lambda'_i)$, and the linear action of the group $G_2 := G/\mathcal{K}$ preserves the summands. Since $\hat{G} \subset \text{Aff}(\Lambda')$, we see that

$$\hat{G} = (\Lambda') \rtimes G'_2,$$

where G'_2 is the isomorphic image of G_2 inside $\text{GL}(\Lambda')$. This shows that the affine group \hat{G} is uniquely determined.

Finally, using the image groups $G_{2,i}$ of G_2 inside $\text{GL}(\Lambda'_i)$, we can define uniquely groups of affine transformations of A_i which fully determine the diagonal action of G on A (up to real affine automorphisms of each A_i). \square

Q.E.D. for Theorem 0.5.

In order to obtain weak rigidity results, one has to introduce a further invariant, called Hodge type, according to the following definition. We shall return to the question of weak rigidity in a sequel to this paper.

Definition 1.3. Let X be an Inoue type manifold (or variety) of special diagonal type, with

$$Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h) \times (M_1 \times \cdots \times M_s).$$

Then an invariant of the integral representation $G \rightarrow \text{Aut}(\Lambda_i)$ is its **Hodge type**, which is the datum, given the decomposition of $\Lambda_i \otimes \mathbb{C}$ as the sum of isotypical components

$$\Lambda_i \otimes \mathbb{C} = \bigoplus_{\chi \in \text{Irr}(G)} U_{i,\chi}$$

of the dimensions

$$\nu(i, \chi) := \dim_{\mathbb{C}} U_{i,\chi} \cap H^{1,0}(A_i)$$

of the Hodge summands for non-real representations.

2. Genus 5 curves having a $(\mathbb{Z}/2\mathbb{Z})^4$ -action

The following is well known (see however section 1 of [BC09b]).

Lemma 2.1. *Let E_1 be a compact curve of genus 1 and assume $G_1 := (\mathbb{Z}/2\mathbb{Z})^n \subset \text{Aut}(E_1)$. Then $n \leq 3$, and, for $n = 3$, $E_1/G_1 \cong \mathbb{P}^1$ with quotient map branched on exactly 4 points P_1, \dots, P_4 . The covering $E_1 \rightarrow E_1/G_1$ factors through multiplication by 2 in E_1 .*

Lemma 2.2. *Let D be a compact curve of genus 5 and assume $H := (\mathbb{Z}/2\mathbb{Z})^n \subset \text{Aut}(D)$. Then $n \leq 4$, and, if $n = 4$, $D/H \cong \mathbb{P}^1$ and the quotient map is branched on exactly 5 points P_1, \dots, P_5 .*

Proof. By the Hurwitz formula, one has, setting $D/H = C$ and $h = \text{genus}(C)$,

$$8 = 2^n(2h - 2 + \frac{m}{2}) \Leftrightarrow 2^{n-4}(4h - 4 + m) = 1,$$

where m is the number of branch points P_1, \dots, P_m . Hence $n \leq 4$. If $n = 4$, then $h = 1$ is not possible, since in this case the abelianization of $\pi_1(C \setminus P_1)$ would equal $\pi_1(C)$, and one would have $m = 0$, a contradiction. Hence $h = 0$ and $m = 5$. \square

The following geometrical game is based on the fact that the 15 intermediate double covers of $D/H = \mathbb{P}^1$ are 5 elliptic curves (each branched on 4 of the 5 branch points) and 10 rational curves (each branched on 2 of the 5 branch points). Let A_i be the elliptic curve branched on all the five points with exclusion of P_i : then $D \rightarrow D/H$ factors as

$$D \rightarrow A_i \rightarrow A_i \rightarrow D/H$$

where the middle map is multiplication by 2, and $D \rightarrow A_i$ is the quotient by an involution with fixed points; the number of fixed points is exactly 8, since, if $g \in H$, the fixed set $\text{Fix}(g)$ is an H -orbit, and has therefore cardinality equal to a multiple of 8. The other 10 involutions have no fixed points, hence they each yield an unramified covering of a curve C_j of genus 3.

We try now to stick to Inoue's original notation, except that we refuse to use the classical symbol for the *Weierstrass \wp -function* to denote the Legendre function \mathcal{L} ; \mathcal{L} is a homographic transform of the Weierstrass function, but not equal to the Weierstrass function.

The Legendre function satisfies the quadratic relation (see [BC09b])

$$y^2 = (\mathcal{L}^2 - 1)(\mathcal{L}^2 - a^2).$$

Let E_1, E_2 be two complex elliptic curves. We assume $E_i = \mathbb{C}/\langle 1, \tau_i \rangle$. Moreover, we denote by z_i a uniformizing parameter on E_i . Then $(\mathbb{Z}/2\mathbb{Z})^3 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ acts on E_i by

- $\gamma_1(z_i) = -z_i$;
- $\gamma_2(z_i) = z_i + \frac{1}{2}$;
- $\gamma_3(z_i) = z_i + \frac{\tau_i}{2}$.

We consider the Legendre \mathcal{L} -function for E_i and denote it by \mathcal{L}_i , for $i = 1, 2$: \mathcal{L}_i is a meromorphic function on E_i and $\mathcal{L}_i: E_i \rightarrow \mathbb{P}^1$ is a double cover ramified in $\pm 1, \pm a_i \in \mathbb{P}^1 \setminus \{0, \infty\}$. It is well known that we have (cf. [In94], lemma 3-2, and also cf. [BC09b], pages 52-54, section 1 for an algebraic treatment):

- $\mathcal{L}_i(\frac{1}{2}) = -1$, $\mathcal{L}_i(0) = 1$, $\mathcal{L}_i(\frac{\tau_i}{2}) = a_i$, $\mathcal{L}_i(\frac{1+\tau_i}{2}) = -a_i$;
- let $b_i := \mathcal{L}_i(\frac{\tau_i}{4})$, then $b_i^2 = a_i$;
- $\frac{d\mathcal{L}_i}{dz_i}(z_i) = 0$ if and only if $z_i \in \{0, \frac{1}{2}, \frac{\tau_i}{2}, \frac{1+\tau_i}{2}\}$, since $\frac{dx_i}{\mathcal{L}_i} = dz_i$.

Moreover

$$\begin{aligned}\mathcal{L}_i(z_i) &= \mathcal{L}_i(z_i + 1) = \mathcal{L}_i(z_i + \tau_i) = \mathcal{L}_i(-z_i) = -\mathcal{L}_i(z_i + \frac{1}{2}), \\ \mathcal{L}_i(z_i + \frac{\tau_i}{2}) &= \frac{a_i}{\mathcal{L}_i(z_i)}.\end{aligned}$$

We consider now the vector space $V_i := H^0(E_i, \mathcal{O}_{E_i}(2[0]))$ (for $i = 1, 2$), and note that $V_i \cong \mathbb{C}^2$ with basis $1, \mathcal{L}_i$, since

$$\operatorname{div}(1 - \mathcal{L}_i) = 2[0] - \operatorname{Poles}(\mathcal{L}_i).$$

Observe that $[-\frac{1}{4}] + [\frac{1}{4}]$ is a $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \gamma_1, \gamma_2 \rangle$ -invariant divisor, hence V_i is a $(\mathbb{Z}/2\mathbb{Z})^2$ -module and splits in its isotypical components as

$$V_i = V_i^{++} \oplus V_i^{+-},$$

since 1 is invariant under $(\mathbb{Z}/2\mathbb{Z})^2$ and \mathcal{L}_i is invariant under γ_1 and is an eigenvector with eigenvalue -1 of γ_2 . If $c \in \mathbb{C} \setminus \{\pm 1, \pm a_i, \pm a_1 a_2\}$, then the divisor

$$D_c := \{(z_1, z_2) \in E_1 \times E_2 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2) = c\}$$

of bidegree $(2, 2)$ is a smooth curve of genus 5. More precisely,

$$\mathcal{O}_{E_1 \times E_2}(D_c) \cong p_1^* \mathcal{O}_{E_1}(2[0]) \otimes p_2^* \mathcal{O}_{E_2}(2[0]).$$

Consider the product action of $(\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$ on $E_1 \times E_2$.

Remark 2.3. 1) It is easy to see that D_c is invariant under the subgroup H of $(\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$ given by

$$(\mathbb{Z}/2\mathbb{Z})^3 \cong H := \langle \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \rangle,$$

where the coordinates are taken with respect to the basis $\gamma_1, \gamma_2, \gamma_3$ on each factor.

2) Moreover, if we choose $c := b_1 b_2$, then we see that for $(z_1, z_2) \in D_{b_1 b_2}$:

$$\mathcal{L}_1(z_1 + \frac{\tau_1}{2})\mathcal{L}_2(z_2 + \frac{\tau_2}{2}) = \frac{a_1 a_2}{\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)} = \frac{a_1 a_2}{b_1 b_2} = b_1 b_2,$$

whence $D_{b_1 b_2}$ is invariant under

$$(\mathbb{Z}/2\mathbb{Z})^4 \cong G := H \oplus \langle \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \rangle \leq (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3.$$

We want to show that the converse holds. More precisely, we prove the following

Proposition 2.4. *Let $f: D \rightarrow \mathbb{P}^1$ be the maximal $G := (\mathbb{Z}/2\mathbb{Z})^4$ -covering branched in 5 given points $p_1, \dots, p_5 \in \mathbb{P}^1$. Then there are two elliptic curves E_1, E_2 such that $D \subset E_1 \times E_2$ is a G -invariant divisor with*

$$\mathcal{O}_{E_1 \times E_2}(D) \cong p_1^* \mathcal{O}_{E_1}(2[0]) \otimes p_2^* \mathcal{O}_{E_2}(2[0]).$$

Choosing appropriate coordinates we can moreover assume that $D = \{(z_1, z_2) \in E_1 \times E_2 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2) = b_1 b_2\}$.

Proof. Let e_1, e_2, e_3, e_4 be a basis of the $\mathbb{Z}/2\mathbb{Z}$ -vectorspace G and let $D \rightarrow \mathbb{P}^1$ branched in p_1, \dots, p_5 be given by the *appropriate orbifold homomorphism*

$$\varphi: \mathbb{T}(2, 2, 2, 2, 2) := \langle x_1, \dots, x_5 \mid \prod_{i=1}^5 x_i, x_1^2, \dots, x_5^2 \rangle \rightarrow (\mathbb{Z}/2\mathbb{Z})^4,$$

where $\varphi(x_i) = e_i$ for $1 \leq i \leq 4$, $\varphi(x_5) = e_5 := e_1 + e_2 + e_3 + e_4$. Then Hurwitz' formula shows that D is a smooth curve of genus 5.

Note that the only elements of G having fixed points on D are the 5 elements e_i ($1 \leq i \leq 5$), and each of them has exactly 8 fixed points on D . Hence, $E_i := D/\langle e_i \rangle$ is an elliptic curve.

We get therefore 5 elliptic curves (as intermediate covers of $D \rightarrow \mathbb{P}^1$), all endowed with a $(\mathbb{Z}/2\mathbb{Z})^3$ -action.

Choose two of these elliptic curves, say E_1, E_2 , and consider the morphism

$$i: D \rightarrow D/\langle e_1 \rangle \times D/\langle e_2 \rangle = E_1 \times E_2.$$

Then $D \cdot E_i = 2$ and i is an embedding of D as a $(\mathbb{Z}/2\mathbb{Z})^4$ -invariant divisor of bidegree 2.

We fix the origin in both elliptic curves so that $D = \{s = 0\}$, where $s \in H^0(E_1 \times E_2, p_1^*(\mathcal{O}_{E_1}(2[0])) \otimes p_2^*(\mathcal{O}_{E_2}(2[0])))^G$.

It remains to show that we can assume D to be of the form $\{(z_1, z_2) \in E_1 \times E_2 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2) = b_1 b_2\}$.

For this we show that

$$H^0(E_1 \times E_2, p_1^*(\mathcal{O}_{E_1}(2[0])) \otimes p_2^*(\mathcal{O}_{E_2}(2[0])))^H \cong \mathbb{C}^2.$$

In fact,

$$\begin{aligned} H^0(E_1 \times E_2, p_1^*(\mathcal{O}_{E_1}(2[0])) \otimes p_2^*(\mathcal{O}_{E_2}(2[0]))) &= V_1 \otimes V_2 = \\ &= (V_1^{+++} \oplus V_1^{++-}) \otimes (V_2^{+++} \oplus V_2^{++-}), \end{aligned}$$

whence

$$\begin{aligned} H^0(E_1 \times E_2, p_1^*(\mathcal{O}_{E_1}(2[0])) \otimes p_2^*(\mathcal{O}_{E_2}(2[0])))^H &= \\ &= (V_1^{+++} \otimes V_2^{+++}) \oplus (V_1^{++-} \otimes V_2^{++-}). \end{aligned}$$

Therefore we have a pencil of H -invariant divisors $D_c := \{(z_1, z_2) \in E_1 \times E_2 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2) = c\}$. It is now obvious that D_c is G -invariant iff $c = \pm b_1 b_2$. The change of sign for b_i is achieved by changing the point $\frac{\tau_i}{4}$ by $\frac{\tau_i}{4} + \frac{1}{2}$. \square

Consider now for the moment the action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \gamma_1, \gamma_2 \rangle$ on E_1 and E_2 , and the induced product action on $E_1 \times E_2$. Assume $H \cong (\mathbb{Z}/2\mathbb{Z})^3$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$, such that

- (1) on each factor E_i it induces the action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \gamma_1, \gamma_2 \rangle$;
- (2) there is an H -invariant pencil in $\mathbb{P}(V_1 \otimes V_2)$, i.e., $\dim(V_1 \otimes V_2)^H = 2$.

Then it is easy to see that we can choose $(\gamma_1, 0)$, $(0, \gamma_1)$, (γ_2, γ_2) as basis of H . Therefore

$$(V_1 \otimes V_2)^{+++} = (V_1^{+++} \otimes V_2^{+++}) \oplus (V_1^{++-} \otimes V_2^{++-}),$$

and the H -invariant pencil is given by $D_c = \{(z_1, z_2) \in E_1 \times E_2 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2) = c\}$.

Consider now $G := H \oplus \langle (\gamma_3, \gamma_3) \rangle \cong (\mathbb{Z}/2\mathbb{Z})^4$. Then $D := \{(z_1, z_2) \in E_1 \times E_2 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2) = b_1b_2\}$.

Proposition 2.5.

- (1) *The restriction map*

$$H^0(E_1 \times E_2, p_1^* \mathcal{O}_{E_1}(2[0]) \otimes p_2^* \mathcal{O}_{E_2}) \rightarrow H^0(D, p_1^* \mathcal{O}_{E_1}(2[0]) \otimes p_2^* \mathcal{O}_{E_2}|D)$$

is an isomorphism of H -modules.

- (2) *There is a pencil of such H -invariant divisors of degree 4 on D . But there is no one which is invariant under $(\mathbb{Z}/2\mathbb{Z})^4$.*

Proof. 1) Let $S := E_1 \times E_2$ and for simplicity, write

$$\mathcal{O}_S(2, 0) := p_1^* \mathcal{O}_{E_1}(2[0]) \otimes p_2^* \mathcal{O}_{E_2}.$$

Then we consider the exact sequence:

$$0 \rightarrow \mathcal{O}_S(-D) \otimes \mathcal{O}_S(2, 0) \rightarrow \mathcal{O}_S(2, 0) \rightarrow \mathcal{O}_S(2, 0) \otimes \mathcal{O}_D \rightarrow 0.$$

By Künneth's formula we get

- i) $h^0(\mathcal{O}_S(-D) \otimes \mathcal{O}_S(2, 0)) = h^0(\mathcal{O}_{E_1})h^0(\mathcal{O}_{E_2}(-2)) = 0$;
- ii) $h^1(\mathcal{O}_S(-D) \otimes \mathcal{O}_S(2, 0)) = 2$.

Therefore

$$r: H^0(S, \mathcal{O}_S(2, 0)) \cong \mathbb{C}^2 \rightarrow H^0(S, \mathcal{O}_S(2, 0) \otimes \mathcal{O}_D)$$

is injective. Since D is not hyperelliptic, it follows by Clifford's theorem that $h^0(S, \mathcal{O}_S(2, 0) \otimes \mathcal{O}_D) \leq 2$. This implies that $h^0(S, \mathcal{O}_S(2, 0) \otimes \mathcal{O}_D) = 2$ and r is an isomorphism (of H -modules).

2) Clear from the previous discussion. □

Remark 2.6. This implies that

$$H^0(S, \mathcal{O}_S(2, 0) \otimes \mathcal{O}_D) \cong \mathbb{C}^2 = V^{+++} \oplus V^{++-}.$$

3. Inoue surfaces with $p_g = 0$ and $K_S^2 = 7$

In [In94] the author describes the construction of a family of minimal surfaces of general type S with $p_g = 0$, $K_S^2 = 7$. We briefly recall the construction of these surfaces and, for lack of reference, we calculate K_S^2 and $p_g(S)$.

For $i \in \{1, 2, 3, 4\}$, let $E_i := \mathbb{C}/\langle 1, \tau_i \rangle$ be a complex elliptic curve. Denoting again by z_i a uniformizing parameter of E_i , we consider the following five involutions on $T := E_1 \times E_2 \times E_3 \times E_4$:

$$\begin{aligned} g_1(z_1, z_2, z_3, z_4) &= (-z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3, z_4), \\ g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2 + \frac{1}{2}, z_3 + \frac{1}{2}, -z_4 + \frac{1}{2}), \\ g_3(z_1, z_2, z_3, z_4) &= (z_1 + \frac{1}{2}, z_2, -z_3 + \frac{1}{2}, -z_4 + \frac{1}{2}), \\ g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, -z_4), \\ g_5(z_1, z_2, z_3, z_4) &= (z_1 + \frac{\tau_1}{2}, z_2 + \frac{\tau_2}{2}, z_3 + \frac{\tau_3}{2}, z_4 + \frac{\tau_4}{2}). \end{aligned}$$

Then $G := \langle g_1, g_2, g_3, g_4, g_5 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^5$. Consider

$$\hat{X} := \{(z_1, z_2, z_3, z_4) \in T \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b_1b_2b_3, \mathcal{L}_3(z_3)\mathcal{L}_4(z_4) = b_3b_4\}.$$

Then

- \hat{X} is a smooth complete intersection of two hypersurfaces in T of respective multidegrees $(2, 2, 2, 0)$ and $(0, 0, 2, 2)$;
- \hat{X} is invariant under the action of G , and G acts freely on \hat{X} .

The above equations show that \hat{X} is the complete intersection of a G -invariant divisor X_1 in the linear system

$$\mathbb{P}(H^0(E_1 \times E_2 \times E_3 \times E_4, p_1^*\mathcal{O}_{E_1}(2[0]) \otimes p_2^*\mathcal{O}_{E_2}(2[0]) \otimes p_3^*\mathcal{O}_{E_3}(2[0]) \otimes p_4^*\mathcal{O}_{E_4})),$$

with a G -invariant divisor X_2 ($\cong E_1 \times E_2 \times D$, where D is a curve of genus 5, a Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^4$ of the projective line ramified in 5 points) in the linear system

$$\mathbb{P}(H^0(E_1 \times E_2 \times E_3 \times E_4, p_1^*\mathcal{O}_{E_1} \otimes p_2^*\mathcal{O}_{E_2} \otimes p_3^*\mathcal{O}_{E_3}(2[0]) \otimes p_4^*\mathcal{O}_{E_4}(2[0]))).$$

Remark 3.1. It is easy to see from the above explicit description that G acts freely on \hat{X} . Note that the action of G has fixed points on T , and even on X_2 .

Remark 3.2. It is immediate to see that the involution g_4 acts freely on \hat{X} and trivially on $(E_1 \times E_2)$. It follows therefore that, setting $\hat{X}_2 := \hat{X}/g_4$, S has another representation as $\hat{X}_2/(\mathbb{Z}/2\mathbb{Z})^4$, where \hat{X}_2 is a divisor in the product $(E_1 \times E_2 \times C)$, and C is a smooth curve of genus 3.

This was the representation of S announced in [BCP11], and it follows from our results that this representation is also unique.

We will show in the next lemma that S is a minimal surface of general type with $K_S^2 = 7$ and $p_g = 0$.

Lemma 3.3. *Let $S = \hat{X}/G$ be as above. Then S is a minimal surface of general type with $K_S^2 = 7$ and $p_g = 0$.*

Proof. Since \hat{X} is of general type, also S is of general type, being an étale quotient of \hat{X} by a finite group. We first remark that \hat{X} is a G -invariant hypersurface of multidegree $(2, 2, 4)$ in $W := E_1 \times E_2 \times D$, where $D \subset E_3 \times E_4$ is a smooth curve of genus 5 given by the equation

$$\{(z_3, z_4) \in E_3 \times E_4 \mid \mathcal{L}_3(z)\mathcal{L}_4(z_4) = b_3b_4\}.$$

By the adjunction formula, the canonical divisor of \hat{X} is the restriction to \hat{X} of a divisor of multidegree $(2, 2, 12)$ on W .

Therefore we can calculate (denoting by F_i the fibre of the projection of W on the (j, k) -th coordinate, with $\{i, j, k\} = \{1, 2, 3\}$):

$$\begin{aligned} K_{\hat{X}}^2 &= ((K_W + [\hat{X}])|_{\hat{X}})^2 = (2F_1 + 2F_2 + 12F_3)^2(2F_1 + 2F_2 + 4F_3) = \\ &= (8F_1F_2 + 48F_1F_3 + 48F_2F_3)(2F_1 + 2F_2 + 4F_3) = \\ &= 32F_1F_2F_3 + 96F_1F_2F_3 + 96F_1F_2F_3 = 224 = 7 \cdot 2^5. \end{aligned}$$

Since G acts freely on \hat{X} , we obtain

$$K_S^2 = \frac{224}{|G|} = \frac{224}{2^5} = 7.$$

Moreover, consider the exact sequence

$$0 \rightarrow \mathcal{O}_W(K_W) \rightarrow \mathcal{O}_W(K_W + [\hat{X}]) \rightarrow \omega_{\hat{X}} \rightarrow 0. \quad (3.1)$$

Using Künneth's formula and Kodaira's vanishing theorem, we get:

- $\dim H^0(W, \mathcal{O}_W(K_W)) = 5$,
- $\dim H^0(W, \mathcal{O}_W(K_W + [X])) = 32$,
- $H^i(W, \mathcal{O}_W(K_W + [X])) = 0$, for $i = 1, 2, 3$,
- $\dim H^1(W, \mathcal{O}_W(K_W)) = 1 + 5 + 5 = 11$,
- $\dim H^2(W, \mathcal{O}_W(K_W)) = q(W) (= q(\hat{X})) = 7$.

Therefore, by the long exact sequence associated to (3.1) we get:

$$p_g(\hat{X}) = h^0(\hat{X}, \omega_{\hat{X}}) = 32 + 11 - 5 = 38,$$

whence

$$\chi(\mathcal{O}_{\hat{X}}) = 1 + p_g(\hat{X}) - q(\hat{X}) = 1 + 38 - 7 = 32.$$

This implies that $\chi(\mathcal{O}_S) = 1$. In order to show that $p_g(S) = 0$, it suffices to show that $q(S) = 0$. Using the fact that

$$i^* : H^0(W, \Omega_W^1) \rightarrow H^0(\hat{X}, \Omega_{\hat{X}}^1)$$

is an isomorphism and that

$$H^0(W, \Omega_W^1) = H^0(E_1, \Omega_{E_1}^1) \oplus H^0(E_2, \Omega_{E_2}^1) \oplus H^0(D, \Omega_D^1),$$

it is easy to see that $H^0(W, \Omega_W^1)^G = H^0(S, \Omega_S^1) = 0$. \square

Definition 3.4. A smooth projective algebraic surface $S := \hat{X}/G$ as above is called an *Inoue surface* with $K_S^2 = 7$ and $p_g = 0$.

3.1. The torsion group of Inoue surfaces with $K_S^2 = 7$. The aim of this section is to prove the following

Theorem 3.5. *Let S be an Inoue surface with $K_S^2 = 7$. Then*

$$H_1(S, \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^4.$$

It is clear from the construction that the fundamental group of an Inoue surface sits in an exact sequence

$$1 \rightarrow \mathbb{Z}^4 \times \pi_5 \rightarrow \pi_1(S) \rightarrow G \cong (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow 1, \quad (3.2)$$

where π_5 denotes the fundamental group of a compact curve of genus five.

Observe that after dividing out by \mathbb{Z}^4 in the exact sequence (3.2) we obtain the orbifold exact sequence (plus a summand $\mathbb{Z}/2\mathbb{Z}$) of the maximal $(\mathbb{Z}/2\mathbb{Z})^4$ -covering of \mathbb{P}^1 ramified in 5 points:

$$1 \rightarrow \pi_5 \rightarrow \pi_1^{orb} \times \mathbb{Z}/2\mathbb{Z} \rightarrow G \cong (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1, \quad (3.3)$$

where $\pi_1^{orb} := \pi_1^{orb}(\mathbb{P}^1 \setminus \{p_1, \dots, p_5\}; 2, 2, 2, 2, 2)$.

The proof of the above theorem will be divided in several steps, and we shall first prove some auxiliary results.

Lemma 3.6. *Assume that there is an exact sequence of groups*

$$1 \rightarrow \Lambda \rightarrow \Gamma' \rightarrow G \rightarrow 1,$$

where Λ and G are abelian.

Assume moreover that

(#) Λ admits a system L of generators with the following property: $\forall h \in L$
 $\exists g \in G$ such that

$$ghg^{-1} = -h.$$

Then $2\Lambda \subset [\Gamma', \Gamma']$ and, in particular, we have an exact sequence

$$\Lambda/2\Lambda \rightarrow \Gamma'^{ab} \rightarrow G \rightarrow 1. \quad (3.4)$$

Proof. Let $h \in \Lambda$, $g \in G$ be such that $ghg^{-1} = -h$. Then $[g, h] = -2h$, whence $2h \in [\Gamma', \Gamma']$. Since this holds for all $h \in L$ and L generates Λ the claim follows. \square

Remark 3.7. i) Assume that there is an exact sequence of groups

$$1 \rightarrow H \rightarrow \Gamma \rightarrow G \rightarrow 1,$$

where G is abelian. Defining $\Gamma' := \Gamma/[H, H]$, we have an exact sequence

$$1 \rightarrow \Lambda := H^{ab} \rightarrow \Gamma' \rightarrow G \rightarrow 1, \quad (3.5)$$

and we have

$$(\Gamma')^{ab} = \Gamma^{ab}.$$

Suppose now that assumption (#) of lemma 3.6 is satisfied for the exact sequence (3.5). Then by lemma 3.6 we have an exact sequence

$$\tilde{\Lambda} := \Lambda/2\Lambda \rightarrow \Gamma^{ab} \rightarrow G \rightarrow 1. \quad (3.6)$$

ii) Define $\Gamma'' := \Gamma'/2\Lambda$. Then we have an exact sequence

$$1 \rightarrow \tilde{\Lambda} \rightarrow \Gamma'' \rightarrow G \rightarrow 1,$$

and $(\Gamma'')^{ab} = (\Gamma')^{ab} = \Gamma^{ab}$.

Choose generators $(g_i)_{i \in I}$ of G and choose for each g_i a lift γ_i to Γ'' . Moreover, let $(\lambda_j)_{j \in J}$ be generators of Λ and denote their images in $\tilde{\Lambda}$ by $\tilde{\lambda}_j$. Then obviously $\Gamma'' = \langle (g_i)_{i \in I}, (\tilde{\lambda}_j)_{j \in J} \rangle$.

Assume also that

$$(\#\#) \quad \gamma_i \lambda_j \gamma_i^{-1} = \pm \lambda_j, \quad \forall i \in I, \forall j \in J.$$

Then (since $2\tilde{\Lambda} = 0$) we have $[\gamma_i, \tilde{\lambda}_j] = 0$ (in Γ''), $\forall i \in I, \forall j \in J$.

In particular this implies that

$$\Gamma'' / \langle [\gamma_i, \gamma_j], i, j \in I \rangle = (\Gamma'')^{ab} = \Gamma^{ab}.$$

Since the genus 5 curve D is an ample divisor $D \subset E_3 \times E_4$, we have by Lefschetz' theorem a surjective morphism

$$\varphi : \pi_5 \cong \pi_1(D) \rightarrow \pi_1(E_3 \times E_4) \cong \mathbb{Z}^4.$$

Defining $K := \ker \varphi$ and $\Lambda := (\pi_5 \times \mathbb{Z}^4) / (K \times \{0\}) \cong \mathbb{Z}^4 \times \mathbb{Z}^4$, we get from (3.2) the exact sequence

$$1 \rightarrow \Lambda \rightarrow \pi_1(S)/K \rightarrow (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow 1. \quad (3.7)$$

Proposition 3.8. $P := (\pi_1(S)/K)^{ab} \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^4$.

Proof. We first verify condition (#) for the exact sequence (3.7).

For this we recall the description of the action of $G := \langle g_1, g_2, g_3, g_4, g_5 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^5$ on $T := E_1 \times E_2 \times E_3 \times E_4$:

$$\begin{aligned} g_1(z_1, z_2, z_3, z_4) &= (-z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3, z_4), \\ g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2 + \frac{1}{2}, z_3 + \frac{1}{2}, -z_4 + \frac{1}{2}), \\ g_3(z_1, z_2, z_3, z_4) &= (z_1 + \frac{1}{2}, z_2, -z_3 + \frac{1}{2}, -z_4 + \frac{1}{2}), \\ g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, -z_4), \\ g_5(z_1, z_2, z_3, z_4) &= (z_1 + \frac{\tau_1}{2}, z_2 + \frac{\tau_2}{2}, z_3 + \frac{\tau_3}{2}, z_4 + \frac{\tau_4}{2}). \end{aligned}$$

Observe that in this situation $\Lambda = \pi_1(T) \subset \mathbb{C}^4$.

We claim now that for each generator $\lambda \in \{\tau_1 e_1, \tau_2 e_2, \tau_3 e_3, \tau_4 e_4, e_1, e_2, e_3, e_4\}$ of the lattice Λ there is a $g \in G$ such that $g\lambda g^{-1} = -\lambda$. For this it suffices to find, for each $k \in \{1, 2, 3, 4\}$, an element $g \in G$ whose linear action has e_k as -1 eigenvector. But this is obvious (e.g. $(g_1 g_3, g_1 g_2, g_4, g_4)$ for $k = (1, 2, 3, 4)$).

Lemma 3.6 implies, setting $P' := \pi_1(S)/K$, that then

$$2\Lambda \subset [P', P'].$$

Denoting by $\gamma_1, \dots, \gamma_5$ lifts to P' of the generators g_1, \dots, g_5 of G , we see that $e_1 = \gamma_3^2$, $e_2 = \gamma_1^2$, $e_3 = \gamma_2^2$ and $e_4 = (\gamma_2 \gamma_4)^2$. Therefore, writing $\lambda_i := \tau_i e_i$, for $1 \leq i \leq 4$, we have that

$$P' = \langle \gamma_1, \dots, \gamma_5, \lambda_1, \dots, \lambda_4 \rangle.$$

Moreover, $[\gamma_i, \lambda_j] = \pm 2\lambda_j$, whence by remark 3.7, we have an exact sequence

$$1 \rightarrow \Lambda/2\Lambda \rightarrow P'' := P'/2\Lambda \rightarrow G \rightarrow 1,$$

and $P = (P')^{ab} = (P'')^{ab} = P''/\langle [\gamma_i, \gamma_j], 1 \leq i, j \leq 5 \rangle$.

A straightforward computation shows the following:

$$\begin{aligned} [\gamma_1, \gamma_2] &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & [\gamma_1, \gamma_3] &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & [\gamma_1, \gamma_4] &= 0, & [\gamma_1, \gamma_5] &= \begin{pmatrix} -\tau_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ [\gamma_2, \gamma_3] &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & [\gamma_2, \gamma_4] &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, & [\gamma_2, \gamma_5] &= \begin{pmatrix} 0 \\ -\tau_2 \\ 0 \\ -\tau_4 \end{pmatrix}, & [\gamma_3, \gamma_4] &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \\ [\gamma_3, \gamma_5] &= \begin{pmatrix} 0 \\ 0 \\ -\tau_3 \\ -\tau_4 \end{pmatrix}, & [\gamma_4, \gamma_5] &= \begin{pmatrix} 0 \\ 0 \\ -\tau_3 \\ -\tau_4 \end{pmatrix}. \end{aligned}$$

This immediately implies that

$$\tilde{\Lambda}/\langle [\gamma_i, \gamma_j] \rangle \cong \mathbb{Z}/2\mathbb{Z},$$

hence we get the exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow P \rightarrow G = (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow 1.$$

It follows that $P \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^4$, where $\mathbb{Z}/4\mathbb{Z}$ is generated by γ_5 , because

$$\gamma_5^2 = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix}. \quad \square$$

Finally we prove

Proposition 3.9. *The natural surjective morphism*

$$H_1(S, \mathbb{Z}) = \pi_1(S)^{ab} \twoheadrightarrow P$$

is an isomorphism.

This implies theorem 3.5.

Proof. Let $\Gamma := \pi_1(S)$. Then from the exact sequence

$$1 \rightarrow H := \mathbb{Z}^4 \times \pi_5 \rightarrow \Gamma \rightarrow G \cong (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow 1,$$

we get by remark 3.7 the exact sequence

$$1 \rightarrow H^{ab} \cong \mathbb{Z}^{10} \times \mathbb{Z}^4 =: \Lambda_1 \oplus \Lambda_2 \rightarrow \Gamma' \rightarrow G \rightarrow 1,$$

and $\Gamma^{ab} = (\Gamma')^{ab}$.

By lemma 3.6 we get an exact sequence

$$\mathbb{Z}^{10} \times (\mathbb{Z}/2\mathbb{Z})^4 \rightarrow \Gamma'' \rightarrow G \rightarrow 1,$$

where again $(\Gamma'')^{ab} = \Gamma^{ab}$.

Note that g_1 acts trivially on the curve D of genus 5, whence it acts trivially on $H_1(D) \cong \mathbb{Z}^{10}$.

Moreover, we have seen in the proof of proposition 3.8 that the commutators $[\gamma_1'', \gamma_i'']$, $2 \leq i \leq 5$, span a subspace V of rank 3 in $\Lambda_2/2\Lambda_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$. Therefore we get (after moding out by V) an exact sequence

$$\mathbb{Z}^{10} \times \mathbb{Z}/2\mathbb{Z} \rightarrow \Gamma''' \rightarrow G \rightarrow 1,$$

where $(\Gamma''')^{ab} = \Gamma^{ab}$. After dividing out $\mathbb{Z}/2\mathbb{Z}$, we obtain the exact sequence

$$H_1(D, \mathbb{Z}) \rightarrow \pi := \Gamma''' / (\mathbb{Z}/2\mathbb{Z}) \rightarrow G \rightarrow 1.$$

We finally get the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \Gamma''' & \longrightarrow & \pi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \\ \mathbb{Z}/2\mathbb{Z} & \longrightarrow & (\Gamma''')^{ab} & \longrightarrow & (\pi)^{ab} = (\pi_1^{orb})^{ab} \oplus \mathbb{Z}/2\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z})^5 & \longrightarrow & 1 \end{array} \quad (3.8)$$

(cf. the exact sequence (3.3)).

This implies that $H_1(S) = \Gamma^{ab} = (\Gamma''')^{ab}$ has cardinality at most $2 \cdot 2^5 = 2^6$. Since $|P| = 2^6$, the surjective morphism $H_1(S, \mathbb{Z}) \rightarrow P$ is then an isomorphism. \square

4. Weak rigidity of Inoue surfaces

In this section we shall prove the following:

Theorem 4.1. *Let S' be a smooth complex projective surface which is homotopically equivalent to an Inoue surface (with $K^2 = 7$ and $p_g = 0$). Then S' is an Inoue surface.*

The same consequence holds under the weaker assumptions that S' has the same fundamental group of an Inoue surface, and that (B) (SAME HOMOLOGY) of theorem 0.5 holds.

Remark 4.2. It is clear from the definition that S is a diagonal CIT (classical Inoue type) manifold.

We can therefore apply theorem 0.5 to this special case. In this special case, we are going to see that the groups $\hat{G}_i = \Lambda'_i \rtimes G_{2,i}$, $i = 1, 2$, are obtained simply by taking $\Lambda'_i := \frac{1}{2}\Lambda_i$, and $G_{2,i} := \{\pm 1\}$.

We shall indeed view things geometrically, as follows. Recall that there is an exact sequence

$$1 \rightarrow \mathbb{Z}^4 \times \pi_5 \rightarrow \pi_1(S) \rightarrow G \cong (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow 1,$$

where π_5 denotes the fundamental group of a compact curve of genus five.

Let $\hat{X} \subset E_1 \times E_2 \times D$ be the étale $(\mathbb{Z}/2\mathbb{Z})^5$ -covering of S . Observe that $H_1 := \langle g_2, g_3, g_4, g_5 \rangle$ acts trivially on $H^0(E_1, \Omega_{E_1}^1)$. This shows that $q(\hat{X}/H_1) \geq 1$, and

$$q(\hat{X}/H_1) = 1 \iff D/H_1 \cong \mathbb{P}^1.$$

But it is obvious that $D/H_1 \cong \mathbb{P}^1$, since $D/G = D/H_1$.

Now, consider instead $H_2 := \langle g_1, g_3, g_4, g_5 \rangle$, which acts trivially on $H^0(E_2, \Omega_{E_2}^1)$. Therefore

$$q(\hat{X}/H_2) = 1 \iff D/H_2 \cong \mathbb{P}^1.$$

$D/H_2 \cong \mathbb{P}^1$ follows from the following lemma 4.4, since H_2 contains three elements having fixed points on D (in fact, the elements $(-z_3 + \frac{1}{2}, -z_4 + \frac{1}{2})$, $(-z_3 + \frac{\tau_3}{2}, -z_4 + \frac{\tau_4}{2})$, $(-z_3 + \frac{1}{2} + \frac{\tau_3}{2}, -z_4 + \frac{1}{2} + \frac{\tau_4}{2})$).

Therefore we have seen:

Proposition 4.3. *Let $S := \hat{X}/G$ be an Inoue surface. Then there are subgroups $H_1, H_2 \leq G$ of index 2, such that $q(\hat{X}/H_1) = q(\hat{X}/H_2) = 1$.*

Let S' be a smooth complex projective surface which has the same fundamental group as S . Then, denoting by \hat{X}' the corresponding étale G -covering of S' , we have:

- there is a smooth curve D' of genus 5 and a surjective morphism $\hat{X}' \rightarrow D'$;
- there are 2 index two subgroups H_1, H_2 of G with $q(\hat{X}'/H_1) = q(\hat{X}'/H_2) = 1$.

Lemma 4.4. *Let $D \rightarrow \mathbb{P}^1$ be the maximal $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^4$ -covering branched in five points. Then:*

- 1) *there are exactly 5 subgroups $H \leq \Gamma$, $H \cong (\mathbb{Z}/2\mathbb{Z})^3$ containing exactly one element having fixed points on D ($\implies g(D/H) = 1$);*
- 2) *there are exactly 10 subgroups $H \leq \Gamma$, $H \cong (\mathbb{Z}/2\mathbb{Z})^3$ containing exactly three elements having fixed points on D ($\implies g(D/H) = 0$).*

Proof. Let $D \rightarrow \mathbb{P}^1$ be the maximal $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^4$ -covering branched in five points p_1, \dots, p_5 , which determines a surjective homomorphism

$$\varphi: \pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_5\}) \rightarrow \Gamma.$$

We denote $\varphi(\gamma_i)$, where γ_i is a geometric loop around p_i , by e_i . Then e_1, e_2, e_3, e_4 is a $\mathbb{Z}/2\mathbb{Z}$ -basis of Γ and $e_1 + e_2 + e_3 + e_4 = e_5$.

Claim 4.5. Let $H \cong (\mathbb{Z}/2\mathbb{Z})^3 \leq \Gamma$. Then either there is a unique $i \in \{1, \dots, 5\}$ such that $e_i \in H$ (and $e_j \notin H$ for $j \neq i$), or there is a subset $\{i, j, k\} \subset \{1, \dots, 5\}$ such that $e_i, e_j, e_k \in H$ (and the other two e_l 's are not in H).

Proof of the claim. It is clear that H contains at least one of the e_i 's. Otherwise $D \rightarrow D/H$ is étale, and by Hurwitz' formula, we get

$$8 = 2g(D) - 2 = |H|(2(g(D/H) - 2) = 8(2g(D/H) - 2),$$

a contradiction. Since any four of the e_i 's are linearly independent, H can contain at most three of them.

Assume now that there are $i \neq j$, such that $e_i, e_j \in H$. W.l.o.g. we can assume $e_1, e_2 \in H$. Then $H = \langle e_1, e_2, \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \rangle$, $\lambda_i \in \{0, 1\}$. Note that $e_1, e_2, \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4$ are linearly independent if and only if $(\lambda_3, \lambda_4) \neq (0, 0)$. Moreover,

- $e_3 \notin H \iff (\lambda_3, \lambda_4) \neq (1, 0)$,
- $e_4 \notin H \iff (\lambda_3, \lambda_4) \neq (0, 1)$,
- $e_5 \notin H \iff (\lambda_3, \lambda_4) \neq (1, 1)$.

This shows: if H contains two of the e_i 's then also a third one.

Q.E.D. for Claim 4.5.

Now, if H contains three of the e_i 's, say e_i, e_j, e_k , then $H = \langle e_i, e_j, e_k \rangle$, and there are exactly $\binom{5}{3} = 10$ such subgroups. The remaining 5 subgroups $H \cong (\mathbb{Z}/2\mathbb{Z})^3$ of Γ therefore contain exactly one of the e_i 's.

Assume now that

$$H_1 := \langle e_1, e_2, e_3 \rangle, \quad H_2 := \langle e_1 + e_2, e_1 + e_3, e_1 + e_4 \rangle.$$

Then it remains to show that $g(D/H_1) = 0$ and $g(D/H_2) = 1$.

Observe that $H_2 = \langle e_5 \rangle \oplus \langle e_1 + e_2, e_1 + e_3 \rangle$, and $H' := \langle e_1 + e_2, e_1 + e_3 \rangle$ acts freely on D . Therefore $g(D/H') = 2$ and e_5 acts on D/H' having two fixed points. By Hurwitz' formula this implies that $g(D/H_2) = 1$.

Now, $g(D/\langle e_1 \rangle) = 1$ and $H_1/\langle e_1 \rangle \cong \langle e_2, e_3 \rangle$ acts with fixed points on $D/\langle e_1 \rangle$. This shows that $D/H_1 \cong \mathbb{P}^1$. \square

Now we are ready to finish the proof of 4.1.

Proof (of thm. 4.1).

Let S' be a smooth complex projective surface which is homotopically equivalent to an Inoue surface S with $K_S^2 = 7$ and $p_g = 0$. In particular $\pi_1(S) \cong \pi_1(S')$ and we take the étale $G := (\mathbb{Z}/2\mathbb{Z})^5$ -covering \hat{X}' , which is homotopically equivalent to \hat{X} (the corresponding covering of the Inoue surface S). By proposition 4.3 we know that \hat{X}' admits a morphism to a curve D' of genus 5, and there are subgroups $H_1, H_2 \leq G$ of index 2 such that $X_i := \hat{X}'/H_i$ has irregularity one.

Therefore there are elliptic curves E'_1, E'_2 and morphisms

$$\hat{X}' \rightarrow X_i \rightarrow E'_i.$$

By the universal property of the Albanese map we get a commutative diagram

$$\begin{array}{ccc} \hat{X}' & \longrightarrow & E'_1 \times E'_2 \times \text{Jac}(D') \\ \downarrow & \nearrow \psi & \\ \text{Alb}(\hat{X}') & & \end{array} \quad (4.1)$$

Lemma 4.6. *Let $E'_i = \mathbb{C}/\Lambda'_i$ and denote by $\Lambda_i := 2\Lambda'_i$. Then ψ corresponds to*

$$H_1(D', \mathbb{Z}) \times \Lambda_1 \times \Lambda_2 \subset H_1(D', \mathbb{Z}) \times \Lambda'_1 \times \Lambda'_2.$$

In particular $E_i := \mathbb{C}/\Lambda_i = E'_i$, ψ restricted to E_i is multiplication by 2, and

$$\text{Alb}(\hat{X}') = \text{Jac}(D') \times E_1 \times E_2.$$

Proof. By the previous discussion ψ is an isogeny, and since the fundamental groups of S and S' are the same, the assertion for \hat{X}' follows from the corresponding statement for \hat{X} . \square

By theorem 0.5 we have that $\varphi: \hat{X}' \rightarrow E_1 \times E_2 \times D'$ is a birational morphism onto its image $W' \subset E_1 \times E_2 \times D' := C_1 \times C_2 \times C_3$. In fact, since the fundamental group of X' is isomorphic to that of an Inoue surface, it follows by lemma 1.2 that G acts on $Z' := E_1 \times E_2 \times D'$ as for an Inoue surface, hence there is no effective divisor Δ of numerical type $(1, 1, 2)$ which is invariant by the action of the group G , as it is easy to verify.

Therefore W' has homology class $2F_1 + 2F_2 + 4F_3$, where F_i is the fibre over a point in the i -th curve C_i , and W' has rational double points as singularities. The linear equivalence class of W' is invariant for the group action. It is the sum of

three classes ξ_i which are respective pull-backs from the projection onto the i -th curve C_i . Hence each class ξ_i is invariant for the action of G on C_i , hence ξ_i is the pull-back from the quotient of C_i by the group G_i , projection of G into the automorphism group of C_i .

By our lemmas 2.1 and 2.2, these quotients are rational curves, hence we conclude that the linear equivalence class of the divisor W' is the same as the one for an Inoue surface S .

It remains to show that W' is given by Inoue's equations, i.e., if we consider the genus 5 curve D' as a hypersurface

$$D' := \{(z_3, z_4) \in E_3 \times E_4 \mid \mathcal{L}_3(z)\mathcal{L}_4(z_4) = b_3b_4\},$$

then

$$W' := \{(z_1, z_2, z_3, z_4) \in E_1 \times \dots \times E_4 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b_1b_2b_3, \\ \mathcal{L}_3(z_3)\mathcal{L}_4(z_4) = b_3b_4\}. \quad (4.2)$$

For this consider the subgroup $H := \langle g_1, g_2, g_3, g_4 \rangle \leq G$. Then there is an H -invariant divisor with class $2F_1 + 2F_2 + 4F_3$ in $Z' := E_1 \times E_2 \times D'$. Therefore $H^0(Z', \mathcal{O}'_{Z'}(2F_1 + 2F_2 + 4F_3)) \cong \mathbb{C}^8$ is an H -module, which decomposes by Künneth's formula and the results in section 2 as follows:

$$\begin{aligned} H^0(Z', \mathcal{O}'_{Z'}(2F_1 + 2F_2 + 4F_3)) &\cong \\ &\cong H^0(E_1 \times E_2 \times D', p_1^*(\mathcal{O}_{E_1}(2[0])) \otimes p_2^*(\mathcal{O}_{E_2}(2[0])) \otimes p_3^*(\mathcal{O}_{D'}(2, 0))) \cong \\ &\cong V_1 \otimes V_2 \otimes V_3 \cong (V_1^{++++} \oplus V_1^{-+++}) \otimes (V_2^{++++} \oplus V_2^{-+++}) \otimes (V_3^{++++} \oplus V_3^{+---}) \cong \\ &\cong V^{++++} \oplus V^{+---} \oplus V^{-+++} \oplus V^{-+++}, \quad (4.3) \end{aligned}$$

where each of the four summands in the last line is isomorphic to \mathbb{C}^2 . In fact, we have

$$\begin{aligned} V^{++++} &= (V_1^{++++} \otimes V_2^{++++} \otimes V_3^{++++}) \oplus (V_1^{-+++} \otimes V_2^{-+++} \otimes V_3^{+---}), \\ V^{+---} &= (V_1^{++++} \otimes V_2^{++++} \otimes V_3^{+---}) \oplus (V_1^{-+++} \otimes V_2^{-+++} \otimes V_3^{++++}), \\ V^{-+++} &= (V_1^{++++} \otimes V_2^{-+++} \otimes V_3^{++++}) \oplus (V_1^{-+++} \otimes V_2^{++++} \otimes V_3^{+---}), \\ V^{-+++} &= (V_1^{-+++} \otimes V_2^{++++} \otimes V_3^{++++}) \oplus (V_1^{++++} \otimes V_2^{-+++} \otimes V_3^{+---}). \end{aligned}$$

The equations of the hypersurfaces in the above pencils are then:

$$\begin{aligned} W'_1(c) &:= \{c = \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)\}, \\ W'_2(c) &:= \{c = \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\frac{1}{\mathcal{L}_3(z_3)}\}, \\ W'_3(c) &:= \{c = \mathcal{L}_1(z_1)\frac{1}{\mathcal{L}_2(z_2)}\mathcal{L}_3(z_3)\}, \\ W'_4(c) &:= \{c = \frac{1}{\mathcal{L}_1(z_1)}\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)\}. \end{aligned}$$

This shows that after possibly replacing one of the elliptic curves E_i with parameter a_i by the elliptic curve E'_i with parameter $\frac{1}{a_i}$ we can w.l.o.g. assume that the pencil of H -invariant hypersurfaces in Z' is given by the equation $\{c = \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)\}$.

Now, we consider g_5 . It is easy to see that $g_5(W'_1(c)) \equiv W'_1(c)$, and if $W'_1(c) = \text{div}(s)$ for $s \in V^{++++}$, then also $g_5(W'_1(c)) = \text{div}(s')$ for $s' \in V^{++++}$. Therefore g_5 is an involution on $\mathbb{P}^1 := \mathbb{P}(V^{++++})$, which is obviously non trivial, whence g_5 has exactly two fixed points. Therefore there are exactly two G -invariant divisors in the pencil $W'_1(c)$. On the other hand, $W'_1(b_1b_2b_3)$ and $W'_1(-b_1b_2b_3)$ are G -invariant.

This shows that W' is of the desired form, hence X' is the canonical model of an Inoue surface. \square

5. Inoue surfaces as bidouble covers and $H^1(S, \Theta_S)$

The aim of this section is to show the following

Theorem 5.1. *Let S be an Inoue surface with $K_S^2 = 7$. Then:*

$$h^1(S, \Theta_S) = 4, \quad h^2(S, \Theta_S) = 8.$$

To prove this result we resort to a result of [ML-P01], where Inoue surfaces are constructed as bidouble covers of the four-nodal cubic.

We briefly recall their description here, for details we refer to [ML-P01], example 4.1 (we keep their notation, even if slightly inconvenient).

We consider a complete quadrilateral Λ in \mathbb{P}^2 and denote the vertices by P_1, \dots, P_6 .

We have labeled the vertices in a way that

- the intersection point of the line $\overline{P_1P_2}$ and the line $\overline{P_3P_4}$ is P_5 ,
- the intersection point of $\overline{P_1P_4}$ and $\overline{P_2P_3}$ is P_6 .

Let $Y \rightarrow \mathbb{P}^2$ be the blow up in P_1, \dots, P_6 , denote by L the total transform of a line in \mathbb{P}^2 , let E_i , $1 \leq i \leq 6$, be the exceptional curve lying over P_i . Moreover, we denote by S_i , $1 \leq i \leq 4$, the strict transforms on Y of the sides $S_i := \overline{P_iP_{i+1}}$ for $1 \leq i \leq 3$, $S_4 := \overline{P_4P_1}$, of the quadrilateral Λ .

The geometry of the situation is that the four (-2) curves S_i come from the resolution of the 4 nodes of the cubic surface Σ which is the anticanonical image of Y , and the curves E_i are the strict transforms of the 6 lines in Σ connecting pairs of nodal points.

The surface Σ contains also a triangle of lines (joining the midpoints of opposite edges of the tetrahedron with sides the lines corresponding to the curves E_i). These are the 3 strict transforms Δ_1 , Δ_2 , Δ_3 of the three diagonals of the complete quadrilateral Λ . Δ_1 is the strict transform of $\overline{P_1P_3}$, Δ_2 of $\overline{P_2P_4}$ and Δ_3 of $\overline{P_5P_6}$.

For each line Δ_i in the cubic surface Σ we consider the pencil of planes containing them, and the base point free pencil of residual conics, which we denote by

$|f_i|$. Hence we have

$$|f_i| = |(-K_Y) - \Delta_i|, \quad \Delta_i + f_i \equiv (-K_Y).$$

In the plane realization we have:

- f_1 is the strict transform on Y of a general element of the pencil of conics Γ_1 through P_2, P_4, P_5, P_6 ,
- f_2 is the strict transform on Y of a general element of the pencil of conics Γ_2 through P_1, P_3, P_5, P_6 ,
- f_3 is the strict transform on Y of a general element of the pencil of conics Γ_3 through P_1, P_2, P_3, P_4 .

It is then easy to see that each curve S_h is disjoint from the other curves S_j ($j \neq h$), Δ_i , and f_i if f_i is smooth. Moreover,

$$\Delta_i \cdot f_i = 2, \quad \Delta_i \cdot f_j = 0 \quad (i \neq j), \quad f_i^2 = 0, \quad f_j f_i = 2 \quad (i \neq j).$$

Definition 5.2. We define the *Inoue divisors* on Y as follows:

- $D_1 := \Delta_1 + f_2 + S_1 + S_2$, where $f_2 \in |f_2|$ smooth;
- $D_2 := \Delta_2 + f_3$, where $f_3 \in |f_3|$ smooth;
- $D_3 := \Delta_3 + f_1 + f'_1 + S_3 + S_4$, where $f_1, f'_1 \in |f_1|$ smooth.

Let $\pi: \tilde{S} \rightarrow Y$ be the bidouble covering with branch divisors D_1, D_2, D_3 (associated to the 3 nontrivial elements of the Galois group).

By the previous remarks we see that over each S_i there are two disjoint (-1) -curves. Contracting these eight exceptional curves we obtain a minimal surface S with $p_g = 0$ and $K_S^2 = 7$. By [ML-P01] these are exactly the Inoue surfaces.

Remark 5.3. We immediately see that there is an open dense subset in the product

$$|f_1| \times |f'_1| \times |f_2| \times |f_3| \cong (\mathbb{P}^1)^4$$

parametrizing the family of Inoue surfaces.

Remark 5.4. The non-trivial character sheaves of this bidouble cover are

- $\mathcal{L}_1 = \mathcal{O}_Y(-K_Y + f_1 - E_4)$;
- $\mathcal{L}_2 = \mathcal{O}_Y(-2K_Y - E_5 - E_6)$;
- $\mathcal{L}_3 = \mathcal{O}_Y(-K_Y + L - E_1 - E_2 - E_3)$.

Lemma 5.5.

- $\dim H^1(\tilde{S}, \Theta_{\tilde{S}})^{inv} = \dim H^1(S, \Theta_S)^{inv} = 4$,
- $\dim H^2(\tilde{S}, \Theta_{\tilde{S}})^{inv} = \dim H^2(S, \Theta_S)^{inv} = 0$.

Proof. It is well known (cf. e.g. [Cat08]) that $H^2(\tilde{S}, \Theta_{\tilde{S}})$ of the $(\mathbb{Z}/2\mathbb{Z})^2$ -covering $\pi: \tilde{S} \rightarrow Y$ decomposes as a direct sum of character spaces

$$H^2(\tilde{S}, \Theta_{\tilde{S}}) \cong H^2(\tilde{S}, \Theta_{\tilde{S}})^{inv} \oplus \bigoplus_{i=1}^3 H^2(\tilde{S}, \Theta_{\tilde{S}})^i,$$

and that the dimensions of the direct summands can be computed as the dimensions of global sections of logarithmic differential forms on the base Y . In fact, we have:

$$\begin{aligned} h^0(Y, \Omega_Y^1(\log D_1, \log D_2, \log D_3)(K_Y)) &= h^2(\tilde{S}, \Theta_{\tilde{S}})^{inv} = h^2(S, \Theta_S)^{inv}; \\ h^0(Y, \Omega_Y^1(\log D_i)(K_Y + \mathcal{L}_i)) &= h^2(\tilde{S}, \Theta_{\tilde{S}})^i = h^2(S, \Theta_S)^i, \quad \forall i \in \{1, 2, 3\}. \end{aligned} \quad (5.1)$$

Note that $|-K_Y| = |3L - \sum_{i=1}^6 E_i|$ is non-empty and does not have a fixed part. Therefore there is a morphism $\mathcal{O}_Y(K_Y) \rightarrow \mathcal{O}_Y$, which is not identically zero on any component of the D_i 's.

We get the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_Y^1(K_Y) & \longrightarrow & \Omega_Y^1(\log D_1, \log D_2, \log D_3)(K_Y) & \longrightarrow & \bigoplus_{i=1}^3 \mathcal{O}_{D_i}(K_Y) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_Y^1 & \longrightarrow & \Omega_Y^1(\log D_1, \log D_2, \log D_3) & \longrightarrow & \bigoplus_{i=1}^3 \mathcal{O}_{D_i} \longrightarrow 0. \end{array} \quad (5.2)$$

From this we get the commutative diagram with injective vertical arrows

$$\begin{array}{ccc} \mathbb{C}^2 \oplus 0 \oplus \mathbb{C}^2 & \xrightarrow{\cong} & H^0(Y, \bigoplus_{i=1}^3 \mathcal{O}_{D_i}(K_Y)) \xrightarrow{\delta} H^1(Y, \Omega_Y^1(K_Y)) \\ & & \downarrow \psi_2 \searrow \varphi \\ \mathbb{C}^4 \oplus \mathbb{C}^2 \oplus \mathbb{C}^5 & \xrightarrow{\cong} & H^0(Y, \bigoplus_{i=1}^3 \mathcal{O}_{D_i}) \xrightarrow{\psi_1} H^1(Y, \Omega_Y^1). \end{array}$$

A standard argument shows that δ is injective (see [Cat84]). In fact, the Chern classes of S_1, S_2, S_3, S_4 are linearly independent, hence φ is injective, which implies that also δ is injective. Therefore

$$h^0(\Omega_Y^1(\log D_1, \log D_2, \log D_3)(K_Y)) = h^2(\Theta_{\tilde{S}})^{inv} = h^2(\Theta_S)^{inv} = 0.$$

Therefore

$$\begin{aligned} h^1(\Theta_{\tilde{S}})^{inv} &= -\chi(\Omega_Y^1(\log D_1, \log D_2, \log D_3)(K_Y)) = \\ &= -(\chi(\Omega_Y^1(K_Y)) + \chi(\bigoplus_{i=1}^3 \mathcal{O}_{D_i}(K_Y))). \end{aligned} \quad (5.3)$$

An easy calculation shows now that $\chi(\bigoplus_{i=1}^3 \mathcal{O}_{D_i}(K_Y)) = 0$, whereas $\chi(\Omega_Y^1(K_Y)) = -4$. \square

We prove now

Proposition 5.6.

- (1) $h^0(Y, \Omega_Y^1(\log D_1)(K_Y + \mathcal{L}_1)) = h^2(\tilde{S}, \Theta_{\tilde{S}})^1 = h^2(S, \Theta_S)^1 \leq 2;$
- (2) $h^0(Y, \Omega_Y^1(\log D_2)(K_Y + \mathcal{L}_2)) = h^2(\tilde{S}, \Theta_{\tilde{S}})^2 = h^2(S, \Theta_S)^2 \leq 3;$
- (3) $h^0(Y, \Omega_Y^1(\log D_3)(K_Y + \mathcal{L}_3)) = h^2(\tilde{S}, \Theta_{\tilde{S}})^3 = h^2(S, \Theta_S)^3 \leq 3.$

In particular, we get $h^2(S, \Theta_S) \leq 8$.

From this result, we easily obtain the following:

Corollary 5.7. *Let S be an Inoue surface with $K_S^2 = 7$, $p_g = 0$. Then:*

$$h^2(S, \Theta_S) = 8, \quad h^1(S, \Theta_S) = h^1(S, \Theta_S)^{inv} = 4.$$

Proof of the corollary. We have by proposition 5.6

$$8 - h^1(S, \Theta_S) \geq h^2(S, \Theta_S) - h^1(S, \Theta_S) = \chi(\Theta_S) = 2K_S^2 - 10\chi(S) = 4,$$

whence $h^1(S, \Theta_S) \leq 4$. On the other hand by lemma 5.5 we know that

$$4 = h^1(S, \Theta_S)^{inv} \leq h^1(S, \Theta_S)$$

and the assertions of the corollary follow. \square

Proof of proposition 5.6. 1) Recall that by definition 5.2 D_1 is the disjoint union of the four curves Δ_1, f_2, S_1, S_2 , and

$$K_Y + \mathcal{L}_1 \equiv f_1 - E_4.$$

We consider the exact sequence (cf. e.g. [EV92], p. 13)

$$\begin{aligned} 0 \rightarrow \Omega_Y^1(\log \Delta_1, \log f_2, \log S_1, \log S_2, \log f_1)(-E_4) \rightarrow \\ \rightarrow \Omega_Y^1((\log \Delta_1, \log f_2, \log S_1, \log S_2)(f_1 - E_4)) \rightarrow \\ \rightarrow \Omega_{f_1}^1(\Delta_1 + f_2 + S_1 + S_2 + f_1 - E_4) \rightarrow 0. \end{aligned} \quad (5.4)$$

Since

$$(\Delta_1 + f_2 + S_1 + S_2 + f_1 - E_4)f_1 = 2 + 2 - 1,$$

we have $\Omega_{f_1}^1(\Delta_1 + f_2 + S_1 + S_2 + f_1 - E_4) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, whence

$$\begin{aligned} h^0(Y, \Omega_Y^1(\log D_1)(K_Y + \mathcal{L}_1)) &= h^0(Y, \Omega_Y^1((\log \Delta_1, \log f_2, \log S_1, \log S_2)(f_1 - E_4))) \\ &\leq h^0(Y, \Omega_Y^1(\log \Delta_1, \log f_2, \log S_1, \log S_2, \log f_1)(-E_4)) + 2. \end{aligned} \quad (5.5)$$

Consider the long exact cohomology sequence of the short exact sequence

$$\begin{aligned} 0 \rightarrow \Omega_Y^1 \rightarrow \Omega_Y^1(\log \Delta_1, \log f_2, \log S_1, \log S_2, \log f_1) \rightarrow \\ \rightarrow \mathcal{O}_{f_1} \oplus \mathcal{O}_{f_2} \oplus \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \oplus \mathcal{O}_{\Delta_1} \rightarrow 0 \end{aligned} \quad (5.6)$$

Since $H^0(Y, \Omega_Y^1) = 0$, $H^0(Y, \Omega_Y^1(\log \Delta_1, \log f_2, \log S_1, \log S_2, \log f_1))$ is the kernel of the connecting homomorphism

$$\delta: H^0(Y, \mathcal{O}_{f_1} \oplus \mathcal{O}_{f_2} \oplus \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \oplus \mathcal{O}_{\Delta_1}) \rightarrow H^1(Y, \Omega_Y^1).$$

By [Cat84] the image of δ is generated by the Chern classes of $\Delta_1, f_2, S_1, S_2, f_1$.

Claim 5.8. $\dim \text{Im}(\delta) = 5$.

Proof of the claim. Assume that

$$\lambda_1 f_1 + \lambda_2 f_2 + a_1 S_1 + a_2 S_2 + \mu \Delta_1 \equiv 0, \quad (5.7)$$

where $\lambda_1, \lambda_2, a_1, a_2, \mu \in \mathbb{C}$. Intersection with E_4 gives $\lambda_1 = 0$, whereas intersection with $S_i, i = 1, 2$ yields $-2a_i = 0$. The equation 5.7 has become

$$\mu \Delta_1 + \lambda_2 f_2 = 0.$$

Intersection with e.g. E_5 gives $\lambda_2 = 0$, whence also $\mu = 0$. □

It follows now that

$$\begin{aligned} h^0(Y, \Omega_Y^1(\log \Delta_1, \log f_2, \log S_1, \log S_2, \log f_1)(-E_4)) &\leq \\ &\leq h^0(Y, \Omega_Y^1(\log \Delta_1, \log f_2, \log S_1, \log S_2, \log f_1)) = 0. \end{aligned}$$

Therefore we have proven

$$h^0(Y, \Omega_Y^1(\log D_1)(K_Y + \mathcal{L}_1)) \leq 2.$$

2) By definition 5.2 D_2 is the disjoint union of the two curves Δ_2, f_3 , and

$$K_Y + \mathcal{L}_2 \equiv -K_Y - E_5 - E_6.$$

Since

$$(K_Y + 2\Delta_2 + (-K_Y - E_5 - E_6))\Delta_2 = (2\Delta_2 - E_5 - E_6)\Delta_2 = -2 < 0,$$

by lemma 5.1 of [BC10] we have

$$H^0(Y, \Omega_Y^1(\log D_2)(K_Y + \mathcal{L}_2)) \cong H^0(Y, \Omega_Y^1(\log f_3)(K_Y + \mathcal{L}_2 + \Delta_2)).$$

Note that

$$K_Y + \mathcal{L}_2 + \Delta_2 \equiv -K_Y - E_5 - E_6 + \Delta_2 \equiv S_1 + S_2 + S_4 + S_3 + E_1 + E_3.$$

Again by lemma 5.1 in [BC10] we see that

$$\begin{aligned} H^0(Y, \Omega_Y^1(\log f_3)(K_Y + \mathcal{L}_2 + \Delta_2)) &\cong \\ &\cong H^0(Y, \Omega_Y^1(\log f_3, \log S_1, \log S_2, \log S_3, \log S_4)(E_1 + E_3)). \end{aligned} \quad (5.8)$$

Since $E_1(f_3 + S_1 + S_2 + S_3 + S_4 + E_1 + E_3) = E_3(f_3 + S_1 + S_2 + S_3 + S_4 + E_1 + E_3) = 2$, we see by the same argument (using the analogous exact sequence 5.4) as in case 1) just applied twice that

$$\begin{aligned} h^0(Y, \Omega_Y^1(\log f_3, \log S_1, \log S_2, \log S_3, \log S_4)(E_1 + E_3)) &\leq \\ &\leq h^0(Y, \Omega_Y^1(\log f_3, \log S_1, \log S_2, \log S_3, \log S_4, \log E_1, \log E_3)) + 2. \end{aligned} \quad (5.9)$$

Claim 5.9. $\dim\langle f_3, S_1, \dots, S_4, E_1, E_3 \rangle = 6$.

Proof of the claim. Note that

$$S_1 + S_4 - S_2 - S_3 \equiv -2E_1 + 2E_3.$$

Therefore, if we show that the Chern classes of $f_3, S_1, \dots, S_4, E_1$ are linearly independent, we have proven the claim.

Assume that

$$\lambda f_3 + a_1 S_1 + a_2 S_2 + a_3 S_3 + a_4 S_4 + \mu E_1 \equiv 0, \quad (5.10)$$

where $\lambda, a_1, a_2, a_3, a_4, \mu \in \mathbb{C}$.

Intersection with S_2, S_3 gives $a_2 = a_3 = 0$, whereas intersection with Δ_1 yields $\mu = 0$. We are left with the equation $\lambda f_3 + a_1 S_1 + a_4 S_4 \equiv 0$. Intersection with E_5 resp. E_6 implies that $a_1 = 0$ resp. $a_4 = 0$, and we conclude that also $\lambda = 0$. \square

Therefore we get

$$h^0(Y, \Omega_Y^1(\log f_3, \log S_1, \log S_2, \log S_3, \log S_4, \log E_1, \log E_3)) = 1,$$

and we have shown 2).

3) D_3 is the disjoint union of the five curves $\Delta_3, f_1, f'_1, S_3, S_4$, and

$$K_Y + \mathcal{L}_3 \equiv L - E_1 - E_2 - E_3.$$

Since

$$(K_Y + 2\Delta_3 + (L - E_1 - E_2 - E_3))\Delta_3 = -2 < 0,$$

by [BC10], lemma 5.1, we have

$$\begin{aligned} H^0(Y, \Omega_Y^1(\log D_3)(K_Y + \mathcal{L}_3)) &\cong \\ &\cong H^0(Y, \Omega_Y^1(\log f_1, \log f'_1, \log S_3, \log S_4)(K_Y + \mathcal{L}_3 + \Delta_3)). \end{aligned} \quad (5.11)$$

Note that

$$K_Y + \mathcal{L}_3 + \Delta_3 \equiv S_1 + S_2 + E_2.$$

Again by lemma 5.1 in [BC10] we see that

$$\begin{aligned} H^0(Y, \Omega_Y^1(\log f_1, \log f'_1, \log S_3, \log S_4)(K_Y + \mathcal{L}_3 + \Delta_3)) &\cong \\ &\cong H^0(Y, \Omega_Y^1(\log f_1, \log f'_1, \log S_3, \log S_4, \log S_1, \log S_2)(E_2)). \end{aligned} \quad (5.12)$$

Since $E_2(f_1 + f'_1 + S_1 + S_2 + S_3 + S_4 + E_2) = 3$, we see by the same arguments as in case 1) that

$$\begin{aligned} h^0(Y, \Omega_Y^1(\log f_1, \log f'_1, \log S_3, \log S_4, \log S_1, \log S_2)(E_2)) &\leq \\ &\leq h^0(Y, \Omega_Y^1(\log f_1, \log f'_1, \log S_3, \log S_4, \log S_1, \log S_2, \log E_2)) + 2. \end{aligned} \quad (5.13)$$

Claim 5.10. The Chern classes of $f_1, S_1, \dots, S_4, E_2$ are linearly independent.

Proof of the claim. Assume that

$$\lambda f_1 + a_1 S_1 + a_2 S_2 + a_3 S_3 + a_4 S_4 + \mu E_2 \equiv 0, \quad (5.14)$$

where $\lambda, a_1, a_2, a_3, a_4, \mu \in \mathbb{C}$.

Intersection with S_3, S_4 gives $a_3 = a_4 = 0$, whereas intersection with E_1 yields then $a_1 = 0$. Intersection with E_3 instead gives $a_2 = 0$. Finally, intersection with E_4 yields $\lambda = 0$, whence also $\mu = 0$. \square

Therefore

$$h^0(Y, \Omega_Y^1(\log f_1, \log f'_1, \log S_3, \log S_4, \log S_1, \log S_2, \log E_2)) = 1,$$

and we have shown 3). \square

Proof of theorem 0.1. (1) has been proved in theorem 4.1.

(3) was proved in section (3.1).

(2) By [In94], page 318, K_S is ample and by corollary 5.7 the tangent space $H^1(S, \Theta_S)$ to the base $\text{Def}(S)$ of the Kuranishi family of S consists of the invariants for the action of the group $(\mathbb{Z}/2\mathbb{Z})^2$. Therefore all the local deformations of S admit a $(\mathbb{Z}/2\mathbb{Z})^2$ -action, hence are $(\mathbb{Z}/2\mathbb{Z})^2$ Galois coverings of the four-nodal cubic surface (the anticanonical image of Y).

Furthermore, the dimension of $H^1(S, \Theta_S)$ is equal to the dimension of the Inoue family containing S in the moduli space $\mathfrak{M}_{1,7}^{can}$, whence the base of the Kuranishi family of S is smooth.

Since the quotient of a smooth variety by a finite group (in our case, the automorphism group $\text{Aut}(S)$) is normal, it follows that the irreducible connected component of the moduli space $\mathfrak{M}_{1,7}^{can}$ corresponding to Inoue surfaces with $K_S^2 = 7, p_g = 0$ is normal and in particular generically smooth.

The family of Inoue surfaces is parametrized by a smooth (4-dimensional) rational variety (cf. e.g. remark 5.3), whence unirationality follows. \square

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