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THE IRREDUCIBLE COMPONENTS OF THE MODULI SPACE OF DIHEDRAL COVERS OF ALGEBRAIC CURVES

FABRIZIO CATANESE, MICHAEL LÖNNE, FABIO PERRONI

Abstract. The main purpose of this paper is to introduce a new invariant for the action of a finite group G on a compact complex curve of genus g. With the aid of this invariant we achieve the classification of the components of the locus (in the moduli space) of curves admitting an effective action by the dihedral group D_n. This invariant could be useful in order to extend the results of Livingston [Liv85], Dunfield and Thurston [Du-Th06] to the ramified case.

1. Introduction

We study moduli spaces of curves that admit an effective action by a given finite group G. These moduli spaces can be seen as closed algebraic subsets $M_g(G)$ of M_g, the moduli space of smooth curves of genus $g > 1$. We are mainly interested in understanding which are the irreducible components of $M_g(G)$.

To a curve C of genus g with an action by G, we can associate several discrete invariants that are constant under deformations, such as the *topological type* of the G-action, which is an homomorphism $\rho\colon G \to Map(C)$ (see Section 2). It turns out that the locus $M_{g,\rho}(G)$ of curves admitting a G-action of topological type ρ is a closed irreducible subset of M_g (Theorem 2.2). On the other hand the action of G on C gives rise to a morphism $p\colon C \to C/G =: C'$, a G-cover, and the geometry of p encodes several numerical invariants that are constant on $M_{g,\rho}(G)$: the genus g' of C', the number d of branch points $y_1, \dots, y_d \in C'$ and the orders $m_1, \dots, m_d$ of the local monodromies. These invariants form the *primary numerical type*. A second numerical invariant is obtained by considering the monodromy $\mu\colon \pi_1(C' \setminus \{y_1, \dots, y_d\}) \to G$ of the restriction of p to $p^{-1}(C' \setminus \{y_1, \dots, y_d\})$. This is the $Aut(G)$ equivalence class of the class function ν which, for each conjugacy class $\mathcal{C}$ in G, counts the number of local monodromies which belong to $\mathcal{C}$ and is called the *ν-type* of the cover.

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By Riemann's existence theorem and the irreducibility of $M_{g',d}$, the irreducible components $M_{g,\rho}(G)$ with a given primary numerical type are in bijection with the quotient of the set of the corresponding monodromies μ modulo the actions by $Aut(G)$ and $Map(g', d)$.
Here $Map(g', d)$ is the full mapping class group of genus g' and d unordered points. Thus a first step toward the general problem consists in finding a fine invariant that distinguishes these orbits, or equivalently the above irreducible components.

In this paper we introduce a new invariant $\hat{\varepsilon}$ for G-actions on smooth curves and we show that when $G = D_n$, the dihedral group of order $2n$, $\hat{\varepsilon}$ distinguishes different irreducible components $M_{g,\rho}(D_n)$, therefore $\hat{\varepsilon}$ is a fine invariant in this case.

Our invariant includes and extends two well known invariants that have been studied in the literature: the data of the conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_d \subset G$ of the local monodromies (modulo the action of $Aut(G)$ and up to permutation), the ν-type of the cover (also called shape in [FV91], cf. Def. 3.9); the class in the second homology group $H_2(G/H, \mathbb{Z})$ (modulo the action of $Aut(G/H)$) corresponding to the unramified cover $p' \colon C/H \to C'$, where H is the normal subgroup of G generated by the local monodromies.
These invariants, which refine the primary numerical type, provide a fine invariant under some restrictions, for instance when G is abelian and when G is the semi-direct product of two finite cyclic groups acting freely (as it follows by combining results from [Cat00], [Cat10], [Edm I] and [Edm II]). However, in general, they are not enough to distinguish irreducible components $M_{g,\rho}(G)$, as one can see already for non-free D_n-actions (see Lemma 5.8).

The construction of $\hat{\varepsilon}$ is similar to the procedure that, using Hopf's theorem, associates an element in $H_2(G, \mathbb{Z})$ to any free G-action on a smooth curve. For any finite group G, let F be the free group generated by the elements of G and let $R \trianglelefteq F$ be the subgroup of relations, that is $G = F/R$. For any $\Sigma \subset G$, union of non trivial conjugacy classes, set G_Σ be the quotient group of F by the subgroup generated by $[F, R]$ and $\hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \in F$, for any $a \in \Sigma$, $b, c \in G$, such that $ab = bc$. Here we denote by $\hat{g} \in F$ the generator corresponding to $g \in G$. To a given G-cover $p \colon C \to C'$ we associate the set Σ of elements which stabilize some point of C. Upon the choice of a geometric basis for the fundamental group of the branch complement $C' \setminus \{y_1, \dots, y_d\}$ our cover is given by an element $v = (c_1, \dots, c_d; a_1, b_1, \dots, a_{g'}, b_{g'}) \in G^{d+2g'}$ satisfying certain conditions (a *Hurwitz generating system*), where the first entries correspond to the local monodromies. Thereby $\Sigma = \Sigma_v$ is the union of the conjugacy classes of the c_i's. The tautological lift $\hat{v}$ of v is $(\widehat{c_1}, \dots, \widehat{c_d}; \widehat{a_1}, \widehat{b_1}, \dots, \widehat{a_{g'}}, \widehat{b_{g'}})$. Finally, define $\varepsilon(v)$ as the class in

G_Σ of

$$\prod_1^d \widehat{c_j} \cdot \prod_1^{g'} [\widehat{a_i}, \widehat{b_i}] .$$

It turns out that the image of $\varepsilon(v)$ in $(G_\Sigma)\,/_{Inn(G)}$ is invariant under the action of $Map(g', d)$ (Proposition 3.6). Moreover the ν-type of v can be deduced from $\varepsilon(v)$, as it is essentially the image of $\varepsilon(v)$ in the abelianized group G_Σ^{ab} (see the Remark after Def. 3.9).

In order to take into account also the automorphism group $Aut(G)$, we define

$$G^\cup = \coprod\nolimits_\Sigma G_\Sigma ,$$

the disjoint union of all the G_Σ's. Now, the group $Aut(G)$ acts on $G^\cup$ and we get a map

$$\hat{\varepsilon}\colon \left(HS(G; g', d))/_{Aut(G)}\right) /_{Map(g',d)} \to (G^\cup)/_{Aut(G)}$$

which is induced by $v \mapsto \varepsilon(v)$. Here we denote by $HS(G; g', d)$ the set of all Hurwitz generating systems of length $d + 2g'$.

Finally, we prove that, when $G = D_n$, the map $\hat{\varepsilon}$ is injective (Theorem 5.1), thus the invariant $\hat{\varepsilon}$ is a fine invariant for D_n-actions. This completes the classification of the irreducible components $M_{g,\rho}(D_n)$ began in [CLP11].

When $g' = 0$ our G_Σ is related to the group $\widehat{G}$ defined in [FV91] (Appendix), where the authors give a proof of a theorem by Conway and Parker. Roughly speaking the theorem says that: if the Schur multiplier $M(G)$ (which is isomorphic to $H_2(G, \mathbb{Z})$) is generated by commutators, then the ν-type is a fine stable invariant, when $g' = 0$. Results of this kind, when $g' > 0$ but for free G-actions and any finite group G, have been proved in [Liv85] and [Du-Th06]. This time the fine stable invariant lives in $H_2(G, \mathbb{Z})/_{Aut(G)}$.
The natural question that arises is whether our $\hat{\varepsilon}$-invariant is a fine stable invariant for any finite group G and any effective G-action on compact curves.

The structure of the paper is the following. In Section 2 we introduce the moduli spaces $M_g(G)$ and $M_{g,\rho}(G)$. Using Riemann's existence theorem, we reduce the problem of the determination of the loci $M_{g,\rho}(G)$ to a combinatorial one. This leads to the concept of topological type and of Hurwitz generating system. In Section 3 we define the function $\hat{\varepsilon}$, the groups $H_{2,\Sigma}(G)$ and we prove some properties. The object of Section 4 is the computation of $H_{2,\Sigma}(D_n)$. These results are all used in Section 5 where we prove the injectivity of $\hat{\varepsilon}$ when $G = D_n$. In the Appendix we collect some results about mapping class groups and their action on fundamental groups. We use these results in the proof of Theorem 5.1.

2. Moduli spaces of G-covers

Throughout this Section g is an integer, $g > 1$. The moduli space of curves of genus g is denoted by M_g. For any finite group G, $M_g(G)$ is the locus of $[C] \in M_g$ such that there exists an effective action of G on C. For any $[C] \in M_g(G)$, the quotient morphism $p\colon C \to C/G = C'$ is a Galois cover with group G, a G-cover, well defined up to isomorphisms.

Riemann's existence theorem allows us to use combinatorial methods to study G-covers, since p determines and is determined by its restriction to $C' \setminus \mathcal{B}$, where $\mathcal{B} = \{ y_1 , \ldots , y_d \} \subset C'$ is the branch locus of p. Fix a base point $y_0 \in C' \setminus \mathcal{B}$ and a point $x_0 \in p^{-1}(y_0)$. Monodromy gives a surjective group-homomorphism

$$\mu\colon \pi_1(C' \setminus \mathcal{B} , y_0) \longrightarrow G \tag{1}$$

that characterizes p up to isomorphism.

Let us recall that a **geometric basis** of $\pi_1(C' \setminus \mathcal{B} , y_0)$ consists of simple non-intersecting (away from the base point) loops

$$\gamma_1, \ldots , \gamma_d, \alpha_1, \beta_1, \ldots , \alpha_{g'}, \beta_{g'}$$

such that we get the presentation

$$\pi_1(C'\setminus\mathcal{B} , y_0) = \langle \gamma_1, \ldots , \gamma_d ; \alpha_1, \beta_1, \ldots , \alpha_{g'}, \beta_{g'} \mid \prod_1^d \gamma_j \cdot \prod_1^{g'} [\alpha_i, \beta_i] = 1 \rangle .$$

Varying a covering in a flat family with connected base, there are some numerical invariants which remain unchanged, the first ones being the respective genera g, g' of the curves C, C', which are related by the Hurwitz formula:

$$2(g-1) = |G|[2(g'-1) + \sum_i (1 - \frac{1}{m_i})], \quad m_i := ord(\mu(\gamma_i)) . \tag{2}$$

Observe moreover that a different choice of the geometric basis changes the generators γ_i, but does not change their conjugacy classes (up to permutation), hence another numerical invariant is provided by the number of elements $\mu(\gamma_i)$ which belong to a fixed conjugacy class in the group G.

We formalize these invariants through the following definition.

Definition 2.1. Let G be a finite group, and let $g', d \in \mathbb{N}$. A g', d-*Hurwitz vector in* G is an element $v \in G^{d+2g'}$, the Cartesian product of G $(d + 2g')$-times. A g', d-Hurwitz vector in G will also be denoted by

$$v = (c_1, \ldots , c_d ; a_1, b_1, \ldots , a_{g'}, b_{g'}) .$$

For any $i \in \{1, \dots, d+2g'\}$, the i-th component v_i of v is defined as usual. The *evaluation* of v is the element

$$ev(v) = \prod_1^d c_j \cdot \prod_1^{g'} [a_i, b_i] \in G\,.$$

A *Hurwitz generating system of length* $d+2g'$ *in* G is a g', d-Hurwitz vector v in G such that the following conditions hold:

(i) $c_i \neq 1$ for all i;
(ii) G is generated by the components v_i of v;
(iii) $\prod_1^d c_j \cdot \prod_1^{g'} [a_i, b_i] = 1$.

We denote by $HS(G; g', d) \subset G^{2g'+d}$ the set of all Hurwitz generating systems in G of length $d+2g'$.

Notice that, once we fix a base point $y_0 \in C' \setminus \mathcal{B}$ and a geometric basis of $\pi_1(C' \setminus \mathcal{B}, y_0)$, there is a one-to-one correspondence between the set of Hurwitz generating systems of length $d+2g'$ in G and the set of monodromies μ as in (1).

Topological type. We recall a result contained in [Cat00], see also [Cat08].

Define the orbifold fundamental group $\pi_1^{orb}(\, C' \setminus \mathcal{B}\,,\, y_0\,; m_1, \dots m_d)$ as the quotient of $\pi_1(C' \setminus \mathcal{B}, y_0)$ by the minimal normal subgroup generated by the elements $(\gamma_i)^{m_i}$. Then, if $p\colon C \to C'$ is a G-covering as above, we have an exact sequence

$$1 \to \pi_1(\, C\,,\, x_0\,) \to \pi_1^{orb}(\, C' \setminus \mathcal{B}\,,\, y_0\,; m_1, \dots m_d) \to G \to 1$$

which is completely determined by the monodromy, and which in turn determines, via conjugation, a homomorphism

$$\rho\colon G \to Out^0(\pi_1(\, C\,,\, x_0\,)) = Map(C) := Diff^+(C)/Diff^0(C)$$

which is fully equivalent to the topological action of G on C.

By Lemma 4.12 of [Cat00], all the curves C of a fixed genus g which admit a given topological action ρ of the group G are parametrized by a connected complex manifold; arguing as in Theorem 2.4 of [Cat10] we get

Theorem 2.2. *The triples (C, G, ρ) where C is a complex projective curve of genus $g \geq 2$, and G is a finite group acting effectively on C with a topological action of type ρ are parametrized by a connected complex manifold $\mathcal{T}_{g;G,\rho}$ of dimension $3(g'-1)+d$, where g' is the genus of $C' = C/G$, and d is the cardinality of the branch locus $\mathcal{B}$.*

The image $M_{g,\rho}(G)$ of $\mathcal{T}_{g;G,\rho}$ inside the moduli space M_g is an irreducible closed subset of the same dimension $3(g'-1)+d$.

Obviously, composing ρ with an automorphism $\varphi \in Aut(G)$, i.e. replacing ρ with $\rho \circ \varphi$, does not change the subgroup $\rho(G) \subset Map(C)$. In particular, $M_{g,\rho}(G) = M_{g,\rho\circ\varphi}(G)$, and similarly $\mathcal{T}_{g;G,\rho} = \mathcal{T}_{g;G,\rho\circ\varphi}$.

Notice that $M_g(G) = \bigcup_\rho M_{g,\rho}(G)$, hence the components of $M_g(G)$ are in one-to-one correspondence with a subset of the different topological types. So, the next question which the above result motivates is: when do two Galois monodromies $\mu_1,\mu_2 : \pi_1^{orb}(C' \setminus \mathcal{B}, y_0; m_1, \dots, m_d) \to G$ have the same topological type?

The answer is theoretically easy: the two covering spaces have the same topological type if and only if they are homeomorphic, hence if and only if μ_1 and μ_2 differ by:

- an automorphism of G;
- and a different choice of a geometric basis. This is performed by the mapping class group

$$Map(g', d) := \frac{Diff^+(C', \mathcal{B})}{Diff^0(C', \mathcal{B})} .$$

To reformulate these conditions in terms of Hurwitz generating systems, notice that $Aut(G)$ acts on $HS(G; g', d)$ componentwise, and $Map(g', d)$ acts on $HS(G; g', d)/_{Aut(G)}$. The latter action is given by the group homomorphism $Map(g', d) \to Out\,(\pi_1(C' \setminus \mathcal{B}, y_0))$ and the identification between monodromies μ and Hurwitz generating systems. Theorem 2.2 then implies that there is a bijection:

$$\{M_{g,\rho}(G) \,\text{with}\, g', d \,\text{fixed}\} \quad \longleftrightarrow \quad \left(HS(G; g', d)/_{Aut(G)}\right) /_{Map(g',d)} .$$

In the next sections, we will also use the action of the unpermuted mapping class group

$$Map^u(g', d+1) := Map^u(C', \mathcal{B} \cup \{y_0\})$$

on $HS(G; g', d)$, where $Map^u(g', d+1)$ consists of diffeomorphisms in $Diff^+(C')$ which are the identity on $\mathcal{B} \cup \{y_0\}$, modulo isotopy. For any $v_1, v_2 \in HS(G; g', d)$, we write $v_1 \sim v_2$ when they are in the same $Map^u(g', d+1)$-orbit. While, $v_1 \approx v_2$ means that they represent the same class in $\left(HS(G; g', d)/_{Aut(G)}\right) /_{Map(g',d)}$. Clearly $v_1 \sim v_2$ implies $v_1 \approx v_2$.

The mapping class group $Map(g', d)$ acts on $HS(G; g', d)$ only up to conjugation, but, since we are interested in classifying Hurwitz generating systems up to $Aut(G)$, we will also use the notation $\varphi \cdot v$, meaning $\varphi \cdot [v]$, with $\varphi \in Map(g', d)$ and $[v] \in HS(G; g', d)/_{Aut(G)}$.

3. The tautological lift

In this section we give the construction of our invariant in several steps. Having defined a suitable group G_Σ, for any $\Sigma \subset G$ union of non-trivial conjugacy classes, we go on to a map ε, which associates to each Hurwitz vector v (with $c_i \in \Sigma$) an element $\varepsilon(v) \in G_\Sigma$. Any automorphism $f \in Aut(G)$ induces an isomorphism $f_\Sigma : G_\Sigma \to G_{f(\Sigma)}$, hence $Aut(G)$ acts on the disjoint union $G^\cup = \coprod_\Sigma G_\Sigma$. We show two key properties of ε:

- it is $Aut(G)$-equivariant (Lemma 3.5), hence it descends to a map
$$\tilde{\varepsilon}\colon HS(G;g',d)/_{Aut(G)} \to G^{\cup}/_{Aut(G)}\,;$$
- $\tilde{\varepsilon}$ is constant on the orbits of the mapping class group $Map(g',d)$ (Proposition 3.6).

Therefore ε descends to our invariant $\hat{\varepsilon}$ which is formalized by the function
$$\hat{\varepsilon}\colon \left(HS(G;g',d)/_{Aut(G)}\right)/_{Map(g',d)} \to G^{\cup}/_{Aut(G)}\,, \quad \forall g', d\,,$$
induced by ε. We conclude the section with the study of general properties of the invariant that are relevant to this work.

Since our construction is inspired by Hopf's description of the second homology group $H_2(G,\mathbb{Z})$ [Hopf], we begin by recalling this. For a finite group G, fix a presentation of G:
$$1 \to R \to F \to G \to 1\,,$$
where F is a free group. Then there is a group isomorphism (cf. [Bro]):
$$H_2(G,\mathbb{Z}) \cong \frac{R\cap[F,F]}{[F,R]}\,. \tag{3}$$
If $v=(a_1,b_1,\dots,a_{g'},b_{g'})\in G^{2g'}$ satisfies $\prod_1^{g'}[a_i,b_i]=1$, then we can associate a class in $H_2(G,\mathbb{Z})$ in the following way: choose liftings $\widehat{a_i},\widehat{b_i}\in F$ of a_i,b_i, then $\prod_1^{g'}[\widehat{a_i},\widehat{b_i}]\in R\cap[F,F]$ and its class in $\frac{R\cap[F,F]}{[F,R]}$ gives an element of $H_2(G,\mathbb{Z})$, according to (3). Clearly, this element does not depend on the various choices, moreover it is invariant under the action of the mapping class group, thus giving a topological invariant of v.

The topological meaning of this invariant is the following. If the G-action on C is free, the covering $p\colon C\to C'$ is étale, and hence it corresponds to a continuous function $Bp\colon C'\to BG$, up to homotopy. Here BG is the classifying space of G. The topological invariant is simply the image $Bp_*([C'])\in H_2(BG,\mathbb{Z})=H_2(G,\mathbb{Z})$ of the fundamental class $[C']\in H_2(C',\mathbb{Z})$ of C' under the homomorphism $Bp_*\colon H_2(C',\mathbb{Z})\to H_2(BG,\mathbb{Z})$ induced by Bp. Now, if we view C' as an Eilenberg-Mac Lane space $K(\pi_1(C'),1)$, then the fundamental class $[C']$ is given by
$$\prod_1^{g'}[\widehat{\alpha_i},\widehat{\beta_i}]\in H_2(\pi_1(C'),\mathbb{Z})\,,$$
where as usual $\alpha_1,\beta_1,\dots,\alpha_{g'},\beta_{g'}$ is a geometric basis of $\pi_1(C')$ and $\widehat{\alpha_i},\widehat{\beta_i}$ are liftings to the free group of a presentation of $\pi_1(C')$. So,
$$Bp_*([C'])=\prod_1^{g'}[\widehat{a_i},\widehat{b_i}]\in H_2(G,\mathbb{Z})\,,$$

where $a_i = \mu(\alpha_i)$, $b_i = \mu(\beta_i)$ and $\mu\colon \pi_1(C') \to G$ is the monodromy of $p\colon C \to C'$.

From now on, $F = \langle \hat{g} \,|\, g \in G \rangle$ is the free group generated by the elements of G. Let $R \trianglelefteq F$ be the normal subgroup of relations, that is $G = \frac{F}{R}$.

Definition 3.1. Let G be a finite group and let F, R be as above. For any union of non-trivial conjugacy classes $\Sigma \subset G$, define

$$R_\Sigma = \langle\langle [F,R], \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \,|\, \forall a \in \Sigma, ab = bc \rangle\rangle\,,$$
$$G_\Sigma = \frac{F}{R_\Sigma}\,.$$

The map $\hat{a} \mapsto a$, $\forall a \in G$, induces a group homomorphism $\alpha\colon G_\Sigma \to G$. Set $K_\Sigma = Ker(\alpha)$.

Lemma 3.2. *With the notation as before, the following holds. $R_\Sigma \subset R$ and $K_\Sigma = \frac{R}{R_\Sigma}$. In particular K_Σ is contained in the center of G_Σ and the short exact sequence*

$$1 \to \frac{R}{R_\Sigma} \to G_\Sigma \to G \to 1$$

is a central extension.

Proof. $[F,R] \subset R$ because R is normal in F and moreover $\hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \in R$ for any $a,b,c \in G$ with $ab = bc$, therefore $R_\Sigma \subset R$. By the definition of α we have that $K_\Sigma = \frac{R}{R_\Sigma}$. Finally, K_Σ is in the center of G_Σ because $[F,R] \subset R_\Sigma$. □

The morphism $\alpha\colon G_\Sigma \to G$ has a *tautological section* $G \to G_\Sigma$, $a \mapsto \hat{a}$. This map is not a group homomorphism in general, but every element $\xi \in G_\Sigma$ can be written as $\hat{g}z = z\hat{g}$, with $g = \alpha(\xi) \in G$ and $z \in K_\Sigma$. Here, by abuse of notation, $\hat{a}$ denotes also the class of $\hat{a} \in F$ in $G_\Sigma = F/R_\Sigma$.

Lemma 3.3. *Let $\hat{a}, \xi \in G_\Sigma$. Suppose that $\hat{a}$ is conjugate to ξ in G_Σ and that $a \in \Sigma$. Then $\xi = \widehat{\alpha(\xi)}$.*

Proof. Let $\hat{b}z$ be a conjugating element, that is $\hat{a}\hat{b}z = \hat{b}z\xi$. As $z \in K_\Sigma$, it commutes with any element, hence

$$\hat{a}\hat{b} = \hat{b}\xi\,. \tag{4}$$

Now apply α and obtain: $ab = b\alpha(\xi)$. By assumption $a \in \Sigma$, hence by definition of G_Σ we have that $\hat{a}\hat{b} = \hat{b}\widehat{\alpha(\xi)}$. Now using (4) we deduce $\xi = \widehat{\alpha(\xi)}$. □

Definition 3.4. Given a g', d-Hurwitz vector

$$v = (c_1, \ldots, c_d; a_1, b_1, \ldots, a_{g'}, b_{g'})$$

in G (cf. Definition 2.1), its *tautological lift*, $\hat{v}$, is the g', d-Hurwitz vector in G_Σ defined by

$$\hat{v} = (\widehat{c_1}, \dots, \widehat{c_d}; \widehat{a_1}, \widehat{b_1}, \dots, \widehat{a_{g'}}, \widehat{b_{g'}})$$

where the factors are the tautological lifts of the factors of v.

Given a g', d-Hurwitz vector v in G with $c_i \neq 1$, $\forall i$, we denote by Σ_v the union of all conjugacy classes of G containing at least one c_i.

For any v as before, set

$$\varepsilon(v) = \prod_1^d \widehat{c_j} \cdot \prod_1^{g'} [\widehat{a_i}, \widehat{b_i}] \in G_{\Sigma_v} \,,$$

the evaluation of the tautological lift $\hat{v}$ of v in G_{Σ_v} (cf. Definition 2.1).

Lemma 3.5. *Let G be any finite group, and let $\Sigma \subset G$ be any union of non trivial conjugacy classes. Then we have:*

i) *any $f \in Aut(G)$ induces an isomorphism $f_\Sigma \colon G_\Sigma \to G_{f(\Sigma)}$;*

ii) *$\varepsilon(f(v)) = f_\Sigma(\varepsilon(v))$, $\forall f \in Aut(G)$ and $\forall v$ a g', d-Hurwitz vector with $c_i \neq 1$, $\forall i$, where $\Sigma = \Sigma_v$.*

Proof. i) $f \in Aut(G)$ lifts to an automorphism $\hat{f} \in Aut(F)$ defined by

$$\hat{f} : \hat{a} \mapsto \widehat{f(a)} \,.$$

We have: $\hat{f}(R) \subset R$, and moreover

$$\hat{f}(\hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1}) \quad = \quad \widehat{f(a)}\widehat{f(b)}\widehat{f(c)}^{-1}\widehat{f(b)}^{-1} \,,$$

for any $a, b, c \in G$. If $a \in \Sigma$, then $f(a) \in f(\Sigma)$ and hence

$$\widehat{f(a)}\widehat{f(b)}\widehat{f(c)}^{-1}\widehat{f(b)}^{-1} \in R_{f(\Sigma)} \,.$$

ii)

$$\begin{aligned}
\varepsilon(f(v)) &= \varepsilon(f(c_1), \dots, f(c_d); f(a_1), \dots, f(b_{g'})) \\
&= \prod_1^d \widehat{f(c_i)} \cdot \prod_1^{g'} [\widehat{f(a_j)}, \widehat{f(b_j)}] \\
&= \prod_1^d \hat{f}(\widehat{c_i}) \cdot \prod_1^{g'} [\hat{f}(\widehat{a_j}), \hat{f}(\widehat{b_j})] = f_\Sigma(\varepsilon(v)) \,.
\end{aligned}$$

□

Now, define

$$G^\cup = \coprod\nolimits_\Sigma G_\Sigma \,,$$

and regard ε as a map $\varepsilon \colon HS(G; g', d) \to G^\cup$, $v \mapsto \varepsilon(v) \in G_{\Sigma_v}$. Then the previous lemma means that ε induces a map

$$\tilde{\varepsilon} \colon HS(G; g', d)/_{Aut(G)} \to (G^\cup)\,/_{Aut(G)} \,.$$

We have the following

Proposition 3.6. *For any $g', d \in \mathbb{N}$, $\tilde{\varepsilon}$ is $Map(g', d)$-invariant, hence it induces a map*

$$\hat{\varepsilon}\colon \left(HS(G; g', d)/_{Aut(G)}\right)/_{Map(g',d)} \to (G^{\cup})/_{Aut(G)} .$$

To prove this proposition we need some preliminary results.

Lemma 3.7. *Let Σ_v be associated to a g', d-Hurwitz vector v as in Definition 3.4. If the Hurwitz vector v' is related to v by an elementary braid move, then $\varepsilon(v) = \varepsilon(v')$.*

Proof. It suffices to consider the case $g = 0$, $d = 2$ and the braid move associated to σ_1. Then

$$v = (c_1, c_2), \quad v' = (c_2, c_2^{-1} c_1 c_2) .$$

In G_{Σ_v} we have, thanks to $c_1 \in \Sigma_v$, $c_1 c_2 = c_2(c_2^{-1} c_1 c_2)$, and the relations of G_{Σ_v}:

$$\varepsilon(v) = \widehat{c_1}\widehat{c_2} = \widehat{c_2}\widehat{c_2^{-1}c_1c_2} = \varepsilon(v') .$$

□

Lemma 3.8. *If $\xi, \eta \in G_{\Sigma}$, then*

$$[\xi, \eta] \quad = \quad \left[\widehat{\alpha(\xi)}, \widehat{\alpha(\eta)}\right] .$$

Proof. Write $\xi = \widehat{\alpha(\xi)}z$ and $\eta = \widehat{\alpha(\eta)}z'$ with z, z' in K_{Σ}, hence central (Lemma 3.2). Then the conclusion is immediate. □

Proof. (Of Proposition 3.6.) Let $\varphi \in Map(g', d)$. Thanks to Lemma 3.7 it suffices to consider the case that φ is a pure mapping class, i.e. that φ does not permute the conjugacy classes associated to the local monodromies. Using Lemma 3.5 ii), we can further ignore the action of G by conjugation and pretend that $Map(g', d)$ acts on $HS(G; g', d)$.

Since φ is a pure mapping class, $v_i \sim (\varphi \cdot v)_i$ and similarly $\hat{v}_i \sim (\varphi \cdot \widehat{v})_i$ (by Lemma 3.3), for $i = 1, \dots, d$, where $\sim$ means conjugation.

By Lemma 3.3, for $i = 1, \dots, d$, we have:

$$(\hat{v})_i \sim (\varphi \cdot \hat{v})_i \Rightarrow (\varphi \cdot \hat{v})_i = \widehat{\alpha((\varphi \cdot \hat{v})_i)} .$$

Now notice that the morphism α (Definition 3.1) is equivariant under the action of the mapping class group in the following sense: consider the factorizations as a map from the free group on $d + 2g'$ generators to G_{Σ}, resp. G, and the mapping class group as a group of automorphisms of this free group. Then α is equivariant, since such automorphisms act by pre-composition.

By the equivariance of α

$$\alpha((\varphi \cdot \hat{v})_i) \quad = \quad (\varphi \cdot v)_i.$$

Hence, for $i = 1, \dots, d$,

$$(\varphi \cdot \hat{v})_i = \widehat{(\varphi \cdot v)_i} = (\widehat{\varphi \cdot v})_i .$$

By Lemma 3.8 we may change also the entries $(\varphi \cdot \hat{v})_i$, $i > d$ in the commutators to $\widehat{\alpha((\varphi\cdot\hat{v})_i)} = (\widehat{\varphi \cdot v})_i$ without changing the value of the commutators. Hence

$$ev(\varphi \cdot \hat{v}) \quad = \quad ev(\widehat{\varphi \cdot v}) \quad = \quad \varepsilon(\varphi \cdot v)\,.$$

By the invariance of the evaluation under the mapping class

$$\varepsilon(v) \quad = \quad ev(\hat{v}) \quad = \quad ev(\varphi \cdot \hat{v})$$

and we have proved our claim. □

Definition 3.9. Let $v \in HS(G; g', d)$ and let $\nu(v) \in \oplus_{\mathcal{C}}\mathbb{Z}\langle\mathcal{C}\rangle$ ($\mathcal{C}$ runs over the set of conjugacy classes of G) be the vector whose $\mathcal{C}$-component is the number of v_j, $j \leq d$, which belong to $\mathcal{C}$.
The map

$$\nu\colon HS(G; g', d) \to \oplus_{\mathcal{C}}\mathbb{Z}\langle\mathcal{C}\rangle$$

obtained in this way induces a map

$$\tilde{\nu}\colon HS(G; g', d)/_{Aut(G)} \to (\oplus_{\mathcal{C}}\mathbb{Z}\langle\mathcal{C}\rangle)\,/_{Aut(G)}$$

which is $Map(g', d)$-invariant, therefore we get a map

$$\hat{\nu}\colon \left(HS(G; g', d)/_{Aut(G)}\right)/_{Map(g',d)} \to (\oplus_{\mathcal{C}}\mathbb{Z}\langle\mathcal{C}\rangle)/_{Aut(G)}\,.$$

For any $v \in HS(G; g', d)$, we call $\hat{\nu}(v)$ the ν-type of v (also called the shape in [FV91]).

Remark 3.10. Let $v \in HS(G; g', d)$ and let $\Sigma_v \subset G$ be the union of the conjugacy classes of v_j, $j \leq d$. The abelianization $G^{ab}_{\Sigma_v}$ of G_{Σ_v} can be described as follows:

$$G^{ab}_{\Sigma_v} \cong \oplus_{\mathcal{C}\subset\Sigma}\mathbb{Z}\langle\mathcal{C}\rangle \oplus \oplus_{g\in G\setminus\Sigma_v}\mathbb{Z}\langle g\rangle\,,$$

where $\mathcal{C}$ denotes a conjugacy class of G. Moreover $\nu(v)$ coincides with the vector whose $\mathcal{C}$-components are the corresponding components of the image in $G^{ab}_{\Sigma_v}$ of $\varepsilon(v) \in G_{\Sigma_v}$ under the natural homomorphism $G_{\Sigma_v} \to G^{ab}_{\Sigma_v}$.

Definition 3.11. Let $\Sigma \subset G$ be a union of non-trivial conjugacy classes of G. We define

$$H_{2,\Sigma}(G) = \ker\left(G_\Sigma \to G \times G^{ab}_\Sigma\right)\,,$$

where $G_\Sigma \to G \times G^{ab}_\Sigma$ is the morphism with first component α (defined in Definition 3.1) and second component the natural morphism $G_\Sigma \to G^{ab}_\Sigma$.

Notice that

$$H_2(G, \mathbb{Z}) \cong \frac{R \cap [F, F]}{[F, R]} \cong \ker\left(\frac{F}{[F, R]} \to G \times G^{ab}_\emptyset\right)\,.$$

In particular, when $\Sigma = \emptyset$, $H_{2,\Sigma}(G) \cong H_2(G, \mathbb{Z})$.

The next result gives a precise relation between $H_2(G, \mathbb{Z})$ and $H_{2,\Sigma}(G)$.

Lemma 3.12. *Let G be a finite group and let $\Sigma \subset G$ be a union of nontrivial conjugacy classes. Write $G = \frac{F}{R}$ and $G_\Sigma = \frac{F}{R_\Sigma}$. Then, there is a short exact sequence*

$$1 \to \frac{R_\Sigma \cap [F,F]}{[F,R]} \to H_2(G,\mathbb{Z}) \to H_{2,\Sigma}(G) \to 1\,.$$

In particular $H_{2,\Sigma}(G)$ is abelian.

Proof. We first define the morphism $H_2(G,\mathbb{Z}) \to H_{2,\Sigma}(G)$.
By Hopf's Theorem we identify $H_2(G,\mathbb{Z})$ with $\frac{R\cap[F,F]}{[F,R]}$ (cf. [Bro]). On the other hand we have:

$$H_{2,\Sigma}(G) = Ker\,(G_\Sigma \to G) \cap Ker\left(G_\Sigma \to G_\Sigma^{ab}\right) = \frac{R}{R_\Sigma} \cap [G_\Sigma, G_\Sigma]\,.$$

By Lemma 3.2, $R_\Sigma \subset R$. The homomorphism $R \cap [F,F] \to \frac{R}{R_\Sigma}$, $r \mapsto rR_\Sigma$, takes values in $H_{2,\Sigma}(G)$. Moreover it descends to a group homomorphism $H_2(G,\mathbb{Z}) \to H_{2,\Sigma}(G)$ because $[F,R] \subset R_\Sigma$.

To prove the surjectivity, let

$$aR_\Sigma \in \frac{R}{R_\Sigma} \cap [G_\Sigma, G_\Sigma]\,.$$

Since $aR_\Sigma \in [G_\Sigma, G_\Sigma] = \frac{[F,F]\cdot R_\Sigma}{R_\Sigma}$, we may assume $a \in [F,F]$. From $aR_\Sigma \in \frac{R}{R_\Sigma}$, we have $aR_\Sigma = rR_\Sigma$, for some $r \in R$. Since $R_\Sigma \subset R$, we deduce that $a \in R$ and so the surjectivity follows.

The kernel of the morphism so defined is $\frac{R_\Sigma \cap [F,F]}{[F,R]}$.

Since $H_2(G,\mathbb{Z})$ is abelian, so is $H_{2,\Sigma}(G)$. □

Proposition 3.13. *Let $v_1, v_2 \in HS(G; g', d)$ be two Hurwitz generating systems in G with the same ν-type. Then $\Sigma_{v_1} = \Sigma_{v_2} =: \Sigma$. Moreover, if $ev(v_1) = ev(v_2) \in G$, then the element*

$$\varepsilon(v_1)^{-1} \cdot \varepsilon(v_2) \;\in\; H_{2,\Sigma}(G)$$

is invariant under the group $\widetilde{Map}(g', d)$ of isotopy classes of orientation-preserving diffeomorphisms of the pair $(C', \mathcal{B})$ that fix y_0. In particular:

1. *if v_1 and v_2 are equivalent then the element is trivial;*
2. *if the element is non-trivial, then v_1 and v_2 are in-equivalent.*

4. Computation of $H_{2,\Sigma}(D_n)$

In this section we derive a complete description of $H_{2,\Sigma}(D_n)$.

Proposition 4.1. *Let $n \in \mathbb{N}$, $n \geq 3$. Then we have:*

- (i) *$H_2(D_n,\mathbb{Z})$ is trivial if n is odd and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if n is even;*
- (ii) *the natural action of $Aut(D_n)$ on $H_2(D_n,\mathbb{Z})$ is trivial.*

Proof. (ii) This claim follows directly from (i) and from the fact that the neutral element of $H_2(D_n, \mathbb{Z})$ is fixed by the action of $Aut(D_n)$.

(i) Identify D_n with the subgroup of $SO(3)$ generated by

$$x := \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $u\colon SU(2) \to SO(3)$ be the homomorphism $q \mapsto R_q$, where we identify $SU(2)$ with the quaternions $q \in \mathbb{H}$ of norm 1, $\mathbb{R}^3$ with $Im\mathbb{H}$, and $R_q(x) = qx\overline{q}$. Consider the binary dihedral group $\tilde{D}_n = u^{-1}(D_n)$. It fits in the following short exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{D}_n \to D_n \to 1\,, \tag{5}$$

from which we get the 5-term exact sequence (see e.g. [Bro], pg. 47, Exercise 6):

(6)

$$H_2(\tilde{D}_n) \to H_2(D_n) \to (H_1(\mathbb{Z}/2\mathbb{Z}))_{D_n} \to H_1(\tilde{D}_n) \to H_1(D_n) \to 0\,,$$

where all the coefficients are in $\mathbb{Z}$ and $(H_1(\mathbb{Z}/2\mathbb{Z}))_{D_n}$ is the group of co-invariants under the D_n-action on $\mathbb{Z}/2\mathbb{Z}$ induced by conjugation by $\tilde{D}_n$, hence $(H_1(\mathbb{Z}/2\mathbb{Z}))_{D_n} = H_1(\mathbb{Z}/2\mathbb{Z})$ since $\mathbb{Z}/2\mathbb{Z}$ is in the center of $\tilde{\mathrm{D}}_n$.

We have that $H_2(\tilde{D}_n) = \{0\}$, since $\tilde{D}_n$ is a finite subgroup of $SU(2) \cong S^3$ (see [Bro] II pg. 47 , Exercise 7). Next, recall that, for any group G, $H_1(G, \mathbb{Z})$ is isomorphic to the abelianization G^{ab} (see [Bro] pg. 36), hence (6) reduces to

$$0 \to H_2(D_n) \to \mathbb{Z}/2\mathbb{Z} \to \tilde{D}_n^{ab} \to \mathrm{D}_n^{ab} \to 0\,.$$

To conclude we show that $Ker(\tilde{D}_n^{ab} \to D_n^{ab}) = \{0\}$ if and only if n is even. With the imaginary units $\underline{i}, \underline{j}, \underline{k} \in \mathbb{H}$ let

$$\xi = \cos\left(\frac{\pi}{n}\right) + \underline{k} \cdot \sin\left(\frac{\pi}{n}\right) \in u^{-1}(x) \quad \text{and} \quad \eta = \underline{j} \in u^{-1}(y)\,.$$

Since $[\xi^\ell, \eta] = \xi^{2\ell}$, $\forall \ell$, we see that, if n is odd, $\xi^n \notin [\tilde{D}_n, \tilde{D}_n]$, but $u(\xi^n) = 1$ and hence $Ker(\tilde{D}_n^{ab} \to D_n^{ab}) \neq \{0\}$. When n is even, $\tilde{D}_n^{ab} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the map $\tilde{D}_n^{ab} \to D_n^{ab}$ is an isomorphism. $\square$

Using Lemma 3.2 from [Wie], we deduce the following

Corollary 4.2. *Let $n \in \mathbb{N}$, $n \geq 4$ even. Then, the binary dihedral group $\tilde{D}_n$ is a Schur cover of D_n and the exact sequence* (5) *identifies $\mathbb{Z}/2\mathbb{Z}$ with $H_2(D_n, \mathbb{Z})$. In particular, for any $(a_1, b_1, \ldots, a_{g'}, b_{g'}) \in (D_n)^{2g'}$ with $\prod_1^{g'} [a_i, b_i] = 1$, the image of $\prod_1^{g'} [\widehat{a_i}, \widehat{b_i}] \in R \cap [F, F]$ in $H_2(D_n, \mathbb{Z}) = \frac{R\cap[F,F]}{[R,F]}$ is given by $\prod_1^{g'} [\tilde{a}_i, \tilde{b}_i]$, where $\tilde{a}_i, \tilde{b}_i \in \tilde{D}_n$ are liftings of a_i, b_i.*

Corollary 4.3. *Let $\Sigma \subset D_n$ be the union of non-trivial conjugacy classes, $\Sigma \neq \emptyset$. Then $H_{2,\Sigma}(D_n) = \{0\}$ in the following cases: n is odd; n is even and Σ contains some reflection; n is even and Σ contains the non-trivial central element. In the remaining case, $H_{2,\Sigma}(D_n) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. If n is odd, then $H_2(D_n, \mathbb{Z}) = \{0\}$ and hence $H_{2,\Sigma}(D_n) = \{0\}$ for any Σ (Lemma 3.12).

If $n = 2k$ and Σ contains some reflection, say $y \in \Sigma$, then $\hat{y}\widehat{x^k}\hat{y}^{-1}\widehat{x^k}^{-1} \in R_\Sigma \cap [F, F]$. But the image of this element in $H_2(D_n, \mathbb{Z})$ is not trivial (Corollary 4.2), hence $H_{2,\Sigma}(D_n) = \{0\}$ (Lemma 3.12). The same argument works if $xy \in \Sigma$.

Assume now $n = 2k$ and $x^k \in \Sigma$. Then $\widehat{x^k}\hat{y}\widehat{x^k}^{-1}\hat{y}^{-1} \in R_\Sigma \cap [F, F]$ and its image in $H_2(D_n, \mathbb{Z})$ is not trivial, hence $H_{2,\Sigma}(D_n) = \{0\}$ also in this case.

Finally, if $n = 2k$ and $\Sigma \subset \mathbb{Z}/n\mathbb{Z} \setminus \{x^k\}$, then

$$R_\Sigma = \langle\langle [F, R], \widehat{x^\alpha}\widehat{x^\beta}\widehat{x^\alpha}^{-1}\widehat{x^\beta}^{-1}, \widehat{x^\alpha}\widehat{x^\beta y}\widehat{x^{n-\alpha}}^{-1}\widehat{x^\beta y}^{-1} \mid x^\alpha \in \Sigma \rangle\rangle .$$

First we note that the image of $\widehat{x^\alpha}\widehat{x^\beta}\widehat{x^\alpha}^{-1}\widehat{x^\beta}^{-1}$ in $H_2(D_n, \mathbb{Z})$ is 0. Second, the elements $\widehat{x^\alpha}\widehat{x^\beta y}\widehat{x^{n-\alpha}}^{-1}\widehat{x^\beta y}^{-1}$ generate an abelian group modulo $[F, R]$. Last, the intersection of this subgroup with $[F, F]/[F, R]$ is generated by elements represented by

$$\widehat{x^\alpha}\widehat{x^\beta y}\widehat{x^{n-\alpha}}^{-1}\widehat{x^\beta y}^{-1} \quad \cdot \quad \widehat{x^{n-\alpha}}\widehat{x^\gamma y}\widehat{x^\alpha}^{-1}\widehat{x^\gamma y}^{-1} .$$

It remains to show that these are trivial modulo $[F, R]$, in fact

$$\begin{aligned}
&\equiv \widehat{x^\beta y}^{-1}\widehat{x^\alpha}\widehat{x^\beta y}\widehat{x^{n-\alpha}}^{-1} \quad \cdot \quad \widehat{x^{n-\alpha}}\widehat{x^\gamma y}\widehat{x^\alpha}^{-1}\widehat{x^\gamma y}^{-1} \\
&\equiv \widehat{x^\beta y}^{-1}\widehat{x^\alpha}\widehat{x^\beta y} \quad \widehat{x^\gamma y}\widehat{x^\alpha}^{-1}\widehat{x^\gamma y}^{-1} \\
&\equiv \widehat{x^\beta y}^{-1}\widehat{x^\alpha} \quad \underbrace{\widehat{x^\beta y} \quad \widehat{x^\gamma y} \quad \widehat{x^{\gamma-\beta}}}_{\in R} \quad \widehat{x^{\gamma-\beta}}^{-1} \quad \widehat{x^\alpha}^{-1}\widehat{x^\gamma y}^{-1} \\
&\equiv \widehat{x^\beta y}^{-1} \quad \widehat{x^\beta y} \quad \widehat{x^\gamma y} \quad \widehat{x^{\gamma-\beta}} \quad \widehat{x^\alpha} \quad \widehat{x^{\gamma-\beta}}^{-1} \quad \widehat{x^\alpha}^{-1}\widehat{x^\gamma y}^{-1} \\
&\equiv \widehat{x^\gamma y} \quad \widehat{x^{\gamma-\beta}} \quad \widehat{x^\alpha} \quad \widehat{x^{\gamma-\beta}}^{-1} \quad \widehat{x^\alpha}^{-1}\widehat{x^\gamma y}^{-1} \\
&\equiv \widehat{x^{\gamma-\beta}} \quad \widehat{x^\alpha} \quad \widehat{x^{\gamma-\beta}}^{-1}\widehat{x^\alpha}^{-1} .
\end{aligned}$$

This last element is trivial modulo $[F, R]$ as noted first. We deduce that $\frac{R_\Sigma \cap [F,F]}{[F,R]} = \{0\}$ and hence $H_{2,\Sigma}(D_n) \cong H_2(D_n, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, by Lemma 3.12.

□

5. The injectivity of $\hat{\varepsilon}$ when $G = D_n$

Recall the following

Notation. For any Hurwitz vector $v = (c_1, \ldots, c_d; a_1, b_1, \ldots, a_{g'}, b_{g'}) \in G^{d+2g'}$,

$$ev(v) = \prod_{i=1}^{d} c_i \cdot \prod_{j=1}^{g'} [a_j, b_j] \in G\,,$$

while, if $c_i \neq 1$, $\forall i$, $\varepsilon(v) = ev(\hat{v}) \in G_{\Sigma_v}$, where $\hat{v} \in (G_{\Sigma_v})^{d+2g'}$ is the tautological lifting (Definition 3.4).

In this section we prove the following

Theorem 5.1. *Let $G = D_n$, the dihedral group of order $2n$. Then, $\forall g', d$, we have:*

(i) $\hat{\varepsilon}\colon \left(HS(G; g', d)/_{Aut(G)}\right)/_{Map(g',d)} \to (G^{\cup})/_{Aut(G)}$ *is injective;*

(ii) *the image $Im(\hat{\varepsilon})$ is the inverse image of $Im(\hat{\nu})$.*

To prove (i), let $[v_1]_{\approx}, [v_2]_{\approx} \in (HS)/_{\approx}$ with $\hat{\varepsilon}([v_1]_{\approx}) = \hat{\varepsilon}([v_2]_{\approx})$. Then there exists an automorphism $f \in Aut(G)$ such that $f(\Sigma_{v_1}) = \Sigma_{v_2}$ and $f(\varepsilon(v_1)) = \varepsilon(v_2)$. Hence, by Lemma 3.5, we assume without loss of generality $\Sigma_{v_1} = \Sigma_{v_2} = \Sigma$ and $\varepsilon(v_1) = \varepsilon(v_2)$, in particular

$$\varepsilon(v_1) \cdot \varepsilon(v_2)^{-1} = 0 \in H_{2,\Sigma}(G)\,.$$

The outline of the proof is now the following. We address the following mutually exclusive cases: $\Sigma = \emptyset$ (the étale case); $\Sigma \neq \emptyset$ and contains some reflection; $\Sigma \neq \emptyset$ and does not contain reflections. In the first case, for each element of HS, we determine a normal form with respect to $\approx$, then we show that two different normal forms are distinguished by $H_2(D_n, \mathbb{Z})$ (recall that $Aut(D_n)$ acts trivially on $H_2(D_n, \mathbb{Z})$). In the second case we will show that all Hurwitz generating systems with the same numerical invariants (n, g' and ν-type) are equivalent with respect to $\approx$ (this agrees with the fact $H_{2,\Sigma}(D_n) = \{0\}$ in this case). In the last case, for every $v \in HS$, we determine a normal form v' with respect to $\approx$. We see that two different normal forms v_1' and v_2' have different invariants, $\varepsilon(v_1') \neq \varepsilon(v_2') \in G_\Sigma$. Finally we prove that $v_1 \approx v_2$ if and only if $\exists f \in Aut(D_n)$ such that $f(\Sigma) = \Sigma$ and $f(\varepsilon(v_1')) = \varepsilon(v_2')$. From this (i) follows. We refer to [CLP11] for a useful description of $Aut(D_n)$.

To prove (ii), we observe that for any $\mathcal{V} \in Im(\hat{\nu})$ there are at most two elements of $(G^{\cup})/_{Aut(G)}$ with image $\mathcal{V}$. This follows from the classification of the possible normal forms. The claim is now a consequence of the fact that any normal form is realized as Hurwitz generating system of some D_n-covering.

Case 1: $\Sigma = \emptyset$ (the étale case). In this case $H_{2,\Sigma}(D_n) = H_2(D_n, \mathbb{Z})$, so $v \in HS(D_n)$ implies $\varepsilon(v) \in H_2(D_n, \mathbb{Z})$. In the following, we identify $H_2(D_n, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, when n is even. Then we have:

Proposition 5.2. *Let $n, g' \in \mathbb{N}$ with $n \geq 3$, $g' > 0$. Then, for any $v \in HS(D_n; g')$, we have:*

(i) $v \approx (y, 1, x, 1, \ldots, 1)$, *if n is odd or if n is even and $\varepsilon(v) = 0$;*

(ii) $v \approx (y, x^{n/2}, x, 1, \ldots, 1)$, *if n is even and $\varepsilon(v) = 1$.*

Proof. Let

$$\overline{v} = v \,(\mathrm{mod}\ \mathbb{Z}/n\mathbb{Z}) \in (\mathbb{Z}/2\mathbb{Z})^{2g'} .$$

Notice that $\overline{v} \in HS(\mathbb{Z}/2\mathbb{Z}; g')$. Since the parameter space for étale $\mathbb{Z}/2\mathbb{Z}$-coverings of curves of a fixed genus is irreducible (see [Cat10], Thm. 2.4, or [Cor87], or [DM] Lemma 5.16), there exists $\varphi \in Map_{g'}$ such that $\varphi \cdot \overline{v} = (1, 0, \ldots, 0)$. Hence

$$\varphi \cdot v = (x^{\ell_1} y, x^{m_1}, \ldots, x^{\ell_{g'}}, x^{m_{g'}}) .$$

The condition $ev(\varphi \cdot v) = 1$ implies that $2m_1 = 0 \pmod n$. Hence $m_1 = 0$ or $m_1 = \frac{n}{2} \pmod n$.
In the first case, which is the only possible if n is odd,

$$\varphi \cdot v = (x^{\ell_1} y, 1, x^{\ell_2}, \ldots, x^{\ell_{g'}}, x^{m_{g'}}) .$$

Consider now $v' := (x^{\ell_2}, x^{m_2}, \ldots, x^{\ell_{g'}}, x^{m_{g'}})$. As $v' \in HS(\mathbb{Z}/n\mathbb{Z}; g'-1)$, from the irreducibility of the parameter space of étale $\mathbb{Z}/n\mathbb{Z}$-coverings of curves of a fixed genus we deduce that $\exists \varphi' \in Map_{g'-1}$ such that $\varphi' \cdot v' = (x^{\lambda}, 1, \ldots, 1)$, with $(\lambda, n) = 1$ ([Cat10], [DM]). Now, from Proposition 6.3 it follows that $\exists \psi \in Map_{g'}$ such that

$$\psi \cdot v = (x^{\ell_1} y, 1, x^{\lambda}, 1, \ldots, 1) .$$

We obtain the normal form (i) after operating with $Aut(D_n)$. The fact that $\varepsilon(v) = 0$ follows from a standard computation (cf. Corollary 4.2). If $m_1 = \frac{n}{2} \pmod n$, we have two subcases: $\langle x^{\ell_2}, x^{m_2}, \ldots, x^{\ell_{g'}}, x^{m_{g'}} \rangle = \mathbb{Z}/n\mathbb{Z}$ (which is the case when $n/2$ is even), or $\langle x^{\ell_2}, x^{m_2}, \ldots, x^{\ell_{g'}}, x^{m_{g'}} \rangle = \langle x^2 \rangle$. Proceeding as in the case $m_1 = 0$ we reach the normal form (ii) in the first subcase, otherwise we obtain $(x^{\ell_1} y, x^{\frac{n}{2}}, x^2, 1, \ldots, 1)$ in the latter subcase. In the latter case, consider the transformation

$$(a_1, b_1, a_2, b_2) \mapsto (a_2 a_1, b_1, b_1 a_2 b_1^{-1}, a_2 b_2 a_2 b_1^{-1}) , \tag{7}$$

which is realized by Map_2 as it preserves the relation $\prod_1^2 [\alpha_i, \beta_i] = 1$, then extend it to $\Pi_{g'}$ using Proposition 6.3 and apply the transformation so obtained to $(x^{\ell_1} y, x^{\frac{n}{2}}, x^2, 1, \ldots, 1)$. We obtain:

$$v \approx (x^{\ell_1 + 2} y, x^{\frac{n}{2}}, x^2, x^{4 + \frac{n}{2}}, 1, \ldots, 1) .$$

Since $\mathbb{Z}/n\mathbb{Z} = \langle x^{\frac{n}{2}}, x^2 \rangle$, there exists $\psi \in Map_{g'}$ such that $\psi \cdot v \approx (x^{\ell_1 + 2} y, x^{\frac{n}{2}}, x, 1, \ldots, 1)$, therefore we obtain the normal form (ii). In both of these subcases we have $\varepsilon(v) = 1$ (cf. Corollary 4.2). ☐

Case 2: $\Sigma \neq \emptyset$ and contains some reflection.

Let $v = (c_1, \ldots, c_d; a_1, b_1, \ldots, a_{g'}, b_{g'})$ be a Hurwitz generating system such that $\{c_1, \ldots, c_d\}$ contains some reflection, actually an even number because any product of commutators in D_n is a rotation. If n is odd, all the reflections belong to the same conjugacy class, while when $n = 2k$ they are divided into two classes. Denote by ν_y (resp. ν_{xy}) the number of c_i's in the class of y (resp. xy). As the pair (ν_y, ν_{xy}) is not $Aut(D_n)$-invariant, we define ν_1, ν_2 by the property that $\{\nu_1, \nu_2\} = \{\nu_y, \nu_{xy}\}$, $\nu_1 \leq \nu_2$ (in [CLP11] we used the notation h for ν_1, k for ν_2). Recall that, under the above hypotheses, $H_{2,\Sigma}(D_n) = \{0\}$ (Corollary 4.3). Indeed we prove that all the v's with fixed g', d, n, Σ and $\{\nu_1, \nu_2\}$ are equivalent each other.

Proposition 5.3. *Let $n, g', d \in \mathbb{N}$ with $n \geq 3$, $g', d > 0$. Then, for any $v \in HS(D_n; g', d)$ such that Σ_v contains some reflection, we have:*

(i) $v \approx (x^{\underline{r}}, x^{1-|\underline{r}|}y, xy, y, \ldots, y\,; x, 1, \ldots, 1, 1)$, *if n is odd;*
(ii) $v \approx (x^{\underline{r}}, \underbrace{x^{\varepsilon-|\underline{r}|}y, xy, \ldots, xy}_{\nu_2}, \underbrace{y, \ldots, y}_{\nu_1}; x, 1, \ldots, 1, 1)$, *if n is even.*

Where $\underline{r} = (r_1, \ldots, r_R)$, $0 < r_i \leq r_{i+1} \leq \frac{n}{2}$, $x^{\underline{r}} = (x^{r_1}, \ldots, x^{r_R})$, $|\underline{r}| = \sum r_i$ (mod n), $\{\nu_1, \nu_2\} = \{\nu_y, \nu_{xy}\}$, $\nu_1 \leq \nu_2$, $\varepsilon \in \{0, 1\}$, $\varepsilon + \nu_2 \equiv 1$ (mod 2).

The idea of the proof is the following. Using the action of the unpermuted mapping class group $Map^u(g', d+1)$ and the fact that at least one c_i is a reflection, we prove that $v \sim (\tilde{c}_1, \ldots, \tilde{c}_d; \tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_{g'}, \tilde{b}_{g'})$, with $\tilde{a}_i, \tilde{b}_i \in \mathbb{Z}/n\mathbb{Z}$, for any i. We collect in the Appendix the relevant facts that will be used about the action of $Map^u(g', d+1)$ on the fundamental group. Then, using results about étale $\mathbb{Z}/n\mathbb{Z}$-covers, we deduce that $v \approx v' := (c'_1, \ldots, c'_d; x, 1, \ldots, 1)$ (Lemma 5.4). At this point we can apply the main theorem of [CLP11] to deduce that, acting with the braid group, it is possible to transform v' to the corresponding normal form. However, we will see that using the entry x in v', the results in the Appendix and Lemma 2.1 of [CLP11], we can transform directly v' in one of the above forms without using the normal forms for the $g' = 0$ case.

Lemma 5.4. *Let v be as in Proposition 5.3. Then*

$$v \approx v' := (c'_1, \ldots, c'_d\,; x, 1, \ldots, 1, 1)\,.$$

Proof. Without loss of generality assume that c_d is a reflection (otherwise act with the braid group). Then, if a_1 is a reflection, by Proposition 6.2 (i), $\exists \varphi \in Map^u(g', d+1)$ such that

$$\varphi \cdot v = (c_1, \ldots, c_{d-1}, (c_d a_1 b_1 a_1^{-1}) c_d (c_d a_1 b_1 a_1^{-1})^{-1}; c_d a_1, b_1, \ldots, a_{g'}, b_{g'})\,.$$

While, if a_1 is a rotation and b_1 is a reflection, by Proposition 6.2 (ii) we have: $\exists \varphi \in Map^u(g', d+1)$ such that

$$\varphi \cdot v = (c_1, \ldots, (c_d [a_1, b_1] a_1^{-1}) c_d (c_d [a_1, b_1] a_1^{-1})^{-1}; a_1, (a_1^{-1} c_d a_1) b_1, \ldots, b_{g'}).$$

Notice that in both cases the d-th entry of $\varphi \cdot v$ is a reflection and that $c_d a_1, (a_1^{-1} c_d a_1) b_1 \in \mathbb{Z}/n\mathbb{Z}$. Proceeding in this way we get $\psi \in Map^u(g', d+1)$ such that $(\psi \cdot v)_i \in \mathbb{Z}/n\mathbb{Z}$, $i = d+1, \ldots, 2g'$.

Next, by the main theorem in [Cat10], we conclude that $\psi \cdot v \approx (\tilde{c}_1, \ldots, \tilde{c}_d; x^\alpha, 1, \ldots, 1)$. We can further assume $(\alpha, n) = 1$. Otherwise, since $D_n = \langle x^\alpha, \tilde{c}_1, \ldots, \tilde{c}_d \rangle$, there exists $x^\beta \in \langle \tilde{c}_1, \ldots, \tilde{c}_d \rangle$ such that $\mathbb{Z}/n\mathbb{Z} = \langle x^{\alpha+\beta} \rangle$. Using Proposition 6.2 (i) and the braid group, we can multiply x^α by any element of $\langle \tilde{c}_1, \ldots, \tilde{c}_d \rangle$. The claim now follows by applying $Aut(D_n)$. □

We now complete the proof of Proposition 5.3. Let v' be as in Lemma 5.4 and let $2N$ be the number of reflections in $\{c'_1, \ldots, c'_d\}$. Applying Lemma 2.1 of [CLP11] we have

$$v' \approx (x^{\underline{r}}, x^\beta y, x^\alpha y, x^{j_{N-1}} y, x^{j_{N-1}} y, \ldots, x^{j_1} y, x^{j_1} y\, ;\, x, 1, \ldots, 1)\,, \tag{8}$$

where $\underline{r} = (r_1, \ldots, r_R)$, $0 < r_i \leq r_{i+1} \leq \frac{n}{2}$, $x^{\underline{r}} = (x^{r_1}, \ldots, x^{r_R})$.

If $N = 1$ the result is clear. Otherwise we conjugate by x simultaneously the entries of each pair $(x^{j_k} y, x^{j_k} y)$ in (8) without changing the other components, hence we obtain:

(9)

$$v' \sim (x^{\underline{r}}, x^\beta y, x^\alpha y, x^{j_{N-1}+2\ell_{N-1}} y, x^{j_{N-1}+2\ell_{N-1}} y, \ldots, x^{j_1+2\ell_1} y, x^{j_1+2\ell_1} y\, ;\, x, \ldots, 1)$$

for any $\ell_1, \ldots, \ell_{N-1} \in \mathbb{Z}$.

The equivalence (9) can be proven as follows. We have:

$$\begin{aligned} v' &\sim (x^{\underline{r}}, x^\beta y, \ldots, x^{j_1} y, (x^{j_1} y x^{-1}) x^{j_1} y (x^{j_1} y x^{-1})^{-1}\, ;\, x, x^{-1} x^{j_1} y x, 1, \ldots, 1) \\ &\sim (x^{\underline{r}}, x^\beta y, \ldots, (x^{-1}) x^{j_1} y (x), x^{j_1} y\, ;\, x, x^{-1} x^{j_1} y x, 1, \ldots, 1) \\ &\sim (x^{\underline{r}}, x^\beta y, \ldots, (x^{-1}) x^{j_1} y (x), (x^{-1}) x^{j_1} y (x)\, ;\, x, 1, \ldots, 1)\,, \end{aligned}$$

where the first and the third equivalences are given by ξ-twists as in Proposition 6.2 (ii), while the second is a braid twist between the last two components. Iterating these steps we can conjugate by any power of x the entries of $(x^{j_1} y, x^{j_1} y)$ simultaneously. By Lemma 2.3 in [CLP11] we can move $(x^{j_k} y, x^{j_k} y)$ to the right and then conjugate its entries by any power of x as before. This proves (9).

If n is odd, choose ℓ_i in (9) such that $j_i + 2\ell_i = \alpha - 1 \pmod{n}$, then apply the automorphism $x^{\alpha-1} y \mapsto y, x \mapsto x$ to obtain (i).

Assume now that n is even. Without loss of generality we have that $\nu_1 = \nu_y \leq \nu_{xy} = \nu_2$ (otherwise apply $Aut(D_n)$). Assume further that $x^\beta y$ in (8) is conjugate to xy.

If $x^\alpha y$ is conjugate to xy, choose ℓ_i such that $j_i + 2\ell_i = \alpha$ or $j_i + 2\ell_i = \alpha - 1 \pmod{n}$, so (9) becomes:

$$v' \sim (x^{\underline{r}}, x^\beta y, x^\alpha y, x^\alpha y, \ldots, x^\alpha y, x^{\alpha-1} y, \ldots, x^{\alpha-1} y\, ;\, x, 1, \ldots, 1)\,.$$

We obtain the normal form (ii) after applying the automorphism $x^\alpha y \mapsto xy$, $x \mapsto x$.
The remaining case, where $x^\alpha y$ is conjugate to y, is similar. □

Notice that (9) follows also from Lemma 2.1 in [Kanev05] (see also [GHS02]), which applies to a more general situation. Since we don't need the whole strength of that result, we preferred to give a complete proof in our case.

Case 3: $\Sigma \neq \emptyset$ and does not contain reflections.

We prove the following

Proposition 5.5. *Let $v, v' \in HS(D_n; g', d)$ with $\Sigma_v, \Sigma_{v'} \subset \mathbb{Z}/n\mathbb{Z}$. Then $v \approx v'$ if and only if there exists $f \in Aut(D_n)$ such that $f(\Sigma_v) = \Sigma_{v'}$ and $f(\varepsilon(v)) = \varepsilon(v')$.*

The "only if" part is clear. So assume $\Sigma_v = \Sigma_{v'} =: \Sigma$ and the existence of f as in the statement. We prove that $v \approx v'$. This is achieved after considering three cases: n is odd; $n = 2k$ and $x^k \in \Sigma$, $n = 2k$ and $x^k \notin \Sigma$. In the first two cases we determine a normal form, with respect to $\approx$, for each such element of $HS(D_n; g', d)$, and then we show that two such elements are equivalent if and only if they have the same normal form. Notice that in both cases $H_{2,\Sigma}(D_n) = \{0\}$ (Corollary 4.3). In the last case, for any such $v \in HS(D_n; g', d)$, we determine a normal form v', with respect to the action of $Map(g', d)$ and then we show that $v_1 \approx v_2$ if and only if $\exists f \in Aut(D_n)$ such that $f(\Sigma) = \Sigma$ and $f(\varepsilon(v_1')) = \varepsilon(v_2')$. Notice that, in this case $H_{2,\Sigma}(D_n) \cong \mathbb{Z}/2\mathbb{Z}$ (Corollary 4.3).

Notation. Let $v = (c_1, \ldots, c_d\,; a_1, b_1, \ldots, a_{g'}, b_{g'}) \in D_n^{d+2g'}$ be a Hurwitz generating system such that $\Sigma_v \subset \mathbb{Z}/n\mathbb{Z}$, i.e. $c_i \in \mathbb{Z}/n\mathbb{Z}$, $\forall i$. We denote by $H = \langle c_1, \ldots, c_d \rangle \subset D_n$ the subgroup generated by the c_i's. Note that, under the above hypotheses, H is normal and it is contained in $\mathbb{Z}/n\mathbb{Z}$. Set $G' := D_n/H$. Then G' is a dihedral group D_m, $m \geq 3$, or is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or to $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.6. *Let $n \in \mathbb{N}$, $n \geq 3$ odd. Let $v \in HS(D_n; g', d)$ with $\Sigma_v \subset \mathbb{Z}/n\mathbb{Z}$. Then*

$$v \approx (x^{\underline{r}}; y, x^h, x, 1, \ldots, 1)\,,$$

where $\underline{r} = (r_1, \ldots, r_d)$, $x^{\underline{r}} = (x^{r_1}, \ldots, x^{r_d})$, $r_1 \leq \ldots \leq r_d < \frac{n}{2}$ and $2h = \sum_1^d r_i$, (mod n).

Proof. Let us consider

$$\overline{v} := (\overline{a_1}, \overline{b_1}, \ldots, \overline{a_{g'}}, \overline{b_{g'}}) \in HS(G'; g', 0)\,,$$

where $\overline{a_i} = a_i$ (mod H), $\overline{b_i} = b_i$ (mod H). By Proposition 5.2 and by the analogous results for cyclic and $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-covers we have:

$$\exists \varphi \in Map_{g'} \quad \text{such that} \quad \varphi \cdot \overline{v} = (\overline{y}, 1, \overline{x}, 1, \ldots, 1)\,.$$

By Proposition 6.3, $\exists\tilde\varphi \in Map(g', d)$ with

$$\tilde\varphi \cdot v = (c_1, \dots, c_d\,;\, x^{\ell_1} y, x^{m_1}, x^{\ell_2}, \dots, x^{m_{g'}})\,,$$

where $x^{m_i} \in H$, $\forall i$, $x^{\ell_2} = x \pmod H$ and $x^{\ell_i} \in H$, $\forall i > 2$.

We now apply the ξ-twists as in Proposition 6.2 (i) with $\ell = 2, 3, \dots, g'$ and we deduce that we can multiply all the x^{ℓ_i}, $i > 1$, by any element of H. Hence $\exists\psi \in Map^u(g', d+1)$ such that

$$\psi \cdot \tilde\varphi \cdot v = (c_1, \dots, c_d\,;\, x^{\ell_1} y, x^{m_1}, x, x^{m_2}, 1, x^{m_3}, \dots, 1, x^{m_{g'}})\,.$$

Similarly, using Proposition 6.2 (ii), we get:

$$v \approx (c_1, \dots, c_d\,;\, x^{\ell_1} y, x^{m_1}, x, 1, \dots, 1, 1)\,.$$

Now, for any $i = 1, \dots, d$, consider $c_i = x^{s_i}$. If $s_i < \frac{n}{2}$, set $r_i = s_i$, otherwise use the braid group to move c_i to the d-th position and then apply Proposition 6.2 (ii) with $\ell = 1$. After this, c_i becomes $c_i^{-1} = x^{n-s_i}$, then set $r_i = n - s_i$. Finally, using the braid group, we can order the c_i's such that $r_i \le r_{i+1}$.

So, we have proved that

$$v \sim (x^{\underline{r}}; x^{\lambda_1} y, x^{\mu_1}, x, 1, \dots, 1)\,,$$

with $r_1 \le \dots \le r_d < \frac{n}{2}$. Now the condition $ev(v) = 1$ implies that $2\mu_1 = \sum_1^d r_i \pmod n$, therefore set $h := \mu_1 \pmod n$.

We reach the normal form after applying the automorphism $x^{\lambda_1} y \mapsto y$, $x \mapsto x$. $\square$

Lemma 5.7. *Let $n = 2k \in \mathbb{N}$ and let $v \in HS(D_n; g', d)$ with $x^k \in \Sigma_v \subset \mathbb{Z}/n\mathbb{Z}$. Then we have:*

$$v \approx (x^{\underline{r}}; y, x^h, x, 1, \dots, 1)\,,$$

where $\underline{r} = (r_1, \dots, r_d)$, $x^{\underline{r}} = (x^{r_1}, \dots, x^{r_d})$, $r_1 \le \dots \le r_d = k$, $2h = \sum_1^d r_i \pmod n$ and $h < k$.

Proof. Proceeding as in the proof of the previous lemma, we have:

$$\exists\varphi \in Map(g', d) \quad \text{such that } \varphi\cdot v = (c_1, \dots, c_d\,;\, x^{\ell_1} y, x^{m_1}, x^{\ell_2}, \dots, x^{m_{g'}})\,,$$

where $x^{m_i} \in H$, $\forall i > 1$, $x^{\ell_2} = x \pmod H$ and $x^{\ell_i} \in H$, $\forall i > 2$.
Since we can multiply all the x^{ℓ_i}, $i > 1$, by any element of H (apply Proposition 6.2 (i) with $\ell = 2, 3, \dots, g'$), we have: $\exists\psi \in Map^u(g', d+1)$ such that

$$\psi \cdot \varphi \cdot v = (c_1, \dots, c_d\,;\, x^{\ell_1} y, x^{m_1}, x, x^{m_2}, 1, x^{m_3}, \dots, 1, x^{m_{g'}})\,.$$

Similarly, using Proposition 6.2 (ii), we get:

$$v \approx (c_1, \dots, c_d\,;\, x^{\ell_1} y, x^{m_1}, x, 1, \dots, 1, 1)\,.$$

Now, for any $i = 1, \dots, d$, consider $c_i = x^{s_i}$. If $s_i \le k$, set $r_i = s_i$, otherwise use the braid group to move c_i to the d-th position and then apply Proposition 6.2 (ii) with $\ell = 1$. In this way c_i becomes c_i^{-1} and so set $r_i = 2k - s_i$.

So, we have proved that

$$v \approx (x^{\underline{r}}; x^{\lambda_1}y, x^{\mu_1}, x, 1, \ldots, 1)\,,$$

with $r_i \leq k$, $\forall i$. Now the condition $ev(v) = 1$ implies that $2\mu_1 = \sum_1^d r_i \pmod n$. If $\mu_1 < k$, set $h = \mu_1$. Otherwise, apply braid group transformations to achieve the ordering $r_i \leq r_{i+1}$, $\forall i \leq d-1$. By hypotheses $r_d = k$ and we apply Proposition 6.2 with $\ell = 1$. Since x^k is central, this operation does not change r_d, while x^{μ_1} becomes x^{μ_1+k}. Set $h = \mu_1 + k \pmod n$.

Finally apply the appropriate element of $Aut(D_n)$ to reach the normal form. □

We now consider the last case.

Lemma 5.8. *Let $n = 2k \in \mathbb{N}$ and let $v \in HS(D_n; g', d)$ with $\Sigma \subset \mathbb{Z}/n\mathbb{Z} \setminus \{x^k\}$. Then we have:*

(i) *$v \approx v' := (x^{\underline{r}}; y, x^h, x, 1, \ldots, 1)$, where $\underline{r} = (r_1, \ldots, r_d)$, $x^{\underline{r}} = (x^{r_1}, \ldots, x^{r_d})$, $r_1 \leq \ldots \leq r_d < k$, $2h = \sum_1^d r_i \pmod n$;*
(ii) *let $v_1' = (x^{\underline{r}}; y, x^h, x, 1, \ldots, 1)$ and $v_2' = (x^{\underline{r}}; y, x^{h+k}, x, 1, \ldots, 1)$, then $\varepsilon(v_1') \neq \varepsilon(v_2') \in (D_n)_\Sigma$;*
(iii) *$v_1' \approx v_2'$ if and only if $\exists f \in Aut(D_n)$ such that $f(\Sigma) = \Sigma$ and $f(\varepsilon(v_1')) = \varepsilon(v_2')$.*

Proof. The proof of (i) is the same as that of the previous lemma. Since in this case $x^k \notin \Sigma$, we can not achieve $h \leq k$.

To prove (ii) recall that $\varepsilon(v) := ev(\hat{v}) \in (D_n)_\Sigma$. So, if $ev(\widehat{v_1'}) = ev(\widehat{v_2'})$, then $ev(\widehat{v_2'})^{-1} \cdot ev(\widehat{v_1'}) = 0 \in H_{2,\Sigma}(D_n)$. But now a direct computation shows that $ev(\widehat{v_2'})^{-1} \cdot ev(\widehat{v_1'}) \neq 0$ (Corollary 4.2), a contradiction.

(iii) The "only if" part is clear. So, assume that $\exists f \in Aut(D_n)$ such that $f(\Sigma) = \Sigma$ and $f(\varepsilon(v_1')) = \varepsilon(v_2')$. Since $f(\varepsilon(v_1')) = \varepsilon(f(v_1'))$, we have $\varepsilon(f(v_1')) = \varepsilon(v_2')$ and so v_2' and $f(v_1')$ have the same ν-type (Remark 3.10). From (i) and (ii) we deduce that

$$f(v_1') \approx (x^{\underline{r}}; x^{\lambda_1}y, x^{h+k}, x, 1, \ldots, 1)\,.$$

Hence, using the automorphism $x^{\lambda_1}y \mapsto y$, $x \mapsto x$, we have that $f(v_1') \approx v_2'$ and so the claim follows. □

6. Appendix. Automorphisms of surface-groups

We collect in this Appendix some facts about mapping class groups and their action on fundamental groups. They should be well known to experts, we include them here for completeness.

Let Y be a compact Riemann surface of genus g' and let $\mathcal{B} = \{y_1, \ldots, y_d\} \subset Y$ be a finite subset of cardinality d. After the choice of a geometric basis of $Y \setminus \mathcal{B}$, we have the following presentation of the fundamental group:

$$\pi_1(Y \setminus \mathcal{B}, y_0) =$$
$$\langle \gamma_1, \ldots, \gamma_d, \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'} \mid \gamma_1 \cdot \ldots \cdot \gamma_d \cdot \Pi_{i=1}^{g'}[\alpha_i, \beta_i] = 1 \rangle .$$

Following [Bir69], there is a short exact sequence

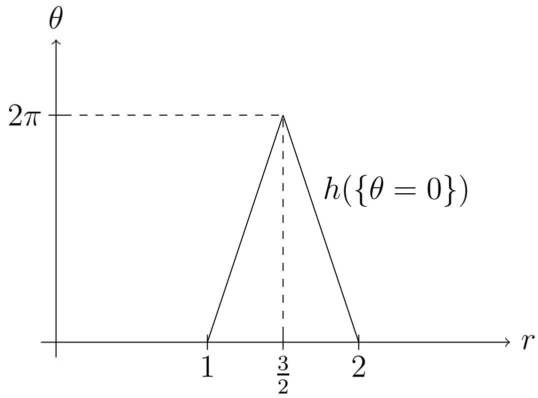
(10)
$$1 \to \pi_1(Y \setminus \mathcal{B}, y_0) \xrightarrow{\Xi} Map^u(Y, \{y_0, y_1, \ldots, y_d\}) \to Map^u(Y, \mathcal{B}) \to 1$$

which induces an injective group homomorphism

$$Map^u(Y, \mathcal{B}) \to Out\,(\pi_1(Y \setminus \mathcal{B}, y_0))$$

([Bir69], Thm. 4). The map Ξ is defined as follows. Let $[c] \in \pi_1(Y \setminus \mathcal{B}, y_0)$ be an element of the geometric basis and let $c\colon [0, 2\pi] \to Y \setminus \mathcal{B}$ be a simple, smooth loop based at y_0, representing $[c]$. Set $\Xi([c])$ be the isotopy class of the ξ-twist, ξ_c. Then extend Ξ to the whole group as an homomorphism. Recall that the ξ-twist, ξ_c, can be defined as follows. Let $N \subset Y \setminus \mathcal{B}$ be a tubular neighborhood of c and let $e\colon A \to N$ be a diffeomorphism between the annulus $A = \{z = re^{i\theta} \in \mathbb{C} \mid 1 \le r \le 2\}$ and N such that $e(\frac{3}{2}, \theta) = c(\theta)$. Define $h\colon A \to A$ as follows

$$h(r, \theta) = \begin{cases} (r, \theta + 4\pi(r-1))\,, & 1 \le r \le \frac{3}{2}; \\ (r, \theta + 4\pi(2-r))\,, & \frac{3}{2} \le r \le 2\,. \end{cases}$$



Then h is a diffeomorphism which is the identity when $r = 1, \frac{3}{2}, 2$. Finally, define $\xi_c\colon Y \to Y$ as the identity on $Y \setminus N$ and as $e \circ h \circ e^{-1}$ on N.

From the sequence (10), it follows that $\pi_1(Y \setminus \mathcal{B}, y_0)$ is isomorphic through Ξ to a normal subgroup of $Map^u(Y, \{y_0, y_1, \ldots, y_d\})$, hence we get an action by conjugation of $Map^u(Y, \{y_0, y_1, \ldots, y_d\})$ to $\pi_1(Y \setminus \mathcal{B}, y_0)$:

$$[f] \cdot [\xi_c] = [f \circ \xi_c \circ f^{-1}]\,.$$

We have:

Lemma 6.1. *For any* $[f] \in Map^u(Y, \{y_0, y_1, \dots, y_d\})$ *and* $[c] \in \pi_1(Y \setminus \mathcal{B}, y_0)$*, we have:*

$$[f] \cdot [\xi_c] = [\xi_{f_\#(c)}]\,,$$

where $f_\#(c)(\theta) = (f \circ c)(\theta)$.

Proof. $f \circ \xi_c \circ f^{-1} = (f \circ e) \circ h \circ (f \circ e)^{-1}$ on N, and coincides with the identity on $Y \setminus N$. The result then follows because $f \circ e\colon A \to Y$ is a tubular neighborhood of $f_\#(c)$. □

One can define, in the same way, ξ-twists with respect to loops that are not based at y_0 and Lemma 6.1 is still valid. In the following result we give the action of ξ-twists around certain loops in terms of a given geometric basis of $\pi_1(Y \setminus \mathcal{B}, y_0)$.

Proposition 6.2. *Let* $\gamma_1, \dots, \gamma_d, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}$ *be a fixed geometric basis of* $\pi_1(Y \setminus \mathcal{B}, y_0)$.

(i) *Let* $c \subset Y \setminus \mathcal{B}$ *be the loop in Figure 1, image of the two sides of the angle inside the polygon with vertex* y_d*. Set* $u = \prod_{k=1}^{\ell-1}[\alpha_k, \beta_k]$*. Then we have:*

$$\begin{aligned}(\xi_c)_*(\alpha_\ell) &= u^{-1}\gamma_d u \alpha_\ell\,;\\ (\xi_c)_*(\gamma_d) &= \left(\gamma_d u \alpha_\ell \beta_\ell \alpha_\ell^{-1} u^{-1}\right)\gamma_d \left(\gamma_d u \alpha_\ell \beta_\ell \alpha_\ell^{-1} u^{-1}\right)^{-1}\,;\end{aligned}$$

$(\xi_c)_*(\alpha_i) = \alpha_i$ $(i \neq \ell)$, $(\xi_c)_*(\beta_i) = \beta_i$ $(\forall\, i)$, $(\xi_c)_*(\gamma_j) = \gamma_j$ $(j \neq d)$.

(ii) *Let* $c \subset Y \setminus \mathcal{B}$ *be the loop in Figure 2, image of the two sides of the angle inside the polygon with vertex* y_d*. Set* $u = \prod_{k=1}^{\ell-1}[\alpha_k, \beta_k]$*. Then we have:*

$$\begin{aligned}(\xi_c)_*(\beta_\ell) &= \alpha_\ell^{-1} u^{-1}\gamma_d u \alpha_\ell \beta_\ell\,;\\ (\xi_c)_*(\gamma_d) &= \left(\gamma_d u [\alpha_\ell, \beta_\ell] \alpha_\ell^{-1} u^{-1}\right)\gamma_d \left(\gamma_d u [\alpha_\ell, \beta_\ell] \alpha_\ell^{-1} u^{-1}\right)^{-1}\,;\end{aligned}$$

$(\xi_c)_*(\beta_i) = \beta_i$ $(i \neq \ell)$, $(\xi_c)_*(\alpha_i) = \alpha_i$ $(\forall\, i)$, $(\xi_c)_*(\gamma_j) = \gamma_j$ $(j \neq d)$.

Proof. (i) The image of α_ℓ under ξ_c is drawn in Figure 3. From this it follows the formula for $(\xi_c)_*(\alpha_\ell)$. Since ξ_c is the identity outside a small tubular neighborhood of c, we have that $(\xi_c)_*(\alpha_i) = \alpha_i$ $(i \neq \ell)$, $(\xi_c)_*(\beta_i) = \beta_i$ $(\forall\, i)$ and $(\xi_c)_*(\gamma_j) = \gamma_j$ $(j \neq d)$. The formula for $(\xi_c)_*(\gamma_d)$ is now a consequence of $\gamma_1 \cdot \ldots \cdot \gamma_d \prod_{i=1}^{g'}[\alpha_i, \beta_i] = 1$, since the product $\gamma_1 \cdot \ldots \cdot \gamma_d \prod_{i=1}^{g'}[\alpha_i, \beta_i]$ must be left fixed.

The proof of (ii) is similar. □

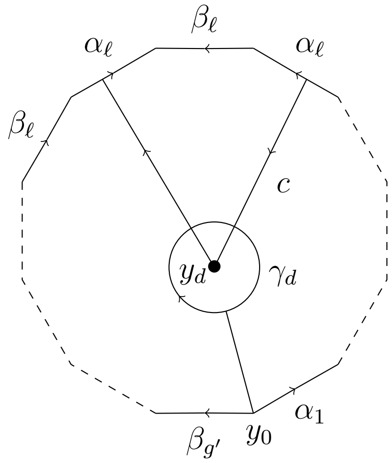


Figure 1.

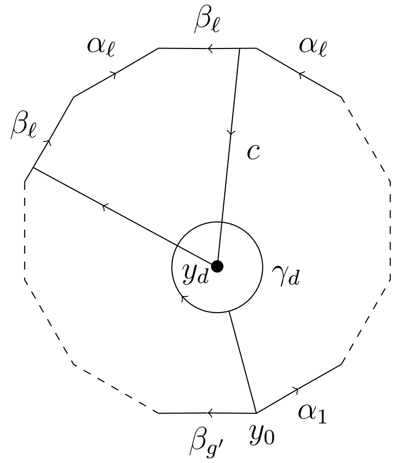


Figure 2.

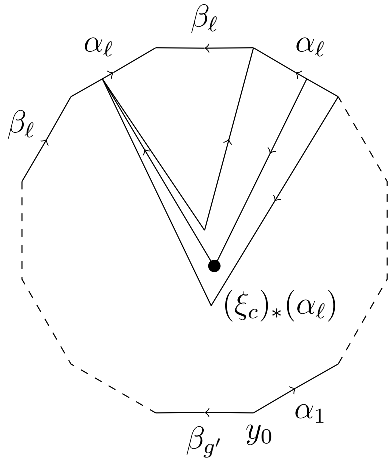


Figure 3.

Proposition 6.3. *Let* $\Pi_{g'} = \langle \alpha_1, \dots, \beta_{g'} \mid \prod_1^{g'} [\alpha_i, \beta_i] \rangle$ *and* $\Pi_{g'-1} = \langle \alpha_2, \dots, \beta_{g'} \mid \prod_2^{g'} [\alpha_i, \beta_i] \rangle$*. Then, for any* $\varphi \in Aut^0(\Pi_{g'-1})$*, there exists* $\psi \in Aut^0(\Pi_{g'})$ *and* $\delta \in \Pi_{g'}$ *such that* $\psi(\alpha_1) = \alpha_1$*,* $\psi(\beta_1) = \beta_1$*,* $\psi(\alpha_i) = \delta\varphi(\alpha_i)\delta^{-1}$*,* $\psi(\beta_i) = \delta\varphi(\beta_i)\delta^{-1}$*,* $i > 1$*.*

Proof. We first extend φ to an automorphism

$$\tilde{\varphi} \in Aut\left(\langle \alpha_2, \dots, \beta_{g'}, \gamma \mid \gamma \cdot \prod_2^{g'} [\alpha_i, \beta_i] \rangle \right)$$

such that $\tilde{\varphi}(\alpha_i) = \varphi(\alpha_i)$, $\tilde{\varphi}(\beta_i) = \varphi(\beta_i)$ and $\tilde{\varphi}(\gamma) = \delta^{-1}\gamma\delta$, $i > 1$. Geometrically this corresponds to representing φ as composition of Dehn twists along curves contained in the complement $Y_{g'-1} \setminus D$ of a closed disk D in a Riemann surface $Y_{g'-1}$ of genus $g' - 1$, where D does not intersect α_i and β_i.

Now simply define $\psi(\alpha_1) = \alpha_1$, $\psi(\beta_1) = \beta_1$, $\psi(\alpha_i) = \delta\tilde{\varphi}(\alpha_i)\delta^{-1}$ and $\psi(\beta_i) = \delta\tilde{\varphi}(\beta_i)\delta^{-1}$, $i > 1$. □

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Authors' Address:

Fabrizio Catanese, Michael Lönne, Fabio Perroni
Lehrstuhl Mathematik VIII,
Mathematisches Institut der Universität Bayreuth
NW II, Universitätsstr. 30
95447 Bayreuth
email: fabrizio.catanese@uni-bayreuth.de, michael.loenne@uni-bayreuth.de, loenne@math.uni-hannover.de, fabio.perroni@uni-bayreuth.de.