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Burniat-type surfaces and a new family of surfaces with $p_g = 0$, $K^2 = 3$

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Abstract The paper is one of a series devoted to the classification, the moduli spaces and the classification of surfaces of general type with $p_g = 0$. Here we generalize a classical construction due to P. Burniat (revised by M. Inoue). Among other results we construct a family of surfaces of general type with $K_S^2 = 3$, $p_g(S) = 0$ realizing a new fundamental group of order 16.

Keywords Surfaces of general type with geometric genus $p_g = 0$ · Action of finite groups · Computer aided constructions

Mathematics Subject Classification 14J29 · 14J25 · 14J10 · 14H30 · 20F05

1 Introduction

The present paper continues a research developed in a series of articles ([3–9, 11]) dedicated to the classification, the moduli spaces and the discovery of new surfaces of general type with geometric genus $p_g = 0$ (the first such having been constructed in [13] and [14]), with particular emphasis on the problem of classifying the possible fundamental groups occurring according to the respective values of $K^2_{S_{min}}$ (see [10, 11, 20, 17–19] for related conjectures and results).

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The construction methods we have been using vary considerably, and in this paper we consider the method originally due to Burniat (Abelian coverings) in the reformulation done by Inoue (quotients by Abelian groups), presenting it in a rather general fashion which seems worthwhile a deeper investigation.

Our general approach consists in considering quotients (cf. [8] for the case of a free action, treated there in greater generality), by some group G of the form $(\mathbb{Z}/m)^r$, of varieties $\hat{X}$ contained in a product of curves $\Pi_i C_i$, where each C_i is a maximal Abelian cover of the projective line with Galois group of the form $(\mathbb{Z}/m)^{n_i}$. Let us explain now the connection with Burniat surfaces.

Burniat surfaces are surfaces of general type with invariants $p_g = 0$ and $K^2 = 6, 5, 4, 3, 2$, whose birational models were constructed by Pol Burniat (cf. [12]) in 1966 as singular bidouble covers of the projective plane. Later these surfaces were reconstructed by Inoue (cf. [16]) as $G := (\mathbb{Z}/2\mathbb{Z})^3$-quotients of a ($G$-invariant) hypersurface $\hat{X}$ of multi degree $(2, 2, 2)$ in a product of three elliptic curves. In the case where G acts freely, this construction and its topological characterization has been largely generalized by the authors in the already cited paper [8].

While Inoue writes the (affine) equation of $\hat{X}$ in terms of the uniformizing parameters of the respective elliptic curves using a variant of the Weierstrass' functions (the Legendre functions), we found it much more useful, especially for a systematic approach to finding all possible such constructions, to write the elliptic curves as the complete intersection of two diagonal quadrics in three space. In fact, we consider first the following diagram of quotient morphisms:

$$\begin{array}{c} E_1 \times E_2 \times E_3 \\ \Big\downarrow {\scriptstyle \mathcal{H}' := (\mathbb{Z}/2)^3}\ \pi' \\ P_1 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \Big\downarrow {\scriptstyle \mathcal{H} := ((\mathbb{Z}/2)^2)^3}\ \pi \\ P_2 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \qquad \begin{array}{l} E_1 := \{x_1^2 + x_2^2 + x_3^2 = 0,\ x_0^2 = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2\} \quad (1) \\ E_2 := \{u_1^2 + u_2^2 + u_3^2 = 0,\ u_0^2 = b_1 u_1^2 + b_2 u_2^2 + b_3 u_3^2\} \\ E_3 := \{z_1^2 + z_2^2 + z_3^2 = 0,\ z_0^2 = c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2\} \end{array}$$

We consider then P_1 with homogeneous coordinates $((s_1 : t_1), (s_2 : t_2), (s_3 : t_3))$ and the pencil of Del Pezzo surfaces of degree 6

$$Y_\lambda := \{s_1 s_2 s_3 = \lambda t_1 t_2 t_3\} \subset P_1.$$

Y_λ is invariant under a subgroup $H_0 \cong (\mathbb{Z}/2\mathbb{Z})^2$ of $\mathcal{H}$ generated by the transformations:

$$t_i \mapsto \epsilon_i t_i, \quad , \epsilon \in \{\pm 1\}, \quad \epsilon_1 \epsilon_2 \epsilon_3 = 1.$$

Therefore $\hat{X}_\lambda := (\pi')^{-1}(Y_\lambda)$ is invariant under a subgroup $G_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \subset (\mathbb{Z}/2\mathbb{Z})^9$. It is now our aim to find all the subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \subset G_1$, having the property that G acts freely on $\hat{X}$.

We give the following

Definition 1.1 Let $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \leq G_1$ be such that G acts freely on $\hat{X}_\lambda$. Then $S_\lambda := \hat{X}_\lambda / G$ is called a *primary Burniat type surface*.

Obviously, primary Burniat surfaces (i.e., Burniat surfaces with $K^2 = 6$) are primary Burniat type surfaces. With the help of the computer algebra system MAGMA we can classify all primary Burniat type surfaces and can prove the following

Theorem 1.2 *Primary Burniat type surfaces are exactly the primary Burniat surfaces.*

We then consider $\hat{X} := (\pi')^{-1}(Y_1)$. Since Y_1 is invariant under a bigger subgroup of $\mathcal{H}$ it turns out that $\hat{X}$ is invariant under $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^6$.

In the second part of the paper we find all the subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^4 \leq G_0$ with the property that there is exactly one element $g_0 \in G$ such that g_0 has isolated fixed points on $\hat{X}$ which are also isolated fixed points on $E_1 \times E_2 \times E_3$, while all other non trivial elements of G act freely on $\hat{X}$. The quotient of $\hat{X}$ by the action of G is then a surface having exactly four ordinary nodes and we give the following

Definition 1.3 Let $G \cong (\mathbb{Z}/2\mathbb{Z})^4 \leq G_0$ be such that $\hat{X}_1 = \hat{X}$ is G-invariant. We shall say that G acts *1-almost freely* on $\hat{X}$, if there is exactly one element $g_0 \in G$ having isolated fixed points on $\hat{X}$ which are also isolated fixed points on $E_1 \times E_2 \times E_3$, while all the other non trivial elements of G act freely.

Then the minimal resolution S of the nodal surface $X := \hat{X}/G$ is called a *4-nodal Burniat type surface*.

Remark 1.4 One can consider, more generally, subgroups

$$G \cong (\mathbb{Z}/2\mathbb{Z})^4 \leq G_0$$

with the property that there is exactly one element $g_0 \in G$ such that g_0 has isolated fixed points on $\hat{X}$, while all other non trivial elements of G act freely on $\hat{X}$. As a matter of fact, if we admit g_0 to have a one-dimensional fixed locus on $E_1 \times E_2 \times E_3$ (which is not contained in $\hat{X}$) we get more examples of surfaces S of general type with $K_S^2 = 3,\ \chi(S) = 1$. It turns out however that these surfaces have $q(S) = 1$. We postpone therefore the study of these surfaces to a further article.

We give here a complete classification of 4-nodal Burniat type surfaces, which turn out to be minimal surfaces of general type with $p_g = 0$ and $K^2 = 3$. This gives us a list of seven subgroups G yielding three (3-dimensional) families of such surfaces. Since these families are nowhere dense in the moduli space, and also in order to determine whether the present construction yields hitherto unknown surfaces, we use the classical result of Armstrong to calculate the fundamental groups of these surfaces.

Hence we see that these families yield three different topological types: one family yields the same fundamental group as the family of Keum-Naie surfaces with $K^2 = 3$ (this is case (i)), one family yields (case ii)) tertiary Burniat surfaces with $K^2 = 3$, and the third family realizes a new fundamental group $P := SmallGroup(16, 13)$ (case iii)). Observe that P is the central product of the dihedral group of order 8 with the cyclic group of order 4.

We summarize our result as follows:

Theorem 1.5 *Let S be a 4-nodal Burniat type surface. Then S is a minimal surface of general type with* $K_S^2 = 3,\ p_g(S) = 0$ *and the fundamental group* [1] *of S is one of the following groups of order* 16*:*

[1] Unlike other authors, when we write 'fundamental group', we mean the topological fundamental group, and not its profinite completion, the algebraic fundamental group.

(i) $\pi_1(S) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$, *or*
(ii) $\pi_1(S) \cong \mathbb{H}_8 \times \mathbb{Z}/2\mathbb{Z}$, *or*
(iii) $\pi_1(S) \cong SmallGroup(16, 13)$.

In a sequel to this paper we shall give other applications of the method considered here, constructing new surfaces as quotients of subvarieties of products of maximal abelian coverings of $\mathbb{P}^1$ having Galois group of the form $(\mathbb{Z}/d)^{m_i}$.

2 Burniat surfaces as reconstructed by Inoue

We briefly recall the construction of the *Burniat surfaces* (cf. [12]) as given by Inoue ([16]). The description given by Inoue is very appropriate in order to calculate the fundamental group.

For $i \in \{1, 2, 3\}$, let $E_i := \mathbb{C}/\langle 1, \tau_i \rangle$ be a complex elliptic curve. Denoting by z_i a uniformizing parameter on E_i, we consider the following three involutions on $T := E_1 \times E_2 \times E_3$:

$$g_1(z_1, z_2, z_3) = \left(-z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3\right),$$
$$g_2(z_1, z_2, z_3) = \left(z_1, -z_2 + \frac{1}{2}, z_3 + \frac{1}{2}\right),$$
$$g_3(z_1, z_2, z_3) = \left(z_1 + \frac{1}{2}, z_2, -z_3 + \frac{1}{2}\right).$$

Then $G := \langle g_1, g_2, g_3 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$.

We consider the Legendre $\mathcal{L}$-function for E_i and denote it by $\mathcal{L}_i$, for $i = 1, 2, 3$: $\mathcal{L}_i$ is a meromorphic function on E_i and $\mathcal{L}_i \colon E_i \to \mathbb{P}^1$ is a double cover ramified in $\pm 1, \pm a_i \in \mathbb{P}^1 \setminus \{0, \infty\}$. It is well known that we have (cf. [16], lemma 3-2, and also cf. [5], pages 52–54, Sect. 2 for an algebraic treatment):

- $\mathcal{L}_i(\frac{1}{2}) = -1$, $\mathcal{L}_i(0) = 1$, $\mathcal{L}_i(\frac{\tau_i}{2}) = a_i$, $\mathcal{L}_i(\frac{1+\tau_i}{2}) = -a_i$;
- let $b_i := \mathcal{L}_i(\frac{\tau_i}{4})$: then $b_i^2 = a_i$;
- $\frac{d\mathcal{L}_i}{dz_i}(z_i) = 0$ if and only if $z_i \in \{0, \frac{1}{2}, \frac{\tau_i}{2}, \frac{1+\tau_i}{2}\}$ since these are the ramification points of $\mathcal{L}_i$.

Moreover

$$\mathcal{L}_i(z_i) = \mathcal{L}_i(z_i + 1) = \mathcal{L}_i(z_i + \tau_i) = \mathcal{L}_i(-z_i) = -\mathcal{L}_i\left(z_i + \frac{1}{2}\right),$$

$$\mathcal{L}_i\left(z_i + \frac{\tau_i}{2}\right) = \frac{a_i}{\mathcal{L}_i(z_i)}.$$

Consider

$$\hat{X}_c := \{(z_1, z_2, z_3)) \in T \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = c, \}.$$

Then

- $\hat{X}_c$ is invariant under the action of G,
- for a general choice of c, $\hat{X}_c$ is a smooth hypersurface in T of multidegree $(2, 2, 2)$ and G acts freely on $\hat{X}_c$, thus $X_c := \hat{X}_c/G$ is a smooth minimal surface of general type with $p_g = 0$, $K^2 = 6$.

– for special values of c and for special choices of the elliptic curves the hypersurface $\hat{X}_c$ has 4, 8, 12, 16 nodes, which are isolated fixed points of G; in these cases X_c gets 1, 2, 3, 4 singularities of type $\frac{1}{4}(1,1)$ and the minimal resolution of singularities of $X_c := \hat{X}_c/G$ is a minimal surface of general type with $p_g = 0$ and $K^2 = 5, 4, 3, 2$.

3 Intersection of diagonal quadrics and $(\mathbb{Z}/2\mathbb{Z})^n$-actions

We consider diagram (1):

$$\begin{array}{ll} E_1 \times E_2 \times E_3 & E_1 := \{x_1^2 + x_2^2 + x_3^2 = 0,\ x_0^2 = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2\} \\ \Big\downarrow {\scriptstyle \mathcal{H}' := (\mathbb{Z}/2)^3}\ \pi' & E_2 := \{u_1^2 + u_2^2 + u_3^2 = 0,\ u_0^2 = b_1 u_1^2 + b_2 u_2^2 + b_3 u_3^2\} \\ P_1 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & E_3 := \{z_1^2 + z_2^2 + z_3^2 = 0,\ z_0^2 = c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2\} \\ \Big\downarrow {\scriptstyle \mathcal{H} := ((\mathbb{Z}/2)^2)^3}\ \pi & \\ P_2 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \end{array}$$

Remark 3.1 (1) π' is given by 'forgetting' the variables x_0, u_0, z_0.

(2) π is given by $x_i^2 = y_i,\ u_i^2 = v_i,\ z_i^2 = w_i,\ i = 1, 2, 3$, where we consider

$$P_2 \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$$

as given by the equations

$$y_1 + y_2 + y_3 = 0, \quad v_1 + v_2 + v_3 = 0, \quad w_1 + w_2 + w_3 = 0.$$

(3) The inverse image of the Del Pezzo surface $Y'_\lambda := \{y_1 v_1 w_1 = \lambda y_2 v_2 w_2\} \subset P_2$ under π splits into two irreducible components:

$$\pi^{-1}(\{y_1 v_1 w_1 = \lambda y_2 v_2 w_2\}) = Y_\lambda^+ \cup Y_\lambda^- \subset P_1,$$

where $Y_\lambda^\pm := \{x_1 u_1 z_1 = \pm\sqrt{\lambda} x_2 u_2 z_2\}$.

(4) If we take homogeneous coordinates

$$((s_1 : t_1), (s_2 : t_2), (s_3 : t_3)),$$

such that the action of $\mathcal{H}$ on $P_1 = (\mathbb{P}^1)^3$ is generated by the transformations:

$$t_i \mapsto \pm t_i, \quad , s_i \mapsto s_i, \quad 1 \le i \le 3,$$
$$t_i \mapsto s_i, \quad s_i \mapsto t_i, \quad 1 \le i \le 3,$$

then we see that the Del Pezzo surface

$$Y_\lambda := \{s_1 s_2 s_3 = \lambda t_1 t_2 t_3\} \subset P_1$$

is invariant under the subgroup $H_0 \cong (\mathbb{Z}/2\mathbb{Z})^2$ of $\mathcal{H}$ generated by the transformations:

$$s_i \mapsto s_i, \quad t_i \mapsto \epsilon_i t_i, \quad , \epsilon \in \{\pm 1\}, \quad \epsilon_1\epsilon_2\epsilon_3 = 1.$$

Then $\hat{X}_\lambda := \pi'^{-1}(Y_\lambda)$ is invariant under $G_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \subset (\mathbb{Z}/2\mathbb{Z})^9$. It is now our aim to find all subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \subset G_1$, such that G acts freely on $\hat{X}_\lambda$.
We obtain in this case a commutative diagram

$$\begin{array}{ccc} \hat{X}_\lambda & \xrightarrow{(\mathbb{Z}/2)^3 \cong G} & \\ {\scriptstyle (\mathbb{Z}/2)^3}\downarrow{\scriptstyle \pi'} & & \\ Y_\lambda \subset P_1 & & S_\lambda := \hat{X}_\lambda / G \\ {\scriptstyle H_0}\downarrow & \swarrow{\scriptstyle (\mathbb{Z}/2)^2} & \\ Z'. & & \end{array} \tag{2}$$

S is then a smooth minimal surface of general type with $K_S^2 = 6,\ p_g = 0$. We shall in fact show that necessarily S is a primary Burniat surface.

(5) If instead we set $\lambda = 1$, we see that the Del Pezzo surface

$$Y := Y_1 = \{s_1 s_2 s_3 = t_1 t_2 t_3\} \subset P_1$$

is invariant under the subgroup $H_1 \cong (\mathbb{Z}/2\mathbb{Z})^3$ of $\mathcal{H}$ generated by the transformations:

$$s_i \mapsto s_i, \quad t_i \mapsto \epsilon_i t_i, \quad , \epsilon \in \{\pm 1\}, \quad \epsilon_1\epsilon_2\epsilon_3 = 1$$

and

$$t_i \mapsto s_i, \quad s_i \mapsto t_i, \quad \forall i.$$

Then $\hat{X} := \pi'^{-1}(Y)$ is invariant under $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^6 \subset (\mathbb{Z}/2\mathbb{Z})^9$. It is now our aim to find all subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^4 \subset G_0$, such that there is exactly one element $g_0 \in G$ which has isolated fixed points on $\hat{X}$ (which are also isolated fixed points on $E_1 \times E_2 \times E_3$), while all other nontrivial elements act freely.
We obtain then a commutative diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{(\mathbb{Z}/2)^4 \cong G} & \\ {\scriptstyle (\mathbb{Z}/2)^3}\downarrow{\scriptstyle \pi'} & & \\ Y \subset P_1 & & S := \hat{X} / G \\ {\scriptstyle H_1}\downarrow & \swarrow{\scriptstyle (\mathbb{Z}/2)^2} & \\ Z. & & \end{array} \tag{3}$$

(6) Note that it is easy to see that Z is the four nodal cubic surface in $\mathbb{P}^3$. In fact, $Y^\pm = \{s_1 s_2 s_3 = \pm t_1 t_2 t_3\}$ is the pull-back of $Z' := \{\sigma_1\sigma_2\sigma_3 = \tau_1\tau_2\tau_3\}$ under the map $s_i^2 = \sigma_i,\ t_i^2 = \tau_i$. Hence $Y^+ \to Z'$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of a Del Pezzo surface of degree 6. On Z', the involution which exchanges σ_i and τ_i has four isolated fixed points. Hence the quotient Z is a four nodal cubic surface.

Observe that in the above remark we described the action of $\mathcal{H}$ on $(\mathbb{P}^1)^3$ in the coordinates $(s_i : t_i)$. We have to rewrite this action in the coordinates x_i, u_i, z_i, and then give the equations of the Del Pezzo surface

$$Y_\lambda = \{s_1 s_2 s_3 = \lambda t_1 t_2 t_3\} \subset P_1$$

in the coordinates x_i, u_i, z_i.

In order to give the action of $\mathcal{H}$ in the coordinates x_i, u_i, z_i and find the equations of the Del Pezzo surfaces $Y_\lambda \subset P_1 \subset (\mathbb{P}^2)^3$, we consider first the following diagram

$$\begin{array}{ccc} E_1 = E & & \\ \downarrow{\scriptstyle \mathbb{Z}/2\mathbb{Z}} & & \\ \mathbb{P}^1 & = & \{x_1^2 + x_2^2 + x_3^2 = 0\} =: C \subset \mathbb{P}^2 \\ \downarrow{\scriptstyle (\mathbb{Z}/2\mathbb{Z})^2} & & \\ \mathbb{P}^1 & = & \{y_1 + y_2 + y_3 = 0\} \subset \mathbb{P}^2. \end{array} \tag{4}$$

Observe that

$$x_1^2 + x_2^2 + x_3^2 = 0 \iff \det \begin{pmatrix} x_1 + i x_2 & -x_3 \\ x_3 & x_1 - i x_2 \end{pmatrix} = 0.$$

Therefore we get a parametrization of C:

$$(s : t) = (x_1 + i x_2 : x_3) = (-x_3 : x_1 - i x_2).$$

With this parametrization, we can rewrite the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^1$ (we use the convenient notation by which all variables not mentioned in a transformation are left unchanged):

(a) $x_1 \mapsto -x_1$ (or equivalently $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} -x_2 \\ -x_3 \end{pmatrix}$) corresponds to $(s : t) \mapsto (t : s)$;
(b) $x_2 \mapsto -x_2$ (or equivalently $\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 \\ -x_3 \end{pmatrix}$) corresponds to $(s : t) \mapsto (-t : s)$;
(c) $x_3 \mapsto -x_3$ (or equivalently $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$) corresponds to $(s : t) \mapsto (s : -t)$.

Remark 3.2 The fixed points of the three involutions above are

(a) $s = \pm t \iff x_1 = x_3 \pm i x_2 = 0$;
(b) $t = \pm i s \iff x_2 = x_1 \pm i x_3 = 0$;
(c) $st = 0 \iff x_3 = x_1 \pm i x_2 = 0$.

The equations for the Del Pezzo surface $Y_\lambda = \{s_1 s_2 s_3 = \lambda t_1 t_2 t_3\}$ in the coordinates x_i, u_i, z_i can now be easily computed.

Lemma 3.3 *Consider*

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2_{(x_1:x_2:x_3)} \times \mathbb{P}^2_{(u_1:u_2:u_3)} \times \mathbb{P}^2_{(z_1:z_2:z_3)},$$

given by the equations

$$x_1^2 + x_2^2 + x_3^2 = 0, \quad u_1^2 + u_2^2 + u_3^2 = 0, \quad z_1^2 + z_2^2 + z_3^2 = 0.$$

Let $Y_\lambda \subset \mathbb{P}^1_{(s_1:t_1)} \times \mathbb{P}^1_{(s_2:t_2)} \times \mathbb{P}^1_{(s_3:t_3)}$ be the Del Pezzo surface given by the equation

$$Y = \{s_1 s_2 s_3 = \lambda t_1 t_2 t_3\}, \quad \lambda \neq 0.$$

Then

$$Y_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2_{(x_1:x_2:x_3)} \times \mathbb{P}^2_{(u_1:u_2:u_3)} \times \mathbb{P}^2_{(z_1:z_2:z_3)}$$

is given by the following eight equations:

(1) $(x_1 + ix_2)(u_1 + iu_2)(z_1 + iz_2) = \lambda x_3 u_3 z_3,$
(2) $(x_1 + ix_2)(u_1 + iu_2)(-z_3) = \lambda x_3 u_3 (z_1 - iz_2),$
(3) $(x_1 + ix_2)(-u_3)(z_1 + iz_2) = \lambda x_3 (u_1 - iu_2) z_3,$
(4) $(-x_3)(u_1 + iu_2)(z_1 + iz_2) = \lambda (x_1 - ix_2) u_3 z_3,$
(5) $(x_1 + ix_2) u_3 z_3 = \lambda x_3 (u_1 - iu_2)(z_1 - iz_2),$
(6) $x_3 (u_1 + iu_2) z_3 = \lambda (x_1 - ix_2) u_3 (z_1 - iz_2),$
(7) $x_3 u_3 (z_1 + iz_2) = \lambda (x_1 - ix_2)(u_1 - iu_2) z_3,$
(8) $-x_3 u_3 z_3 = \lambda (x_1 - ix_2)(u_1 - iu_2)(z_1 - iz_2).$

Proof We have seen that each $\mathbb{P}^1$ (written as a conic in $\mathbb{P}^2$) has a birational map to $\mathbb{P}^1$ given by:

$$(s_1 : t_1) = (x_1 + ix_2 : x_3) = (-x_3 : x_1 - ix_2),$$
$$(s_2 : t_2) = (u_1 + iu_2 : u_3) = (-u_3 : u_1 - iu_2),$$
$$(s_3 : t_3) = (z_1 + iz_2 : z_3) = (-z_3 : z_1 - iz_2).$$

Since each birational map is well defined at each point either through the second or through the third ratio, this implies immediately that the divisorial equation of the Del Pezzo surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to the above eight equations in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. □

Let $\hat{X} \subset E_1 \times E_2 \times E_3$ be the inverse image of the Del Pezzo surface $Y_\lambda \subset P_1$ given by the above eight equations. Then we have:

Lemma 3.4 (1) $\underline{\lambda \neq 0}$*: then* $\hat{X}_\lambda$ *is invariant under the group* $G_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \leq (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$, *where*

$$G_1 := \{(\epsilon_0, \epsilon_1, \eta_0, \epsilon_2, \zeta_0, \epsilon_3) \subset (\mathbb{Z}/2\mathbb{Z})^6 | \epsilon_1 \epsilon_2 \epsilon_3 = 1\}.$$

The action of G_1 *on* $E_1 \times E_2 \times E_3$ *is given by:*

$$x_0 \mapsto \epsilon_0 x_0, \quad u_0 \mapsto \eta_0 u_0, \quad z_0 \mapsto \zeta_0 z_0,$$
$$x_3 \mapsto \epsilon_1 x_3, \quad u_3 \mapsto \epsilon_2 u_3, \quad z_3 \mapsto \epsilon_3 z_3, \quad \epsilon_1 \epsilon_2 \epsilon_3 = 1.$$

(2) $\underline{\lambda = 1}$*: then* $\hat{X} := \hat{X}_1$ *is invariant under the group* $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^6 \leq (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$, *where*

$$G_0 := \{(\epsilon_0, \eta_1, \epsilon_1, \eta_0, \epsilon_2, \zeta_0, \epsilon_3) \subset (\mathbb{Z}/2\mathbb{Z})^7 | \epsilon_1 \epsilon_2 \epsilon_3 = 1\}.$$

The action of G_0 *on* $E_1 \times E_2 \times E_3$ *is given by:*

$$x_0 \mapsto \epsilon_0 x_0, \quad u_0 \mapsto \eta_0 u_0, \quad z_0 \mapsto \zeta_0 z_0, \quad \begin{pmatrix} x_1 \\ u_1 \\ z_1 \end{pmatrix} \mapsto \eta_1 \begin{pmatrix} x_1 \\ u_1 \\ z_1 \end{pmatrix},$$
$$x_3 \mapsto \epsilon_1 x_3, \quad u_3 \mapsto \epsilon_2 u_3, \quad z_3 \mapsto \epsilon_3 z_3, \quad \epsilon_1 \epsilon_2 \epsilon_3 = 1.$$

Proof Just observe that multiplication of the variables x_1, u_1, z_1 by -1 correspond to exchanging, for each $i = 1, 2, 3$, s_i with t_i. □

Definition 3.5 (1) Let $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \leq G_1$ have the property that G acts freely on $\hat{X}_\lambda$. Then $S_\lambda := \hat{X}_\lambda / G$ is is called a *primary Burniat type surface*.

(2) Let $G \cong (\mathbb{Z}/2\mathbb{Z})^4 \leq G_0$, such that $\hat{X}_1 = \hat{X}$ is invariant under G. We say that G acts *1-almost freely* on $\hat{X}$, if there is exactly one element $g_0 \in G$ having isolated fixed points on $\hat{X}$ which are also isolated fixed points on $E_1 \times E_2 \times E_3$ while all other nontrivial elements of G act freely.

Then the minimal resolution of singularities S of $X := \hat{X}/G$ is called a *4-nodal Burniat type surface*.

Observe that a primary Burniat type surface S_λ is a smooth minimal surface of general type with $\chi(S) = 1$ and $K_S^2 = 6$. In particular, primary Burniat surfaces are primary Burniat type surfaces.

The minimal resolution of a 4-nodal Burniat type surface is then a minimal surface of general type with $K_S^2 = 3,\ \chi(S) = 1$.

4 The fixed points of G_0 on $\hat{X}$

Remark 4.1 Fix $a_1, a_2, a_3 \in \mathbb{C}$ distinct so that the curve

$$E := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 | x_1^2 + x_2^2 + x_3^2 = 0, x_0^2 = a_1x_1^2 + a_2x_2^2 + a_3x_3^2\}$$

is smooth (hence an elliptic curve). Then:

$$g(x_0 : x_1 : x_2 : x_3) := (\alpha_0x_0, \alpha_1x_1, x_2, \alpha_3x_3),\ \alpha_i \in \{\pm 1\},$$

has fixed points on E if and only if either

- $\alpha_0 = \alpha_1 = \alpha_3 = -1$, or
- exactly one $\alpha_i = -1$, the others are equal to 1.

Note that the group of automorphisms that we consider is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3 \cong \{(m_1, m_2, m_3, m_4) \in (\mathbb{Z}/2\mathbb{Z})^4 | m_3 = 0\}$.

4.1 Elements of G_0 having a fixed locus of dimension 2 on $E_1 \times E_2 \times E_3$

Let $g \in G_0$ be an element leaving a surface

$$S \subset T := E_1 \times E_2 \times E_3$$

pointwise fixed. Then we have the following three possibilities:

(i) $g = \mathrm{id}_{E_1} \times \mathrm{id}_{E_2} \times g_3$, where g_3 has fixed points on E_3;
(ii) $g = \mathrm{id}_{E_1} \times g_2 \times \mathrm{id}_{E_3}$, where g_2 has fixed points on E_2;
(iii) $g = g_1 \times \mathrm{id}_{E_2} \times \mathrm{id}_{E_3}$, where g_1 has fixed points on E_1.

(i) $g = \mathrm{id}_{E_1} \times \mathrm{id}_{E_2} \times g_3$: this implies $\epsilon_0 = \eta_1 = \epsilon_1 = 1$ and $\eta_0 = \epsilon_2 = 1$. This implies $\epsilon_3 = 1$, whence we have for g_3 only one possibility:

$$g_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

By symmetry we get for the cases ii) and iii) the following two respective possibilities:

$$(ii)\quad g_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad (iii)\quad g_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

4.2 Elements of G_0 having a fixed locus of dimension 1 on $E_1 \times E_2 \times E_3$

Let $g \in G_0$ be an element leaving a curve $C \subset T := E_1 \times E_2 \times E_3$ pointwise fixed. Then we have the following three possibilities:

(i) $g = \mathrm{id}_{E_1} \times g_2 \times g_3$, where g_2, g_3 have fixed points on E_2 resp. E_3;
(ii) $g = g_1 \times \mathrm{id}_{E_2} \times g_3$, where g_1, g_3 have fixed points on E_1 resp. E_3;
(iii) $g = g_1 \times g_2 \times \mathrm{id}_{E_3}$, where g_1, g_2 have fixed points on E_1 resp. E_2.

(i) $g = \mathrm{id}_{E_1} \times g_2 \times g_3$: then $\epsilon_0 = \eta_1 = \epsilon_1 = 1$, in particular, $\epsilon_2 = \epsilon_3$. We have therefore:

$$g = (g_1, g_2, g_3) = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \eta_0 \\ 1 \\ 1 \\ \epsilon_2 \end{pmatrix}, \begin{pmatrix} \zeta_0 \\ 1 \\ 1 \\ \epsilon_3 \end{pmatrix} \right).$$

This shows that we have the following two possibilities for g_2:
(a) $\eta_0 = 1$ and $\epsilon_2 = \epsilon_3 = -1$,
(b) $\eta_0 = -1$ and $\epsilon_2 = \epsilon_3 = 1$.
(a) The first possibility for g is:

$$g = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right).$$

(b) The second possibility for g is:

$$g = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

(ii) $g = g_1 \times \mathrm{id}_{E_2} \times g_3$: by symmetry of E_1 and E_2, we get the following two possibilities for g:

$$g = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right), \text{ or } g = \left(\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

(iii) $g = g_1 \times g_2 \times \mathrm{id}_{E_3}$: again by symmetry we have two possibilities for g:

$$g = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right), \text{ or } g = \left(\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

Table 1 The elements of G_0 having fixed points on T, written additively

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ϵ_0	0	0	1	0	0	0	1	0	1	1	1	0	0	0	0	1	1
η_1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1
ϵ_1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	1	1	1
η_0	0	1	0	0	1	0	0	0	1	1	0	0	1	1	0	0	1
ϵ_2	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	1
ζ_0	1	0	0	0	1	0	1	0	0	1	0	0	1	0	1	1	0
ϵ_3	0	0	0	1	0	1	0	0	0	0	1	0	1	1	0	1	0

Remark 4.2 Note that $\hat{X} \subset T$ is an ample divisor, hence the fixed locus of the above elements has non trivial intersection with $\hat{X}$.

4.3 Elements of G_0 having isolated fixed points on $E_1 \times E_2 \times E_3$

We still have to find all elements of G_0 which have isolated fixed points on T.

An element $g = (g_1, g_2, g_3) \in G_0$ has isolated fixed points on $T = E_1 \times E_2 \times E_3$ if and only if g_i ($\neq$ id) has fixed points on E_i. Therefore, on each E_i, g_i is one of the four elements

$$g_i \in \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

We will list in Table 1 all the elements of G_0 which have fixed points on T.

Observe that, unlike before, we write the group additively.

More precisely, the elements 1, 2, 3 have a fixed locus of dimension 2, the elements 4–9 have a fixed curve and the elements 10–17 have isolated fixed points on T.

We shall prove now the following

Proposition 4.3 *The elements* $11 - 17$ *do have fixed points on* $\hat{X}$, *whereas the fixed points of the element 10 do not intersect* $\hat{X}$.

Proof We recall that we have the Del Pezzo surface $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, given in the coordinates $(s_i : t_i)$ by $Y := \{s_1 s_2 s_3 = t_1 t_2 t_3\}$, or in the coordinates (x_i, u_i, z_i) as the subvariety of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by the equations in Lemma 3.3.

We have to check whether the fixed points of the elements $10 - 17$ listed in Table 1 are contained in the pull back $\hat{X}$ of Y.

10) The fixed points are given by $x_0 = u_0 = z_0$, i.e. they are of the form

$$((0 : \pm i\mu_1 x_2 : x_2 : \pm\lambda_1 x_2), (0 : \pm i\mu_2 u_2 : u_2 : \pm\lambda_2 u_2)(0 : \pm i\mu_3 z_2 : z_2 : \pm\lambda_3 z_2)),$$

where $\mu_i = \sqrt{1 + \lambda_i^2}$, and λ_i depends on a_i (resp.b_i, resp c_i).

It is now easy to check that, for a general choice of the elliptic curves E_1, E_2, E_3, points of this form never fulfill the 8 equations of Y.

11) The fixed points are given by $x_0 = u_3 = z_3 = 0$. By Remark 3.2 $u_3 = z_3 = 0$ corresponds to $s_2 t_2 = 0 = s_3 t_3 = 0$. Whence e.g. all points of the form

$$((s_1 : t_1), (0 : t_2), (s_3 : 0)),$$

$(s_1 : t_1)$ arbitrary, lie on Y. This implies that the pull-back of Y contains fixed points of G corresponding to number 11 in Table 1.

12) The fixed points are given by $x_1 = u_1 = z_1 = 0$. By Remark 3.2 this correspond to $s_i = \pm t_i$. This implies that the points $s_i = \epsilon_i t_i$, $\epsilon_i \in \{\pm 1\}$, $\epsilon_1\epsilon_2\epsilon_3 = 1$, are contained in the pull-back of Y.

13) Here we have $x_1 = u_2 = z_2 = 0$, or in the coordinates $(s_i : t_i)$:

$$s_1 = \pm t_1,\ t_2 = \pm i s_2,\ t_3 = \pm i s_3.$$

Again it is obvious that some of these fixed points are contained in the pull-back of Y.

14), 15) $x_3 = u_0 = z_3 = 0$ resp. $x_3 = u_3 = z_0 = 0$: these cases are equal to case 11 by symmetry on the three elliptic curves. Hence also here the fixed points are contained in the pull-back of Y.

16), 17) $x_2 = u_1 = z_2 = 0$ resp. $x_2 = u_2 = z_1 = 0$: these cases are symmetric to case 13. □

In the remaining part of the section we briefly sketch the analogous results for G_1, i.e., we exhibit the elements $g \in G_1$, which have fixed points on $E_1 \times E_2 \times E_3$. Recall that $G_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \leq (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$, where

$$G_1 := \{(\epsilon_0, \epsilon_1, \eta_0, \epsilon_2, \zeta_0, \epsilon_3) \subset (\mathbb{Z}/2\mathbb{Z})^6 | \epsilon_1\epsilon_2\epsilon_3 = 1\}.$$

Remark 4.4 The calculations are quite the same as before for the group G_0, just note that here we always have $\eta_1 = 1$. Then it is easy to see that the elements of G_1 having a fixed surface or a fixed curve are the same as for G_0.

For the elements having isolated fixed points there is a small difference.

4.4 Elements of G_1 having isolated fixed points on $E_1 \times E_2 \times E_3$

We have to find all elements of G_1, which have isolated fixed points on T. We have to exclude those elements of G_1 from G, where some of the fixed points are contained in the base locus of the pencil $\hat{X}_\lambda$.

An element $g = (g_1, g_2, g_3) \in G_1$ has isolated fixed points on $T = E_1 \times E_2 \times E_3$ if and only if $g_i (\neq \text{id})$ has fixed points on E_i. On each E_i we have the two elements

$$g_i = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

Table 2 The elements of G_1 having fixed points on T

	1	2	3	4	5	6	7	8	9	10	11	12	13
ϵ_0	0	0	1	0	0	0	1	0	1	1	1	0	0
η_1	0	0	0	0	0	0	0	0	0	0	0	0	0
ϵ_1	0	0	0	0	0	1	0	1	0	0	0	1	1
η_0	0	1	0	0	1	0	0	0	1	1	0	1	0
ϵ_2	0	0	0	1	0	0	0	1	0	0	1	0	1
ζ_0	1	0	0	0	1	0	1	0	0	1	0	0	1
ϵ_3	0	0	0	1	0	1	0	0	0	0	1	1	0

We will list now all elements of G_1 having fixed points on T in the following Table 2.

Note that again we write the group additively in the sequel.

Note that the elements 1, 2, 3 have a fixed locus of dimension 2, the elements 4–9 have a fixed curve and the elements 10–13 have isolated fixed points on T.

The following is easy to verify

Proposition 4.5 *The elements* $11 - 13$ *do have fixed points on the base locus of the pencil* $\hat{X}_\lambda$*, whereas the fixed points of the element 10 do not lie on the base locus of* $\hat{X}_\lambda$.

We can now prove the following

Theorem 4.6 *Let S be a primary Burniat type surface. Then S is a primary Burniat surface.*

Proof The following MAGMA script shows that there are two subgroups $(\mathbb{Z}/2\mathbb{Z})^3 \cong G \leq G_1$ acting freely on $\hat{X}_\lambda$, for $\lambda \in \mathbb{C}$ general.

```
K:=FiniteField(2); V5:=VectorSpace(K,5); V2:=VectorSpace(K,2);
H:=Hom(V5,V2);
U1:=sub<V5|[0,0,0,0,1]>; U2:=sub<V5|[0,0,1,0,0]>;
U3:=sub<V5|[1,0,0,0,0]>; U4:=sub<V5|[0,0,0,1,0]>;
U5:=sub<V5|[0,0,1,0,1]>; U6:=sub<V5|[0,1,0,0,0]>;
U7:=sub<V5| [0,1,0,1,0] >; U8:=sub<V5| [1,0,1,0,0] >;
U9:=sub<V5| [1,0,0,0,1] >; U10:=sub<V5| [1,0,0,1,0] >;
U11:=sub<V5| [0,1,1,0,0] >; U12:=sub<V5| [0,1,0,1,1] >;
N:=sub<V5|[0,0,0,0,0]>;
w1:=V5![1,0,0,0,0];
w2:=V5![0,0,1,0,0];
x:=V2![1,0]; y:=V2![0,1];
M:={@ @};
 for a in H do
   if a(w1) eq x then
     if a(w2) eq y then
       if Kernel(a) meet U1 eq N then
         if Kernel(a) meet U2 eq N then
           if Kernel(a) meet U3 eq N then
              if Kernel(a) meet U4 eq N then
                if Kernel(a) meet U5 eq N then
                  if Kernel(a) meet U6 eq N then
                    if Kernel(a) meet U7 eq N then
                      if Kernel(a) meet U8 eq N then
                        if Kernel(a) meet U9 eq N then
                          if Kernel(a) meet U10 eq N then
                            if Kernel(a) meet U11 eq N then
                              if Kernel(a) meet U12 eq N then
                           Include(~M,a);
end if;end if;end if;end if;end if;end if;end if;
end if;end if;end if;end if;end if;end if;end if;
end for;
M;
{@
   [1 0]
   [1 1]
```

```
    [0 1]
    [0 1]
    [1 1],

    [1 0]
    [1 0]
    [0 1]
    [1 1]
    [1 1]
@}
```

It is now easy to see that the two cases are equivalent under the symmetry exchanging E_1 and E_2. Therefore they yield the same surfaces. □

5 4-Nodal Burniat type surfaces

In this section we shall give a complete classification of 4-nodal Burniat type surfaces.

$$\begin{array}{c} E_1 \times E_2 \times E_3 \\ \mathcal{H}' := (\mathbb{Z}/2)^3 \left\downarrow \pi' \right. \\ P_1 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \mathcal{H} := ((\mathbb{Z}/2)^2)^3 \left\downarrow \pi \right. \\ P_2 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

$$E_1 := \{x_1^2 + x_2^2 + x_3^2 = 0,\ x_0^2 = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2\}$$

$$E_2 := \{u_1^2 + u_2^2 + u_3^2 = 0,\ u_0^2 = b_1 u_1^2 + b_2 u_2^2 + b_3 u_3^2\}$$

$$E_3 := \{z_1^2 + z_2^2 + z_3^2 = 0,\ z_0^2 = c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2\}$$

Using the notation in Sect. 3 we see that $\hat{X} := \pi'^{-1}(Y)$ is invariant under $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^6 \subset (\mathbb{Z}/2\mathbb{Z})^9$. We find now subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^4 \subset G_0$ such that there is exactly one element $g_0 \in G$ having (isolated) fixed points on $\hat{X}$ and on T, while all the other nontrivial elements of G act freely.

Remark 5.1 We shall see then that this unique element $g_0 \in G$ has 32 fixed points on $\hat{X}$, whence $X := \hat{X}/G$ has 4 nodes (this fact justifies our terminology).

It is clear that the minimal model S of X is a surface of general type with invariants $K_S^2 = 3$ and $\chi(S) = 1$. Looking in fact at the respective groups G , we see that in all cases $q(S) = 0$, whence $p_g(S) = 0$.

The following MAGMA script has as output bases of subgroups $G \leq G_0$ as $\mathbb{F}_2$-vectorspaces, which contain exactly one element g_0 having isolated fixed points on $\hat{X}$ which are also isolated fixed points on T.

```
K:=FiniteField(2);
V6:=VectorSpace(K,6); V2:=VectorSpace(K,2); H:=Hom(V6,V2);
U1:=sub<V6|[0,0,0,0,0,1]>; U2:=sub<V6|[0,0,0,1,0,0]>;
```

```
U3:=sub<V6|[1,0,0,0,0,0]>; U4:=sub<V6|[0,0,0,0,1,0]>;
U5:=sub<V6|[0,0,0,1,0,1]>; U6:=sub<V6|[0,0,1,0,0,0]>;
U7:=sub<V6|[1,0,0,0,0,1]>; U8:=sub<V6|[0,0,1,0,1,0]>;
U9:=sub<V6|[1,0,0,1,0,0]>; U10:=sub<V6|[1,0,0,0,1,0]>;
U11:=sub<V6|[0,0,1,1,0,0]>; U12:=sub<V6|[0,0,1,0,1,1]>;
U13:=sub<V6|[0,1,0,0,0,0]>;  U14:=sub<V6|[0,1,0,1,1,1]>;
U15:=sub<V6|[1,1,1,0,0,1]>; U16:=sub<V6|[1,1,1,1,1,0]>;
N:=sub<V6|[0,0,0,0,0,0]>;
w1:=V6![1,0,0,0,0,0]; w2:=V6![0,0,0,1,0,0];
x:=V2![1,0]; y:=V2![0,1];
M:={@ @};
for a in H do
 if a(w1) eq x then
   if a(w2) eq y then
     if Kernel(a) meet U1 eq N then
       if Kernel(a) meet U2 eq N then
         if Kernel(a) meet U3 eq N then
           if Kernel(a) meet U4 eq N then
             if Kernel(a) meet U5 eq N then
               if Kernel(a) meet U6 eq N then
                 if Kernel(a) meet U7 eq N then
                   if Kernel(a) meet U8 eq N then
                     if Kernel(a) meet U9 eq N then Include(~M,a);
end if; end if;end if;end if; end if; end if;
end if;end if;end if;end if;end if;
end for;
F:={@ V6! [1,0,0,0,1,0],V6! [0,0,1,1,0,0],
V6! [0,0,1,0,1,1],V6! [0,1,0,0,0,0],
V6![0,1,0,1,1,1],V6![1,1,1,0,0,1],V6![1,1,1,1,1,0] @};

M1:={@ @};
 for i in [1..#M] do K:={@ @};
   for x in Kernel(M[i]) do  Include(~K,x);
 end for;
   if  #(K meet F) eq 1 then Include(~M1,i);
 end if; end for;
M1;
{@ 7, 8, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24 @}
MM:={@ M[7],M[8],M[10],M[11],M[13],M[14],M[15],M[16],M[17],
M[18],M[19],M[20],M[22],M[24] @};

MB:={@ MM[1], MM[2],MM[3],MM[5],MM[6],MM[7],MM[8] @};
L:={@ @};
 for x in MB do Include(~L,Kernel(x));
end for;
```

Remark 5.2 We want to observe that MM contains 14 subgroups, which split into seven pairs of equivalent subgroups under the symmetry obtained by exchanging E_1 and E_2.

There are seven groups in the set L. We list generators for each of these in Table 3.

Remark 5.3 In each of the seven subgroups $G \leq G_0$ there is exactly one (non trivial) element having fixed points on $\hat{X}$. These elements are:

(A) $g_0 = (1, 0, 0, 0, 0, 1, 0, 0, 1)$,

Table 3 Generators of $G \cong (\mathbb{Z}/2\mathbb{Z})^4$

	ϵ_0	η_1	ϵ_1	η_0	η_1	ϵ_2	ζ_0	η_1	ϵ_3		ϵ_0	η_1	ϵ_1	η_0	η_1	ϵ_2	ζ_0	η_1	ϵ_3
A	1	0	0	0	0	1	0	0	1	B	1	0	0	0	0	1	0	0	1
	0	1	0	0	1	1	1	1	1		0	1	0	0	1	0	1	1	0
	0	0	1	0	0	0	1	0	1		0	0	1	0	0	0	1	0	1
	0	0	0	1	0	1	1	0	1		0	0	0	1	0	1	1	0	1
C	1	0	0	0	0	1	1	0	1	D	1	0	0	0	0	1	1	0	1
	0	1	0	0	1	1	1	1	1		0	1	0	0	1	0	0	1	0
	0	0	1	0	0	1	1	0	0		0	0	1	0	0	0	1	0	1
	0	0	0	1	0	1	0	0	1		0	0	0	1	0	1	0	0	1
E	1	0	0	0	0	1	1	0	1	F	1	0	0	0	0	1	1	0	1
	0	1	0	0	1	1	1	1	1		0	1	0	0	1	1	0	1	1
	0	0	1	0	0	0	1	0	1		0	0	1	0	0	0	1	0	1
	0	0	0	1	0	1	0	0	1		0	0	0	1	0	1	0	0	1
G	1	0	0	0	0	1	1	0	1										
	0	1	0	0	1	0	1	1	0										
	0	0	1	0	0	0	1	0	1										
	0	0	0	1	0	1	0	0	1										

(B) $g_0 = (1, 0, 0, 0, 0, 1, 0, 0, 1)$,
(C) $g_0 = (0, 0, 1, 0, 0, 1, 1, 0, 0)$,
(D) $g_0 = (0, 1, 0, 0, 1, 0, 0, 1, 0)$,
(E) $g_0 = (1, 1, 1, 0, 1, 0, 1, 1, 1)$,
(F) $g_0 = (1, 1, 1, 1, 1, 1, 0, 1, 0)$,
(G) $g_0 = (0, 1, 0, 1, 1, 1, 1, 1, 1)$.

In order to calculate the fundamental groups of the corresponding quotient surfaces it is convenient to rewrite the action of G_i, $i = A, B, C, D, E, F, G$, on $T := E_1 \times E_2 \times E_3$ in terms of uniformizing parameters z_i for E_i.

For $i \in \{1, 2, 3\}$, let $E_i := \mathbb{C}/\langle 1, \tau_i \rangle$ be a complex elliptic curve. Then we choose as basis for the $(\mathbb{Z}/2\mathbb{Z})^3$-action on E_i:

- $(z_i \mapsto -z_i) = (1, 0, 0)$,
- $(z_i \mapsto -z_i + \frac{\tau_i}{2}) = (0, 1, 0)$,
- $(z_i \mapsto -z_i + \frac{1}{2}) = (0, 0, 1)$.

Then we can rewrite the generators of G_i, $i \in \{A, B, C, D, E, F, G\}$, in Table (3) in the following way.

We would like to point out that in the cases C, D, E, F, G we choose a different basis from the one in Table (3).

(1) G_A is generated by:

$$g_1(z_1, z_2, z_3) = (-z_1, -z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}),$$
$$g_2(z_1, z_2, z_3) = (-z_1 + \tfrac{\tau_1}{2}, z_2 + \tfrac{1}{2} + \tfrac{\tau_2}{2}, -z_3 + \tfrac{1}{2} + \tfrac{\tau_3}{2}),$$
$$g_3(z_1, z_2, z_3) = (-z_1 + \tfrac{1}{2}, z_2, z_3 + \tfrac{1}{2}),$$
$$g_4(z_1, z_2, z_3) = (z_1, z_2 + \tfrac{1}{2}, z_3 + \tfrac{1}{2}).$$

(2) G_B is generated by:

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (-z_1, -z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}),\\ g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{\tau_1}{2}, -z_2 + \tfrac{\tau_2}{2}, z_3 + \tfrac{\tau_3}{2}),\\ g_3(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, z_2, z_3 + \tfrac{1}{2}),\\ g_4(z_1, z_2, z_3) &= (z_1, z_2 + \tfrac{1}{2}, z_3 + \tfrac{1}{2}). \end{aligned}$$

(3) G_C is generated by:

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, -z_3),\\ g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{\tau_1}{2}, z_2 + \tfrac{1}{2} + \tfrac{\tau_2}{2}, -z_3 + \tfrac{1}{2} + \tfrac{\tau_3}{2}),\\ g_3(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, z_2, -z_3 + \tfrac{1}{2}),\\ g_4(z_1, z_2, z_3) &= (z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}). \end{aligned}$$

(4) G_D is generated by:

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, z_3),\\ g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, z_2, z_3 + \tfrac{1}{2}),\\ g_3(z_1, z_2, z_3) &= (z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}),\\ g_4(z_1, z_2, z_3) &= (-z_1 + \tfrac{\tau_1}{2}, -z_2 + \tfrac{\tau_2}{2}, -z_3 + \tfrac{\tau_3}{2}). \end{aligned}$$

(5) G_E is generated by:

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, z_3),\\ g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, z_2, z_3 + \tfrac{1}{2}),\\ g_3(z_1, z_2, z_3) &= (z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}),\\ g_4(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2} + \tfrac{\tau_1}{2}, -z_2 + \tfrac{\tau_2}{2}, -z_3 + \tfrac{1}{2} + \tfrac{\tau_3}{2}). \end{aligned}$$

(6) G_F is generated by:

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, z_3),\\ g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, z_2, z_3 + \tfrac{1}{2}),\\ g_3(z_1, z_2, z_3) &= (z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}),\\ g_4(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2} + \tfrac{\tau_1}{2}, -z_2 + \tfrac{1}{2} + \tfrac{\tau_2}{2}, -z_3 + \tfrac{\tau_3}{2}). \end{aligned}$$

(7) G_G is generated by:

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, z_3),\\ g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, z_2, z_3 + \tfrac{1}{2}),\\ g_3(z_1, z_2, z_3) &= (z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}),\\ g_4(z_1, z_2, z_3) &= (-z_1 + \tfrac{\tau_1}{2}, -z_2 + \tfrac{1}{2} + \tfrac{\tau_2}{2}, -z_3 + \tfrac{1}{2} + \tfrac{\tau_3}{2}). \end{aligned}$$

Remark 5.4 (1) We have $G_i \le G_0 \le (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$. Denote by K_i, $i = 1, 2, 3$, the kernel of the projection on the i-th factor. Then we have:

(i) $K_3 \subset K_1 \oplus K_2$, for the groups G_i, $i = A, B, C$;
(ii) $K_i \cap (K_j \oplus K_l) = \{0\}$, where $\{i, j, l\} = \{1, 2, 3\}$, for the groups G_i, $i = D, E, F, G$.

(2) The unique element in G_i having fixed points on $\hat{X}$ is

(i) $g_1(z_1, z_2, z_3) = (-z_1, -z_2 + \frac{1}{2}, -z_3 + \frac{1}{2})$, for $i = A, B$,
(ii) $g_1(z_1, z_2, z_3) = (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2}, -z_3)$, for $i = C$,
(iii) g_4, for $i = D, E, F, G$.

Recall that we have written $E_i = \mathbb{C}/\langle e_i, \tau_i e_i\rangle$, $i = 1, 2, 3$.

Denote by Λ the fundamental group of $E_1 \times E_2 \times E_3$, so that, setting $\Lambda_i = \langle e_i, \tau_i e_i \rangle$, we have $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$.

At this moment we invoke the hyperplane section theorem of Lefschetz, which we apply to the ample divisor $\hat{X} \subset E_1 \times E_2 \times E_3$: it follows that $\pi_1(\hat{X}) \cong \pi_1(E_1 \times E_2 \times E_3) = \Lambda$.

Hence the universal covering $\tilde{X}$ of $\hat{X} \subset E_1 \times E_2 \times E_3$ has a natural inclusion $\tilde{X} \subset \mathbb{C}^3$.

Now the affine group

$$\Gamma_i := \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, e_1, \tau_1 e_1, e_2, \tau_2 e_2, e_3, \tau_3 e_3 \rangle \leq \mathbb{A}(3, \mathbb{C}), \tag{5}$$

where the γ_k are lifts of the generators g_k of G_i as affine transformations, acts on $\mathbb{C}^3$ leaving $\tilde{X}$ invariant.

Moreover, $X_i = \hat{X}/G_i = \tilde{X}/\Gamma_i$.

Then by Armstrong's result (cf. [1,2]) we have

$$\pi_1(X_i) = \Gamma_i / \operatorname{Tors}(\Gamma_i), \tag{6}$$

where $\operatorname{Tors}(\Gamma_i)$ is the normal subgroup of Γ_i generated by all elements of Γ_i having finite order (indeed they have order equal to 2): since these are precisely the elements which have fixed points on $\tilde{X}$.

Remark 5.5 Denote by $g_0 \in G$ the unique element which has fixed points on $\hat{X}$, and denote by $\gamma_0 \in \Gamma_i$ a lift of g_0 to $\mathbb{A}(3, \mathbb{C})$. Observe that

$$\gamma_0 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 + \mu_1 \\ -z_2 + \mu_2 \\ -z_3 + \mu_3 \end{pmatrix},$$

where $\mu_i = \frac{1}{2}\epsilon_i \in \frac{1}{2}\Lambda_i$.

(1) Assume that $\gamma \in \Gamma_i$ has a fixed point on the universal covering $\tilde{X}$ of $\hat{X}$. Then there is a $\lambda \in \Lambda$ such that $\gamma = \gamma_0 t_\lambda$.

(2) Let $z = (z_1, z_2, z_3) \in \tilde{X} \subset \mathbb{C}^3$. Then z yields a fixed point of g_0 on $\hat{X}$ if and only if there exists $\hat{\lambda} \in \Lambda$ such that

$$2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} + \hat{\lambda} \iff z = \frac{1}{4}\epsilon + \frac{1}{2}\hat{\lambda},$$

where $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$.

We need the following

Lemma 5.6 *$z = \frac{1}{4}\epsilon + \frac{1}{2}\hat{\lambda} \in \tilde{X}$ is a fixed point of $\gamma = \gamma_0 t_\lambda$ if and only if $\lambda = -\hat{\lambda}$.*

Proof

$$\begin{aligned}\gamma(z) = \gamma_0(z + \lambda) &= -z - \lambda + \frac{1}{2}\epsilon = -\frac{1}{4}\epsilon - \frac{1}{2}\hat{\lambda} - \lambda + \frac{1}{2}\epsilon \\ &= \frac{1}{4}\epsilon + \frac{1}{2}\hat{\lambda} - \hat{\lambda} - \lambda = z - \hat{\lambda} - \lambda = z \iff \lambda = -\hat{\lambda}.\end{aligned} \tag{7}$$

□

Note that g_0 has 64 fixed points on $T = E_1 \times E_2 \times E_3$, but only 32 lie on $\hat{X}$. These 32 points are divided in four G_i-orbits. Let $P_1, \ldots, P_4 \in \tilde{X}$ be four representatives of the four orbits. Then we have $P_i = \frac{1}{4}\epsilon + \frac{1}{2}\hat{\lambda}_{P_i}$. Then:

$$\mathrm{Tors}(\Gamma_i) = \langle\langle \gamma_0 t_{\hat{\lambda}_{P_1}}, \gamma_0 t_{\hat{\lambda}_{P_2}}, \gamma_0 t_{\hat{\lambda}_{P_3}}, \gamma_0 t_{\hat{\lambda}_{P_4}} \rangle\rangle.$$

Moreover, since the point z can be changed modulo Λ, the above argument shows that $2\Lambda \subset \mathrm{Tors}(\Gamma_i)$, hence $\pi_1(S) = \pi_1(X)$ is a quotient of the 2-step nilpotent group Π_G such that

$$1 \to \Lambda/2\Lambda \to \Pi_G \to G \to 1.$$

We can now prove the following theorem:

Theorem 5.7 *Let S_i, $i \in \{A, B, C, D, E, F, G\}$ be the minimal resolution of the surface $X_i := \hat{X}/G_i$ (having four ordinary nodes). Then S_i is a minimal surface of general type with $K^2_{S_i} = 3$, $p_g(S_i) = 0$, with fundamental group*

(i) $\pi_1(S_i) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$, *for $i = A, B, C$;*
(ii) $\pi_1(S_i) \cong \mathbb{H} \times \mathbb{Z}/2\mathbb{Z}$, *for $i = D$;*
(iii) $\pi_1(S_i) \cong SmallGroup(16, 13)$, *for $i = E, F, G$.*

Remark 5.8 (1) Cases D,E,F,G are obviously quotients of a primary Burniat surface by an involution having four isolated fixed points. Since primary Burniat surfaces satisfy Bloch's conjecture asserting that the group of zero cycles of degree 0 modulo rational equivalence is trivial (cf. [15]), it follows that also the surfaces S_D, S_E, S_F, S_G satisfy Bloch's conjecture.
Cases A, B, C have the same fundamental group as the Keum-Naie surfaces with $K^2 = 3$. Each of the cases E, F, G yields a (3-dimensional) family, which is new. Actually, the fundamental group $SmallGroup(16, 13)$, which is the central product of the dihedral group of order 8 with the cyclic group of order 4, has not yet been realized by a surface with $K^2 = 3$, $p_g = 0$.

(2) Denote by $\hat{S}_i$ the double cover of X_i branched exactly in the four nodes. Then

- $\hat{S}_i$ is a surface of general type with $K^2_S = 6$, $p_g = q = 1$ if $i = A, B, C$,
- $\hat{S}_i$ is a primary Burniat surface for $i = D, E, F, G$.

(3) It is easy to see that the groups G_A, G_B, G_C yield the same family of surfaces. Indeed, exchanging E_2 with E_3 has the effect of exchanging G_A and G_B, whereas exchanging E_1 with E_3 has the effect of exchanging G_B and G_C.

The same holds for the groups G_E, G_F and G_G. Therefore, in order to prove the above theorem, it suffices to calculate the fundamental group in the cases A, D, E.

Proof (A) The fixed points of $g_0(z_1, z_2, z_3) = (-z_1, -z_2 + \frac{1}{2}, -z_3 + \frac{1}{2})$ are the points $(z_1, z_2, z_3) \in E_1 \times E_2 \times E_3$ such that

$$z_1 \in \left\{0, \frac{1}{2}, \frac{\tau_1}{2}, \frac{1}{2} + \frac{\tau_1}{2}\right\},$$

$$z_i \in \left\{\frac{1}{4}, \frac{1}{4} + \frac{1}{2}, \frac{1}{4} + \frac{\tau_i}{2}, \frac{1}{4} + \frac{1}{2} + \frac{\tau_i}{2}\right\}, \; i = 2, 3.$$

These are 64 points, but only 32 of these are on $\hat{X}$, namely:

$$(z_1, z_2, z_3),\ z_1 \in \left\{0, \frac{1}{2}, \frac{\tau_1}{2}, \frac{1}{2} + \frac{\tau_1}{2}\right\}, \quad (z_2, z_3) \in \left\{\left(\frac{1}{4}, \frac{1}{4} + \frac{\tau_3}{2}\right), \left(\frac{1}{4}, \frac{1}{4} + \frac{1}{2} + \frac{\tau_3}{2}\right),\right.$$
$$\left(\frac{1}{4} + \frac{1}{2}, \frac{1}{4} + \frac{\tau_3}{2}\right), \left(\frac{1}{4} + \frac{1}{2}, \frac{1}{4} + \frac{1}{2} + \frac{\tau_3}{2}\right), \left(\frac{1}{4} + \frac{\tau_2}{2}, \frac{1}{4}\right), \left(\frac{1}{4} + \frac{\tau_2}{2}, \frac{1}{4} + \frac{1}{2}\right),$$
$$\left.\left(\frac{1}{4} + \frac{1}{2} + \frac{\tau_2}{2}, \frac{1}{4}\right), \left(\frac{1}{4} + \frac{1}{2} + \frac{\tau_2}{2}, \frac{1}{4} + \frac{1}{2}\right)\right\}. \tag{8}$$

In fact, recall that the affine equation of $\hat{X}$ (cf. [16]) is

$$\hat{X} = \{(z_1, z_2, z_3) \in T | \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b_1 b_2 b_3\},$$

where $b_i = \mathcal{L}_i(\frac{\tau_i}{4})$. Observe that $b_i^2 = a_i$. Let $(\mathcal{L}_i(z_i)_0 : \mathcal{L}_i(z_i)_1)$ be homogeneous coordinates of the point $\mathcal{L}_i(z_i)$. The equation of $\hat{X}$ is then:

$$\mathcal{L}_1(z_1)_0\mathcal{L}_2(z_2)_0\mathcal{L}_3(z_3)_0 = b_1 b_2 b_3 \mathcal{L}_1(z_1)_1\mathcal{L}_2(z_2)_1\mathcal{L}_3(z_3)_1.$$

It follows easily from the properties of the Legendre function that

$$\left(\mathcal{L}_i\left(z_i + \frac{\tau_i}{2}\right)_0 : \mathcal{L}_i\left(z_i + \frac{\tau_i}{2}\right)_1\right) = (a_i\mathcal{L}_i(z_i)_1 : \mathcal{L}_i(z_i)_0).$$

In particular, we have

$$\left(\mathcal{L}_i\left(\frac{1}{4}\right)_0 : \mathcal{L}_i\left(\frac{1}{4}\right)_1\right) = (0:1), \quad \left(\mathcal{L}_i\left(\frac{1}{4} + \frac{\tau_i}{2}\right)_0 : \mathcal{L}_i\left(\frac{1}{4} + \frac{\tau_i}{2}\right)_1\right) = (1:0).$$

Now it follows easily that a fixed point (z_1, z_2, z_3) of g_0 on T lies in fact on $\hat{X}$ if and only if it satisfies the equations

$$\mathcal{L}_1(z_1)_0\mathcal{L}_2(z_2)_0\mathcal{L}_3(z_3)_0 = \mathcal{L}_1(z_1)_1\mathcal{L}_2(z_2)_1\mathcal{L}_3(z_3)_1 = 0.$$

Therefore a fixed point $(z_1, z_2, z_3) \in T$ of g_0 lies on $\hat{X}$ if and only if $z_1 \in \left\{0, \frac{1}{2}, \frac{\tau_1}{2}, \frac{1}{2} + \frac{\tau_1}{2}\right\}$ and

$$(z_2, z_3) \in \left\{\left(\frac{1}{4}, \frac{1}{4} + \frac{\tau_3}{2}\right), \left(\frac{1}{4}, \frac{1}{4} + \frac{1}{2} + \frac{\tau_3}{2}\right), \left(\frac{1}{4} + \frac{1}{2}, \frac{1}{4} + \frac{\tau_3}{2}\right), \left(\frac{1}{4} + \frac{1}{2}, \frac{1}{4} + \frac{1}{2} + \frac{\tau_3}{2}\right),\right.$$
$$\left.\left(\frac{1}{4} + \frac{\tau_2}{2}, \frac{1}{4}\right), \left(\frac{1}{4} + \frac{\tau_2}{2}, \frac{1}{4} + \frac{1}{2}\right), \left(\frac{1}{4} + \frac{1}{2} + \frac{\tau_2}{2}, \frac{1}{4}\right), \left(\frac{1}{4} + \frac{1}{2} + \frac{\tau_2}{2}, \frac{1}{4} + \frac{1}{2}\right)\right\}.$$

These points fall into 4 G_A- orbits, and it is easy to verify that we can choose as representatives the four points:

$$P_1 = \left(0, \frac{1}{4}, \frac{1}{4} + \frac{\tau_3}{2}\right), \quad P_2 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4} + \frac{\tau_3}{2}\right),$$
$$P_3 = \left(\frac{\tau_1}{2}, \frac{1}{4}, \frac{1}{4} + \frac{\tau_3}{2}\right), \quad P_4 = \left(\frac{1}{2} + \frac{\tau_1}{2}, \frac{1}{4}, \frac{1}{4} + \frac{\tau_3}{2}\right).$$

Writing as above $P_i = \frac{1}{4}\epsilon + \frac{1}{2}\hat{\lambda}_{P_i}$, we see that $\epsilon = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and

$$\hat{\lambda}_{P_1} = \begin{pmatrix} 0 \\ 0 \\ \tau_3 \end{pmatrix}, \quad \hat{\lambda}_{P_2} = \begin{pmatrix} 1 \\ 0 \\ \tau_3 \end{pmatrix}, \quad \hat{\lambda}_{P_3} = \begin{pmatrix} \tau_1 \\ 0 \\ \tau_3 \end{pmatrix}, \quad \hat{\lambda}_{P_4} = \begin{pmatrix} 1 + \tau_1 \\ 0 \\ \tau_3 \end{pmatrix}.$$

Therefore

$$\pi_1(X_j) = \Gamma_i/\langle\langle \gamma_0 t_{\hat{\lambda}_{P_i}} : i = 1, 2, 3, 4\rangle\rangle, \ j = A, B.$$

The following MAGMA script gives $\pi_1(X_j) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$.

```
G1:=DirectProduct([CyclicGroup(2),CyclicGroup(2),CyclicGroup(2)]);
G2:=DirectProduct([CyclicGroup(2),CyclicGroup(2),CyclicGroup(2)]);
G3:=DirectProduct([CyclicGroup(2),CyclicGroup(2),CyclicGroup(2)]);

H:=DirectProduct([G1,G2,G3]);
PolyGroup:=func<seq|Group<a1,a2,a3,a4|
          a1^seq[1], a2^seq[2],a3^seq[3],a4^seq[4], a1*a2*a3*a4>>;
P1:=PolyGroup([2,2,2,2]);
P2:=PolyGroup([2,2,2,2]);
P3:=PolyGroup([2,2,2,2]);
P:=DirectProduct([P1,P2,P3]);
f:=Homomorphism(P,H, [P.1,P.2,P.3,P.4,P.5,P.6,P.7,P.8,P.9,
P.10,P.11,P.12],[H!(1,2),H!(3,4),H!(5,6),H!(1,2)(3,4)(5,6),
H!(7,8),H!(9,10),H!(11,12),H!(7,8)(9,10)(11,12),H!(13,14),
H!(15,16),H!(17,18),H!(13,14)(15,16)(17,18)]);
R:=Rewrite(P,Kernel(f));
R;
Finitely presented group R on 6 generators
Generators as words in group P
   R.1 = (P.2 * P.1)^2   /* =  e_1
   R.2 = (P.3 * P.1)^2   /* = \tau_1
   R.3 = (P.6 * P.5)^2   /*= e_2
   R.4 = (P.7 * P.5)^2   /*= \tau_2
   R.5 = (P.10 * P.9)^2  /* = e_3
   R.6 = (P.11 * P.9)^2  /*= \tau_3
Relations
    (R.1, R.2^-1) = Id(R)
    (R.3, R.4^-1) = Id(R)
    (R.5, R.6^-1) = Id(R)
    (R.4^-1, R.6^-1) = Id(R)
    (R.1^-1, R.5^-1) = Id(R)
    (R.5, R.2) = Id(R)
    (R.1^-1, R.3^-1) = Id(R)
    (R.2^-1, R.4^-1) = Id(R)
    (R.1^-1, R.6^-1) = Id(R)
    (R.3^-1, R.6^-1) = Id(R)
    (R.4^-1, R.5^-1) = Id(R)
    (R.2^-1, R.6^-1) = Id(R)
    (R.3^-1, R.5^-1) = Id(R)
    (R.4, R.1) = Id(R)
    (R.2^-1, R.3^-1) = Id(R)
   R.6^-1 * R.5 * R.2^-1 * R.1 * R.5^-1 * R.6 *
   R.1^-1 * R.2 = Id(R)
   R.1^-1 * R.2 * R.3^-1 * R.4 * R.2^-1 *
   R.1 * R.4^-1 * R.3 = Id(R)
   R.3^-1 * R.4 * R.5^-1 * R.6 * R.4^-1 *
   R.3 * R.6^-1 * R.5 = Id(R)
CASE A:
```

```
***********
GG1:=sub<H|H!(1,2)(11,12)(17,18),
H!(3,4)(9,10)(11,12)(13,14)(15,16)(17,18),
H!(5,6)(13,14)(17,18),H!(7,8)(11,12)(13,14)(17,18)>;

/*The only element of GG1 having fixed points is
(1,2)(11,12)(17,18).*/

Pi1:=Rewrite(P,GG1@@f);
Q1:=quo<Pi1|P.1*P.7*P.11, P.1*P.7*P.11*(P.11*P.9)^2,
P.1*P.7*P.11*(P.2*P.1)^2*(P.11*P.9)^2,
P.1*P.7*P.11*(P.3*P.1)^2*(P.11*P.9)^2,
P.1*P.7*P.11*(P.2*P.1)^2*(P.3*P.1)^2*(P.11*P.9)^2 >;
IdentifyGroup(Q1);
<16, 10>
```

D) Here we have $g_0 = (-z_1 + \frac{\tau_1}{2}, -z_2 + \frac{\tau_2}{2}, -z_3 + \frac{\tau_3}{2})$. The 64 fixed points of g_0 on $T := E_1 \times E_2 \times E_3$ are:

$$z \in \left\{ \frac{1}{4} \begin{pmatrix} \pm\tau_1 \\ \pm\tau_2 \\ \pm\tau_3 \end{pmatrix} + \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^3 \right\}.$$

Here it suffices again to look at the affine equation of $\hat{X}$ and we see that all the above points satisfy

$$\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = \pm b_1 b_2 b_3.$$

They lie on $\hat{X}$ (i.e., they fulfill the equation $\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b_1 b_2 b_3$) if and only if

$$z \in \left\{ \frac{1}{4} \begin{pmatrix} \pm\tau_1 \\ \pm\tau_2 \\ \pm\tau_3 \end{pmatrix} + \frac{1}{2} \left\{ 0, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \right\}.$$

It is easy to see that we can choose as representatives for the 4 G_D-orbits:

$$P_1 = \left(\frac{\tau_1}{4}, \frac{\tau_2}{4}, \frac{\tau_3}{4}\right), \ P_2 = \left(\frac{\tau_1}{4} + \frac{1}{2}, \frac{\tau_2}{4} + \frac{1}{2}, \frac{\tau_3}{4}\right),$$
$$P_3 = \left(\frac{\tau_1}{4} + \frac{1}{2}, \frac{\tau_2}{4}, \frac{\tau_3}{4} + \frac{1}{2}\right), \ P_4 = \left(\frac{\tau_1}{4}, \frac{\tau_2}{4} + \frac{1}{2}, \frac{\tau_3}{4} + \frac{1}{2}\right).$$

Hence we have:

$$\hat{\lambda}_{P_1} = 0, \quad \hat{\lambda}_{P_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{\lambda}_{P_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{\lambda}_{P_4} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

And the MAGMA script

```
CASE D
**********
GG4:=sub<H|H!(1,2)(11,12)(13,14)(17,18), H!(3,4)(9,10)(15,16),
H!(5,6)(13,14)(17,18),H!(7,8)(11,12)(17,18)>;

/*The only element of GG4 having fixed points is
(3,4)(9,10)(15,16).*/
```

```
Pi4:=Rewrite(P,GG4@@f);

Q4:=quo<Pi4| P.2*P.6*P.10, P.2*P.6*P.10*(P.2 * P.1)^2*(P.6 * P.5)^2,
P.2*P.6*P.10*(P.2 * P.1)^2 *(P.10 * P.9)^2,
P.2*P.6*P.10*(P.6 * P.5)^2 *(P.10 * P.9)^2>;
IdentifyGroup(Q4);
<16, 12>
```

gives $\pi_1(X_D) \cong \mathbb{H} \times \mathbb{Z}/2\mathbb{Z}$.

E) Here we have $g_0 = (-z_1 + \frac{1}{2} + \frac{\tau_1}{2}, -z_2 + \frac{\tau_2}{2}, -z_3 + \frac{1}{2} + \frac{\tau_3}{2})$. The 64 fixed points of g_0 on T are:

$$z \in \left\{ \frac{1}{4} \begin{pmatrix} \pm(1+\tau_1) \\ \pm\tau_2 \\ \pm(1+\tau_3) \end{pmatrix} + \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^3 \right\}.$$

Observe now that

$$\mathcal{L}_i \left(\frac{1}{4} + \frac{\tau_i}{4} \right)^2 = \mathcal{L}_i \left(\frac{1}{4} + \frac{\tau_i}{4} + \frac{1}{2} \right)^2 = -a_i,$$

whence $\left\{ \mathcal{L}_i \left(\frac{1}{4} + \frac{\tau_i}{4} \right), \mathcal{L}_i \left(\frac{1}{4} + \frac{\tau_i}{4} + \frac{1}{2} \right) \right\} = \left\{ \sqrt{-1} b_i, -\sqrt{-1} b_i \right\}$.

Then we see that the points

$$z \in \left\{ \frac{1}{4} \begin{pmatrix} \pm(1+\tau_1) \\ \pm\tau_2 \\ \pm(1+\tau_3) \end{pmatrix} + \frac{1}{2} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right\}$$

lie on $\hat{X}$, whereas the other 32 points satisfy the equation $\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = -b_1 b_2 b_3$.

We again can choose as representatives of the four G_E-orbits the following points:

$$P_1 = \frac{1}{4} \begin{pmatrix} (1+\tau_1) \\ \tau_2 \\ (1+\tau_3) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad P_2 = \frac{1}{4} \begin{pmatrix} (1+\tau_1) \\ \tau_2 \\ (1+\tau_3) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$P_3 = \frac{1}{4} \begin{pmatrix} (1+\tau_1) \\ \tau_2 \\ (1+\tau_3) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_4 = \frac{1}{4} \begin{pmatrix} (1+\tau_1) \\ \tau_2 \\ (1+\tau_3) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

whence we have

$$\hat{\lambda}_{P_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{\lambda}_{P_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \hat{\lambda}_{P_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\lambda}_{P_4} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

And the MAGMA script

```
CASE E
***********
GG5:=sub<H|H!(1,2)(11,12)(13,14)(17,18),
H!(3,4)(9,10)(11,12)(13,14)(15,16)(17,18),
H!(5,6)(13,14)(17,18),H!(7,8)(11,12)(17,18)>;

/*The only element of GG5 having fixed points is
(1, 2)(3, 4)(5, 6)(9, 10)(13, 14)(15, 16)(17, 18).*/
```

```
Pi5:=Rewrite(P,GG5@@f);
Q5:=quo<Pi5| P.1*P.2*P.3*P.6*P.9*P.10*P.11*(P.10*P.9)^2,
P.1*P.2*P.3*P.6*P.9*P.10*P.11*(P.2 * P.1)^2*(P.6 * P.5)^2*(P.10*P.9)^2,
P.1*P.2*P.3*P.6*P.9*P.10*P.11*(P.2 * P.1)^2 ,
P.1*P.2*P.3*P.6*P.9*P.10*P.11*(P.6 * P.5)^2>;
IdentifyGroup(Q5);
<16, 13>
```

gives $\pi_1(X_D) \cong SmallGroup(16, 13)$. □

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