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A characterization of varieties whose universal cover is a bounded symmetric domain without ball factors ☆

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We give two characterizations of varieties whose universal cover is a bounded symmetric domain without ball factors in terms of the existence of a holomorphic endomorphism σ of the tensor product $T \otimes T'$ of the tangent bundle T with the cotangent bundle T'. To such a curvature type tensor σ one associates the first Mok characteristic cone $\mathcal{CS}$, obtained by projecting on T the intersection of $ker(\sigma)$ with the space of rank 1 tensors. The simpler characterization requires that the projective scheme associated to $\mathcal{CS}$ be a finite union of projective varieties of given dimensions and codimensions in their linear spans which must be skew and generate.

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Bounded symmetric domains
Ample canonical bundle

1. Introduction

A central problem in the theory of complex manifolds is the one of determining the compact complex manifolds X whose universal covering $\tilde{X}$ is biholomorphic to a bounded domain $\Omega \subset \mathbb{C}^n$.

A first important restriction is given by theorems by Siegel and Kodaira [14,19] extending to several variables a result of Poincaré, and asserting that necessarily such a manifold X is projective and has ample canonical divisor K_X.

A restriction on Ω is given by another theorem of Siegel ([18], cf. also [10]) asserting that Ω must be holomorphically convex.

The question concerning which domains occur was partly answered by Borel [4] who showed that, given a bounded symmetric domain $\Omega \subset \mathbb{C}^n$, there exists a properly discontinuous group $\Gamma \subset \mathrm{Aut}(\Omega)$ which acts freely on Ω and is cocompact (i.e., is such that $X =: \Omega/\Gamma$ is a compact complex manifold with universal cover $\cong \Omega$).

We consider the following question: given a bounded domain $\Omega \subset \mathbb{C}^n$, how can we tell when a projective manifold X with ample canonical divisor K_X has Ω as universal covering?

The question was solved by Yau [21] in the case of a ball, using the theorem of Aubin and Yau (see [22,1]) asserting the existence of Kähler–Einstein metrics for varieties with ample canonical bundle. The existence of such metrics, joint to some deep knowledge of the differential geometry of bounded symmetric domains, allows to obtain more general results (see also [3,8,12] as general references).

Together with Franciosi [5] we took up the question for the case of a polydisk, and a fully satisfactory answer was found in [6] for the special case where the bounded symmetric domain has all factors of tube type, i.e., the domain is biholomorphic, via the Cayley transform, to some **tube domain**

$$\Omega = V + i\mathfrak{C},$$

where V is a real vector space and $\mathfrak{C} \subset V$ is an open self-dual cone containing no lines.

The main results in the tube case are as follows:

Theorem 1.1. *(See [6].) Let X be a compact complex manifold of dimension n with K_X ample.*

Then the following two conditions (1) *and* (1′)*, resp.* (2) *and* (2′) *are equivalent:*

(1) *X admits a slope zero tensor $0 \neq \psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$ (for some positive integer m);*

(1′) *$X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of* $\mathrm{Aut}(\Omega)$ *acting freely.*

(2) *X admits a semi-special tensor $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$, where η is a 2-torsion invertible sheaf, such that there is a point $p \in X$ for which the corresponding hypersurface $F_p =: \{\phi_p = 0\} \subset \mathbb{P}(TX_p)$ is reduced.*

(2′) *The universal cover of X is a polydisk.*

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\tilde{X} = \Omega$.

The main purpose of the present paper is to extend these results to the more general case of locally symmetric varieties X whose universal cover is a bounded symmetric domain without irreducible factors which are isomorphic to a ball of dimension at least two.

In the case where there are no ball factors, we get the following result, using the concept of an algebraic curvature type tensor σ.

Theorem 1.2. *Let X be a compact complex manifold of dimension n with K_X ample.*

Then the universal covering $\tilde{X}$ is a bounded symmetric domain without factors isomorphic to higher dimensional balls if and only if there is a holomorphic tensor $\sigma \in H^0(End(T_X \otimes T_X^\vee))$ enjoying the following properties:

1) there is a point $p \in X$, and a splitting of the tangent space $T = T_{X,p}$

$$T = T'_1 \oplus \cdots \oplus T'_m$$

such that the first Mok characteristic cone $\mathcal{CS}$ of σ is $\neq T$ and moreover $\mathcal{CS}$ splits into m irreducible components $\mathcal{CS}'(j)$ with

2) $\mathcal{CS}'(j) = T'_1 \times \cdots \times \mathcal{CS}'_j \times \cdots \times T'_m$

3) $\mathcal{CS}'_j \subset T'_j$ is the cone over a smooth non-degenerate (that is, the cone $\mathcal{CS}'_j$ spans the vector space T'_j) projective variety $\mathcal{S}'_j$ unless $\mathcal{CS}'_j = 0$ and $\dim(T'_j) = 1$.

Moreover, we can recover the universal covering of $\tilde{X}$ from the sequence of pairs $(\dim(\mathcal{CS}'_j), \dim(T'_j))$.

As we shall recall later, the first Mok characteristic cone $\mathcal{CS} \subset T_X$ is defined as the (closure of the) projection on the first factor of the intersection of $ker(\sigma)$ with the cone of rank 1 tensors:

$$ker(\sigma) \cap \{t \otimes t^\vee \in (T_X \otimes T_X^\vee)\}.$$

The above result can be simplified if we restrict to locally symmetric varieties X whose universal covering $\tilde{X}$ is a bounded symmetric domain without factors of rank one.

Theorem 1.3. *Let X be a compact complex manifold of dimension n with K_X ample.*

Then the universal covering $\tilde{X}$ is a bounded symmetric domain without factors of rank one if and only if there is $p \in X$ such that, setting $T = T_{X,p}$, *A) there is a holomorphic tensor* $\sigma \in H^0(End(T_X \otimes T_X^\vee))$ *such that the first Mok characteristic variety* $\mathcal{S}^1 \subset \mathbb{P}(T)$ *is* $\neq \mathbb{P}(T)$ *and moreover* $\mathcal{S}^1$ *is the disjoint union of smooth projective varieties* $\mathcal{S}'_j$ *whose projective spans are projectively independent and generate* $\mathbb{P}(T)$*.*

In other words, iff

A1) there is a point $p \in X$ *and a splitting of the tangent space* $T = T_{X,p}$

$$T = T'_1 \oplus \cdots \oplus T'_m$$

such that the first Mok characteristic cone $\mathcal{CS}$ *of* σ *is* $\neq T$ *and moreover* $\mathcal{CS}$ *splits into* m *irreducible components* $\mathcal{CS}'(j)$ *with*

A2) $\mathcal{CS}'(j) \subset T'_j$ *and* $\mathcal{CS}'(j)$ *generates* T'_j

A3) the projective variety $\mathcal{S}'_j := \mathbb{P}(\mathcal{CS}'_j)$ *is smooth (and non-degenerate, as required in A2)).*

Moreover, we can recover the universal covering of $\tilde{X}$ *from the sequence of pairs* $(\dim(\mathcal{CS}'_j), \dim(T'_j))$*.*

The above characterizations are important in order to obtain a more precise formulation of a result of Kazhdan [9].

Corollary 1.4. *Assume that* X *is a projective manifold with* K_X *ample, and that the universal covering* $\tilde{X}$ *is a bounded symmetric domain without irreducible factors which are higher dimensional balls.*

Let $\tau \in \mathrm{Aut}(\mathbb{C})$ *be an automorphism of* $\mathbb{C}$*.*

Then the conjugate variety X^τ *has universal covering* $\tilde{X}^\tau \cong \tilde{X}$*.*

It is worthwhile observing that balls of dimension higher than one are taken care of, once one allows a finite unramified covering, by the Yau inequality for summands of the tangent bundle; hence one can combine the present results with those of [23] and [20], and obtain full results for the general case where $\tilde{X}$ is any bounded symmetric domain.

A couple of words about the strategy of the proof:

(1) knowing that K_X is ample, we have a Kähler–Einstein metric h, and we consider the Levi-Civita connection
(2) parallel transport defines then the restricted holonomy group H (the connected component of the identity in the holonomy group)
(3) by the theorems of De Rham and Berger (see [2] and also [17]), the universal covering $\tilde{X}$ of X splits as a product $\tilde{X} = D_1 \times D_2$ where
(4) D_1 is a bounded symmetric domain without factors of ball type and D_2 is the product of the irreducible factors of dimension $\geqslant 2$ for which the holonomy group is the unitary group;
(5) we observe that by the Bochner principle [11] the tensor σ is parallel, hence if we restrict the tensor at any point $p \in X$ we observe that σ_p is H-invariant.

(6) We decompose the holonomy group as a product and accordingly the vector space T, the tangent space to X at p.
(7) We use elementary representation theory to derive some restrictions which the tensor σ must satisfy (this is done in Section 3).
(8) We associate to any such tensor σ its first Mok characteristic variety (see Section 2), having in mind two standard possibilities constructed using the algebraic curvature tensors of irreducible bounded domains defined by Kobayashi and Ochiai in [13].
(9) The rest of the proof (Sections 4 and 5) is projective geometry, using Mok's description [16,15] of the orbits on $T_i \otimes T_i^\vee$ of the complexified holonomy group of an irreducible bounded domain.

2. Algebraic curvature-type tensors and their first Mok characteristic varieties

In this section we consider the following situation. We are given a direct sum

$$T = T_1 \oplus \cdots \oplus T_k$$

of irreducible representations T_i of a group H_i (the unusual notation is due to the fact that T in the application shall be the tangent space to a projective manifold at one point, and $H = H_1 \times \cdots \times H_k$ shall be the restricted holonomy group).

Definition 2.1. 1) An algebraic curvature-type tensor is a nonzero element

$$\sigma \in End\big(T \otimes T^\vee\big).$$

2) Its first Mok characteristic cone $\mathcal{CS} \subset T$ is defined as the projection on the first factor of the intersection of $ker(\sigma)$ with the set of rank 1 tensors, plus the origin:

$$\mathcal{CS} := \big\{t \in T \mid \exists t^\vee \in T^\vee \setminus \{0\},\ \big(t \otimes t^\vee\big) \in ker(\sigma)\big\}.$$

3) Its **first Mok characteristic variety** is the subset $\mathcal{S} := \mathbb{P}(\mathcal{CS}) \subset \mathbb{P}(T)$.

4) More generally, for each integer h, consider

$$\big\{A \in T \otimes T^\vee \mid A \in ker(\sigma),\ \mathrm{Rank}(A) \leqslant h\big\},$$

and consider the algebraic cone which is its projection on the first factor

$$\mathcal{CS}^h := \big\{t \in T \mid \exists A \in ker(\sigma),\ \mathrm{Rank}(A) \leqslant h,\ \exists t' \in T\colon\ t = At'\big\},$$

and define then $\mathcal{S}^h := \mathbb{P}(\mathcal{CS}^h) \subset \mathbb{P}(T)$ to be the **h-th Mok characteristic variety.**

5) We define then the **full characteristic sequence** as the sequence

$$\mathcal{S} = \mathcal{S}^1 \subset \mathcal{S}^2 \subset \cdots \subset \mathcal{S}^{k-1} \subset \mathcal{S}^k = \mathbb{P}(T).$$

Remark 2.2.

(1) In the case where σ is the curvature tensor of an irreducible symmetric bounded domain $\mathcal{D}$, Mok [16] proved that the difference sets $\mathcal{S}^h \setminus \mathcal{S}^{h-1}$ are exactly all the orbits of the parabolic subgroup P associated to the compact dual $\mathcal{D}^\vee = G/P$. In particular, the algebraic cone $\mathcal{CS}^h$ is irreducible and H_i-invariant.
(2) More generally, if $\mathcal{S}$ is an H_i-invariant algebraic cone, and H_i is the holonomy group of an irreducible symmetric bounded domain, then necessarily $\mathcal{S}$ is irreducible and indeed equal to one of the $\mathcal{S}^h$.
(3) In the case instead where H_i acts as the full unitary group on T_i, then any H_i-invariant algebraic cone in T_i is trivial, that is, either equal to T_i or just equal to $\{0\}$, where $0 \in T_i$ is the origin.

Lemma 2.3. *If σ is the curvature tensor of an irreducible symmetric bounded domain $\mathcal{D}$, $\mathcal{S}^h$ is smooth if and only if $h = 1$.*

Proof. That $\mathcal{S}^1$ is smooth follows form the above remark, since $\mathcal{S}^1$ is a single orbit.

Conversely, observe that we have a sequence of inclusions for the Mok characteristic varieties:

$$\mathcal{S} = \mathcal{S}^1 \subset \mathcal{S}^2 \subset \cdots \subset \mathcal{S}^{k-1} \subset \mathcal{S}^k = \mathbb{P}(T).$$

Let then $P \in \mathcal{S}^1$, and let G be the stabilizer of P. The tangent space V of $\mathbb{P}(T)$ at P is a G-representation, and the Zariski tangent spaces to $\mathcal{S}^h$ yield a flag of G-invariant subspaces of V,

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V.$$

Our assertion follows then from the claim that V_1 and V are the unique invariant subspaces; since then, for $k > h > 1$, we have $V_h \neq V_1 \Rightarrow V_j = V$ therefore $\mathcal{S}^h$ is singular at every point of $\mathcal{S}^1$.

Let us prove the claim.

The first characteristic variety $\mathcal{S}^1$ is homogeneous for the compact subgroup K which stabilizes the origin of the bounded domain $\mathcal{D}$. The stabilizer K_P, as proven in Theorem 2.3 and Proposition 2.4 at page 5 of [7], acts irreducibly on the normal space to $\mathcal{S}^1$ at P. Therefore V splits as $V_1 \oplus N_P$, where N_P is K_P-irreducible. Now, G contains K_P, hence if W is G-invariant and strictly contains V_1, then W is also K_P-invariant and $W = V$. □

One geometric situation we have particularly in mind is the one where

$$\sigma = \bigoplus_{i=1}^{k} \sigma_i \in \bigoplus_{i=1}^{k} \bigl(End\bigl(T_i \otimes T_i^\vee\bigr)\bigr) \subset End\bigl(T \otimes T^\vee\bigr). \qquad (**)$$

In this case it is clear that

$$ker(\sigma) = \bigoplus_{i=1}^{k} ker(\sigma_i) \oplus \left(\bigoplus_{i \neq j} (T_i \otimes T_j^\vee) \right),$$

and if we intersect the Kernel of σ with the set of rank 1 tensors, we obtain

$$ker(\sigma) \cap \{t \otimes t^\vee\} = \left\{ t \otimes t^\vee \;\middle|\; t = \sum_{i=1}^{k} t_i,\ t^\vee = \sum_{j=1}^{k} t_j^\vee,\ \forall i\ (t_i \otimes t_i^\vee) \in ker(\sigma_i) \right\}.$$

Defining now

$$\mathcal{CS}_i := \{t_i \in T_i \mid \exists t_i^\vee \in T_i^\vee \setminus \{0\},\ (t_i \otimes t_i^\vee) \in ker(\sigma_i)\},$$

and similarly $\mathcal{S}_i := \mathbb{P}(\mathcal{CS}_i) \subset \mathbb{P}(T_i)$ we see therefore that under hypothesis (∗∗) we have an inclusion $\bigoplus_{i=1}^{k} \mathcal{CS}_i \subset \mathcal{CS}$.

But indeed, since

$$\mathcal{CS} = \left\{ t = \sum_j t_j \;\middle|\; \exists t^\vee \neq 0,\ t_i \otimes t_i^\vee \in ker(\sigma_i) \forall i \right\}$$
$$= \left\{ t = \sum_j t_j \;\middle|\; \exists t_i^\vee \neq 0,\ t_i \otimes t_i^\vee \in ker(\sigma_i) \right\},$$

we have that

$$\mathcal{CS} = \bigcup_{i=1}^{k} T_1 \oplus T_2 \oplus \cdots \oplus T_{i-1} \oplus \mathcal{CS}_i \oplus T_{i+1} \oplus \cdots \oplus T_k.$$

The above formula yields a decomposition of the Zariski closed projective set $\mathcal{S}$ as the union of the Zariski closed projective sets

$$\mathcal{S}(i) := \mathbb{P}(T_1 \oplus T_2 \oplus \cdots \oplus T_{i-1} \oplus \mathcal{CS}_i \oplus T_{i+1} \oplus \cdots \oplus T_k).$$

The latter sets are the join of the linear subspace

$$\mathbb{P}(T_{\hat{i}}) := \mathbb{P}(T_1 \oplus T_2 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus \cdots \oplus T_k)$$

with $\mathcal{S}_i$.

Remark 2.4. The next question is: when is the above an irredundant decomposition?

It is a necessary condition that each $\mathcal{CS}_i \neq T_i$ (i.e., $\mathcal{S}_i \neq \mathbb{P}(T_i)$), otherwise $\mathcal{CS}(i) = T$.

This condition is also sufficient. In fact, the irreducible components of $\mathcal{S}(i)$ are the joins of $\mathbb{P}(T_{\hat{i}})$ with the irreducible components of $\mathcal{S}_i$, hence for each component of $\mathcal{S}(i)$

the projection onto $\mathbb{P}(T_j)$ is surjective whenever $j \neq i$: therefore this component cannot be contained in any $\mathcal{S}(j)$ when $j \neq i$.

Remark 2.5. At the other extreme, if $\sigma_i = Id_{T_i}$, or σ_i is invertible, then $ker(\sigma_i) = 0$, hence in this case $\mathcal{S}(i) = \mathbb{P}(T_{\hat{i}})$.

More generally, it can happen that $\mathcal{S}(i)$ is a linear subspace, iff $\mathcal{CS}_i$ is a linear subspace. In the sequel, we shall assume that each σ_i is H_i-invariant: hence $\mathcal{CS}_i$ shall be an invariant subspace of T_i: by the irreducibility of T_i, the only possibility is either that $\mathcal{CS}_i = 0$, or that $\mathcal{CS}_i = T_i$.

We can avoid both possibilities by requiring (the second case should only occur for factors T_i of dimension $= 1$)

(1) $\mathcal{S} \neq \mathbb{P}(T)$
(2) if $\mathcal{S}$ has a component $\mathcal{S}^0$ which is a linear subspace, then $\mathcal{S}^0$ must be a hyperplane.

3. Holonomy invariant curvature-type tensors

In this section we continue our consideration of a curvature-type tensor σ assuming that it is invariant by the natural action of the group $H = H_1 \times \cdots \times H_k$.

We can naturally write σ as a direct sum $\sigma = \bigoplus_{(i,j),\,(h,k)} \sigma_{(i,j),(h,k)}$,

$$\sigma_{(i,j),(h,k)} : T_i \otimes T_j^\vee \to T_h \otimes T_k^\vee.$$

Lemma 3.1. *$\sigma_{(i,j),(h,k)} = 0$ if $i \neq h$ or $j \neq k$, while for $i \neq j$*

$$\sigma_{(i,j)} := \sigma_{(i,j),\,(i,j)} : T_i \otimes T_j^\vee \to T_i \otimes T_j^\vee$$

is a multiple of the identity.

Proof. The second assertion is a consequence of Schur's lemma once we show that $T_i \otimes T_j^\vee$ is, for $i \neq j$, an irreducible representation of the compact group $H_i \times H_j$.

We use moreover that T_i is an irreducible representation of H_i. H_i being compact, if χ_i is the character of the representation T_i, and $d\mu_i$ is the Haar measure of H_i, we have that irreducibility is equivalent to

$$\int_{H_i} \chi_i \overline{\chi_i}\, d\mu_i = 1$$

Since the character $\chi_{i,j}$ of $T_i \otimes T_j^\vee$ on $H_i \times H_j$ is

$$\chi_{i,j}(x, y) := \chi_i(x)\overline{\chi_j(y)}$$

and

$$\int_{H_i\times H_j} \left|\chi_{i,j}(x,y)\right|^2 d\mu_{i,j} = [\text{by Fubini}] = \int_{H_i} \chi_i\overline{\chi_i}\, d\mu_i \cdot \int_{H_j} \overline{\chi_j}\chi_j\, d\mu_j = 1$$

we conclude that $T_i \otimes T_j^\vee$ is, for $i \neq j$, an irreducible representation of $H_i \times H_j$.

For the first assertion, assume now that $i \neq h$ and let

$$\sigma' := \sigma_{(i,j),\,(h,k)} : T_i \otimes T_j^\vee \to T_h \otimes T_k^\vee.$$

Since σ is H-invariant, σ' is $H_i \times H_h$-invariant. By what we have seen $Hom(T_i, T_h) \cong T_h \otimes T_i^\vee$ is an irreducible nontrivial representation of $H_i \times H_h$, hence there are no $H_i \times H_h$ invariant homomorphisms in $Hom(T_i, T_h)$. A fortiori $\sigma' = 0$.

A completely analogous argument yields $\sigma' = 0$ if $j \neq k$. □

Using the previous lemma we consider the first characteristic variety in the case where σ is H-invariant.

In this case it is clear that

$$ker(\sigma) = \bigoplus_{i=1}^{k} ker(\sigma_i) \oplus \left(\bigoplus_{i\neq j,\, \sigma_{i,j}=0} \left(T_i \otimes T_j^\vee\right)\right),$$

and if we intersect the Kernel of σ with the set of rank 1 tensors, we obtain

$$ker(\sigma) \cap \left\{t \otimes t^\vee\right\} = \left\{ t \otimes t^\vee \;\middle|\; t = \sum_{i=1}^{k} t_i,\ t^\vee = \sum_{j=1}^{k} t_j^\vee,\ \forall i\ \left(t_i \otimes t_i^\vee\right) \in ker(\sigma_i), \right.$$
$$\left. t_i \otimes t_j^\vee = 0\ \forall i \neq j \text{ s.t. } \sigma_{i,j} \neq 0 \right\}.$$

Its projection on the first factor is the set

$$\mathcal{CS} = \bigcup_j \mathcal{CS}(j) := \bigcup_j \left\{ t = \sum_{i=1}^{k} t_i \in T \;\middle|\; \exists t_j^\vee \in T_j^\vee \setminus \{0\}, \right.$$
$$\left. \left(t_j \otimes t_j^\vee\right) \in ker(\sigma_j),\ t_i = 0\ \forall i \neq j \text{ s.t. } \sigma_{i,j} \neq 0 \right\},$$

and we have a corresponding set $\mathcal{S}(j) := \mathbb{P}(\mathcal{CS}(j)) \subset \mathbb{P}(T)$.

We can also write

$$\mathcal{CS}(j) = \hat{T}_{1,j} \oplus \hat{T}_{2,j} \oplus \cdots \oplus \hat{T}_{j-1,j} \oplus \mathcal{CS}_j \oplus \hat{T}_{j+1,j} \oplus \cdots \oplus \hat{T}_{k,j},$$

where $\hat{T}_{i,j} = T_i$ if $\sigma_{i,j} = 0$, $\hat{T}_{i,j} = 0$ if $\sigma_{i,j} \neq 0$. In other words, if one forgets about the ordering,

$$\mathcal{CS}(j) = \mathcal{CS}_j \oplus \Big(\bigoplus_{\sigma_{i,j}=0} T_i \Big).$$

Our first remark is that, in case where there exists a $\sigma_{i,j} \neq 0$ with $i \neq j$, then necessarily $\mathcal{S}(j)$ is contained in a hyperplane.

Our second observation is that however in this case the decomposition $\mathcal{CS} = \bigcup_j \mathcal{CS}(j)$ does not need to be irredundant, as one sees already in the case $k = 2$, $\sigma_{1,2} \neq 0$, $\sigma_{2,1} = 0$.

We end this section with an important, even if trivial, remark.

Remark 3.2. If V is an H-invariant linear subspace of T, then there is a subset $\mathcal{I} \subset \{1, \dots, k\}$ such that $V = \bigoplus_{i \in \mathcal{I}} T_i$.

4. Proof of Theorem 1.2

One implication follows right away from Section 3, since we may take $\sigma := \bigoplus_{i=1}^k \sigma_i$, letting σ_i be the algebraic curvature tensor of a bounded symmetric domain of rank greater than one (cf. [13], Lemma 2.9), and the identity of $T_i \otimes T_i^\vee$ for the factors of dimension equal to one. Then

$$\mathcal{CS} = \bigcup_{i=1}^{k} (T_1 \oplus T_2 \oplus \dots \oplus T_{i-1} \oplus \mathcal{CS}_i \oplus T_{i+1} \oplus \dots \oplus T_k).$$

The converse implication follows since K_X is ample, hence we may consider the Kähler–Einstein metric of X, for which the tensor σ is parallel, as proven by Kobayashi in [11] (since it is a tensor of covariant type two and contravariant type also two).

Hence the (irredundant) irreducible decomposition

$$\mathcal{CS} = \bigcup_{j=1}^{m} \mathcal{CS}'(j) = \bigcup_{j=1}^{m} (T'_1 \times \dots \times \mathcal{CS}'_j \times \dots \times T'_m)$$

is invariant under the holonomy group H.

Observe that, since K_X is ample, all the irreducible holonomy factors H_i are either equal to $U(T_i)$, or H_i acts on T_i as the holonomy of a bounded symmetric domain. This implies that if $\mathcal{CS}_j \subset T_j$ is a proper H_j-invariant subset, then $\mathcal{CS}_j$ is (see Remark 2.2, and also [16], and also [15], page 252) a Mok characteristic variety and T_j is a bounded symmetric domain factor.

The holonomy invariance implies, as shown in Section 4, that there is another (possibly redundant) irreducible (by what we have just observed) decomposition

$$\mathcal{CS} = \bigcup_{i=1}^{k} \mathcal{CS}(i) = \bigcup_{i=1}^{k} \Big(\mathcal{CS}_i \oplus \Big(\bigoplus_{\sigma_{j,i}=0} T_j \Big) \Big).$$

It follows immediately that $k \geqslant m$.

On the other hand, by our assumption and by Lemma 7.1, the linear subspace

$$\tilde{T'}_j := \left(T'_1 \oplus \cdots \oplus T'_{j-1} \oplus T'_{j+1} \oplus \cdots \oplus T'_m\right)$$

is the maximal vector subspace V such that $V + \mathcal{CS}'(j) \subset \mathcal{CS}'(j)$, hence these linear subspaces are holonomy invariant, in particular their mutual intersections are holonomy invariant.

We conclude that each subspace T'_j is holonomy invariant. By Remark 3.2 each T'_j is a sum of a certain number of T_i's.

Comparing the two decompositions, it follows that each $\mathcal{CS}'(j)$ equals some $\mathcal{CS}(i)$, and the hypothesis that the linear span of $\mathcal{CS}'(j)$ equals T implies that

$$\mathcal{CS}'(j) = \mathcal{CS}(i) = \mathcal{CS}_i \oplus \left(\bigoplus_{j \neq i} T_j\right) =: \mathcal{CS}_i \oplus \tilde{T}_i.$$

Once more, by Lemma 7.1 $\tilde{T'}_j$ is the maximal linear subspace V such that $V + \mathcal{CS}'(j) \subset \mathcal{CS}'(j)$, and the above equality shows that this subspace contains $\tilde{T}_i$. Since all the subspaces T'_j yield a direct sum, and are holonomy invariant, it follows that $T'_j = T_i$, and $\mathcal{CS}'_j = \mathcal{CS}_i$.

Therefore we finally obtain that $m = k$ and that, when $\mathcal{CS}'_j = \mathcal{CS}_i \neq 0$, then $\mathcal{S}'_j = \mathcal{S}_i$ is a smooth projective variety.

Since the only smooth characteristic variety is the first Mok characteristic variety (as shown in Lemma 2.3), it follows that the cones $\mathcal{CS}_i$ are just the origin when $\dim(T_i) = 1$, or they are the cones over the first Mok characteristic variety.

To finish the proof, we must only show the following claim.

Claim. *The dimension and codimension of the first Mok characteristic variety determines the irreducible bounded symmetric domain $\mathcal{D}$ of rank $\geqslant 2$.*

Proof. This follows from the following table.

Let $\mathcal{D}$ be an irreducible Hermitian symmetric space of rank > 1.

The following table follows from Mok's enumeration of the characteristic variety $\mathcal{S}^1(\mathcal{D})$, see Mok's Book [15], page 250.

$\mathcal{D}$	$\dim(\mathcal{D})$	$\dim(\mathcal{S}^1(\mathcal{D}))$
$I_{p,q}$	pq	$p+q-2$
II_n	$\frac{n(n-1)}{2}$	$2(n-2)$
III_n	$\frac{n(n+1)}{2}$	$n-1$
IV_n	n	$n-2$
V	16	10
VI	27	16

Let $\eta : \mathcal{IHSS} \to \mathbb{N} \times \mathbb{N}$ be defined as

$$\eta(\mathcal{D}) := \left(\dim(\mathcal{D}), \dim\left(\mathcal{S}^1(\mathcal{D})\right)\right)$$

Fact: η is injective.

Proof. The proof is obtained by direct inspection of the above table. Indeed, it is not difficult to check that the pairs $(27, 16)$ and $(16, 10)$ come just from the domains of VI and V respectively. To show that the pair comings from domains IV_n come just from the domains of type IV_n it is necessary to recall the following isomorphisms:

$$IV_3 \cong III_2, \qquad IV_6 \cong II_4, \qquad IV_4 \cong I_{2,2}$$

□

5. Proof of Theorem 1.3

In one direction, if X is locally symmetric without factors of rank 1, consider the tensor σ such that σ_i is the curvature tensor for all i, and $\sigma_{i,j}$ is the identity on $T_i \otimes T_j^\vee$ $\forall i \neq j$.

We saw then in Section 4 that

$$\mathcal{CS}(j) = \mathcal{CS}_j \oplus (0)$$

and then $\mathcal{S}(j) \subset \mathbb{P}(T_j)$ is the first Mok characteristic variety, which is smooth, hence A1), A2) and A3) hold.

Conversely, all the components $\mathcal{CS}(j)$ of the cone $\mathcal{CS}$ contain no nontrivial vector subspace, since the cone over a projective variety is singular unless the variety is a linear subspace. Hence, by the observations made in Section 4 it follows that the holonomy invariant tensor σ is such that all the components $\sigma_{i,j}$ are a nonzero multiple of the identity on $T_i \otimes T_j^\vee$ $\forall i \neq j$.

Then $\mathcal{CS}(j) = \mathcal{CS}_j \oplus (0)$ and the projective variety $\mathcal{S}_j$ is smooth and holonomy invariant, therefore we conclude as for Theorem 1.2 that $\mathcal{S}_j$ is the first Mok characteristic variety, and that we recover the universal cover from the variety $\mathcal{S}$.

6. Proof of Kazhdan's type corollary

Consider the conjugate variety X^τ: since K_X is ample we may assume that X is projectively embedded by $H^0(X, \mathcal{O}_X(mK_X))$.

τ carries X to X^τ and K_X to K_{X^τ}, hence also X^τ has ample canonical divisor.

Moreover, τ carries the algebraic curvature type tensor σ to a similar tensor σ^τ. The equations of the Mok characteristic varieties are defined over $\mathbb{Z}$, hence we obtain that τ transforms each variety $\mathcal{S}^i(X, \sigma)$ into $\mathcal{S}^i(X^\tau, \sigma^\tau)$, in particular respecting their dimension and codimension.

We conclude then immediately by the last assertion of our main theorems that the universal covering of X^τ is $\tilde{X}$. □

7. Elementary lemmas

We collect here, for the readers' benefit, some trivial but important observations.

Lemma 7.1. *Let $\mathcal{S} \subset \mathbb{P}(V) = \mathbb{P}^n$ be a non-degenerate projective variety, $\mathcal{S} \neq \mathbb{P}(V)$, and consider the join $Z := \mathcal{S} * \mathbb{P}(W) \subset \mathbb{P}(V \oplus W) = \mathbb{P}^{n+m}$. Then Z is smooth $\Leftrightarrow$ $W = 0$ and $\mathcal{S}$ is smooth.*

Proof. Let I be the homogeneous ideal of $\mathcal{S}$. Since $\mathcal{S}$ is non-degenerate, each $0 \neq f \in I$ has degree $\geqslant 2$, and moreover I contains some nonzero polynomial (since $\mathcal{S} \neq \mathbb{P}(V)$).

We shall show that $\mathbb{P}(W) \subset Sing(Z)$, observing that $\mathbb{P}(W) \neq \emptyset$ unless $W = 0$, which is exactly what we have to proof.

Observe that

$$Z = \{(v, w) \mid f(v) = 0, \ \forall f \in I\}.$$

Hence

$$Sing(Z) = \left\{(v, w) \;\middle|\; \frac{\partial f}{\partial v_j}(v) = 0, \ \forall f \in I\right\}.$$

Since however $deg(f) \geqslant 2$, $\forall f \in I$, $f \neq 0$, $\frac{\partial f}{\partial v_j}(v)$ vanishes for $v = 0$, hence $\mathbb{P}(W) = \{(0, w) \mid w \in W\} \subset Sing(Z)$. □

In the next lemma we use our standard notation, introduced in Section 2.

Lemma 7.2. *Let $\mathcal{CS} \subset T_i$ be an H_i-invariant algebraic cone.*

Then there is no nontrivial linear subspace V_i such that $V_i + \mathcal{CS} \subset \mathcal{CS}$, unless $\mathcal{CS} = T_i$.

Proof. If V_i is nontrivial, then $W_i := \{v \mid v + \mathcal{CS} \subset \mathcal{CS}\}$ is a nontrivial linear subspace, which is H_i invariant. But T_i is an irreducible representation, hence if $W_i \neq \{0\}$, then $\mathcal{CS} = T_i$. □

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