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Unirationality of Ueno-Campana's threefold

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Abstract. We shall prove that the threefold studied in the paper " Remarks on an Example of K. Ueno" by F. Campana is unirational. This gives an affirmative answer to a question posed in the paper above and also in the book by K. Ueno, "Classification theory of algebraic varieties and compact complex spaces".

1. Introduction

Let k be any field of characteristic $\neq 2$ containing a primitive fourth root of unity $\sqrt{-1}$. We shall work over k unless otherwise stated. Let $[x : y : z]$ be the homogeneous coordinates of $\mathbb{P}^2$ and let

$$C := \left(y^2z = x(x^2 - z^2)\right) \subseteq \mathbb{P}^2$$

be the harmonic elliptic curve, having an automorphism g of order 4 defined by

$$g^*(x : y : z) = \left(-x : \sqrt{-1}y : z\right)$$

whose quotient is $\mathbb{P}^1$. When k is the complex number field $\mathbb{C}$, we have

$$(C, g) \simeq \left(E_{\sqrt{-1}}, \sqrt{-1}\right) ,$$

where $E_{\sqrt{-1}} = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, the elliptic curve of period $\sqrt{-1}$, and $\sqrt{-1}$ is the automorphism induced by multiplication by $\sqrt{-1}$ on $\mathbf{C}$. This is because the complex elliptic curve with an automorphism of order 4 acting on the space of global holomorphic 1-forms as $\sqrt{-1}$ is unique up to isomorphism.

Let (C_j, g_i) $(j = 1, 2, 3)$ be three copies of (C, g). Let

$$Z = C_1 \times C_2 \times C_3.$$

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For simplicity, we denote the automorphism of Z defined by (g_1, g_2, g_3) by the same letter g. Then g is an automorphism of Z of order 4 and the quotient threefold

$$Y := (C_1 \times C_2 \times C_3)/\langle g \rangle$$

has 8 singular points of type $(1, 1, 1)/4$ and 28 singular points of type $(1, 1, 1)/2$. Let X be the blow up of Y at the maximal ideals of these singular points. Then X is a smooth projective threefold defined over k. In his paper, Campana [1] proved that X is a rationally connected threefold when $k = \mathbb{C}$. We shall call X the *Ueno-Campana's threefold*.

In [1, Question 4], Campana asked whether X is rational or unirational (at least over $\mathbb{C}$). See also [4, Page 208] for this Question and [3] for a relevant example and applications to complex dynamics. The aim of this short note is to give an affirmative answer to this question:

Theorem 1.1. *Ueno-Campana's threefold X is unirational, i.e., there is a dominant rational map $\mathbb{P}^3 \cdots \to X$.*

We shall show that X is birationally equivalent to the Galois quotient of a conic bundle over $\mathbb{P}^2$ *with* a rational section, while X itself is birationally equivalent to a conic bundle over $\mathbb{P}^2$ *without* any rational section.

2. Proof of Theorem 1.1

The curves (C_i, g_i) $(i = 1, 2, 3)$ are birationally equivalent to (C_i^0, g_i), where C_i^0 is the curve in the affine space $\mathbb{A}^2 = \operatorname{Spec} k[X_i, Y_i]$, and g_i is the automorphism of C_i^0, defined by

$$Y_i^2 = X_i\left(X_i^2 - 1\right) \ , \ g_i^* Y_i = \sqrt{-1} Y_i \ , \ g_i^* X_i = -X_i .$$

The affine coordinate ring $k[C_i^0]$ of C_i^0 is

$$k\left[C_i^0\right] = k\left[X_i, Y_i\right] / \left(Y_i^2 - X_i\left(X_i^2 - 1\right)\right).$$

We set $x_i := X_i \bmod (Y_i^2 - X_i(X_i^2 - 1))$, $y_i := Y_i \bmod (Y_i^2 - X_i(X_i^2 - 1))$. We note that $y_i^2 = x_i(x_i^2 - 1)$, $g^* y_i = \sqrt{-1} y_i$, $g^* x_i = -x_i$ in $k[C_i^0]$.

Then $(Z = C_1 \times C_2 \times C_3, g = (g_1, g_2, g_3))$ is birationally equivalent to the affine threefold

$$V := C_1^0 \times C_2^0 \times C_3^0$$

with automorphism (g_1, g_2, g_3), which we denote by the same letter g, and with affine coordinate ring

$$k[V] = k[C_i^0] \otimes k[C_2^0] \otimes k[C_3^0] \text{ generated by } x_1, x_2, x_3, y_1, y_2, y_3.$$

The rational function field $k(Z)$ of Z is

$$k(Z) = k(V) = k(x_1, x_2, x_3, y_1, y_2, y_3).$$

In both $k[V]$ and $k(Z)$, we have

$$y_i^2 = x_i\left(x_i^2 - 1\right), \tag{2.1}$$

$$g^* y_i = \sqrt{-1} y_i \ , \ g^* x_i = -x_i . \tag{2.2}$$

Since X is birationally equivalent to $V/\langle g\rangle$, the rational function field $K(X)$ of X is identified with the invariant subfield $k(Z)^g$ of $k(Z)$, i.e.,

$$k(X) = k(Z)^g = \{f \in k(Z)\,|g^*f = f\}.$$

Consider the following elements in $k(Z)$:

$$b_2 := \frac{x_2}{x_1}, \quad b_3 := \frac{x_3}{x_1}, \quad a_2 := \frac{y_2}{y_1}, \quad a_3 := \frac{y_3}{y_1}, \tag{2.3}$$

$$u_1 := x_1^2, \quad w_1 := y_1^4, \quad \lambda_1 := x_1 y_1^2, \tag{2.4}$$

and define the subfield L of $k(Z)$ by

$$L := k(b_2, b_3, a_2, a_3, u_1, w_1, \lambda_1).$$

Here we used the fact that $x_1 \neq 0$, $y_1 \neq 0$ in $k(Z)$.

Lemma 2.1. *$k(X) = L$ in $k(Z)$.*

Proof. By (2.2) and (2.3), $b_2, b_3, a_2, a_3, u_1, w_1, \lambda_1$ are g-invariant. Hence

$$L \subseteq k(X) \subseteq k(Z). \tag{2.5}$$

Note that $k(Z) = L(y_1)$. This is because

$$x_1 = \frac{\lambda_1}{y_1^2}\ , \quad x_2 = b_2 x_1, x_3 = b_3 x_1\ , \quad y_2 = a_2 y_1\ , \quad y_3 = a_3 y_1,$$

by (2.3) and (2.4). Since $y_1^4 = w_1$ and $w_1 \in k(Z)$, it follows that

$$[k(Z) : L] \leq 4, \tag{2.6}$$

where $[k(Z) : L]$ is the degree of the field extension $L \subseteq k(Z)$, i.e., the dimension of $k(Z)$ being naturally regarded as the vector space over L.

On the other hand, the group $\langle g\rangle = \mathrm{Gal}(k(Z)/k(X))$ is of order 4. Thus, by the fundamental theorem of Galois theory, we have that

$$[k(Z) : k(X)] = [K(Z) : k(Z)^g] = \text{ ord}\,(g) = 4. \tag{2.7}$$

The result now follows from (2.5), (2.6), (2.7). Indeed, by (2.5), we have

$$[k(Z) : L] = [k(Z) : k(X)][k(X) : L].$$

On the other hand, $[k(Z) : L] \leq 4$ by (2.6), and $[k(X) : L] \geq 1$. Hence $[k(X) : L] = 1$ by (2.7). This means that $L = k(X)$ in $k(Z)$, as claimed. □

Lemma 2.2. *$L = k(u_1, b_2, b_3, a_2, a_3)$ in $k(Z)$.*

Proof. Since $u_1, b_2, b_3, a_2, a_3 \in L$, it follows that $k(u_1, b_2, b_3, a_2, a_3) \subseteq L$. Let us show that $L \subseteq k(u_1, b_2, b_3, a_2, a_3)$. For this, it suffices to show that $w_1, \lambda_1 \in k(u_1, b_2, b_3, a_2, a_3)$.

Recall that, by (2.1), $y_1^2 = x_1(x_1^2 - 1)$; hence, squaring both sides of the equality and using (2.6), we obtain that

$$w_1 = y_1^4 = x_1^2\left(x_1^2 - 1\right)^2 = u_1(u_1 - 1)^2. \tag{2.8}$$

Hence $w_1 \in k(u_1, b_2, b_3, a_2, a_3)$. From $y_1^2 = x_1(x_1^2 - 1)$ again, we have that

$$\lambda_1 = x_1 y_1^2 = x_1^2\left(x_1^2 - 1\right) = u_1(u_1 - 1). \tag{2.9}$$

Hence $\lambda_1 \in k(u_1, b_2, b_3, a_2, a_3)$ as well. □

Lemma 2.3. *Let $j = 2, 3$. Then, $a_j^2 - b_j \neq 0$ in both $k(Z)$ and $k(X)$.*

Proof. By using (2.1), we obtain that

$$a_j^2 - b_j = \frac{y_j^2}{y_1^2} - \frac{x_j}{x_1} = \frac{x_j\left(x_j^2 - 1\right)}{x_1\left(x_1^2 - 1\right)} - \frac{x_j}{x_1} = \frac{x_j}{x_1}\left(\frac{x_j^2 - 1}{x_1^2 - 1} - 1\right), \tag{2.10}$$

in $k(Z)$. Recall that $x_i \neq 0$ in $k(Z)$. Thus, if $a_j^2 - b_j = 0$ in $k(Z)$, then we would have $(x_j^2 - 1)/(x_1^2 - 1) = 1$ in $K(Z) = k(V)$ from the equality above, and therefore, $x_j = \pm x_1$ in $k[V]$. However, this contradicts the fact that x_1 is identically 0 on the set of $\overline{k}$-valued points $(\{0\} \times C_2 \times C_3)(\overline{k})$ but $\pm x_j$ $(j = 2, 3)$ are not identically 0 on it. This contradiction implies that $a_j^2 - b_j \neq 0$ in $k(Z)$. Since $a_j^2 - b_j \in k(Z)^g = k(X)$ and $k(X)$ is a subfield of $k(Z)$, it follows that $a_j^2 - b_j \neq 0$ in $k(X)$ as well. □

Proposition 2.4. *$k(X) = L = k(b_2, b_3, a_2, a_3)$ in $k(Z)$. More precisely, in $k(Z)$, we have*

$$u_1 = \frac{a_2^2 - b_2}{a_2^2 - b_2^3} = \frac{a_3^2 - b_3}{a_3^2 - b_3^3}. \tag{2.11}$$

Proof. By Lemmas 2.1 and 2.2, it suffices to show the equality (2.10) in $k(Z)$. Observe that, for $j = 2, 3$:

$$y_j^2 = x_j\left(x_j^2 - 1\right), \quad y_1^2 a_j^2 = x_1 b_j\left(x_1^2 b_j^2 - 1\right)$$

hence multiplication by x_1 yields

$$x_1^2 b_j\left(x_1^2 b_j^2 - 1\right) = x_1 y_1^2 a_j^2 = x_1^2\left(x_1^2 - 1\right) a_j^2,$$

and dividing by x_1^2 and observing that $u_1 = x_1^2$ we obtain

$$b_j\left(u_1 b_j^2 - 1\right) = (u_1 - 1)a_j^2$$

i.e.,

$$(**)\ u_1\left(a_j^2 - b_j^3\right) = a_j^2 - b_j.$$

Using the previous lemma and (**) we obtain $(a_j^2 - b_j^3) \neq 0$, so we can divide and obtain (2.11). □

Proposition 2.5. *X is birationally equivalent to the affine hypersurface H in $\mathbb{A}^4 =$ Spec $k[a, b, \alpha, \beta]$, defined by*

$$\left(a^2 - b\right)\left(\alpha^2 - \beta^3\right) = \left(\alpha^2 - \beta\right)\left(a^2 - b^3\right),$$

or equivalently defined by

$$a^2\beta\left(1 - \beta^2\right) = \alpha^2 b\left(1 - b^2\right) + b\beta\left(b^2 - \beta^2\right),$$

Proof. By Lemma (2.1) and Proposition (2.4), $k(X) = k(a_2, a_3, b_2, b_3)$ in $k(Z)$, with a relation

$$\left(a_2^2 - b_2\right)\left(a_3^2 - b_3^3\right) = \left(a_3^2 - b_3\right)\left(a_2^2 - b_2^3\right). \tag{2.12}$$

Expanding both sides and subtracting then the common term $a_2^2 a_3^2$, we obtain

$$-a_2^2 b_3^3 - b_2 a_3^2 + b_2 b_3^3 = -a_3^2 b_2^3 - b_3 a_2^2 + b_3 b_2^3.$$

Solving this relation in terms of a_2, we obtain that

$$a_2^2 b_3\left(1 - b_3^2\right) = a_3^2 b_2\left(1 - b_2^2\right) + b_2 b_3\left(b_2^2 - b_3^2\right). \tag{2.13}$$

Since $b_3 = x_3/x_1$ is not a constant in $k(Z)$, it follows that $b_3(1 - b_3^2) \neq 0$ in $k(Z)$, whence also not 0 in $k(X)$. Thus

$$a_2^2 = \frac{a_3^2 b_2\left(1 - b_2^2\right) + b_2 b_3\left(b_2^2 - b_3^2\right)}{b_3\left(1 - b_3^2\right)}. \tag{2.14}$$

Therefore a_2 is algebraic over $k(a_3, b_2, b_3)$ of degree at most 2. Since X is of dimension 3 over k, it follows that a_3, b_2, b_3 form a transcendence basis of $k(X)$ over k. Thus, the subring $k[a_3, b_2, b_3]$ of $k(X)$ is isomorphic to the polynomial ring over k of Krull-dimension 3. Moreover, the right hand side of 2.14 is not a square in $k(a_3, b_2, b_3)$. Indeed, the multiplicity of b_3 in the denominator is 1 while the numerator is not in k and the multiplicity of b_3 in the numerator is 0. Thus Eq. (2.14) is the minimal polynomial of a_2 over $k(a_3, b_2, b_3)$. Hence X is birationally equivalent to the double cover of $\mathbb{A}^3 = \operatorname{Spec} k[a_3, b_2, b_3]$, defined by (2.14). This means that X is birationally equivalent to the hypersurface in the affine space $\mathbb{A}^4 = \operatorname{Spec} k[a, \alpha, b, \beta]$, defined by (2.14) or, equivalently, defined by (2.13), or by (2.12), in which (a_2, a_3, b_2, b_3) are replaced by (a, α, b, β). □

Corollary 2.6. *Let $H \subseteq \mathbb{A}^4 = \operatorname{Spec} k[a, \alpha, b, \beta]$ be the same as in Proposition 2.5. Consider the affine plane $\mathbb{A}^2 = \operatorname{Spec} k[b, \beta]$ and the natural projection*

$$\pi : \mathbb{A}^4 \to \mathbb{A}^2$$

defined by

$$(a, b, \alpha, \beta) \mapsto (b, \beta).$$

Then the natural restriction map

$$p := \pi|H : H \to \mathbb{A}^2$$

is a conic bundle over $\mathbb{A}^2$. In particular, the graph Γ of the rational map $\tilde{p} : X \cdots \to \mathbb{P}^2$ naturally induced by p forms a conic bundle on Γ over $\mathbb{P}^2$. We note that Γ is projective and birationally equivalent to X over k.

Proof. The fibre $\pi^{-1}(\eta)$ of π over the generic point $\eta \in \mathbb{A}^2 = \operatorname{Spec} k[b, \beta]$ is the affine space $\mathbb{A}^2_\eta = \operatorname{Spec} k(b, \beta)[a, \alpha]$ defined over $\kappa(\eta) = k(b, \beta)$. Thus by the second equation in Proposition 2.5, the generic fibre $X_\eta := (\pi|H)^{-1}(\eta)$ is the conic in $\mathbb{A}^2_\eta$, defined by

$$a^2\beta\left(1 - \beta^2\right) = \alpha^2 b\left(1 - b^2\right) + b\beta\left(b^2 - \beta^2\right).$$

This implies the result. □

Remark 2.7. The conic X_η in the proof of Proposition 2.6 has no rational point over $\kappa(\eta) = k(b, \beta)$, i.e., the set $X_\eta(k(b, \beta))$ is empty.

Proof. Suppose to the contrary that $(a(b, \beta), \alpha(b, \beta)) \in X_\eta(k(b, \beta))$. We can write

$$a(b, \beta) = \frac{P(b, \beta)}{Q(b, \beta)}\ , \quad \alpha(b, \beta) = \frac{R(b, \beta)}{Q(b, \beta)},$$

where $P(b, \beta), Q(b, \beta), R(b, \beta) \in k[b, \beta]$ with no non-constant common factor, possibly after replacing the denominators by their product. Then substituting the above into the equation of X_η and clearing the denominator, we would have the following identity in $k[b, \beta]$:

$$P\,(b, \beta)^2\,\beta\left(1-\beta^2\right) = R(b, \beta)^2 b\left(1-b^2\right) + Q(b, \beta)^2 b\beta\left(b^2-\beta^2\right).$$

Since $k[b, \beta]$ is a polynomial ring, in particular, it is a UFD, it would follow that $P(b, \beta)$ is divisible by b and $R(b, \beta)$ is divisible by β in $k[b, \beta]$. Thus $P(b, \beta) = P_1(b, \beta)b$ and $R(b, \beta) = R_1(b, \beta)\beta$ for some $P_1(b, \beta), R_1(b, \beta) \in k[b, \beta]$. Substituting these two into the equality above and dividing by $b\beta \neq 0$, it follows that

$$P_1(b, \beta)^2 b\left(1-\beta^2\right) = R_1(b, \beta)^2\beta\left(1-b^2\right) + Q(b, \beta)^2\left(b^2-\beta^2\right).$$

Substitute $b = 0$ into this equation: we obtain $R_1(0, \beta)^2\beta + Q(0, \beta)^2\beta^2 = 0$, which, by the parity of the degree, implies that $R_1(0, \beta) = Q(0, \beta) = 0$. This means that both $R_1(b, \beta)$ and $Q(b, \beta)$ are divisible by b. Similarly, if we substitute $\beta = 0$ into the above equation we find that both $P_1(b, \beta)$ and $Q(b, \beta)$ are divisible by β. Thus we can write

$$P_1(b, \beta) = \beta P_2(b, \beta),\ \ R_1(b, \beta) = bR_2(b, \beta),\ \ Q(b, \beta) = b\beta Q_2(b, \beta),$$

where $P_2(b, \beta),\ R_2(b, \beta),\ Q(b, \beta) \in k[b, \beta]$. But this implies that all $P(b, \beta), Q(b, \beta), R(b, \beta)$ are divisible by $b\beta$, a contradiction. □

The next corollary completes the proof of Theorem (1.1):

Corollary 2.8. *Let* $H \subseteq \mathbb{A}^4 = \operatorname{Spec} k[a, \alpha, b, \beta]$*,* $p : H \to \mathbb{A}^2 = \operatorname{Spec} k[b, \beta]$ *be the same as in Proposition* 2.5 *and Corollary* 2.6*. Consider another affine space* $\operatorname{Spec} k[s, t]$ *and the (finite Galois) morphism of degree* 4

$$f : \operatorname{Spec} k[s, t] \to \operatorname{Spec} k[b, \beta]$$

defined by

$$f^*b = s^2, \quad f^*\beta = t^2.$$

Consider then the fibre product

$$Q := H \times_{\operatorname{Spec} k[b,\beta]} \operatorname{Spec} k[s, t]$$

and the natural second projection $p_2 : Q \to \operatorname{Spec} k[s, t]$*. Then* p_2 *is a conic bundle with a rational section and* Q *is a rational threefold. In particular,* H*, hence* X*, is unirational.*

Proof. Recall that H is the hypersurface in $\operatorname{Spec} k[a, b, \alpha, \beta]$ defined by

$$a^2\beta\left(1-\beta^2\right)=\alpha^2 b\left(1-b^2\right)+b\beta\left(b^2-\beta^2\right),$$

or equivalently by

$$\left(a^2-b\right)\left(\alpha^2-\beta^3\right)=\left(\alpha^2-\beta\right)\left(a^2-b^3\right).$$

Thus, by definition of the fibre product, Q is a hypersurface in the affine space $\mathbb{A}^4=\operatorname{Spec} k[a, \alpha, s, t]$, defined by

$$a^2t^2\left(1-t^4\right)=\alpha^2 s^2\left(1-s^4\right)+s^2t^2\left(s^4-t^4\right),$$

or equivalently by

$$\left(a^2-s^2\right)\left(\alpha^2-t^6\right)=\left(\alpha^2-t^2\right)\left(a^2-s^6\right).$$

Then the natural projection $p_2 : Q \to \operatorname{Spec} k[s, t]$ is a conic bundle with generic fibre

$$Q_{\eta'}=\left(a^2t^2(1-t^4\right)=\alpha^2 s^2\left(1-s^4\right)+s^2t^2\left(s^4-t^4\right)\right)\subseteq \operatorname{Spec} k(s,t)[a,\alpha]=\mathbb{A}^2_{\eta'},$$

where η' is the generic point of $\operatorname{Spec} k[s, t]$. Then $Q_{\eta'}$ has a rational point $(a, \alpha) = (s, t) \in Q(k(s, t))$ over $\kappa(\eta') = k(s, t)$. Hence $Q_{\eta'}$ is isomorphic to $\mathbb{P}^1_{\eta'}$ over $k(s, t)$. Thus, denoting the affine coordinate of $\mathbb{P}^1_{\eta'}$ by v, we obtain that

$$k(Q)=k(s,t)(Q_{\eta'})\simeq k(s,t)\left(\mathbb{P}^1_{\eta'}\right)=k(s,t)(v)=k(s,t,v).$$

Since Q is of dimension 3 over k, it follows that s, t, v are algebraically independent over k. Hence, $k(Q)$ is isomorphic to the rational function field of $\mathbb{P}^3$ over k. Hence Q is a rational threefold over k, i.e., birationally equivalent to $\mathbb{P}^3$ over k. Since the natural morphism $p_1 : Q \to H$, i.e., the first projection morphism in the fibre product, is a finite dominant morphism of degree 4, Q is birational to $\mathbb{P}^3$ and H is birationally equivalent to X, all over k, we obtain a rational dominant map $q : \mathbb{P}^3 \cdots \to X$ over k, from the natural projection $p_1 : Q \to H$. Hence X is unirational. □

Remark 2.9. (1) Colliot-Thélène finally proved in [2] that the hypersurface in Proposition (2.5) is rational, whence Ueno-Campana's threefold X is actually rational.

(2) Hence Ueno-Campana's threefold X provides the second explicit example of a complex smooth rational threefold admitting primitive automorphisms of positive entropy. Actually, automorphisms of $E^3_{\sqrt{-1}}$ of the same shape as those in Lemma 4.3 of [3] induce such automorphisms of X.

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