

Caustics of Plane Curves, Their Birationality and Matrix Projections

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Dedicated to Klaus Hulek on the occasion of his 60th birthday.

Abstract After recalling the notion of caustics of plane curves and their basic equations, we first show the birationality of the caustic map for a general source point S in the plane. Then we prove more generally a theorem for curves D in the projective space of 3×3 symmetric matrices B . For a general 3×1 vector S the projection to the plane given by $B \rightarrow BS$ is birational on D , unless D is not a line and D is contained in a plane of the form $\Delta_{v} := \{B \mid Bv = 0\}$.

1 Introduction and Setup

Given a plane curve C and a point S , a source of light (which could also lie at infinity, as the sun), the light rays L_P originating in S , and hitting the curve C in a point P , are reflected by the curve, and the caustic \mathcal{C} of C is the envelope of the family of reflected rays Λ_P .

Our first Theorem 3 says that the correspondence between the curve C and the caustic curve \mathcal{C} is birational, i.e., it is generically one to one, if the light source point S is chosen to be a general point.

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We learnt about this problem in [4], to which we refer for an account of the history of the theory of caustics and for references to the earlier works of von Tschirnhausen, Quetelet, Dandelin, Chasles, and more modern ones (as [2, 3]).

Our methods are from algebraic geometry, so we got interested in a generalization of this result, in which the special form of a certain curve D plays no role: we achieve this goal in Theorem 4.

Let us now describe the mathematical set up for the description of caustics.

Let $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}^2$ and let $C \subset \mathbb{P}^2$ be a plane irreducible algebraic curve, whose normalization shall be denoted by C' .

Choose an orthogonality structure in the plane, i.e. two points, called classically the cyclic points, and let \mathbb{P}_{∞}^1 be the line ('at infinity') joining them.

The two cyclic points determine a unique involution ι on \mathbb{P}_{∞}^1 for which the cyclic points are fixed, hence an involution, called orthogonality, on the pencils of lines passing through a given point of the affine plane $\mathbb{P}^2 \setminus \mathbb{P}_{\infty}^1$.

Without loss of generality, we choose appropriate projective coordinates such that

$$\iota : (x, y, 0) \mapsto (-y, x, 0), \text{Fix}(\iota) = \{(1, \pm\sqrt{-1}, 0)\}.$$

Let $S \in \mathbb{P}^2$ be a light source point, and to each point $P \in \mathbb{P}^2 \setminus \{S\}$ associate the line $L_P := \overline{PS}$. In the case where $P \in C$, we define Λ_P , the reflected light ray, as the element of the pencil of lines through P determined by the condition that the cross ratio

$$CR(N_P, T_P, L_P, \Lambda_P) = -1,$$

ensuring the existence of a symmetry with centre P leaving the tangent line T_P to C at P and the normal line $N_P := \iota(T_P)$ fixed, and exchanging the incoming light ray L_P with the reflected light ray Λ_P .

We thus obtain a rational map of the algebraic curve C to the dual projective plane:

$$\Lambda : C \dashrightarrow (\mathbb{P}^2)^{\vee}.$$

Definition 1. The Caustic \mathcal{C} of C is defined as the envelope of the family of lines $\{\Lambda_P\}$: in other words, setting $\Gamma := \Lambda(C)$, $\mathcal{C} = \Gamma^{\vee}$.

Remark 1. since the biduality map $\Gamma \dashrightarrow \Gamma^{\vee}$ is birational (cf. [7], pages 151–152), the map $C \dashrightarrow \mathcal{C}$ is birational iff $\Lambda : C \dashrightarrow \Gamma$ is birational. Moreover, by the biduality theorem, the class of the caustic \mathcal{C} is the degree of Γ , and the degree of \mathcal{C} is the class of Γ .

We shall quickly see in the next section the basic calculations which give the class of \mathcal{C} , i.e. the degree of Γ , in the case where C and S are general (more precise Plücker type formulae which show how the singularities of the curve C

and the special position of S make these numbers decrease are to be found in [4] and [5]).

In Sect. 3 we show our first result, that Λ is birational onto its image for general choice of the source point S , if C is not a line (in this case Γ is a line, and the caustic is a point). The next section recalls a well known lemma about lines contained in the determinantal variety Δ which is the secant variety of the Veronese surface V .

This lemma plays a crucial role in the proof of our main result, which says the following (see Theorem 4 for more details):

Theorem 1. *Let $D \subset \mathbb{P} := \mathbb{P}(\text{Sym}^2(\mathbb{C}^3))$ be a curve.*

Then, for general $S \in \mathbb{P}^2$, the projection $\pi_S : \mathbb{P} = \mathbb{P}(\text{Sym}^2(\mathbb{C}^3)) \dashrightarrow \mathbb{P}^2$ given by $\pi_S(B) := BS$ has the property that its restriction to D , $\pi_S|_D$, is birational onto its image, unless (and this is indeed an exception) D is a curve contained in a plane $\Delta(S') = \{B \mid BS' = 0\}$ (contained in the determinantal hypersurface $\Delta = \{B \mid \det(B) = 0\}$) and D is not a line.

This result suggests the investigation of a more general situation concerning the birationality of linear projections given by matrix multiplications.

Problem 1. Given a linear space \mathbb{P} of matrices B , and a linear space \mathbb{P}' of matrices S , consider the matrix multiplication $\pi_S(B) = BS$. For which algebraic subvarieties $D \subset \mathbb{P}$ is the restriction of the projection $\pi_S|_D$ birational onto its image for a general choice of $S \in \mathbb{P}'$?

2 Equations in Coordinates

Let $f(x_0, x_1, x_2) = 0$ be the equation of C in the appropriate system of homogeneous coordinates, let $d := \text{deg}(f)$, and let $F := (f_0(x), f_1(x))$ be the first part of the gradient of f . For a point $x = (x_0, x_1, x_2)$ we define

$$(F, x) := f_0(x)x_0 + f_1(x)x_1, \quad \{F \wedge x\} := f_0(x)x_1 - f_1(x)x_0.$$

Then the tangent line T_P at a point P with coordinates x is the transpose of the row vector $(f_0(x), f_1(x), f_2(x))$.

The normal line N_P is orthogonal to the tangent line, hence it has the form $N_P = {}^t(-f_1(x), f_0(x), f_3(x))$, and the condition that $P \in N_P$ forces the unknown rational function $f_3(x)$ to fulfill $-f_1(x)x_0 + f_0(x)x_1 + f_3(x)x_2 \equiv 0$, thus

${}^t N_P$ is the row vector

$${}^t N_P = (-x_2 f_1(x), x_2 f_0(x), -\{F \wedge x\}).$$

We find now the line L_P as the line in the pencil spanned by T_P and N_P passing through S : as such the line L_P is a column vector which is a linear combination

$\lambda T_P + \mu N_P$; the condition that $S \in L_P$ then determines $\lambda = -{}^t N_P \cdot S$, $\mu = {}^t T_P \cdot S$, where S is the transpose of the vector (s_0, s_1, s_2) .

Hence we get

$$L_P(S) = A(P)S, \quad A(P) := -T_P {}^t N_P + N_P {}^t T_P,$$

in particular the matrix $A(P)$ is skew symmetric.

To obtain the reflected ray $\Lambda(P)$ it is sufficient, by definition, to change the sign of λ , and we get therefore:

$$\Lambda_P(S) = B(P)S, \quad B(P) := T_P {}^t N_P + N_P {}^t T_P.$$

Remark 2. (1) The matrices $A(P)$ and $B(P)$ are functions which are defined for all general points P of the plane.

(2) The matrix $B(P)$ is symmetric and has rank at most two, since its image is generated by N_P and T_P ; moreover we have

$$B(P)P = 0, A(P)P = 0, \quad \forall P \in C.$$

(3) Assume that C is not a line passing through a cyclic point: then the matrix $B(P)$ has precisely rank two on the non empty open set where $f_1^2 + f_0^2 \neq 0$ and $x_2 \neq 0$; the former condition clearly holds for a general point $P \in C$, otherwise the dual curve of C would be contained in a line $y_0 = \pm\sqrt{-1}y_1$.

(4) The entries of the matrix $B(x)$ are given by polynomials of degree $2d - 1$.

By the preceding remark follows easily the classical theorem asserting that

Theorem 2. *The class of the caustic, i.e., the degree of Γ , equals $d(2d - 1)$, for a general curve C and a general choice of S .*

In fact C has degree d , and $B(x)S$ is given by polynomials of degree $2d - 1$ in x , which have no base points on a general curve C .

3 Birationality of the Caustic Map

Theorem 3. *If C is not a line, then the caustic map $C \dashrightarrow \mathcal{C}$ is birational, for general choice of S .*

Proof. As already remarked, the caustic map is birational iff the map $\Lambda : C \dashrightarrow \Gamma$ is birational. Observe that Λ defines a morphism $C' \rightarrow \Gamma$ which we also denote by Λ .

The matrix B , whose entries are polynomials of degree $2d - 1$, yields a map

$$B : C' \rightarrow D \subset \mathbb{P}^5 = \mathbb{P}(\text{Sym}^2(\mathbb{C}^3)).$$

Lemma 1. $B : C' \rightarrow D := \Phi(C)$ is birational.

Proof. It suffices to recall Remark 2: for a general point $P \in C$, $B(P)$ has rank exactly two, and $B(P)P = 0$. Hence $P = \ker(B(P))$, and the matrix $B(P)$ determines the point $P \in \mathbb{P}^2$. ■

We have now a projection $\mathbb{P}(\text{Sym}^2(\mathbb{C}^3)) \dashrightarrow \mathbb{P}^2$ given by

$$\pi_S(B) := BS.$$

Consider the linear subspace

$$W := \{B \mid B_{0,0} + B_{1,1} = 0\}.$$

We observe preliminarily that the curve D is contained in the linear subspace W since, setting for convenience $f_i := f_i(x)$, the matrix $B(x)$ has the following entries:

$$B_{0,0} = -2x_2 f_0 f_1, \quad B_{1,1} = 2x_2 f_0 f_1.$$

Then our main result follows from the next assertion, that, for a general choice of $S \in \mathbb{P}^2$, the projection π_S yields a birational map of D onto $\Gamma := \pi_S(D)$.

In order to prove this, we set up the following notation:

$$\Delta_S := \{B \mid BS = 0\}, \quad \Delta := \{B \mid \det(B) = 0\} = \cup_S \Delta_S.$$

Observe that Δ is the secant variety of the Veronese surface

$$V := \{B \mid \text{rank}(B) = 1\}.$$

Observe that the curve D is contained in the linear subspace W , is contained in Δ but not contained in the Veronese surface V .

We are working inside the subspace W , and we observe first of all that the centre of the projection π_S restricted to W is the linear space

$$W_S := \Delta_S \cap W.$$

Observe moreover that $\Delta \cap W = \cup_S W_S$.

Now, the projection π_S is not birational on D if and only if, for a general $B \in D$, there exists another $B' \in D$, $B \neq B'$, such that the chord (i.e., secant line) $B * B'$ intersects W_S in a point B'' (observe that the general point $B \in D$ does not lie in the line W_S).

There are two possible cases:

Case I: B'' is independent of the point $B \in D$.

Case II: B'' moves as a rational function of the point $B \in D$, hence the points B'' sweep the line W_S .

Lemma 2. *The assumption that case I holds for each $S \in \mathbb{P}^2$ leads to a contradiction.*

Proof of the Lemma. Under our assumption, for each S there is a point $B''(S)$ such that infinitely many chords of D meet W_S in $B''(S)$.

Let us see what happens if we specialize S to be a general point $P \in C$.

The first alternative is

(I-1) $B''(P) = B(P)$: in this case, for each point $B^1 \in D$ there is $B^2 \in D$ such that $B(P), B^1, B^2$ are collinear. Since this happens for each choice of $B(P), B^1$, every secant is a trisecant, hence, by the well known trisecant lemma (cf. [1], page 110), D is a plane curve of order at least three.

Take now a general $S \in \mathbb{P}^2$: since $B''(S)$ is on a secant to D , $B''(S)$ belongs to the secant variety Σ of D (here a plane Π), but we claim that it is not in D . In fact, if there were a point $P \in C'$ such that $B''(S) = B(P)$, then $B(P)S = 0$ contradicting that S is a general point. Hence we obtain that the plane Π intersects Δ in a bigger locus than D : since Δ is a cubic hypersurface, it follows that $\Pi \subset \Delta$.

By Proposition 1 it follows that either there is a point S' such that $S' \in \ker(B), \forall B \in \Pi$, or there is a line $L \in \mathbb{P}^2$ such that $\ker(B) \in L, \forall B \in \Pi$: both cases imply that the curve C must be contained in a line, a contradiction.

The second alternative is

(I-2) $B'' := B''(P) \neq B(P)$. Then there is a point $B' \in D$ (possibly infinitely near) such that B' is a linear combination of B'' and $B := B(P)$.

However, since $BP = 0, B''P = 0$, and $B \neq B''$, then also for their linear combination B' we have $B'P = 0$. The consequence is, since $B'P = B'P' = 0$, that B' has rank one. Therefore, if B' is not infinitely near, B' cannot be a general point of D , hence B' is independent of P : but then $C \subset \ker(B')$, and since we assume that C is not a line, we obtain $B' = 0$, a contradiction.

If P' is infinitely near to the point $P \in C$, i.e., P, P' span the tangent line to C at P , and B, B' span the tangent line to D at $B = B(P)$, we work over the ring of tangent vectors $\mathbb{C}[\epsilon]/(\epsilon^2)$, and we observe that

$$(B + \epsilon B')(P + \epsilon P') = 0 \Rightarrow BP' = 0.$$

For $P \in C$ general this is a contradiction, since $BP' = 0, BP = 0$ imply that $B = B(P)$ has rank one. ■

Lemma 3. *The assumption that case II holds for general $S \in \mathbb{P}^2$ leads to a contradiction.*

Proof of the Lemma. As we already observed, for general S , B'' moves as a rational function of the point $B \in D$, hence the points B'' sweep the line W_S . Therefore the line W_S is contained in the secant variety Σ of the curve D . As this happens for general S , and $\Delta \cap W = \cup_S W_S$, it follows that the threefold $\Delta \cap W$ is contained in the secant variety Σ .

Since Σ is irreducible, and has dimension at most three, it follows that we have equality

$$\Delta \cap W = \Sigma.$$

We conclude that, for P_1, P_2 general points of C , the line joining $B(P_1)$ and $B(P_2)$ is contained in Δ .

By Proposition 1, and since $\ker(B(P_1)) = P_1, \ker(B(P_2)) = P_2$, we have that the matrices in the pencil $\lambda_1 B(P_1) + \lambda_2 B(P_2)$ send the span of P_1, P_2 to its orthogonal subspace.

This condition is equivalent to

$${}^t P_1(B(P_2))P_1 = 0 \quad \forall P_1, P_2 \in C$$

(${}^t P_2(B(P_1))P_2 = 0$ follows in fact since P_1, P_2 are general).

Fix now a general point P_2 : then we have a quadratic equation for C , hence C is contained in a conic.

A little bit more of attention: the matrix $B(P_2)$ has rank two, hence the quadratic equation defines a reducible conic, and, C being irreducible, C is a line, a contradiction. ■

4 Linear Subspaces Contained in the Determinantal Cubic

$\Delta := \{B \mid \det(B) = 0\}$

Proposition 1. *Let $\lambda B_0 + \mu B_1$ be a line contained in the determinantal hypersurface Δ of the projective space of symmetric 3×3 matrices.*

Then the line is contained in a maximal projective subspace contained in Δ , which is either of the type

$$\Delta_S := \{B \mid BS = 0\},$$

for some $S \in \mathbb{P}^2$, or of the type

$$\Delta(L) := \{B \mid BL \subset L^\perp\} = \{B \mid B|_L \equiv 0\},$$

for some line $L \subset \mathbb{P}^2$.

Proof. A pencil of reducible conics either has at most one (non infinitely near) base point $S \in \mathbb{P}^2$, or it has a line L as fixed component.

In the first case the pencil is $\subset \Delta_S$, in the second case it is contained in the subspace $\Delta(L)$ consisting of the conics of the form $L + L'$, where L' is an arbitrary line in the plane. ■

Remark 3. Even if the result above follows right away from the classification of pencils of conics, it is useful to recall the arguments which will be used in the sequel.

For instance, we observe that the hyperplane sections of the Veronese surface V are smooth conics, hence no line is contained in V .

5 Birationality of Certain Matrix Projections of Curves

In this final section we want to show the validity of a much more general statement:

Theorem 4. *Let $D \subset \mathbb{P} := \mathbb{P}(\text{Sym}^2(\mathbb{C}^3))$ be a curve and $B : C' \rightarrow D \subset \mathbb{P}$ be its normalization.*

Then, for general $S \in \mathbb{P}^2$, the projection $\pi_S : \mathbb{P} = \mathbb{P}(\text{Sym}^2(\mathbb{C}^3)) \dashrightarrow \mathbb{P}^2$ given by $\pi_S(B) := BS$ has the property that its restriction to D , $\pi_S|_D$ is birational onto its image, unless D is a curve contained in a plane $\Delta(S')$ and is not a line.

In the latter case, each projection $\pi_S|_D$ has as image the line $(S')^\perp$ and is not birational.

Proof. Let $\mathcal{G} := \text{Gr}(1, \mathbb{P})$ be the Grassmann variety of lines $\Lambda \subset \mathbb{P}$: \mathcal{G} has dimension 8.

Define, for $S \in \mathbb{P}^2$, $\mathcal{G}_S := \{\Lambda \in \mathcal{G} | \Lambda \cap \Delta_S \neq \emptyset\}$. Indeed, these 6-dimensional submanifolds of \mathcal{G} are the fibres of the second projection of the incidence correspondence

$$I \subset \mathcal{G} \times \mathbb{P}^2, I := \{(\Lambda, S) | \Lambda \cap \Delta_S \neq \emptyset\}.$$

In turn I is the projection of the correspondence

$$J \subset \mathcal{G} \times \Delta \times \mathbb{P}^2, J := \{(\Lambda, B, S) | B \in \Lambda, BS = 0\}.$$

Recall further that $\Delta \setminus V$ has a fibre bundle structure

$$\mathcal{K} : \Delta \setminus V \rightarrow \mathbb{P}^2$$

such that $\mathcal{K}(B) := \ker(B)$, and with fibre over S equal to $\Delta_S \setminus V$.

Remark 4. 1. Observe that for matrices $B \in V$ we can write them in the form

$$B = x \ ^t x, \text{ for a suitable vector } x, \text{ and in this case } \ker(B) = x^\perp, \text{Im}(B) = \langle \{x\} \rangle.$$

2. In any case, since the matrices B are symmetric, we have always

$$\text{Im}(B) = \ker(B)^\perp.$$

Consider now the fibres of $I \rightarrow \mathcal{G}$: for a general line Λ , its fibre $\mathcal{S}(\Lambda)$ is

1. If $\Lambda \cap \Delta \neq \Lambda$, $\Lambda \cap \Delta \subset \Delta \setminus V$, then $\mathcal{S}(\Lambda)$ consists of at most three points;

2. If $\Lambda \cap \Delta \neq \Lambda$, $(\Lambda \cap V) \neq \emptyset$, then $\mathcal{S}(\Lambda)$ consists of a line x^\perp and at most one further point;
3. If $\Lambda \subset \Delta$ is of the form $\Lambda \subset \Delta_S$, then $\mathcal{S}(\Lambda)$ consists of one or two lines containing S ;
4. If $\Lambda \subset \Delta$ is of the form $\Lambda \subset \Delta(L)$, $\mathcal{S}(\Lambda)$ consists of the line L .

Since, if $\Lambda \subset \Delta(L)$, the conics in Λ consist of L plus a line L' moving in the pencil of lines through a given point P .

We let

$$U \subset \mathcal{G} \times \mathbb{P}, U := \{(\Lambda, B) | B \in \Lambda\}$$

be the universal tautological \mathbb{P}^1 -bundle, and we denote by $p : U \rightarrow \mathbb{P}$ the second projection.

Recall now that the secant variety Σ of D is defined as follows: we have a rational map $\psi : C' \times C' \dashrightarrow \mathcal{G}$ associating to the pair (s, t) the line $B(s) * B(t)$ joining the two image points $B(s), B(t)$.

Then one denotes by U' the pull back of the universal bundle, and defines Σ as the closure of the image $p(U')$.

The condition that for each $S \in \mathbb{P}^2$ the projection π_S is not birational on D means that, if Y is the closure of the image of ψ , then $Y \cap \mathcal{G}_S$ has positive dimension.

This implies that the correspondence

$$I_D := \{(\Lambda_y, S) | y \in Y, \Lambda_y \cap \Delta_S \neq \emptyset\} \subset Y \times \mathbb{P}^2$$

has dimension at least three and surjects onto \mathbb{P}^2 .

Projecting I_D on the irreducible surface Y , we obtain that all the fibres have positive dimension, and we infer that each secant line Λ_y has a fibre $\mathcal{S}(\Lambda_y)$ of positive dimension.

There are two alternatives:

- (i) A general secant Λ_y is not contained in Δ , but intersects the Veronese surface V .
- (ii) Each secant line $\Lambda_y \subset \Delta$.

Step (I): the theorem holds true if $D \subset V$.

Proof of step I.

In this case any element of D is of the form $B(t) = x(t)^t x(t)$, and

$$\pi_S(B(t)) = x(t)^t [x(t)S] = (x(t), S)x(t) = x(t).$$

Hence, for each S , the projection π_S is the inverse of the isomorphism

$$\phi : x \in \mathbb{P}^2 \rightarrow V, \phi(x) = x^t x.$$



We may therefore assume in the sequel that D is not contained in V .

Step (II): the theorem holds in case (i).

Proof of step II.

Observe preliminarily that, in case (i), $D \not\subset \Delta$; else we could take two smooth points $B_1, B_2 \in D \cap (\Delta \setminus V)$, and the secant line $B_1 * B_2$ could not fulfill (i).

Choose then a point $B_0 \in D$, $B_0 \in \mathbb{P} \setminus \Delta$, hence w.l.o.g. we may assume that B_0 is the identity matrix I .

Since any other point $B(t) \in D$ is on the line joining B_0 with a point $x(t)^t x(t) \in V$, we may write locally around a point of C'

$$B(t) = I + \xi(t)^t \xi(t),$$

where $\xi(t)$ is a vector valued holomorphic function.

Now, for each s, t , the secant line $B(t) * B(s)$ meets the Veronese surface V .

Since $B(t)$ cannot have rank equal to 1, there exists λ such that

$$\lambda B(t) + B(s) = \lambda(I + \xi(t)^t \xi(t)) + (I + \xi(s)^t \xi(s))$$

has rank equal to 1, i.e.,

$$\begin{aligned} K_\lambda := \ker(\lambda B(t) + B(s)) &= \{v | [\lambda(I + \xi(t)^t \xi(t)) + (I + \xi(s)^t \xi(s))]v = 0\} = \\ &= \{v | (\lambda + 1)v + \lambda \xi(t)(\xi(t), v) + \xi(s)(\xi(s), v) = 0\} \end{aligned}$$

has dimension 2.

Let us now make the assumption:

(**) two general points $\xi(t), \xi(s)$ are linearly independent.

The above formula shows however that, under assumption (**), it must be that v is a linear combination of $\xi(t), \xi(s)$. This is clear if $\lambda + 1 \neq 0$, otherwise v is orthogonal to the span of $\xi(t), \xi(s)$, contradicting that the kernel has dimension 2.

Hence $K_\lambda = \langle \xi(t), \xi(s) \rangle$ and the condition that $\xi(t) \in K_\lambda$ yields

$$(\lambda + 1)\xi(t) + \lambda \xi(t)(\xi(t), \xi(t)) + \xi(s)(\xi(s), \xi(t)) = 0$$

and implies

$$(***) \forall s, t \quad (\xi(s), \xi(t)) = 0.$$

(***) says that $K_\lambda = \langle \xi(t), \xi(s) \rangle$ is an isotropic subspace, which can have at most dimension 1.

Hence assumption (**) is contradicted, and we conclude that it must be:

$$(***) \xi(t) = f(t)u,$$

where u is an isotropic vector and $f(t)$ is a scalar function.

Even if this situation can indeed occur, we are done since in this case the matrix in V is unique, $u^t u$, each secant Λ_y contains $u^t u$, hence $\mathcal{S}(\Lambda_y) = u^\perp \cup T_y$ where T_y is a finite set. Therefore, for general S , the fibre $\{y | \Lambda_y \cap \Delta_S \neq \emptyset\}$ is a finite set. ■

Step III: the theorem holds true in case (ii).

Proof of Step III.

Consider the general secant line Λ_y . We have two treat two distinct cases.

Case (3): $\Lambda_y \subset \Delta$ is of the form $\Lambda_y \subset \Delta_S$ (then $\mathcal{S}(\Lambda_y)$ consists of one or two lines containing S).

Case (4): $\Lambda_y \subset \Delta$ is of the form $\Lambda_y \subset \Delta(L)$ (then $\mathcal{S}(\Lambda_y)$ consists of the line L).

In case (3), this means that two general matrices $B(s), B(t)$ have a common kernel $S(s, t)$. Since the general matrix $B(t)$ is in $\Delta \setminus V$, its rank equals 2 and $S(s, t) = S(t) \forall s$.

Hence the curve D is contained in a plane Δ_S . In this case however $Im B(t) \subset S^\perp$ and every projection $\pi_{S'}(B(t)) = B(t)S'$ lands in the line S^\perp , so that the projection cannot be birational, unless our curve D is a line.

In case (4) for two general matrices $B(s), B(t)$ there exists a line $L = L(s, t)$ such that $B(s), B(t) \in \Delta(L)$.

Since two such general matrices have rank equal to 2, and $B(t)L \subset L^\perp, B(s)L \subset L^\perp$, if $v(t) \in \ker B(t)$ it follows that $v(t) \in L$ (since $\ker B(t) \cap L \neq \emptyset$). Therefore, if $B(t) \neq B(s)$, then $L(t, s) = \langle\langle v(t), v(s) \rangle\rangle$.

However, the above conditions $B(t)L \subset L^\perp, B(s)L \subset L^\perp$ are then equivalent to

$$(B(t)v(s), v(s)) = {}^t v(s)B(t)v(s) = 0, \forall t, s.$$

Fixing t this is a quadratic equation in $v(s)$, but, since the curve D is irreducible, and $B(t)$ has rank equal to 2, we see that the vectors $v(s)$ belong to a line. Therefore the line $L = L(s, t)$ is independent of s, t and the conclusion is that the curve D is contained in the plane $\Delta(L)$.

In suitable coordinates for \mathbb{P}^2 , we may assume that $L = \langle\langle e_2, e_3 \rangle\rangle$ and $L^\perp = \langle\langle e_1 \rangle\rangle$.

Choosing then $S = e_1$, we obtain an isomorphic projection, since for a matrix

$$B = \begin{pmatrix} a & b & c \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

we have

$$B(e_1) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

■

Remark 5. The referee suggested some arguments to simplify the proofs.

For Theorem 3, this is the proposal:

- (a) Firstly, in the case of the caustic, the curve D parametrizes the reducible conics of the form $T_P + N_P$, where T_P is the tangent to the curve C at P , and N_P is the normal.

If S is a general point in \mathbb{P}^2 , then the degree of D equals the number of such conics passing through S , hence, if ν is the degree of the curve \mathcal{N} of normal lines, μ is the degree of $C' \rightarrow \mathcal{N}$, then

$$\deg(D) = \deg(C^\vee) + \mu\nu.$$

The above formula shows that $\deg(D) \geq 4$ if C is not a line.

In fact, then $\deg(C^\vee) \geq 2$, while in general $\nu \geq 1$ (the normal N_P contains P). But, if $\nu = 1$, then the dual curve of \mathcal{N} , the evolute, is a point, so C is a circle, but in this case $\mu = 2$.

- (b) Therefore, if one shows that D is contained in a plane π , then the plane π is contained in the cubic hypersurface Δ , hence we can apply Proposition 1.
- (c) In turn, to show that D is a plane curve, it is necessary and sufficient to show that two general tangent lines to D meet, which follows if one proves that:
- (d) For each secant line there is a cone over D and with vertex a point B'' , such that the secant line passes through B''

(since then the two tangent lines are coplanar).

In case (I), (d) follows since then, for each general S , there is a point $B''(S)$ such that a curve of secants passes through $B''(S)$, and we get a cone over D with vertex $B''(S)$. Varying S , the point $B''(S)$ must vary, since $B''(S)S = 0$; hence the cone varies, and we get that for each secant (d) holds true.

In case (II), as we have shown, the secant variety of D equals $W \cap \Delta$, which is the secant variety of the rational normal quartic $W \cap V$: but the singular locus of the secant variety of $W \cap V$ equals $W \cap V$ and contains D , hence $W \cap V = D$, a contradiction.

The argument suggested for Theorem 4 requires some delicate verification, so we do not sketch it here.

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At the moment of writing up the references for the present article, I became aware, by searching on the arXiv, that they have written an independent and different proof of birationality of the caustic map for general source, in [6].

Thanks to the referee for helpful comments, and for sketching alternative arguments, which are reproduced in the remark above.

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