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Vector bundles on curves coming from variation of Hodge structures

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Fujita's second theorem for Kähler fibre spaces over a curve asserts, that the direct image V of the relative dualizing sheaf splits as the direct sum $V = A \oplus Q$, where A is ample and Q is unitary flat. We focus on our negative answer [F. Catanese and M. Dettweiler, Answer to a question by Fujita on variation of Hodge structures, to appear in *Adv. Stud. Pure Math.*] to a question by Fujita: is V semiample? We give here an infinite series of counterexamples using hypergeometric integrals and we give a simple argument to show that the monodromy representation is infinite. Our counterexamples are surfaces of general type with positive index, explicitly given as abelian coverings with group $(\mathbb{Z}/n)^2$ of a Del Pezzo surface Z of degree 5 (branched on the union of the lines of Z, which form a bianticanonical divisor), and endowed with a semistable fibration with only three singular fibres. The simplest such surfaces are the three ball quotients considered in [I. C. Bauer and F. Catanese, A volume maximizing canonical surface in 3-space, *Comment. Math. Helv.* **83**(1) (2008) 387–406.], fibred over a curve of genus 2, and with fibres of genus 4. These examples are a larger class than the ones corresponding to Shimura curves in the moduli space of Abelian varieties.

Keywords: Relative dualizing sheaf; semiampleness; monodromy; semistable fibration.

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0. Introduction

In this paper, we first begin recalling previous results [21, 22, 9, 10] concerning Fujita's first and second theorem for Kähler fibre spaces over a curve, asserting that the direct image V of the relative dualizing sheaf splits as the direct sum

$V = A \oplus Q$, where A is ample and Q is unitary flat. Then, we focus on our negative answer [9, 10] to a question posed by Fujita 30 years ago: V does not need to be semiample.

We show here, that the two examples of [9] fits into an infinite series of counterexamples, again based on the use of hypergeometric integrals *à la* Deligne–Mostow, for each positive number n and each way to write n as a sum of four positive integers, and yielding a family of cyclic coverings of the line parametrized by $\mathbb{P}^1$.

Following Beukers and Heckman, we can show that the monodromy group of Q is infinite without resorting to the classification by Schwarz.

Under some mild restrictions on n and the four integers (for n the restriction boils down to the fact, that n should be coprime to six), we give a very simple explicit description of fibred surfaces $f : S \to B$, which are obtained from the above family via a cyclic $\mathbb{Z}/n$-base change $B \to \mathbb{P}^1$, and which have the following remarkable properties:

(1) The Albanese map $\alpha : S \to \mathrm{Alb}(S)$ has as image a curve of genus $b \geq 2$, and coincides with the fibration $f : S \to B$;
(2) all the fibres of α are smooth, except three singular fibres which are constituted of two smooth curves of genus b meeting transversally in one point;
(3) the surfaces S have all positive index, indeed $K_S^2 > 2.5e(S)$,
(4) the direct image $V = f_*(\omega)$ of the relative dualizing sheaf splits as the direct sum $V = A \oplus Q$, where A is ample and Q is unitary flat, and Q corresponds to an infinite monodromy representation of $\pi_1(B)$: hence V is not semiample (since, by the results of [9], a unitary flat bundle is semiample, if and only if the monodromy representation is finite).

In the previous two examples [9, 10], we had $n = 7$, but we used another method to produce a base change yielding a semistable fibration; as a consequence the degree of the base change, that we needed was much larger than seven (42 in the easier case), and the semistable fibrations were not described with full details. The description, we give here was motivated by a question by Fujino, who asked whether we could give a completely explicit example of a semistable fibration satisfying property (4).

To underline the simplicity of the present geometric construction, let us observe that the simplest surfaces in our series correspond to writing $5 = 2 + 1 + 1 + 1$, and are therefore surfaces S fibred over a curve of genus 2, and with fibres of genus 4. It turns out, that these surfaces are among the ball quotients which were previously considered in [3].

The following is our main result.

Theorem 0.1. *There exists an infinite series of surfaces with ample canonical bundle, whose Albanese map is a fibration $f : S \to B$ onto a curve B of genus $b = \frac{1}{2}(n-1)$, and with fibres of genus $g = 2b = n - 1$, where n is any integer relatively prime with* 6.

These Albanese fibrations yield negative answers to Fujita's question about the semiampleness of $V := f_*\omega_{S|B}$, *since here* $V := f_*\omega_{S|B}$ *splits as a direct sum* $V = A \oplus Q$, *where* A *is an ample vector bundle, and* Q *is a unitary flat bundle with infinite monodromy group.*

The fibration f *is semistable: indeed all the fibres are smooth, with the exception of three fibres, which are the union of two smooth curves of genus* b, *which meet transversally in one point.*

For $n = 5$, *we get three surfaces which are rigid, and are quotient of the unit ball in* $\mathbb{C}^2$ *by a torsion free cocompact lattice* Γ. *The rank of* A, *respectively* Q, *is in this case equal to* 2.

Finally, we end surveying quite briefly relations with existing literature concerning Shimura curves in the moduli space of Abelian varieties: this is the work of several people, but especially the work of Moonen [39] is related to our easiest examples.

1. Fujita's Theorems and Questions on Vector Bundles on Curves Arising from Variation of Hodge Structures

An important progress in classification theory was stimulated by a theorem of Fujita, who showed [21], that if X is a compact Kähler manifold and $f : X \to B$ is a fibration onto a projective curve B (i.e. f has connected fibres), then the direct image sheaf

$$V := f_*\omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^*K_B))$$

is a nef vector bundle on B, where 'nef' means that each quotient bundle Q of V has degree $\deg(Q) \geq 0$; sometimes the word 'nef' is replaced by the word 'numerically semipositive'.

In the note [22], Fujita announced the following quite stronger result.

Theorem 1.1 (Fujita [22]). *Let* $f : X \to B$ *be a fibration of a compact Kähler manifold* X *over a projective curve* B, *and consider the direct image sheaf*

$$V := f_*\omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^*K_B)).$$

Then V *splits as a direct sum* $V = A \oplus Q$, *where* A *is an ample vector bundle and* Q *is a unitary flat bundle.*

Fujita sketched the proof, but referred to a forthcoming paper concerning the positivity of the so-called local exponents (this paper was never written, see [2]).

Soon afterwards, using Griffihts' [23–25] results on variation of Hodge structures, since the fibre of $V := f_*\omega_{X|B}$ over a point $b \in B$, such that $X_b := f^{-1}(b)$ is smooth is the vector space $V_b = H^0(X_b, \Omega^{n-1}_{X_b})$, Kawamata [31, 32] improved on Fujita's result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

$$\mathrm{Kod}(X) \geq \mathrm{Kod}(B) + \mathrm{Kod}(F)$$

(here F is a general fibre). Kawamata did this by showing the semipositivity also for the direct image of higher powers of the relative dualizing sheaf

$$W_m := f_*(\omega_{X|B}^{\otimes m}) = f_*(\mathcal{O}_X(m(K_X - f^*K_B))).$$

Kawamata also extended his result to the case, where the dimension of the base variety B is > 1 in [31], giving later a simpler proof of semipositivity in [33]. There has been a lot of literature on the subject ever since, see the references we cited (see [17] for the ampleness of W_m, when $m \geq 2$ and when the fibration is not birationally isotrivial, see also [20, 19, 36, 37, 41, 47, 48] and [49]). Kawamata introduced a simple lemma, concerning the degree of line bundles on a curve, whose metric grows at most logarithmically around a finite number of singular points, which played a crucial role for the proof.

The missing details concerning the proof of the second theorem of Fujita, using Kawamata's lemma and some crucial estimates given by Zucker [47] for the growth of the norm of sections of the L^2-extension of Hodge bundles, were provided in [9], where also a negative answer was given to the following question posed by Fujita in 1982 ([20, Problem 5, p. 600, Proceedings of the 1982 Taniguchi Conference]).

To understand this question, it is not only important to have in mind Fujita's second theorem, but it is also very convenient to recall the following classical definition used by Fujita in [21, 22].

Let V be a holomorphic vector bundle over a projective curve B.

Definition 1.1. Let $p : \mathbb{P} := \mathrm{Proj}(V) = \mathbb{P}(V^\vee) \to B$ be the associated projective bundle, and let H be a hyperplane divisor (s.t. $p_*(\mathcal{O}_{\mathbb{P}}(H)) = V$).

Then V is said to be:

(NP) numerically semipositive, if and only if every quotient bundle Q of V has degree $\deg(Q) \geq 0$,

(NEF) nef, if and only if H is nef on $\mathbb{P}$,

(A) ample, if and only if H is ample on $\mathbb{P}$,

(SA) semiample, if and only H is semiample on $\mathbb{P}$ (there is a positive multiple mH such that the linear system $|mH|$ is base point free).

Remark 1.1. Recall that $(A) \Rightarrow$ (SA) $\Rightarrow$ (NEF) $\Leftrightarrow$ (NP), the last follows from the following result due to Hartshorne [28].

Proposition 1.1. *A vector bundle V on a curve is nef, if and only it is numerically semipositive*, i.e. *if and only if every quotient bundle Q of V has degree* $\deg(Q) \geq 0$, *and V is ample, if and only if every quotient bundle Q of V has degree* $\deg(Q) > 0$.

Moreover, we have also:

Definition 1.2. A flat holomorphic vector bundle on a complex manifold M is a holomorphic vector bundle $\mathcal{H} := \mathcal{O}_M \otimes_{\mathbb{C}} \mathbb{H}$, where $\mathbb{H}$ is a local system of complex vector spaces associated to a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$,

$$\mathbb{H} := (\tilde{M} \times \mathbb{C}^r)/\pi_1(M)$$

$\tilde{M}$ being the universal cover of M (so that $M = \tilde{M}/\pi_1(M)$).

We say that $\mathcal{H}$ is unitary flat, if it is associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$.

Question 1.1 (Fujita). Is the direct image $V := f_* \omega_{X|B}$ semiample?

In [9], we established a technical result which clarifies how Fujita's question is very closely related to Fujita's II theorem.

Theorem 1.2. *Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold M, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semiample if and only if $\mathrm{Im}(\rho)$ is finite.*

Hence in our particular case, where $V = A \oplus Q$ with A ample and Q unitary flat, the semiampleness of V, simply means that the flat bundle has finite monodromy (this is another way of wording the fact that the representation of the fundamental group $\rho : \pi_1(B) \to U(r, \mathbb{C})$ associated to the flat unitary rank-r bundle Q has finite image).

The main new result in our joint work [9] was to provide a negative answer to Fujita's question in general.

Theorem 1.3. *There exist surfaces X of general type endowed with a fibration $f : X \to B$ onto a curve B of genus ≥ 3, and with fibres of genus 6, such that $V := f_* \omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where A is an ample rank-2 vector bundle, and the flat unitary rank-2 summands Q_1, Q_2 have infinite monodromy group* (i.e. *the image of ρ_j is infinite*). *In particular, V is not semiample.*

Recall however that in special cases, one can conclude that V is semiample.

Corollary 1.1. *Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B. Then $V := f_* \omega_{X|B}$ is a direct sum $V = A \oplus (\bigoplus_{i=1}^h Q_i)$, with A ample and each Q_i unitary flat without any nontrivial degree zero quotient. Moreover,*

(I) *if Q_i has rank equal to 1, then it is a torsion bundle ($\exists\, m$ such that $Q_i^{\otimes m}$ is trivial) (Deligne);*
(II) *if the curve B has genus 1, then rank $(Q_i) = 1$, $\forall\, i$;*
(III) *In particular, if B has genus at most 1, then V is semiample.*

Proof. The idea of the proof is as follows:

(I) was proven by Deligne [13] (and by Simpson using the theorem of Gelfond–Schneider), while;
(II) Follows since $\pi_1(B)$ is abelian, if B has genus 1: hence every representation splits as a direct sum of one-dimensional ones. $\square$

In our construction for Theorem 1.3, we started from hypergeometric integrals associated to a cyclic group of order 7, and we derived the nonfiniteness of the monodromy as a consequence of the classification due to Schwarz [43].

However, in order to provide a semistable fibration, we first resolved the singularities of the resulting surface fibres over $\mathbb{P}^1$, then applied blow ups in order to achieve that the reduced divisors associated to the fibres would be normal crossing divisors, and then applied the general method in order to construct a semistable base change.

The final result was that these examples had a base of much larger genus, and the description given was not fully detailed. The novelty of this paper, answering a question by Osamu Fujino, is to provide an explicit semistable fibration without having to take a base change, where the genus of the base curve B becomes too large. This will be discussed in the fourth section, where we shall also give a simpler proof.

An interesting observation, concerning the crucial difference of the roles played by unitary flat bundles vs. flat bundles in our context, is given by the following result. While a unitary flat bundle is nef, the same does not hold for a flat bundle. This is no surprise, as communicated to the first author by Kollár, in view of the following old theorem of Weil [45], reproven by Atiyah in [1].

Theorem 1.4 (Weil–Atiyah). *A vector bundle V over a projective curve is* (*isomorphic to*) *a flat holomorphic bundle if and only if*, *in its unique decomposition as a direct sum $V = \oplus_i V_i$ of indecomposable bundles*, *each of the summands V_i has degree zero.*

In our situation, we proved (again in [9], see also [35]).

Theorem 1.5. *Let $f : X \to B$ be a Kodaira fibration*, i.e. *X is a surface and all the fibres of f are smooth curves not all isomorphic to each other. Then the direct image sheaf $V := f_*\omega_{X|B}$ has strictly positive degree hence $\mathcal{H} := R^1 f_*(\mathbb{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not nef* (i.e. *not numerically semipositive*).

1.1. *Semistable reduction*

Assume now that $f : X \to B$ is a fibration of a compact Kähler manifold X, over a projective curve B, and consider the invertible sheaf $\omega := \omega_{X|B} = \mathcal{O}_X(K_X - f^*K_B)$.

By Hironaka's theorem, there is a sequence of blow-ups with smooth centers $\pi : \hat{X} \to X$, such that

$$\hat{f} := f \circ \pi : \hat{X} \to B$$

has the property that all singular fibres F are such that $F = \sum_i m_i F_i$, and $F_{\mathrm{red}} = \sum_i F_i$ is a normal crossing divisor.

Since $\pi_*\mathcal{O}_{\hat{X}}(K_{\hat{X}}) = \mathcal{O}_X(K_X)$, we obtain

$$\hat{f}_*\omega_{\hat{X}|B} = \hat{f}_*\mathcal{O}_{\hat{X}}(K_{\hat{X}} - \hat{f}^*K_B) = f_*\mathcal{O}_X(K_X - f^*K_B) = f_*\omega_{X|B}.$$

Therefore, one can assume wlog that all the fibres of f have reduction, which is a normal crossing divisor, and the well-known semistable reduction theorem, whose statement is here reproduced, shows that one can reduce to the case, where the fibration is semistable, i.e. all fibres are reduced and yield normal crossing divisors.

Theorem 1.6 (Semistable reduction theorem [34]). *There exists a cyclic Galois covering of B, $B' \to B = B'/G$, such that the normalization X'' of the fibre product $B' \times_B X$ admits a resolution $X' \to X''$, such that the resulting fibration $f' : X' \to B'$ has the property that all the fibres are reduced and normal crossing divisors.*

$$\begin{tikzcd} X' \arrow[r, "v'"] \arrow[d, "f'"] & X'' \arrow[r, "v''"] \arrow[d] & X \arrow[d, "f"] \\ B' \arrow[r, "Id"] & B' \arrow[r, "u"] & B. \end{tikzcd}$$

The following proposition was used in [9], while reducing the proof of Fujita's second theorem to the semistable case.

Proposition 1.2. *The sheaf $V' := f'_* \omega_{X'|B'}$ is a subsheaf of the sheaf $u^*(V)$, where $V := f_* \omega_{X|B}$, and the cokernel $u^*(V)/V'$ is concentrated on the set of points corresponding to singular fibres of f.*

The proposition shows indeed that, when the fibration is not semistable, then certain unitary flat summands on B' may yield ample summands on B; and the precise calculation given in its proof helps to decide exactly when this happens.

2. Fujita's Second Theorem

The tools used for the proof of Fujita's second theorem involve differential geometric notions of positivity, which we now recall.

Definition 2.1. Let (E, h) be a Hermitian vector bundle on a complex manifold M. Take the canonical Chern connection associated to the Hermitian metric h, and denote by $\Theta(E, h)$ the associated Hermitian curvature, which gives a Hermitian form on the complex vector bundle $T_M \otimes E$.

Then, one says that E is Nakano positive (respectively: semipositive), if there exists a Hermitian metric h, such that the Hermitian form associated to $\Theta(E, h)$ is strictly positive definite (respectively: semipositive definite).

Remark 2.1. Umemura proved [44] that a vector bundle V over a curve B is positive (i.e. Griffiths positive, or equivalently Nakano positive), if and only if V is ample.

One of the principal positivity property can be summarized through the well-known slogan: 'curvature decreases in sub-bundles'.

Except that one has to formulate the statement properly as follows: curvature decreases in Hermitian sub-bundles. Indeed the example of Kodaira fibrations produces sub-bundles of a flat bundle (they have zero curvature), which are positively curved.

We pass now to sketch the ideas used in the proof of Fujita's second theorem.

Theorem 2.1 (Fujita [22]). *Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf*

$$V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B)).$$

Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.

2.1. *Sketch of proof of Fujita's theorem*

(I) Thanks to the auxiliary results shown in the previous section, using the semistable reduction theorem (yielding a base change $B' \to B$, such that all fibres of the pull back $X' \to B'$ are reduced with normal crossings) and in particular Proposition 1.2, giving a comparison of the pull back of V with the analogously defined V', it suffices to prove the theorem in the semistable case, i.e. where each fibre is reduced and a normal crossing divisor (see [9, Proposition 2.9] for details).

(II) Idea of the proof in the case of no singular fibres.

V is a holomorphic sub-bundle of the holomorphic vector bundle $\mathcal{H}$ associated to the local system $\mathbb{H}_{\mathbb{Z}} := \mathcal{R}^m f_*(\mathbb{Z}_X)$, $m := \dim(X) - 1$ (i.e. $\mathcal{H} = \mathbb{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$).

The bundle $\mathcal{H}$ is flat, hence the curvature $\Theta_{\mathcal{H}}$ associated to the flat connection satisfies $\Theta_{\mathcal{H}} \equiv 0$.

We view V as a holomorphic sub-bundle of $\mathcal{H}$, while

$$V^\vee \cong R^m f_* \mathcal{O}_X, \quad m = \dim(X) - 1$$

is a holomorphic quotient bundle of $\mathcal{H}$.

The curvature formula for sub-bundles gives (σ is the II fundamental form)

$$\Theta_V = \Theta_{\mathcal{H}}|_V + \bar{\sigma}\ {}^t\sigma = \bar{\sigma}\ {}^t\sigma$$

and Griffiths ([24], see also [26] and [48]) proves that the curvature of $V^\vee$ is seminegative, since its local expression is of the form $ih'(z) d\bar{z} \wedge dz$, where $h'(z)$ is a semipositive definite Hermitian matrix.

In particular, we have that the curvature Θ_V of V is semipositive and, moreover, that the curvature vanishes identically, if and only if the second fundamental form σ vanishes identically, i.e. if and only if V is a flat sub-bundle.

However, by semipositivity, we get that the curvature vanishes identically, if and only its integral, the degree of V, equals zero. Hence V is a flat bundle, if and only if it has degree 0.

The same result then holds true, by a similar reasoning, for each holomorphic quotient bundle Q.

(III) The more difficult part of the proof uses some crucial estimates given by Zucker (using Schmid's asymptotics for Hodge structures) for the growth of the norm of sections of the L^2-extension of Hodge bundles, and the following lemma by Kawamata ([32], see also [41, Proposition 3.4, p. 11]).

Lemma 2.1. *Let L be a holomorphic line bundle over a projective curve B, and assume that L admits a singular metric h which is regular outside of a finite set S and has at most logarithmic growth at the points $p \in S$.*

Then the first Chern form $c_1(L, h) := \Theta_h$ is integrable on B, and its integral equals $\deg(L)$.

The above lemma shows that in the semistable case, singularities are influent, and the argument runs as in the case of no singular fibres.

(IV) The existence of such a metric follows from the results of Schmid in [42] and Zucker in [47], leading to the following lemma.

Lemma 2.2. *For each point $s \in B$, there exists a basis of V given by elements σ_j, such that their norm in the flat metric outside the punctures grows at most logarithmically.*

In particular, for each quotient bundle Q of V, its determinant admits a metric with growth at most logarithmic at the punctures $s \in S$, and the degree of Q is given by the integral of the first Chern form of the singular metric.

3. Cyclic Coverings of the Projective Line Branched on Four Points

In this section, we explain how we obtain explicit examples of fibrations, where $V = f_*\omega$ has a flat summand. Let $\zeta_n := e^{\frac{2\pi i}{n}}$. Consider a cyclic covering of the projective line with group $\mathbb{Z}/n$, branched on four points. Hence a curve $C = C_x$ described by an equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - x y_0)^{m_3}, \quad x \in \mathbb{C} \backslash \{0, 1\}, \tag{3.1}$$

where, of course, $\gcd(m_0, \dots, m_3, n) = 1$.

The above equation describes a singular curve inside the line bundle over $\mathbb{P}^1$ whose sheaf of sections is the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$, and we denote by C the normalization of this curve. Then C admits a Galois cover $\phi : C \to \mathbb{P}^1$ with cyclic Galois group equal to the group of nth roots of unity in $\mathbb{C}$,

$$G = \{\zeta \in \mathbb{C}^* \mid \zeta^n = 1\}$$

acting by scalar multiplication on z_1. The choice of a generator in G yields an isomorphism $G \cong \mathbb{Z}/n$, for instance, we have $G = \langle \epsilon \rangle$, where ϵ acts as $z_1 \mapsto \zeta_n z_1$. The cover ϕ is branched at $\mathcal{S} = \{s_1 = 0, s_2 = 1, s_3 = x, s_0 = \infty\}$ (where, in projective coordinates $[y_0, y_1]$, one has $0 = [1, 0]$, $1 = [1, 1]$, $x = [1, x]$, $\infty = [0, 1]$).

We shall make the restrictive assumption that

$$0 < m_j \leq n-3 \quad \text{and} \quad m_0+m_1+m_2+m_3 = n.$$

Remark 3.1. We want to point out that the above is indeed a restriction, even if we allow a change of the generator of G taking the residue class of a number h coprime to n. This change of generator has the effect of replacing m_j with the rest modulo n of hm_j, which we denote by $[hm_j]$.

Now, take the example, where $n=8$ and $m_1=m_2=4$, $m_3=3$, $m_4=5$. Then, however, we change the generator of G, the residue class $[h4]$ shall always be equal to 4. Hence the sum $[hm_0]+[hm_1]+[hm_2]+[hm_3] > n = 8$ always, and indeed the sum is then always equal to $2n=16$ (observe in fact that $\Sigma := ([hm_0]+[hm_1]+[hm_2]+[hm_3]) \in \{n, 2n.3n\}$, and changing m_j to $n-m_j$, $\Sigma \mapsto 4n-\Sigma$).

Now, for $j \in \mathbb{Z}/n\mathbb{Z}$, let $\chi_j : G \to \mathbb{C}^*, \zeta \mapsto \zeta^j$ and let $\mathbb{L}_j$ denote the rank-one local system on $\mathbb{P}^1\backslash\mathcal{S}$, whose monodromy matrix α_s at s_i is given by $\zeta_n^{m_i\cdot j}$.

Let $H^1(C,\mathbb{C})_j$ be the subspace of $H^1(C,\mathbb{C})$ on which G acts as χ_j. Then, one has an isomorphism

$$H^1(C,\mathbb{C})_j = H^1(\mathbb{P}^1\backslash\mathcal{S}, \mathbb{L}_j)$$

and moreover

$$H^{1,0}(\mathbb{P}^1\backslash\mathcal{S}, \mathbb{L}_j) = H^{1,0}(C)_j,$$

where $H^{1,0}(C)_j$ is again the part of $H^{1,0}(C)$ on which G acts by the character χ_j (cf. [14, Sec. 2.23]). For $j \neq 0$, one has

$$\dim(H^1(C,\mathbb{C})_j) = \dim(H^1(\mathbb{P}^1\backslash\mathcal{S}, \mathbb{L}_j)) = 2.$$

For $j \neq 0$, let $\mu_{i,j} = \frac{[m_i\cdot j]}{n}$. By [14, Eq. (2.20.1)],

$$\dim H^{1,0}(\mathbb{P}^1\backslash\mathcal{S}, \mathbb{L}_j) = -1 + \sum_{i=0}^{3} \mu_{i,j} \quad \text{for } j \in \mathbb{Z}/n\mathbb{Z},\ j \neq 0. \tag{3.2}$$

Hence, under the above assumption (3.1), we have

$$H^1(C,\mathbb{C})_{-1} = H^{1,0}(C,\mathbb{C})_{-1} \simeq H^{1,0}(\mathbb{P}^1\backslash\mathcal{S}, L_{-1}).$$

By [14, Proposition 2.20], the Hermitian form H_j on $H^1(\mathbb{P}^1\backslash\mathcal{S}, \mathbb{L}_j)$ is positive definite on $H^{(1,0)}(P\backslash\mathcal{S}, \mathbb{L}_j)$ and negative definite on $H^{(0,1)}(\mathbb{P}^1\backslash\mathcal{S}, \mathbb{L}_j)$. Hence the positivity and the negativity index of H_j are given by

$$\left(-1 + \sum_{i=0}^{3} \mu_{i,j}, 3 - \sum_{i=0}^{3} \mu_{i,j}\right). \tag{3.3}$$

Varying $x \in \mathbb{C}\backslash\{0,1\}$, one obtains a family of curves $\pi : \mathcal{C} \to \mathbb{C}\backslash\{0,1\}$ with fibre $\pi^{-1}(x) = C_x$, equipped with a compatible action of G. For each $j \in \mathbb{Z}/n\mathbb{Z}$, one also obtains a local system $\mathbb{L}'_j$ on

$$M := \{(x,y) \in \mathbb{C}^2 \mid x, y \neq 0, 1, x \neq y\}$$

which extends $\mathbb{L}_j$ to M. Let

$$f : M \to \mathbb{C}\backslash\{0,1\}, \quad (x,y) \mapsto x.$$

The higher direct image $\hat{\mathbb{H}} = R^1\pi_*(\mathbb{C})$ decomposes then with respect to the G-action into χ_j-equivariant parts

$$\hat{\mathbb{H}} = \bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}} \hat{\mathbb{H}}_j, \quad \text{where } \hat{\mathbb{H}}_j = R^1 f_*\mathbb{L}'_j.$$

If $j \neq 0$, then the monodromy representation of $\hat{\mathbb{H}}_j$ by (3.4) respects a Hermitian form H_j of index

$$\left(-1 + \sum_{i=0}^{3} \mu_{i,j}, 3 - \sum_{i=0}^{3} \mu_{i,j}\right). \tag{3.4}$$

We shall prove in the Appendix, that this monodromy representation is irreducible, if the m_i's are coprime to n (as we shall assume in the sequel). This result and the following lemma shall be used to show the existence of a flat unitary summand with infinite monodromy: but we shall also give a self-contained and more elementary proof of the main theorem, which only uses the second Fujita theorem.

The index of H_j is related to finiteness properties of the monodromy of $\hat{\mathbb{H}}_j$ by the following result which is a straightforward generalization of [4, Theorem 4.8] (cf. [27]). We remark that the lemma applies in many other contexts, e.g. for more general rigid local systems or motivic local systems, whose Hodge numbers can be calculated (cf. [30, 15, 16]).

Lemma 3.1. *Choose an embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$. Let K be either a finite abelian extension of $\mathbb{Q}$ or a totally real Galois extension of $\mathbb{Q}$. Denote by* $\mathrm{Gal} := \mathrm{Gal}(K/\mathbb{Q})$ *and let $\mathcal{O}_K$ denote the ring of integers of K. Let Γ be a finitely generated group and let $\rho : \Gamma \to \mathrm{GL}_n(\mathcal{O}_K)$ be an absolutely irreducible representation, whose image shall be denoted by $H := \mathrm{Im}(\rho)$. Suppose that ρ respects a Hermitian form*, i.e. *there exists a Hermitian matrix $M = (m_{i,j}) \in K^{n\times n}$ with*

$$\bar{A}^T MA = M \quad \forall\, A \in H.$$

For $\sigma \in$ Gal, let $M^\sigma = (m^\sigma_{i,j})$. Then H is finite if and only if M^σ is a definite Hermitian form for all $\sigma \in$ Gal.

Proof. If H is finite, then H leaves the positive definite unitary form

$$\bar{v}^T Mw := \sum_{h\in H} \bar{v}^T \bar{h}^T hw \quad v,w \in K^n$$

invariant. By our assumptions on K, any $\sigma \in$ Gal commutes with complex conjugation, hence H^σ leaves the form defined by the matrix M^σ invariant. Moreover, M^σ is determined up to a constant, since H^σ is again irreducible. Since H^σ is also finite, the matrix M^σ must be definite.

Let now the form M^σ be definite for any $\sigma \in \mathrm{Gal}$. By the additive isomorphism $\mathcal{O}_K \simeq \mathbb{Z}^d$ ($d = |\mathrm{Gal}|$), the representation ρ gives rise to a representation

$$\tilde{\rho} : \Gamma \to \mathrm{GL}_{nd}(\mathbb{Z})$$

such that the trace of $\tilde{\rho}(g)$ coincides with the relative trace of $\rho(g)$:

$$\mathrm{Trace}(\tilde{\rho}(g)) = \mathrm{Trace}_{K/\mathbb{Q}}(\mathrm{Trace}(\rho(g)) = \sum_{\sigma \in \mathrm{Gal}} \mathrm{Trace}(\rho(g))^\sigma.$$

Extending the scalars from $\mathbb{Z}$ to $\mathbb{C}$, we obtain from $\tilde{\rho}$ a representation

$$\tilde{\rho} \otimes \mathbb{C} : \Gamma \to \mathrm{GL}_{nd}(\mathbb{C}).$$

Since any semisimple representation with values in $\mathbb{C}$ is determined up to isomorphy by its trace by the theorem of Brauer–Nesbitt, there exists a matrix $g \in \mathrm{GL}_n(\mathbb{C})$, such that

$$(\tilde{\rho} \otimes \mathbb{C})^g = \prod_{\sigma \in \mathrm{Gal}} \rho^\sigma \otimes \mathbb{C},$$

where $\rho^\sigma \otimes \mathbb{C}$ denotes the extension of scalars of ρ^σ from $\mathcal{O}_K$ to $\mathbb{C}$. By the definiteness of the forms M^σ ($\sigma \in \mathrm{Gal}$), the latter representation takes its values in the product $\prod_{\sigma \in \mathrm{Gal}} U(M^\sigma)$ of the compact unitary groups $U(M^\sigma)$ associated to the Hermitian forms M^σ ($\sigma \in \mathrm{Gal}$). We conclude that the image of $\tilde{\rho}$ is contained in the compact and discrete group

$$\left(\prod_{\sigma \in \mathrm{Gal}} U(M^\sigma)^{g^{-1}} \right) \cap \mathrm{GL}_{nd}(\mathbb{Z})$$

and is hence finite. Therefore the image of ρ is finite. □

The next result is a straightforward consequence of the above lemma (of course, it may also be derived by Schwarz' list of hypergeometric differential equations with finite monodromy [43]). For simplicity, we restrict ourselves to the cases considered in the next section.

Corollary 3.1. *Assume that*

$$n \in \mathbb{N},\ n \geq 5, \quad \textit{such that } \mathrm{GCD}(n, 6) = 1, \quad m_0, m_1, m_2, m_3 \in \mathbb{N},$$
$$\textit{with } 1 \leq m_j \leq n - 1, \quad m_0 + m_1 + m_2 + m_3 = n,$$
$$m_k,\ (m_i + m_3) \in (\mathbb{Z}/n\mathbb{Z})^*, \quad \forall\, i = 0, 1, 2, \quad k = 0, 1, 2, 3.$$

Then there is $j \in (\mathbb{Z}/n\mathbb{Z})^*$, such that the monodromy of the local system $\hat{\mathbb{H}}_j$ is infinite.

Proof. Since $n - 1$ and $m_0 + m_3$ are invertible in $\mathbb{Z}/n\mathbb{Z}$, there is a j, such that $j(m_0 + m_3) \equiv -1 \pmod{n}$.

Define now $m_i' := [m_i j]$, so that $m_0' + m_3' = n - 1$.

We have the obvious inequalities $2 \leq m'_1 + m'_2 \leq 2n - 2$. Hence

$$n + 1 \leq m'_0 + \cdots + m'_3 \leq 3n - 3$$

and therefore

$$m'_0 + \cdots + m'_3 = 2n.$$

Hence the underlying unitary form is indefinite by formula (3.4), hence the monodromy is infinite by Lemma 3.1 and Proposition A.1. $\square$

4. Abelian Coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ Yielding Surfaces which are Counterexamples to Fujita's Question

In this section, we shall provide an infinite series of examples of surfaces fibred over a curve, whose fibres are curves with a symmetry of $G := \mathbb{Z}/n$ (and with quotient $\mathbb{P}^1$).

To avoid too many technicalities, we make the following simplifying assumptions, part of which were already mentioned in Corollary 3.1:

$$n \in \mathbb{N},\ n \geq 5, \quad \text{such that } \mathrm{GCD}(n, 6) = 1, \quad m_0, m_1, m_2, m_3, n_0, n_1, n_2 \in \mathbb{N},$$
$$\text{with } 1 \leq n_i, m_j \leq n - 1, \quad m_0 + m_1 + m_2 + m_3 = n, n_0 + n_1 + n_2 = n$$
$$m_j,\ n_i,\ (m_i + m_3) \in (\mathbb{Z}/n)^*, \quad \forall\, i = 0, 1, 2, \quad j = 0, 1, 2, 3.$$

Lemma 4.1. *Such integers n_i, m_j satisfying the above properties exist, if and only if n and 6 are coprime.*

Proof. If n is even, then if m_j is a unit in $\mathbb{Z}/n$, then m_j is odd, but then $m_i + m_3$ is even, and cannot be a unit.

If instead $3 \mid n$, then without loss of generality

$$m_0 \equiv m_3 \pmod 3, \quad m_1 \equiv m_2 \equiv -m_3 \pmod 3$$

but then $m_1 + m_3$ is not a unit in $\mathbb{Z}/n$.

Finally, if $\mathrm{GCD}(n, 6) = 1$, then we can simply choose

$$m_0 = m_1 = m_2 = 1, \quad m_3 = n - 3, \quad n_0 = n_1 = 1, \quad n_2 = n - 2. \qquad \square$$

Definition 4.1. We shall refer to the choice

$$m_0 = m_1 = m_2 = 1, \quad m_3 = n - 3, \quad n_0 = n_1 = 1, \quad n_2 = n - 2$$

as the standard case.

We consider again the equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - x y_0)^{m_3}, \quad x \in \mathbb{C} \backslash \{0, 1\}$$

but, we homogenize it to obtain the equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (x_0 y_1 - x_1 y_0)^{m_3} x_0^{n-m_3}.$$

The above equation describes a singular surface Σ', which is a cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $G := \mathbb{Z}/n$; Σ' is contained inside the line bundle $\mathbb{L}_1$ over $\mathbb{P}^1 \times \mathbb{P}^1$ whose sheaf of holomorphic sections $\mathcal{L}_1$ equals $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$.

The first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induces a morphism $p : \Sigma' \to \mathbb{P}^1$ and we consider the curve B, normalization of the covering of $\mathbb{P}^1$ given by

$$w_1^n = x_0^{n_0} x_1^{n_1} (x_1 - x_0)^{n_2}.$$

We consider the normalization Σ of the fibre product $\Sigma' \times_{\mathbb{P}^1} B$.

Σ is an abelian covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $(\mathbb{Z}/n)^2$, and the local monodromies are as follows:

$$\{y = \infty\} = \{y_0 = 0\} \mapsto (m_0, 0), \quad \{y = 0\} = \{y_1 = 0\} \mapsto (m_1, 0),$$
$$\{y = 1\} = \{y_1 = y_0\} \mapsto (m_2, 0),$$
$$\{x = \infty\} \mapsto (n - m_3, n_0), \quad \{x = 0\} \mapsto (0, n_1), \quad \{x = 1\} \mapsto (0, n_2),$$
$$\Delta := \{(x_0 y_1 - x_1 y_0) = 0\} \mapsto (m_3, 0).$$

Since the branch divisor is a not a normal crossing divisor, we blow-up the three points $P_0 := \{x_0 = y_0 = 0\}$, $P_1 := \{x_1 = y_1 = 0\}$, $P_2 = \{x_0 - x_1 = y_0 - y_1 = 0\}$.

We obtain in this way a del Pezzo surface, which we denote by Z, we denote by E_i the exceptional (-1)-curve inverse image of the point P_i, and we notice that the pull back of the branch divisor is now a normal crossing divisor.

The local monodromies around the three exceptional divisors are now

$$E_0 \mapsto (m_0, n_0), \quad E_1 \mapsto (m_1 + m_3, n_1), \quad E_2 \mapsto (m_2 + m_3, n_2).$$

We finally define S to be the normalization of the pull back $\Sigma \times_{\mathbb{P}^1 \times \mathbb{P}^1} Z$.

Proposition 4.1. *The surface S is smooth, and each irreducible component of the branch locus $\mathcal{B}$ has local monodromy of order n.*

Proof. Given two irreducible components $\mathcal{B}_i$, $\mathcal{B}_j$, they are smooth and they intersect transversally in exactly one point, or are disjoint. Hence, it is sufficient to show

- that the inertia subgroups (image of the local monodromy) are cyclic of order n;
- if $\mathcal{B}_i$, $\mathcal{B}_j$ intersect, the corresponding inertia subgroups generate $(\mathbb{Z}/n)^2$.

By our assumptions, all the local monodromies are elements of order n, hence the first assertion.

The second assertion follows from the following fact: $(\mathbb{Z}/n)^2$ is generated by pairs of the form

$$(a,0),(0,b), a,b \in (\mathbb{Z}/n)^*,$$
$$(a,0),(c,b), a,b \in (\mathbb{Z}/n)^*,$$
$$(0,a),(b,c), a,b \in (\mathbb{Z}/n)^*$$

or of the form

$$(n-m_3,n_0), \quad (m_0,n_0)$$

since their span is the span of $(n-m_3-m_0,0)$, (m_0,n_0) and n_0, m_3+m_0 are units in $(\mathbb{Z}/n)$. $\square$

Proposition 4.2. *Let $f : S \to B$ be the morphism induced by the projection $\Sigma \to B$. Then*

(1) *The genus of the fibres F equals $g = n-1$.*
(2) *The genus of the base curve B equals $b = \frac{n-1}{2}$.*
(3) *All the fibres are smooth, except the fibres over $x=0$, $x=1$, $x=\infty$, which consists of two smooth curves of genus b intersecting transversally in exactly one point.*
(4) *f is the Albanese map of S,* i.e. $b = q := h^1(\mathcal{O}_S) = h^1(\mathcal{O}_S(K_S))$.

Proof. (1) and (2) follow from Hurwitz' formulae:

$$2(g-1) = -2n + 4(n-1), \quad 2(b-1) = -2n + 3(n-1),$$

since the general fibre is a $(\mathbb{Z}/n)$-cyclic cover of $\mathbb{P}^1$ totally ramified in four points, while B is a $(\mathbb{Z}/n)$-cyclic cover of $\mathbb{P}^1$ totally ramified in three points.

(3) The fibres over $x=0$, $x=1$, $x=\infty$ are the inverse images of two smooth curves meeting transversally in exactly one point P and which are part of the branch locus $\mathcal{B}$. The covering is totally branched on P, hence these special fibres consist of two smooth curves meeting transversally in exactly one point P'. Both are $(\mathbb{Z}/n)$-cyclic covers of $\mathbb{P}^1$ totally ramified in three points, hence their genus equals b.

The other fibres are the inverse image of a $\mathbb{P}^1$ intersecting the branch locus transversally in four points, hence they are all smooth of genus $g = n-1$.

(4) There are several ways to prove that $b = q$, some more explicit, the following one is in the spirit of this paper.

We want to calculate

$$q := h^1(\mathcal{O}_S) = h^1(\mathcal{O}_S(K_S)) = h^1(\omega_{S|B}(K_B))$$

and we denote $\omega_{S|B}$ for short by ω.

We use the Leray spectral sequence for f, saying that

$$q = h^0(B, \mathcal{R}^1 f_*(\omega)(K_B)) + h^1(f_*(\omega)(K_B)).$$

The first term, by relative duality, equals $b = h^0(B, \mathcal{O}_B(K_B))$, while the second vanishes, as $f_*(\omega) = V = A \oplus Q$ by Fujita's second theorem. Then $h^1(V(K_B)) =$

$h^0(V^\vee)$ by Serre duality, and $h^0(A^\vee) = 0$ since A is ample, while $h^0(Q^\vee) = 0$, else the monodromy of some summand of the unitary flat bundle Q would have trivial monodromy, contradicting the irreducibility of the monodromy representation. □

Proposition 4.3. *The smooth surface S is minimal of general type with K_S ample, and with invariants*

$$e(S) = c_2(S) = 3 + 2(n-2)(n-3) = 2n^2 - 10n + 15; \quad K_S^2 = 5(n-2)^2.$$

They have positive index $\sigma(S) = \frac{1}{3}(K_S^2 - 2e(S)) > 0$ and indeed their slope $\frac{K_S^2}{e(S)} \geq 2, 5$.

We have that the universal cover of S is the unit ball in $\mathbb{C}^2$, if and only if $n = 5$, which corresponds to the case of three distinct surfaces S', S'', S'''.

Proof. The calculation for the topological Euler–Poincaré characteristic $e(S)$ follows from the Zeuthen–Segre formula asserting that $e(S)$ equals the sum of the product $e(B)e(F) = 4(b-1)(g-1)$ with the number μ of singular fibres counted with multiplicity: here therefore $\mu = 3$ and we get $e(S) = 3+2(n-2)(n-3) = 2n^2-10n+15$.

To calculate K_S^2, we observe that K_S is numerically equivalent to the pull back of $K_Z + \frac{n-1}{n}\mathcal{B}$.

Since $\mathcal{B} \equiv 4L_1 + 4L_2 - 2\Sigma_i E_i$, where L_1, L_2 are the total transforms of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, and $K_Z = -2L_1 - 2L_2 + \Sigma_i E_i$, we obtain that $\mathcal{B} \equiv -2K_Z$, hence K_S is numerically equivalent to the pull back of $K_Z + \frac{n-1}{n}\mathcal{B} = -\frac{n-2}{n}K_Z$.

Since $-K_Z$ is ample, and $K_Z^2 = 5$, we easily obtain that S has ample canonical divisor K_S, and

$$K_S^2 = 5(n-2)^2.$$

The surface S is minimal since K_S is ample.

We now calculate the slope as

$$\frac{5(n-2)^2}{3 + 2(n-2)(n-3)} = \frac{5}{2}\frac{n-2}{n-3+\frac{3}{2(n-2)}} > \frac{5}{2}.$$

The same formula shows that the slope is a strictly decreasing function of n, tending to $\frac{5}{2}$ as $n \to \infty$, and beginning with slope $= 3$ for $n = 5$. But, by the theorem of Yau, slope equal to 3 is equivalent to having the ball as universal cover.

Consider now the case $n = 5$: the four-tuple of residue classes modulo 5 is equivalent, modulo simultaneous multiplication by a unit, to $1 + 1 + 1 + 2 = 5$, and this is the only representation via integer rests which add up to 5. Also the n_i are uniquely determined as $1 + 2 + 2 = 5$.

There are two different cases: $m_3 \neq m_i$, or (up to renumbering) $m_3 = m_0 = m_1$; in this second case there are two subcases, according to $n_0 = n_1$ or $n_0 \neq n_1$. □

Remark 4.1. The above three surfaces, which occur for $n = 5$ have already been constructed in [3].

Being ball quotients, they are rigid.

In joint work of the first author together with Ingrid Bauer, it was recently shown that the above surfaces S for $n \geq 5$ are rigid.

Another interesting question is whether the surfaces S are always $K(\pi, 1)$'s, i.e. whether their universal covering is always contractible.

Recall now the following algebraic formula for the Euler number, the so-called Zeuthen–Segre formula (see the lecture notes [6]).

Definition 4.2. Let $f : S \to B$ be a fibration of a smooth algebraic surface S onto a curve of genus b, and consider a fibre $F_t = \sum n_i C_i$, where the C_i are irreducible curves.

Then the divisorial singular locus of the fibre is defined as the divisorial part of the critical scheme, $D_t := \sum (n_i - 1) C_i$, and the Segre number of the fibre is defined as

$$\mu_t := \deg \mathcal{F} + D_t K_S - D_t^2,$$

where the sheaf $\mathcal{F}$ is concentrated in the singular points of the reduction of the fibre, and is the quotient of $\mathcal{O}_S$ by the ideal sheaf generated by the components of the vector $d\tau/s$, where $s = 0$ is the equation of D_t, and where τ is the pull back of a local parameter at the point $t \in B$.

More concretely,

$$\tau = \prod_j f_j^{n_j}, \quad s = \tau \Big/ \left(\prod_j f_j \right)$$

and the logarithmic derivative yields

$$d\tau = s \left[\sum_j n_j \left(df_j \prod_{h \neq j} f_h \right) \right].$$

The following is the refined Zeuthen–Segre formula.

Theorem 4.1. *Let $f : S \to B$ be a fibration of a smooth algebraic surface S onto a curve of genus b, and with fibres of genus g.*

Then

$$c_2(S) = 4(g-1)(b-1) + \mu,$$

where $\mu = \sum_{t \in B} \mu_t$, and $\mu_t \geq 0$ is defined as above. Moreover, μ_t is strictly positive, except if the fibre is smooth or a multiple of a smooth curve of genus $g = 1$.

Proposition 4.4. *Let S be one of the surfaces considered in this section. Then any surface X, which is homeomorphic to S has Albanese map which is a fibration onto a curve B of the same genus $b = 1/2(n-1)$ as the Albanese image of S. If moreover X is diffeomorphic to S, the Albanese fibres have the same genus $g = 2b = n - 1$ and, if the number of singular points on the fibres is finite, there are only three* *singularity on the fibres, counted with multiplicity. In particular, there are at most three singular fibres.*

Proof. The first two statements follow directly from [7], Theorem A.

For the last statement, we invoke the above refined Zeuthen–Segre formula

$$e(X) = 4(b-1)(g-1) + \mu.$$

Since b, g are the same for S and X, it follows that $\mu = 3$, which shows the third assertion.

The refined version of the Zeuthen–Segre formula implies in particular that, if D is the divisorial part of the critical locus, then $3 = \mu \geq DK_X - D^2$, where $DK_X - D^2 = 2p(D) - 2 - 2D^2$ is a positive even number.

Each nonreduced fibre $F_t = \Sigma_i n_i C_i$ gives a contribution $D_t := \Sigma_i (n_i - 1) C_i$ to D, and Zariski's lemma says that, if $D_t \neq 0$, then $D_t^2 < 0$ unless F_t is a multiple fibre.

If we had a multiple fibre $F_t = mC$, then we would have $D_t K_X - D_t^2 = D_t K_X = (m-1)/mFK_X = (2g-2)(m-1)/m \geq (g-1) \geq 4$, which is a contradiction. Hence there are no multiple fibres.

Assume that F_t is a nonreduced fibre, so that $D_t \neq 0$: then $D_t K_X - D_t^2 = 2$, since it is a strictly positive even integer which is not greater than 3.

So, if there are infinitely many singular points on the fibres, then there is exactly one nonreduced fibre and at most one more singular point; in particular, there are at most two singular fibres. □

We can summarize our main result in the following theorem, for which we give two proofs, one self-contained and based on Fujita's second theorem, the other based on the theory of hypergeometric integrals.

Theorem 4.2. *There exists an infinite series of surfaces with ample canonical bundle, whose Albanese map is a fibration $f : S \to B$ onto a curve B of genus $b = 1/2(n-1)$, with fibres of genus $g = 2b = n-1$; here $n \geq 5$ can be any integer relatively prime with 6 and f is as in Proposition 4.2.*

These Albanese fibrations yield negative answers to Fujita's question about the semiampleness of $V := f_* \omega_{S|B}$, since here $V := f_* \omega_{S|B}$ splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle, and Q is a unitary flat bundle with infinite monodromy group.

The fibration f is semistable: indeed all the fibres are smooth, with the exception of three fibres which are the union of two smooth curves of genus b, which meet transversally in one point.

For $n = 5$, we get three surfaces which are rigid, and are quotient of the unit ball in $\mathbb{C}^2$ by a torsion free cocompact lattice Γ.

The rank of A, respectively Q is in this case equal to 2.

Proof. By Propositions 4.1–4.3, the only assertion, which needs to be shown is that, if we consider the splitting of $V := f_*\omega_{S|B}$ as a direct sum $V = A \oplus Q$, where A is an ample vector bundle, and Q is a unitary flat bundle, then Q has infinite monodromy group.

We observe that the group $G = \mathbb{Z}/n$ acts on the fibration, thus we have a splitting according to the characters of G, $j \in \mathbb{Z}/n$,

$$V = \bigoplus_{j\in\mathbb{Z}/n} V_j.$$

The fact that all the fibres are smooth and that the only singular fibres are two smooth curves intersecting transversally in one point shows that the vanishing cycles are homologically trivial. Hence the local monodromies in cohomology are trivial, thus we have a flat vector bundle $\mathcal{H} := R^1 f_*(\mathbb{C})$, which is a holomorphic flat bundle having V as a holomorphic sub-bundle.

Similarly, we have a splitting

$$\mathcal{H} = \bigoplus_{j\in\mathbb{Z}/n} \mathcal{H}_j,$$

where the flat bundles $\mathcal{H}_j$ have all rank 2 for $j \neq 0$, as observed in section three.

Moreover, we have the following direct sum of complex vector bundles $\mathcal{H}_j = V_j \oplus \overline{V_{-j}}$, and there are *a priori* several possible cases:

- $\mathcal{H}_j = V_j$ and $V_{-j} = 0$, hence V_j is a flat holomorphic bundle; in this case the bundle $\mathcal{H}_j$ carries a flat Hermitian form which is positive definite;
- $\mathcal{H}_j = \overline{V_{-j}}$ and $V_j = 0$, hence V_{-j} is a flat holomorphic bundle; in this case the bundle $\mathcal{H}_j$ carries a flat Hermitian form which is negative definite;
- $\mathcal{H}_j = V_j \oplus \overline{V_{-j}}$, both summands have rank 1, and here the bundle $\mathcal{H}_j$ carries a flat Hermitian form which is indefinite. This case could *a priori* bifurcate in the cases V_j is flat, or V_j is ample (i.e. it has strictly positive degree).

First Proof:

Step 1: V is not flat.

In fact, otherwise (see for instance [9, Theorem 4])

$$0 = 12\deg(V) = K_S^2 - 8(g-1)(b-1);$$

hence $K_S^2 = 8(g-1)(b-1) = 2e(S) - 6$, contradicting Proposition 4.3.

Step 2: Hence $V = \oplus_j V_j$ admits an ample rank 1 summand $V_j = A_j$.

Step 3: It suffices to prove the theorem in the case, where n is prime.

In fact, if k divides n, $n = hk$, we have an anologous fibration $f_k : S_k \to B_k$ for the surface S_k obtained by taking the associated $(\mathbb{Z}/k)^2$ covering.

Pulling back the fibration to B, under $\psi : B \to B_k$, we obtain a surface $S' = S/G'$, where $G' = \mathbb{Z}/h$; and the fibration f factors through $f' : S' \to B$. Hence

$V' = \psi^*(V_k)$ is a direct summand of V and we are done, since V' has a unitary flat summand Q' with infinite monodromy.

Step 4: There is an eigenbundle $\mathcal{H}_j$ with infinite monodromy.

This follows from Step 2 and the following lemma.

Lemma 4.2. *If $V_j = A_j$ is an ample rank 1 summand, then $\mathcal{H}_j$ is irreducible and with infinite monodromy.*

Proof (of the Lemma). We first show that the rank two flat vector bundle is irreducible. Otherwise, there would be an exact sequence of flat vector bundles

$$0 \to \mathcal{H}' \to \mathcal{H}_j \to \mathcal{H}'' \to 0,$$

where both $\mathcal{H}', \mathcal{H}''$ have rank 1.

Since $\mathcal{H}_j = V_j \oplus \overline{V_{-j}}$, we get a nontrivial homomorphism $V_j \to \mathcal{H}_j$ which realizes V_j as a holomorphic sub-bundle. Composing with the above surjection $\mathcal{H}_j \to \mathcal{H}''$, we get a holomorphic homomorphism $V_j \to \mathcal{H}''$, which must be zero since the target has degree zero, while $V_j = A_j$ has positive degree. We deduce a nontrivial holomorphic homomorphism $V_j \to \mathcal{H}'$, which must be zero by the same argument, and we have found a contradiction to the fact that $V_j \to \mathcal{H}_j$ is injective. $\square$

Step 4. Observe preliminarily that our surfaces, the Albanese map f and all the bundles V, Q are defined over $\mathbb{Z}$.

Now, by construction we have a flat rank 2 summand $V_{-1} = \mathcal{H}_{-1}$ (since $m_0 + m_1 + m_2 + m_3 = n$). Hence, when n is prime, $\mathcal{H}_{-1}$ and $\mathcal{H}_j$ are Galois conjugate. The condition that the monodromy is infinite is obviously invariant under Galois conjugation (since a finite group of matrices transforms to a finite group under a field automorphism).

Hence also $V_{-1} = \mathcal{H}_{-1}$ has infinite monodromy, and it is a direct summand of Q with infinite monodromy.

Second Proof: By Corollary 3.1, there is $j \in (\mathbb{Z}/n)^*$, such that $\hat{\mathbb{H}}_j$ carries a monodromy invariant indefinite Hermitian form H_j, and is irreducible with infinite monodromy.

Therefore also $\mathcal{H}_j$ has infinite monodromy. Since j is a unit, it follows that $\mathcal{H}_{-1}$ and $\mathcal{H}_j$ are Galois conjugate. Hence $V_{-1} = \mathcal{H}_{-1}$ has also infinite monodromy, and the same holds for Q, of which V_{-1} is a direct summand.

Remark 4.2. In the standard case, V has a lot of flat summands.

In fact, $V_j = 0$ for $j \leq \frac{n}{3}$ (since $3j \leq n$ implies $j + j + j + [j(n-3)] < 2n \Rightarrow j + j + j + [j(n-3)] = n$); hence V_{-j} i flat for $j \leq \frac{n}{3}$.

On the other hand, for $n = 11$, we can take $m_0 = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 5$ and then only V_{10} is a flat summand, of course with infinite monodromy.

5. General Observations and Relation with Shimura Curves

Consider our surfaces $S \to B$ as yielding a curve inside the compactified moduli space of curves of genus $g = n-1$. The image of B inside $\overline{\mathfrak{M}_g}$ intersects the boundary only in points belonging to the divisor $\Delta_{g/2,g/2}$.

Moreover, under the Torelli map $\mathfrak{M}_g \to \mathfrak{A}_g$, the image does not go to the boundary, since the singular fibres have compact Jacobian.

For $n = 5$, we obtain a rigid curve inside $\overline{\mathfrak{M}_g}$, a phenomenon which is not new: compare the examples provided by double Kodaira fibrations [8].

Now, B parametrizes all the curves with an action of $\mathbb{Z}/n$, whose quotient is $\mathbb{P}^1$, and with branch locus $\mathcal{S}$ consisting of four points: because all deformations preserving the symmetry come from $H^1(C, \Theta_C)^G$, which is isomorphic to $H^1(\mathbb{P}^1, \Theta(-\mathcal{S}))$, the space of logarithmic deformations of the pair consisting of $\mathbb{P}^1$ and the four points on it.

B parametrizes, via the Torelli map, also principally polarized Abelian surfaces with such a symmetry.

The question is whether the symmetry-preserving deformations of these Abelian varieties are just the ones parametrized by B.

The main point is that (see [11] and [18], especially for more details concerning the relation with Shimura curves) the dual of $H^1(C, \Theta_C)^G$ equals $H^0(2K_C)^G$, while the tangent space to the symmetry-preserving deformations of the Abelian varieties is given by

$$\mathrm{Sym}^2(H^0(K_C))^G = \mathrm{Sym}^2\left(\bigoplus_j V_j\right)^G = \bigoplus_{j \le n/2} (V_j \otimes V_{-j}).$$

Observe that $V_0 = 0$, while, for a character j, writing as usual $\mu_{i,j} = \frac{1}{n}[m_i j]$, the condition $\sum_i \mu_{i,j} = 2$ is equivalent to dim $(V_j \otimes V_{-j}) = 1$, else one has dim $(V_j \otimes V_{-j}) = 0$.

In other words, the number of parameters for the symmetry-preserving deformations of these Abelian varieties is just the number of rank 2 ample bundles in the direct image sheaf $f_*(\omega)$.

If there is only one such ample summand, then this means that, we have a Shimura curve in $\mathfrak{A}_g$. This situation leads to a finite number of cases, which were classified by Moonen in [39] (see [18] for groups more general than cyclic groups).

Interest in these Shimura curves is due to a conjecture by Oort, that there should not be such curves as soon as g is bigger than 7, see [38] and references therein for results in this direction.

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Appendix

Proposition A.1. *Let* $m_0, m_1, m_2, m_3, n \in \mathbb{Z}$ *with* $0 < m_k \leq n - 3$ $(0 \leq k \leq 3)$ *and* $m_0 + m_1 + m_2 + m_3 = n$. *For* $j \in 1, \dots, n-1$, *let* $\hat{\mathbb{H}}_j$ *be the local system as in Sec.* 3. *Assume additionally that each of the numbers* $m_0, \dots, m_3$ *is coprime to* n (*respectively*, *assume that* j *is coprime to* n *with no further assumption on* $m_0, \dots, m_3$). *Then the local systems* $\hat{\mathbb{H}}_j$ *are irreducible for* $j = 1, \dots, n-1$ (*respectively*, *for* j *prime to* n).

Proof. By construction, the local sections of $\hat{\mathbb{H}}_{-j}$ are variations in x of periods on the desingularizations of the curves

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - x y_0)^{m_3}, \quad x \in \mathbb{C}\backslash\{0,1\}$$

of the form (given on the affine part belonging to $y_1 = 1$)

$$\int_\gamma \frac{y_0^s (1-y_0)^t (1 - x y_0)^u}{z_1^j} dy_0,$$

where s, t, u are integers, cf. [46, Sec. 2]. It is convenient to introduce integers A, B, C and rational numbers a, b, c by the following conditions:

$$A = (1-b)n = m_0, \quad B = (b + 1 - c)n = m_2,$$
$$C = an = m_3, \quad n - A - B - C = m_1.$$

Therefore

$$a = \frac{m_3}{n}, \quad b = 1 - \frac{m_0}{n}, \quad c = 2 - \frac{m_0}{n} - \frac{m_2}{n}.$$

If γ denotes integration from 0 to 1, then the above integral can be expressed as a hypergeometric function as follows:

$$\int_0^1 \frac{y_0^s (1-y_0)^t (1 - x y_0)^u}{z_1^j} dy_0$$
$$= D \cdot {}_2F_1(ja - u, jb - j + 1 + s, jc - 2j + 2 + t + s; x),$$

where D is a constant in $\mathbb{C}$, cf. [46, Formula (7)]. Hence, in order to show that $\hat{\mathbb{H}}_j$ is irreducible, it suffices to show that the hypergeometric differential equation belonging to ${}_2F_1(ja - u, jb - j + 1 + s, jc - 2j + 2 + t + s; x)$ is irreducible. This is the case, if and only if the values $ja - u$, $jb - j + 1 + s$ and the differences $(jc - 2j + 2 + t + s) - (ja - u)$, $(jc - 2j + 2 + t + s) - (jb - j + 1 + s)$ are not contained in $\mathbb{Z}$, cf. [5, Corollary 3.10]. Obviously, the latter condition holds if and only if the values $ja = \frac{jm_3}{n}$, $jb = -\frac{jm_0}{n} + j$ and

$$ja - jc = \frac{jm_3}{n} + \frac{jm_0}{n} + \frac{jm_2}{n} - 2j = -\frac{jm_1}{n} - j$$

as well as

$$jb - jc = j - \frac{jm_0}{n} + \frac{jm_0}{n} + \frac{jm_2}{n} - 2j = \frac{jm_2}{n} - j$$

are not contained in $\mathbb{Z}$. This holds by our assumptions. □

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