HIGHER DIMENSIONAL LEMNISCATES: THE GEOMETRY OF r PARTICLES IN n-SPACE WITH LOGARITHMIC POTENTIALS

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Contents

Introduction

What is a lemniscate?

If the reader looks on the web, or in the classical literature, as an answer, he will find the lemniscate of Bernoulli, which is the singular level set of the function $|z^2 - c^2|$.

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The level sets of such a function are called *Cassini's ovals*: they are just the sets of points z in the complex plane $\mathbb{C}$ such that the product $|z-c||z+c|$ of the respective distances of z from the points $+c, -c$, where $c \in \mathbb{R}$, is equal to a constant (whereas ellipses are defined by the condition that the sum of the distances from two given points is constant).

Since any univariate polynomial $P(z) \in \mathbb{C}[z]$ is a product of linear factors, $P(z) = \prod_1^r (z - w_j)$, we see that if we take r points $w_1, \dots, w_r$ in the plane, the locus of points z such that the product of the r respective distances $|z - w_j|$ is equal to a constant, is just a level set of the absolute value $|P(z)|$ of the complex polynomial $P(z)$.

Using this analogy, the second author and Marco Paluszny in [CP91] defined a big lemniscate as a singular level set of the absolute value $|P(z)|$ of the complex polynomial $P(z)$, and a small lemniscate as a singular connected component of a big lemniscate.

If we take the square $F(z) = |P(z)|^2 = P(z)\overline{P(z)}$, we obtain a real polynomial $F \in \mathbb{R}[x, y]$ and if the points $w_1, \dots, w_r$ are distinct, they are absolute minima with nondegenerate Hessian, and the major results of [CP91] consisted in describing the topological configurations of the union of the big lemniscates (resp.: of the small lemniscates) in the special situation where $f := \log(F)$ is a global Morse function, i.e, a function whose critical points y_i all have a non-degenerate Hessian, and such that all the critical values $v_i := f(y_i)$ are different.

Here, the critical points of F are just the absolute minima $w_1, \dots, w_r$, and the roots $y_1, \dots, y_{r-1}$ of the complex derivative $P'(z)$: the points y_i have (negativity) index 1, and are thus saddle points.

There is a beautiful order which governs the pictures of these lemniscates, and leads to nice and interesting generating functions. The key idea is to enumerate the components where the topological configuration is fixed as the orbits of a subgroup of the braid group acting on the set of edge labelled trees.

While the first two authors ([BC97]) generalized these investigations to the case of an algebraic function on a Riemann surface (i.e., a connected complex manifold of complex dimension equal to 1), Marco Paluszny ([PMO05] and [AMOP06]) defined and started to investigate the similar loci in $\mathbb{R}^3$, producing nice pictures of the corresponding configurations.

The first purpose of the present paper is to lay the foundation of the theory of lemniscates in $\mathbb{R}^N$, proving some rather strong basic results.

Definition 0.1.

(1) *Let $w_1, \dots, w_r \in \mathbb{R}^N$ be distinct points, and consider the functions*

$$F : \mathbb{R}^N \to \mathbb{R}^+, F(x) := \prod_1^r |x - w_j|^2, \quad f(x) := \log F(x) = \sum_1^r \log(|x - w_j|^2).$$

(2) *A* big lemniscate *(for $w_1, \ldots, w_r \in \mathbb{R}^N$) is defined to be a singular level set Γ_c of $f(x)$. The big lemniscate configuration of f is the union $\Gamma(f)$ of the singular level sets Γ_c.*

(3) *A* small lemniscate *(for $w_1, \ldots, w_r \in \mathbb{R}^N$) is defined to be a connected component Λ_c of a level set $\Gamma_c = \{x | f(x) = c\}$, which is singular. The small lemniscate configuration of f is the union $\Lambda(f)$ of the small lemniscates.*

(4) *The configuration $\Lambda(f)$ of small lemniscates is said to be* weakly generic *if the function $f(x)$ is a (local) Morse function, i.e., its critical points y_i all have a non-degenerate Hessian.*

(5) *The configuration of big lemniscates $\Gamma(f)$ is said to be* generic *if the function $f(x)$ is a global Morse function, i.e., it is a Morse function and the critical values $f(y_i)$ are all different (notice that the absolute minima for $F(x)$ are just the zeros of $F(x)$, i.e., the points w_j, which are all automatically non degenerate).*

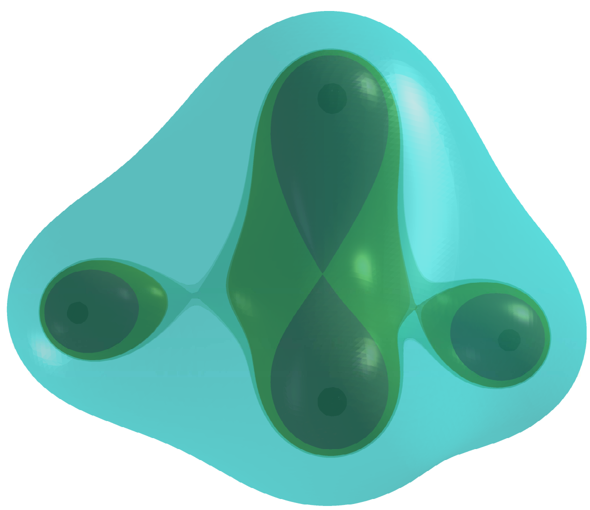

FIGURE 1. A generic big lemniscate configuration for $r = 4$ surrounded by a non singular level set. The four points are still visible at the interior of the lemniscates.

In the case where the points $w_1, \ldots, w_r$ lie in an affine plane contained in $\mathbb{R}^N$, the situation is easy to analyse, see corollary 4.3, and, like in the case $N = 2$, if f is a Morse function, we have just $r - 1$ critical points of (negativity) index 1.

The following are our main theorems.

Theorem 0.2. *Let $w_1, \dots, w_r \in \mathbb{R}^N$ be distinct points, not lying in a (real) affine plane, and consider the functions*

$$F : \mathbb{R}^N \to \mathbb{R}^+, F(x) := \prod_1^r |x - w_j|^2, \quad f(x) := \log F(x) = \sum_1^r \log(|x - w_j|^2).$$

Then at every critical point of $f(x)$ (resp.: of $F(x)$) the Hessian has positivity at least $N - 1$.
In particular, the index of negativity can only be 0 or 1.
Moreover, the set of critical points consists of isolated points, plus possibly a finite set of circles (compact real submanifolds of dimension 1) consisting of local minima.

Theorem 0.3. *Let $w_1, \dots, w_r \in \mathbb{R}^N$ and let*

$$F : \mathbb{R}^N \to \mathbb{R}^+, F(x) := \prod_1^r |x - w_j|^2, \quad f(x) := \log F(x) = \sum_1^r \log(|x - w_j|^2)$$

be as in theorem 0.2.

(1) *Assume that F is a (local) Morse function: then $F(x)$ has r absolute minima, h local minima, and exactly $r + h - 1$ critical points of (negativity) index 1.*
(2) *There are examples already in $\mathbb{R}^3$, where h can be arbitrarily large.*

Remark 0.4. Part (1) of the above theorem is a direct consequence of theorem 0.2 and standard Morse theory. Part (2) is shown in section 8, see in particular proposition 9.1.

The results are based, once again, on elementary complex analysis: but this time in several variables.

The first idea is to take an isometric embedding of $\mathbb{R}^N$ into $\mathbb{C}^n$ ($n = [\frac{N-1}{2}] + 1$). This is crucial, since on a complex vector space any real bilinear form can be written as the sum $Q + \mathcal{L} + \bar{Q}$ where Q is a complex bilinear form, and $\mathcal{L}$ (the Levi form) is Hermitian: the easiest case being $n = 1$, where, if $z = x + iy$, $a, b, c \in \mathbb{R}$,

$$(a + ib)z^2 + (a - ib)\bar{z}^2 + cz\bar{z} = 2a(x^2 - y^2) - 4bxy + c(x^2 + y^2).$$

Then we prove (under the assumption that $w_1, \dots, w_r$ are not contained in a complex line), using the classical Fubini-Study form, that the function f is strictly plurisubharmonic, i.e., its Levi form $\mathcal{L}$ is strictly positive definite.
The second trick is then to choose an appropriate isometric embedding as above, in order to prove the statement about the positivity of the Hessian at each critical point. The rest is a consequence of the generalised Morse lemma.

For theorem 0.2, we just use topology and Morse theory, and we exploit symmetry in order to produce 'extra' local (but not global) minima.

The topological configuration is then easily described by the graph whose edges are the saddle points of F, in a similar fashion to in [CP91].

The second main purpose of this paper is to propose as a theme of investigation the description of the configurations of generic big and small lemniscate configurations. Even if we cannot use the Riemann existence theorem as in real dimension two, our first main theorem yields a very strong information: the critical points have positivity at least $N-1$, hence, by the generalized Morse lemma, there are local analytic coordinates $u_1, \dots, u_N$ such that f has one of these two normal forms:

$$f(u) = u_1^2 + \cdots + u_{N-1}^2 \pm u_N^k; \tag{1}$$

$$f(u) = u_1^2 + \cdots + u_{N-1}^2. \tag{2}$$

In the first case the critical points are isolated, in the second case, since the critical set is shown to be compact, we obtain smooth curves diffeomorphic to circles.

The compactness of the set of critical points is shown by the generalization of a theorem of Gauss: lemma 4.1 asserts that the critical points lie in the convex hull of the points $w_1, \dots, w_r$.

Definition 0.5.

(1) *Define $\mathcal{GL}(r,N)$ as the open set of the space $(\mathbb{R}^N)^r$ of r (distinct) points in $\mathbb{R}^N$ such that the function $f(x) = \sum_{k=1}^r \log|x-w_k|^2$ is a global Morse function, i.e. we have a generic big lemniscate configuration $\Gamma(f)$.*

(2) *We say that two big lemniscate configurations $\Gamma(f_1)$, $\Gamma(f_2)$ have the same topological type if there is a homeomorphism of the pair $(\mathbb{R}^N, \Gamma(f_1))$ with the pair $(\mathbb{R}^N, \Gamma(f_2))$.*
We have then a map from the set $\pi_0(\mathcal{GL}(r,N))$ of the connected components of the set of lemniscate generic r-tuples of points to the set of topological types.

(3) *Denote by $b(r,N)$ the number of connected components of $\mathcal{GL}(r,N)$ and by $a(r,N)$ the corresponding number of topological types of the big lemniscate configurations, by $c(r,N)$ the corresponding number of topological types of the small lemniscate configurations.*
We define the corresponding generating functions as

$$B_N(t) := \sum_r \frac{b(r+1,N)}{r!} t^r, \;\; A_N(t) := \sum_r \frac{a(r+1,N)}{r!} t^r,$$

$$C_N(t) := \sum_r c(r+1,N) t^r.$$

Similarly, define $b(r,N,h)$ the number of connected components of $\mathcal{GL}(r,N)$ where f has h local minima, and define similarly

$$a(r,N,h), \;\; c(r,N,h), \;\; B_{N,h}(t), A_{N,h}(t), C_{N,h}(t).$$

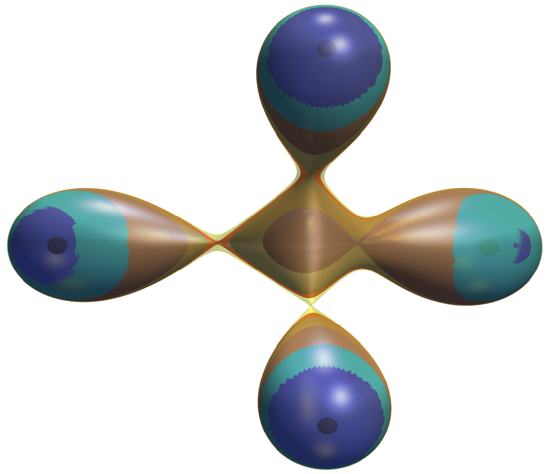

FIGURE 2. Another generic big lemniscate configuration for $r = 4$.

In the appendix to [CP91] it is shown that, in dimension $m = 2$, where $h = 0$, $b(r, 2) = a(r, 2)$ and the function

$$A_2(t) = B_2(t) = \frac{1}{1 - \sin t}.$$

More complicated results were shown for the number of small lemniscate configurations.

In this view, these are the questions which we would like to pose.

Question 0.6.

(1) *Given (r, N), which is the maximal number $M(r, N) = max\{h\}$ of (non global) local minima for the function f?*
(2) *What is the form of the generating functions $A_N(t), B_N(t), B_{N,h}(t), A_{N,h}(t)$?*
(3) *Is it true that the $A_N = B_N$ as in the case $N = 2$?*

To shed light on the above questions, let us observe that the connected components of the configuration space of r distinct points consist of several domains, separated by walls of two different types.

The first type of walls contain as general points r-tuples $w_1, \ldots, w_r$ such that the associated function f has a simple singularity of the form $u_1^2 + \cdots + u_{N-1}^2 + u_N^3$; a pair of nondegenerate critical points, of respective indices 0, 1, go to disappear when crossing the wall in one direction: we shall call these *walls of quantitative type,* since the number of critical points changes.

The second type of walls are those where two critical values become equal: here crossing the wall the topological type of the big lemniscate configuration may change, hence we shall call these *walls of qualitative type.*

Finally, concerning the first question, we get examples with 4 points in $\mathbb{R}^3$ and $h = 1$, and, for every $h \geq 2$, with $r = 3h$ points in $\mathbb{R}^3$ and h non global minima. These examples suggest the conjecture that h may be bounded by a constant times r.

Let's end this introduction by describing a straightforward but potentially quite useful application of the Gauss-type lemma 4.1, showing that the critical points lie in the convex hull of the points $w_1, \dots, w_r$.

Theorem 0.7. *1) Let $\Omega \in \mathbb{R}^N$ be a bounded domain and let $w_1, \dots, w_r \in \mathbb{R}^N$ be r pairwise distinct points. Consider*

$$f \colon \bar{\Omega} \to \mathbb{R} \cup \{-\infty\}, \quad f(x) = \sum_{k=1}^{r} \log |x - w_k|^2.$$

Then all maxima of f (in $\bar{\Omega}$) are contained in $\partial\Omega$.
2) Moreover, if the closure $\bar{\Omega}$ euqals the convex hull $Conv(\{w_1, \dots, w_r\})$ of the points $w_1, \dots, w_r$, then all maxima of $f|\bar{\Omega}$ are contained in $\partial\Omega \setminus \mathcal{F}$, where $\mathcal{F}$ is the union of the interior parts of the faces of $\partial\Omega$ of dimension at least two ($\Longleftrightarrow$ all maxima are contained in one dimensional faces of $\partial\Omega$).

Remark 0.8. Of course it would be interesting from the point of view of physics to consider also the case of non logarithmic potentials, of the type

$$f(x) = \sum_{k=1}^{r} |x - w_k|^{\alpha}.$$

But it is not clear that one can use methods from complex analysis in this more general situation.

1. The Hermitian Levi form

Definition 1.1. Let $U \subset \mathbb{C}^n$ be a domain and let $f \colon U \to \mathbb{R}$ be a function, which is twice continuously differentiable. Moreover, let $z_0 \in U$ be a point. The Hermitian form $\mathcal{L}_{f,z_0}$ given by

$$\mathcal{L}_{f,z_0}(w) := \sum_{i,j=1}^{n} \frac{\partial^2 f(z_0)}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{pmatrix} \in \mathbb{C}^n,$$

is called the *(Hermitian) Levi form of* f *at* z_0.

Here we use the standard formalism: if $z = x + iy$, then

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \ , \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) \ .$$

Then we have the following result related to the Fubini-Study metric.

Lemma 1.2. *Let* $U \subset \mathbb{C}^n \backslash \{0\}$. *Then the Hermitian Levi form* $\mathcal{L}_{f,z}$ *of* $f(z) := \log |z|^2$ *is positive semidefinite for each* $0 \neq z \in U$.
Moreover, at each point $0 \neq z \in U$, *the Levi form* $\mathcal{L}_{f,z}$ *of* f *at* z *has positivity* $n-1$, *and the kernel is the line* $\mathbb{C}z$.

Proof. We denote, for $v, w \in \mathbb{C}^n$, the *standard Hermitian product* by

$$\langle v, w \rangle := \sum_{k=1}^{n} v_k \bar{w}_k.$$

Then for $z \in U$ and $w \in \mathbb{C}^n$ we have

$$(3) \quad \mathcal{L}_{f,z}(w) = \sum_{i,j=1}^{n} \frac{\partial^2 (\log(\sum_{k=1}^{n} z_k \bar{z}_k))}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j = \sum_{i,j=1}^{n} \frac{\partial}{\partial z_i} (\frac{z_j}{|z|^2}) w_i \bar{w}_j =$$

$$= \sum_{i,j=1}^{n} \frac{|z|^2 \delta_{ij} - \bar{z}_i z_j}{|z|^4} w_i \bar{w}_j = \frac{1}{|z|^4} (|z|^2 |w|^2 - |\langle w, z \rangle|^2).$$

From the Cauchy-Schwartz inequality it follows now

- $\mathcal{L}_{f,z} \geq 0$ for all, $0 \neq z \in U$, and
- $\mathcal{L}_{f,z}(w) = 0$ if and only if $w \in \mathbb{C}z$.

□

We consider now the following situation: let $w_1, \dots, w_r \in \mathbb{C}^n$ be r different points and consider the functions

$$F \colon \mathbb{C}^n \setminus \{w_1, \dots, w_r\} \to \mathbb{R},$$

respectively

$$f \colon \mathbb{C}^n \setminus \{w_1, \dots, w_r\} \to \mathbb{R}$$

given by

- $F(z) := \prod_{i=1}^{r} |z - w_i|^2 = \prod_{i=1}^{r} F_i$, respectively
- $f(z) := \sum_{i=1}^{r} \log |z - w_i|^2 = \sum_{i=1}^{r} f_i$.

Then we have the following:

Lemma 1.3.

(1) $f(z) := \sum_{i=1}^{r} \log |z - w_i|^2$ *is plurisubharmonic, i.e.,* $\mathcal{L}_{f,z} \geq 0$ *for all* $z \neq 0$.

(2) *$\mathcal{L}_{f,z} > 0$ for all $z \neq 0$ except if all the points $w_1, \dots, w_r$ are contained in an affine complex line.*
(3) *If $w_1, \dots, w_r$ are contained in an affine complex line L, then for all $z \in L$ the Levi form $\mathcal{L}_{f,z}$ has exactly nullity 1, in the direction of L.*

Proof. If $z \neq w_k$, then the Levi form $\mathcal{L}_{f_k,z}$ of f_k is ≥ 0 in z and its kernel is the line $\mathbb{C}(z - w_k)$ by lemma 1.2. Since

$$\mathcal{L}_{f,z} = \sum_{k=1}^{r} \mathcal{L}_{f_k,z},$$

we see that

- $\mathcal{L}_{f,z} \geq 0$ for all $z \neq w_1, \dots, w_r$, and
- $v \in \ker(\mathcal{L}_{f,z}) \iff v \in \mathbb{C}(z - w_k)\ \forall\ k = 1, \dots, r.$

Therefore $\mathcal{L}_{f,z}$ has non trivial kernel if and only if all the vectors $z - w_1, \dots, z - w_r$ are proportional. This holds if and only if there is an element $u \in \mathbb{C}^n$ such that $w_1, \dots, w_r \in z + \mathbb{C}u$.

3) Moreover, if $v \in \ker(\mathcal{L}_{f,z}) \setminus \{0\}$, $v \in \mathbb{C}u$, i.e., v lies in the direction of L. □

2. Linear complex structures

Consider now $X = \mathbb{R}^{2n}$ together with an $\mathbb{R}$-valued symmetric bilinear form

$$H \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}.$$

Remark 2.1. A *complex structure* (or a $\mathbb{C}$-*structure*) on X is a Hodge decomposition

$$X \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \bar{V},$$

where $V \subset X \otimes_{\mathbb{R}} \mathbb{C}$ is a complex sub(-vector-)space, such that

$$X = \{v + \bar{v} | v \in V\}.$$

We can extend H to a symmetric $\mathbb{C}$-bilinear form on $X \otimes_{\mathbb{R}} \mathbb{C}$, which we denote by $H_{\mathbb{C}}$. I.e., for $x = v + \bar{v},\ y = w + \bar{w} \in X$ we have: $H(x, y) = H_{\mathbb{C}}(v + \bar{v}, w + \bar{w})$.

More precisely, we get:

$$H(x, y) = H_{\mathbb{C}}(v + \bar{v}, w + \bar{w}) = H_{\mathbb{C}}(v, w) + H_{\mathbb{C}}(\bar{v}, \bar{w}) + H_{\mathbb{C}}(v, \bar{w}) + H_{\mathbb{C}}(\bar{v}, w). \tag{4}$$

Using the above formula, we have the following well known

Lemma 2.2. *Let H be a symmetric (real) bilinear form on $\mathbb{R}^{2n}$, endowed with a given complex structure (i.e., $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Then there is a decomposition*

$$H = Q + \bar{Q} + \mathcal{L},$$

where Q is a symmetric $\mathbb{C}$-bilinear form and $\mathcal{L}$ is a Hermitian form.

Proof. This follows from equation (4) setting $Q := H_{\mathbb{C}}(v, w)$ and $\mathcal{L} := H_{\mathbb{C}}(v, \bar{w}) + H_{\mathbb{C}}(\bar{v}, w)$. Then $\bar{Q}(v, w) = \overline{H_{\mathbb{C}}(v, w)} = H_{\mathbb{C}}(\bar{v}, \bar{w})$. □

Remark 2.3. Let $U \subset \mathbb{C}^n$ be a domain and let $g\colon U \to \mathbb{R}$ be a function, which is twice continuously differentiable. Applying lemma 2.2 to the Hessian H_g of g in $z \in U$, we get:

$$H_{g,z} = Q_{g,z} + \bar{Q}_{g,z} + 2\mathcal{L}_{g,z},$$

where the matrix of $Q_{g,z}$ is given by $(\frac{\partial^2 g}{\partial z_j \partial z_k})_{1 \leq j,k \leq n}$ and $\mathcal{L}_{g,z}$ is the Levi form of g in z.

Remark 2.4. If the points $w_1, \ldots, w_r$ are contained in a complex line, then from the formula above (remark 2.3) it follows that the non degenerate critical points of $f(z) := \sum_{i=1}^r \log |z - w_i|^2$ have negativity 1 (cf. also lemma 1.1. in [CP91]).

3. The case where the real affine span of $w_1, \ldots, w_r$ has dimension ≥ 3

We can prove the following

Proposition 3.1. *Consider $X = \mathbb{R}^{2n}$ with the standard Euclidean metric and let H be a symmetric bilinear form with positivity $p \leq 2n - 2$. Then there is a $\mathbb{C}$-structure on X such that*

- *there is a $\mathbb{C}$-basis $v_1, \ldots, v_n$ (for this $\mathbb{C}$-structure),*
- *$\{v_1, \bar{v}_1, \ldots, v_n, \bar{v}_n\}$ is a unitary basis for the standard Hermitian product on $X \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2n}$,*
- *the Levi form $H_{\mathbb{C}}(v, \bar{v})$ is not positive definite.*

Proof. Since the positivity p of H fulfills $p \leq 2n - 2$, there is an orthonormal basis $e_1, \ldots, e_{2n}$ (w.r.t. the Euclidean metric) such that

- $H(e_i, e_j) = 0$ for $i \neq j$,
- if $\lambda_j := H(e_j, e_j)$, then $\lambda_1, \lambda_2 \leq 0$.

Set

- $\hat{v}_j := e_{2j-1} + ie_{2j}$ for $1 \leq j \leq n$, and
- $v_j := \frac{1}{\sqrt{2}} \hat{v}_j$, $1 \leq j \leq n$.

Then $v_1, \bar{v}_1, \ldots, v_n, \bar{v}_n$ is a unitary basis of $\mathbb{C}^{2n} = X \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the standard Hermitian product. Moreover, we have

$$H_{\mathbb{C}}(v_j, \bar{v}_j) = \frac{1}{2} H_{\mathbb{C}}(e_{2j-1} + ie_{2j}, e_{2j-1} - ie_{2j}) = \frac{1}{2}(\lambda_{2j-1} + \lambda_{2j}). \tag{5}$$

In particular $H_{\mathbb{C}}(v_1, \bar{v}_1) \leq 0$. □

As a consequence of the above considerations we get the following:

Proposition 3.2. *Assume that the real affine span of the points $w_1, \dots, w_r \in \mathbb{R}^{2n}$ has (real) dimension ≥ 3. Then at each critical point x of*

$$f(z) := \sum_{i=1}^{r} \log |z - w_i|^2 = \sum_{i=1}^{n} f_i$$

the index of positivity is at least $2n - 1$.

Proof. Consider the Hessian H_f of f at the critical point $x \in \mathbb{R}^{2n}$ and assume that for the positivity p it holds $p \leq 2n - 2$. Then by proposition 3.1 there is a complex structure on X and a $\mathbb{C}$-basis $v_1, \dots, v_n$ such that $\{v_1, \bar{v}_1, \dots, v_n, \bar{v}_n\}$ is a unitary basis for the standard Hermitian product on $X \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2n}$ and the Hermitian form $H_{f,\mathbb{C}}(v, \bar{v})$ in x is not positive definite. But $H_{f,\mathbb{C}}(v, \bar{v}) = 2\mathcal{L}_{f,x}(v, \bar{v})$, contradicting lemma 1.3. □

Remark 3.3. In particular, if $f := \sum_{i=1}^{r} \log |z - w_i|^2$ is a local Morse function, it has only h local minima in $\mathbb{R}^{2n} \setminus \{w_1, \dots, w_r\}$ and exactly $(h + r - 1)$ other critical points, each with positivity $(2n - 1)$ and negativity 1.

Remark 3.4. If we assume that the r points $w_1, \dots, w_r \in \mathbb{R}^{2n}$ are contained in a real affine plane, then without loss of generality we may assume:

- $X = \mathbb{C} \times \mathbb{C}^{n-1}$,
- $w_1, \dots, w_r \in \mathbb{C} \times \{0\}$, $w_j = (\xi_j, 0)$, $\xi_j \in \mathbb{C}$.

Then

$$f(z) := \sum_{i=1}^{r} \log |z - w_i|^2 = \sum_{k=1}^{n} f_k,$$

where $f_k(z) = f_k(z_1, z_2) = \log(|z_1 - \xi_k|^2 + |z_2|^2)$, $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}^{n-1}$.

4. Geometric properties of $f(x) := \sum_{i=1}^{r} \log |x - w_i|^2$

We have the following extension of a classical result due to Gauss.

Lemma 4.1. *Let $w_1, \dots, w_r$ be r different points in $\mathbb{R}^N$. Then the critical points of*

$$f(x) = \sum_{k=1}^{r} \log |x - w_k|^2 = \sum_{k=1}^{r} f_k$$

lie in the convex hull of $w_1, \dots, w_r$.

Proof. Let $x \in \mathbb{R}^N \setminus \{w_1, \dots, w_r\}$ and set $g_k(x) := |x - w_k|^2$. Then $f_k = \log g_k$ and

$$\operatorname{grad}(f_k)(x) = \frac{2(x - w_k)}{|x - w_k|^2} = 2(x_k - w_k) g_k^{-1}(x).$$

Therefore x is a critical point of f if an only if

$$(6)\quad \sum_{k=1}^{r}(x-w_k)g_k^{-1}(x)=0 \iff x\sum_{k=1}^{r} g_k^{-1}(x)=\sum_{k=1}^{r} w_k g_k^{-1}(x) \iff$$
$$\iff x=\frac{\sum_{k=1}^{r} w_k g_k^{-1}(x)}{\sum_{k=1}^{r} g_k^{-1}(x)}=\sum_{k=1}^{r}\Big(\frac{g_k^{-1}(x)}{\sum_{k=1}^{r} g_k^{-1}(x)}\Big)w_k$$

Note that for $x \in \mathbb{R}^N \setminus \{w_1,\dots,w_r\}$ we have $g_k(x)>0$, hence also

$$t_k := \frac{g_k^{-1}(x)}{\sum_{k=1}^{r} g_k^{-1}(x)} > 0,$$

and $\sum_{k=1}^{r} t_k = 1$. This proves the claim. □

Lemma 4.2. *Let $w_1,\dots,w_r$ be r different points in $\mathbb{R}^N$. The Hesse matrix of $f(x)=\sum_{k=1}^{r}\log|x-w_k|^2$ in x is given by*

$$H_{f,x}=2\sum_{k=1}^{r}\frac{|x-w_k|^2E_N-2(x-w_k)(x-w_k)^T}{|x-w_k|^4},$$

where E_N is the $N\times N$ identity matrix and $(x-w_k)^T=(x_1-w_{k,1},\dots,x_N-w_{k,N})$.

Proof. In fact,

$$(7)\quad (H_{f,x})_{ij}=\sum_{k=1}^{r}\frac{\partial^2(\log|x-w_k|^2))}{\partial x_i\partial x_j}=\sum_{k=1}^{r}\frac{\partial}{\partial x_i}\Big(\frac{2(x_j-w_{k,j})}{|x-w_k|^2}\Big)=$$
$$=2\sum_{k=1}^{r}\frac{|x-w_k|^2\delta_{ij}-2(x_j-w_{k,j})(x_i-w_{k,i})}{|x-w_k|^4}.$$

□

Corollary 4.3. *Assume that the points $w_1,\dots,w_r$ lie in an affine subspace V. Without loss of generality we may assume that we have a decomposition of $\mathbb{R}^N$ in an orthogonal direct sum, i.e., $\mathbb{R}^N=V\oplus V^\perp$ where $w_1,\dots,w_r\in V$.*

Then the critical points x of f lie in V and the Hessian of f is of the form $H_f=H_{f|V}+H'$, where $H_{f|V}$ is the Hessian of $f|V$, and H' is a positive definite quadratic form on $V^\perp$.

Proof. Since the critical points of f lie in the convex hull of the points $w_1,\dots,w_r\in V$, it follows that they also lie in V. It is easy to see that for the statement about the Hessian it suffices to prove the analogous statement for each summand f_k of f. We have that the Hessian of f_k equals to

$$H_{f_k,x} = 2(\frac{|x-w_k|^2 E_N - 2(x-w_k)(x-w_k)^T}{|x-w_k|^4}).$$

If x is a critical point of f, then $x \in V$, i.e., $x = v + 0 \in V \oplus V^\perp$. Then (since also $w_k \in V$, the negative part of the Hessian of f_k is zero on $V^\perp$)

$$H_{f_k,x}(u_1, u_2) = H_{f|V}(u_1) + 2|x-w_k|^2(|u_2|^2).$$

□

The above considerations as well as proposition 3.2 allow us now to prove the following result

Theorem 4.4. *Let $w_1, \ldots, w_r \in \mathbb{R}^N$ be r distinct points and let*

$$f\colon \mathbb{R}^N \setminus \{w_1, \ldots, w_r\} \to \mathbb{R}$$

be given by

$$f(x) = \sum_{k=1}^{r} \log|z-w_k|^2.$$

Then f has only critical points of negativity 0 or 1.

Proof. If $N = 2n$ and the real affine span of $w_1, \ldots, w_r$ has dimension ≥ 3 this follows from proposition 3.2. If $N = 2n-1$ and the real affine span of $w_1, \ldots, w_r$ has dimension ≥ 3 then, we embed $\mathbb{R}^N$ in $\mathbb{R}^{2n} = \mathbb{R}^N \times \mathbb{R}$ and the claim follows from proposition 3.2 and corollary 4.3. If instead the real affine span of $w_1, \ldots, w_r$ has dimension ≤ 2, then we use lemma 1.1. of [CP91] and corollary 4.3.

□

Proof of Theorem 0.2. The first part follows from the above theorem.

To prove the second part we need the following generalized Morse Lemma (cf. e.g. Satz 5.5 in [Cat99]):

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \to \mathbb{R}$ be a real analytic function. Let $p \in \Omega$ be a critical point of f. Assume that the Hessian $H_{f,p}$ has positivity index $\geq N-1$. Then there are local coordinates $u_1, \cdots, u_N$ centered at p such that:*

$$f(u_1, \cdots, u_N) = u_1^2 + \cdots + u_{N-1}^2 + F(u_N),$$

where either $F(u_N) = c \pm u_N^d$, $d \geq 2$, or $F(u_N) \equiv c, c \in \mathbb{R}$.

Denote by $\mathcal{C}$ the set of critical points of f and let $p \in \mathcal{C}$. Since the positivity of $H_{f,p}$ is at least $N-1$, we have to consider four cases for p:

- either p is a non-degenerate critical point with negativity 1 or
- p is a non-degenerate local minimum or

- $H_{f,p}$ is positive semi-definite, with positivity $N-1$, and we can apply the above generalized Morse lemma since F is real analytic.

  Since the critical points near p are solutions of the system of equations

$$\begin{cases} u_1 = 0\,, \\ \vdots \\ u_{N-1} = 0\,, \\ F'(u_N) = 0 \end{cases}$$

  p is an isolated critical point if $F'(u_N)$ is not identically zero, and a local minimum iff d is even and the sign equals $+1$;
- if $F'(u_N) \equiv 0$ the set $\mathcal{C}$ is near p a 1-dimensional embedded submanifold defined by the equations $u_1 = \cdots = u_{N-1} = 0$. Moreover, it is clear that the points of this 1-dimensional submanifold are local minima. Hence a connected component of $\mathcal{C}$ is either an isolated point or a 1-dimensional embedded submanifold consisting of local minima. Since $\mathcal{C}$ is compact (because it is closed and bounded because contained in the convex hull of the points $w_1, \cdots, w_r$) the 1-dimensional connected components of $\mathcal{C}$ are embedded circles.

□

The following result, which proves a conjecture in computer vision, is now quite obvious.

Corollary 4.6. *(= Theorem 0.7).*
1) Let $\Omega \in \mathbb{R}^N$ be a bounded domain and let $w_1, \dots, w_r \in \mathbb{R}^N \setminus \partial\Omega$ be r different points. Consider

$$f\colon \bar{\Omega} \to \mathbb{R} \cup \{-\infty\}, \quad f(x) = \sum_{k=1}^{r} \log |x - w_k|^2.$$

Then all maxima of f (in $\bar{\Omega}$) are contained in $\partial\Omega$.
2) Moreover, if $\bar{\Omega} = \mathrm{Conv}(\{w_1, \dots, w_r\})$ is the convex hull of the points $w_1, \dots, w_r$, then all maxima of $f|\bar{\Omega}$ are contained in $\partial\Omega \setminus \mathcal{F}$, where $\mathcal{F}$ is the interior of the union of the faces of $\partial\Omega$ of dimension at least two ($\iff$ all maxima are contained in one dimensional faces of $\partial\Omega$).

Proof. Without lost of generality we can assume N to be even. The proof of 1) follows from the fact that f is a plurisubharmonic function w.r.t. any complex structure on $\mathbb{R}^N$ compatible with the standard metric. The proof of 2) is by contradiction. Assume that a local maximum x_0 of $f|\bar{\Omega}$ is in the interior of a face A of dimension greater than one. Then there is a 2-dimensional affine plane Π through x_0 contained in A. As in Proposition 3.2 we can construct a complex structure J compatible with the metric of $\mathbb{R}^N$ such that Π is a complex line w.r.t. J. Then the restriction of f to $\Pi \cap A$ is subharmonic and has a local maximum at the interior point x_0. Thus f must be constant on the affine plane Π. But this is not possible since $f(x)$ goes to infinity as $x \in \Pi$ goes to infinity. □

5. Symmetries give rise to local minima

Assume that $G \leq \mathbb{O}(N) := \{A \in \mathrm{Mat}(N, N, \mathbb{R}) | A^T = A^{-1}\}$ is a finite subgroup and that $\mathbb{R}^N$ is an irreducible G-representation.

We choose a set of points $\Sigma = \{w_1, \ldots, w_r\} \subset \mathbb{R}^N \setminus \{0\}$ which is a union of G-orbits. Hence the two functions

- $F(x) := \prod_{i=1}^r |x - w_i|^2 = \prod_{i=1}^n F_i$, respectively
- $f(x) := \sum_{i=1}^r \log |x - w_i|^2 = \sum_{i=1}^n f_i$.

are G-invariant functions.

Since the origin $0 \in \mathbb{R}^N$ is a fixed point of G, and since f is G-invariant, it follows that $Df(0)$ is also G-invariant. By the irreducibility of $(\mathbb{R}^N)^\vee \cong \mathbb{R}^N$ as G-representation, it follows that $Df(0) = 0$, i.e., 0 is a critical point of f.

Under the above assumptions we can now prove the following:

Proposition 5.1. *Let $G \leq \mathbb{O}(N)$ be a finite subgroup such that $\mathbb{R}^N$ is an irreducible G-representation. Suppose that $\Sigma = \{w_1, \ldots, w_r\} \subset \mathbb{R}^N \setminus \{0\}$ is a union of G-orbits and that Σ is not contained in an affine plane. Then 0 is a local minimum of $F(x) := \prod_{i=1}^r |x - w_i|^2$ resp. of $f(x) := \sum_{i=1}^r \log |x - w_i|^2$.*

Proof. If the Hessian $H_{f,0}$ of f in 0 is identically zero, then by remark 2.3 the Levi form $\mathcal{L}_{f,0}$ is identically zero. Lemma 1.3 implies then that Σ is contained in an affine plane, a contradiction. Therefore $H_{f,0}$ is not identically zero, whence it is non-degenerate, since otherwise $\ker H_{f,0}$ is a non-trivial G-invariant subspace of $\mathbb{R}^N$, contradicting the irreducibility of the G-representation.

We know by proposition 3.2 that the positivity of the Hessian H_f at a critical point is at least $N - 1$. This means that either

i) $H := H_{f,0} > 0$, or
ii) the positivity of H is $N - 1$ and the negativity is 1.

In the second case, there are Euclidean coordinates $(x_1, \ldots, x_{N-1}, y)$ such that (up to a multiplicative constant), we have

$$H(x_1, \ldots, x_{N-1}, y) = \sum_{i=1}^{N-1} a_i x_i^2 - y^2, \ a_i > 0.$$

Then G leaves the cone

$$\mathcal{C} := \{x \in \mathbb{R}^N | H(x) = 0\} = \{(x_1, \dots, x_{N-1}, y) \in \mathbb{R}^N | y^2 = \sum_{i=1}^{N-1} a_i x_i^2\}$$

invariant. This implies that G leaves invariant the central line $L := \{(0, \dots, 0, y) \in \mathbb{R}^N | y \in \mathbb{R}\}$ of $\mathcal{C}$, contradicting the irreducibility of the G-representation. Therefore, we have seen that $H > 0$, i.e., f has a local minimum in 0.

□

Remark 5.2. It is not clear that for this choice of points $w_1, \dots, w_r$ the function $f(z) := \sum_{i=1}^r \log |z - w_i|^2$ is a local Morse function. At any rate, f is never a global Morse function, since the critical points appear as orbits of the symmetry group, hence they do not have different values.

But if f has a local minimum in 0 (as seen above), we shall see in the next section that, if we perturb the points $w_1, \dots, w_r$ a little bit, obtaining points $w'_1, \dots, w'_r$, we can achieve that

- $f(x) := \sum_{i=1}^r \log |z - w'_i|^2$ is a global Morse function,
- f has a local and not global minimum.

6. The configuration space

We go back to the notation defined in the introduction, see definition 0.5.
$\mathcal{GL}(r, N)$ is the open set in the space $(\mathbb{R}^N)^r$of r (distinct) points in $\mathbb{R}^N$, such that the function $f(x) = \sum_{k=1}^r \log |x - w_k|^2$ is a global Morse function, i.e. we have a generic big lemniscate configuration $\Gamma(f)$.

Proposition 6.1. *The complement*

$$\mathcal{Y} := (\mathbb{R}^N)^r \setminus \mathcal{GL}(r, N)$$

is a real semi-algebraic set different from $(\mathbb{R}^N)^r$*, in particular the open set* $\mathcal{GL}(r, N)$ *is non empty.*

Proof. It is sufficient to consider the conditions that say that $w := (w_1, \dots, w_r) \in \mathcal{GL}(r, N)$. The condition that the points w_j are pairwise distinct amount to the fact that $w \notin \Delta_{i,j} := \{w | w_i = w_j\}$.
Observe that $\Delta_{i,j}$ is a linear subspace of codimension N.

The condition that F is a (local) Morse function is the condition that all critical points are nondegenerate. To this purpose, we consider as customary the critical variety:

$$\mathcal{CR} := \{(x, w) | x \in \mathbb{R}^N, w \in (\mathbb{R}^N)^r \setminus \cup_{i<j} \Delta_{i,j}, \frac{\partial F}{\partial x_j}(x, w) = 0 \ \forall j = 1, \dots, N.\}$$

Here $F(x,w)$ is the real polynomial $\prod_1^r |x-w_j|^2$.

Clearly $\mathcal{CR}$ is defined by N polynomial equations, so it is a real algebraic set, and its projection to $(\mathbb{R}^N)^r \setminus \Delta := (\mathbb{R}^N)^r \setminus \cup_{i<j}\Delta_{i,j}$ is proper by lemma 4.1. Now, the equation $\frac{\partial F}{\partial x_j}(x,w) = 0 \ \forall j = 1,\dots,n$ is equivalent to the equation $\frac{\partial f}{\partial x_j}(x,w) = 0 \ \forall j = 1,\dots,n$ if $x \notin \Sigma = \{w_1,\dots,w_r\}$, hence to the vanishing of the gradient $\mathrm{grad}_x f(x,w)$ with respect to the variable x of the function $f(x,w)$.

We change now variables setting $-u_j := (x-w_j)$.

Given, the function $\mathrm{grad}_x f(x,w)$, the derivative with respect to the variable w_i of this vector valued function,

$$\frac{\partial}{\partial w_i}[\mathrm{grad}_x f(x,w)] = \frac{\partial}{\partial w_i}[\frac{x-w_i}{|x-w_i|^2}]$$

equals to the derivative with respect to the variable u_i of the function $\frac{u_i}{|u_i|^2}$.

But the derivative of the vector valued function $\frac{u}{|u|^2}$ is given by the matrix of the quadratic form (on tangent vectors v) $\frac{1}{|u|^4}[(u,u)(v,v) - 2(u,v)(u,v)]$. We restrict to the open set $u \neq 0$, and without loss of generality, by homogeneity, we can assume $|u| = 1$ and indeed, after a change of orthonormal basis, that $u = e_1$. Then the quadratic form becomes

$$|v|^2 - 2(e_1,v)^2 = -v_1^2 + v_2^2 + \ldots + v_N^2,$$

which is non degenrate.

We have therefore established that $\mathcal{CR}$ is smooth of codimension N outside of the locus where $x = w_i$. However, the points $x = w_i$ are isolated critical points of F.

Hence the locus of $\mathcal{CR}$ where the projection $\pi : \mathcal{CR} \to (\mathbb{R}^N)^r$ is not a submersion (being a submersion means that the derivative is surjective) is a closed algebraic set and its image in $(\mathbb{R}^N)^r$, by the Tarski-Seidenberg theorem (cf. [Jac74], page 323 for an elementary proof), and by Sard's theorem, is a semialgebraic set of dimension strictly smaller than Nr.

Now, the key well known fact is that the isolated critical points of the function F are exactly the points of $\mathcal{CR}$ where the projection $\pi : \mathcal{CR} \to (\mathbb{R}^N)^r$ is a submersion.

The final condition that f be a global Morse function runs as follows: we have a non empty open set for which F, hence f, is a local Morse function; this set is the complement of the above semialgebraic set, that we denote by $\mathcal{L}(r,N)$ (observe that for an r-tuple of points in $\mathcal{L}(r,N)$ the singular level sets may contain more than one singular point).

Over the open set $\mathcal{L}(r,N)$ the critical points are a finite set, and the condition that f_w ($f_w(x) = f(x,w)$) is a global Morse function is that the values of F_w on the critical points which are not in Σ are pairwise distinct. We are thus removing another closed semialgebraic set.

That $\mathcal{GL}(r,N)$ is non empty follows by considering points $w_1,\dots,w_r$ which lie in $\mathbb{R}^2$. Using a complex structure where $\mathbb{R}^2=\mathbb{C}$, we reduce using corollary 4.3 to the case of polynomial lemniscates on $\mathbb{C}$, dealt with in [CP91].

□

Proposition 6.2. *Assume that $\mathcal{GL}(r,N)$ is everywhere dense, equivalently, its complementary set is a semialgebraic set of real dimension $< Nr$.*

Then, given an r-tuple $w\in(\mathbb{R}^N)^r$ of points $w_1,\dots,w_r\in\mathbb{R}^N$ (yielding a set of points $\Sigma_w:=\{w_1,\dots,w_r\}\subset\mathbb{R}^N$) such that

- *$f_w(x):=\sum_{i=1}^r \log|z-w_i|^2$ is a local Morse function,*
- *f_w has h local minima in $\mathbb{R}^N\setminus\Sigma_w$,*

then for each $\delta>0$, there is another r-tuple of points $w'_1,\dots,w'_r$, with $|w'_i-w_i|<\delta$ such that

- *$f_{w'}(x):=\sum_{i=1}^r \log|z-w'_i|^2$ is a global Morse function,*
- *$f_{w'}$ has h local minima in $\mathbb{R}^N\setminus\Sigma_{w'}$,*

Proof. The second assertion follows from the first, since we can find an r-tuple w' very close to w and lying in $\mathcal{GL}(r,N)$.

Then, for δ sufficiently small, the difference $|f_w-f_{w'}|$ is smaller, on any given compact K containing the convex hull of the set Σ_w, than any given $\epsilon>0$, provided $|w'_i-w_i|<\delta$.

Let now y be a local minimum for f_w which is not a global minimum. Then there is a constant r such that the closed ball $\overline{B(y,r)}$ contains no other critical points, and, for $x\in\partial B(y,r/2)$, $f_w(x)>f_w(y)+2\epsilon(y)$, where $\epsilon(y)>0$ is a constant.

Set $\epsilon:=min_y\epsilon(y)$ and choose δ as above: then $f_{w'}$ still possesses a local minimum inside $\overline{B(y,r/2)}$.

□

Proposition 6.3. *$\mathcal{GL}(r,N)$ is everywhere dense.*

Proof. Assume the contrary: then there is a connected component U of $\mathcal{L}(r,N)$ which has empty intersection with $\mathcal{GL}(r,N)$ and such that it has a common boundary M with a connected component U' of $\mathcal{GL}(r,N)$.

Take a general point w of the common boundary M, and take an analytic arc I transversal to M at w; apply the following two lemmas 6.4 and 6.5. Then the inverse image J of I inside the critical variety $\mathcal{CR}$ consists of several arcs J_h which map homeomorphically to I, plus there is (possibly) another arc J' which maps with degree 2 to the part of I lying in one of the two domains U, respectively U'.

Set $I_U:=I\cap U$, and similarly $I_{U'}:=I\cap U'$. By the hypothesis, there are two arcs A,B of I_U on which the critical values are the same; by analytic continuation, these arcs A,B cannot both respectively be part of two arcs of the form J_h. Hence one, say A, of them lies in the arc J'. By lemma 6.5 B cannot lie in an arc of the form J_h, otherwise, by analytic continuation, the critical values on $J_h \cap I_{U'}$ would be imaginary. Finally, the possibility that both arcs A, B lie in J' contradicts lemma 6.5 .

□

Lemma 6.4. *Assume that we have a degenerate critical point of the function $f = f_w$, where there are local coordinates $(u_1, \dots, u_N)$ such that*

$$f(u) = u_1^2 + \cdots + u_{N-1}^2 + c + \eta u_N^3,$$

where $\eta = \pm 1$.

Then for w' in a neighbourhood of w we have a deformation of f of the form

$$f_{w'}(u) = u_1^2 + \cdots + u_{N-1}^2 + c' + \eta u_N^3 + t(w') u_N,$$

for $t(w')$ an appropriate analytic function of w'.

The critical point $u = 0$ deforms to two real nondegenerate critical points

$$u_1 = u_2 = \cdots = u_{N-1} = 0, \sqrt{3} u_N = \sqrt{-\eta t(w')}$$

for the points w' where the function $\eta t(w')$ is negative, and to two imaginary critical points for the points w' where the function $\eta t(w')$ is positive.

In particular, the critical values of the function $f_{w'}$ at the two real critical points are distinct as soon as the points are distinct (i.e., for $t(w') \neq 0$); while the critical values at the imaginary critical points are non real.

Lemma 6.5. *The locus of r-tuples w such that f_w has at least two degenerate singular points (counted with multiplicity) has codimension at least 2.*

We shall provide the proof of the above lemmas in another paper.

Remark 6.6. In the next section we shall give explicit examples, first of the situation in Proposition 5.1, then of cases where one has many local (non global) minima, and we show that in these examples f is in fact a local Morse function (but never a global Morse function).

7. Examples

In this section we give some explicit examples of points $\{w_1, \dots, w_r\} \subset \mathbb{R}^N$, such that $f(x) := \sum_{i=1}^r \log |z - w_i|^2$ has one or more local minima in $\mathbb{R}^N \setminus \Sigma$.

7.1. **The hypercube.** Let $N \geq 3$ be a natural number and consider the midpoints of the big faces of the hypercube, i.e., the points $P_1, P_2, \ldots, P_{2N} \in \mathbb{R}^N$, with

$$P_i := e_i,\ P_{N+i} := -e_i,\ 1 \leq i \leq N.$$

Here e_i is the i-th standard basis vector of $\mathbb{R}^N$. Let

$$F(x) := \prod_{i=1}^{N} |x - e_i|^2 |x + e_i|^2.$$

Proposition 7.1. *Then F has $2N$ absolute minima in the points $x = \pm e_i$, $1 = 1, \ldots, n$, a local minimum in $x = 0$ and $2N$ non degenerate critical points of negativity 1 in $x = \pm\sqrt{\frac{N-2}{N}} e_i$.*

In fact,

(8)

$$\begin{aligned} F(x) = F(x_1, \ldots, x_N) &= ((x_1-1)^2 + x_2^2 + \ldots + x_N^2)((x_1+1)^2 + x_2^2 + \ldots + x_N^2)\cdot \\ &\cdot (x_1^2 + (x_2-1)^2 + \ldots + x_N^2)(x_1^2 + (x_2+1)^2 + \ldots + x_N^2) \cdot \ldots \cdot \\ &\cdot (x_1^2 + x_2^2 + \ldots + (x_N-1)^2)(x_1^2 + x_2^2 + \ldots + (x_N-1)^2) = \\ &= (x_1^2 - 2x_1 + 1 + x_2^2 + \ldots + x_N^2)(x_1^2 + 2x_1 + 1 + x_2^2 + \ldots + x_N^2) \cdot \ldots \\ &\cdot \ldots \cdot (x_1^2 + x_2^2 + \ldots + x_N^2 - 2x_N + 1)(x_1^2 + x_2^2 + \ldots + x_N^2 + 2x_N + 1) = \\ &= 1 + 2(N-2)(x_1^2 + x_2^2 + \ldots + x_N^2) + f_{\geq 3}(x). \end{aligned}$$

Here $f_{\geq 3}(x) \in \mathbb{R}[x_1, \ldots, x_N]$ is a sum of monomials of order ≥ 3. This implies that for $N \geq 3$, F has a local minimum in $x = 0$.

In order to prove the above proposition, it suffices to prove the following two lemmata.

Lemma 7.2. *If $x = \lambda e_i$, then x is a critical point of F if and only if $\lambda = \pm 1$ or $\lambda = \pm\sqrt{\frac{N-2}{N}}$.*

Proof. We define for $1 \leq i \leq N$: $F_i(x) := |x - e_i|^2 |x + e_i|^2$. Then it is easy to verify that

$$\frac{\partial F_i}{\partial x_j}(x) = \begin{cases} 4x_i(x_1^2 + \ldots + x_N^2 - 1),\ i = j, \\ 4x_j(x_1^2 + \ldots + x_N^2 + 1),\ i \neq j. \end{cases}$$

This implies that

$$\frac{\partial F}{\partial x_i}(x) = 4x_i \left[(|x|^2 + 1) \sum_{k=1}^{N} F_1 \cdot \ldots \cdot \hat{F}_k \cdot \ldots \cdot F_N - 2F_1 \cdot \ldots \cdot \hat{F}_i \cdot \ldots \cdot F_N \right],$$

for $i = 1, \ldots, N$.

Remark 7.3. It is clear that $\operatorname{grad} F(x) = 0$ for $x = 0$ and $x = \pm e_i$.

Therefore from now on we assume that $x \neq 0$ and $\lambda \neq \pm 1$.
Observe that for $x_i \neq 0$, $\frac{\partial F}{\partial x_i}(x) = 0$ if and only if

$$A_i(x) := (|x|^2 + 1) \sum_{k=1}^{N} F_1 \cdot \ldots \cdot \hat{F}_k \cdot \ldots \cdot F_N - 2F_1 \cdot \ldots \cdot \hat{F}_i \cdot \ldots \cdot F_N = 0.$$

Due to the symmetry we can assume wlog that $x = \lambda e_1$, $\lambda \neq 0, 1, -1$. Then $\frac{\partial F}{\partial x_j}(x) = 0$, for $j = 2, \ldots, N$ and

$$(9) \quad A_1(x) = 0 \iff (1-\lambda^2) F_2 \cdot \cdots \cdot F_N = (1+\lambda^2) \sum_{k=2}^{N} F_1 \cdot \ldots \cdot \hat{F}_k \cdot \ldots \cdot F_N$$

$$\iff (1-\lambda^2)(\lambda^2+1)^{2(N-1)} = (N-1)(1+\lambda^2)((\lambda-1)^2(\lambda+1)^2(\lambda^2+1)^{2(N-2)}$$

$$\iff \lambda^2 = \pm \frac{N-2}{N}.$$

□

Lemma 7.4. *Let $x \in \mathbb{R}^N$ and assume that there are $1 \leq i < j \leq N$ such that $x_i \neq 0 \neq x_j$. Then* $\operatorname{grad} F(x) \neq 0$.

Proof. We assume that $x \in \mathbb{R}^N$ such that there is $k \geq 2$ there are $1 \leq i_1 < \ldots < i_k \leq N$ such that $x_{i_j} \neq 0$, $x_j = 0$ otherwise. For reasons of symmetry we again suppose wlog that $x_i \neq 0$ for $1 \leq i \leq k$ and $x_i = 0$ otherwise, i.e.,

$$x = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Then

$$\frac{\partial F}{\partial x_i}(x) = 0 \iff (|x|^2 + 1) \sum_{k=1}^{N} F_1 \cdot \ldots \cdot \hat{F}_k \cdot \ldots \cdot F_N = 2F_1 \cdot \ldots \cdot \hat{F}_i \cdot \ldots \cdot F_N,$$

for $i = 1, \ldots, k$. Since $F_i(x) \neq 0$ for all i, this implies that $F_1(x) = \ldots = F_k(x)$. We have

$$F_i(x) = |x|^4 + 2|x|^2 - (4x_i^2 - 1),$$

whence $a := x_1^2 = \ldots = x_k^2$. Now we can write

$$F_i(x) = \begin{cases} (ka)^2 + (2k-4)a + 1 =: P, \ 1 \leq i \leq k, \\ (ka+1)^2 =: Q, \ k < i \leq N. \end{cases}$$

This implies

$$A_1(x) = (1 + ka)\left(kP^{k-1}Q^{N-k} + (N-k)P^kQ^{N-k-1}\right) - 2P^{k-1}Q^{N-k} =$$

$$= ((1 + ka)k - 2)P^{k-1}Q^{N-k} + (1 + ka)(N - k)P^kQ^{N-k-1}.$$

The last expression is clearly strictly positive for $k \geq 2$, since $a > 0$ and $P, Q > 0$.
□

Here we compute the Hesse matrix of $f = \log(F)$ at the critical points $\pm\lambda e_i$, $\lambda = \sqrt{\frac{N-2}{N}}$. Actually, due to the symmetry, it is enough to compute $H_{f,x}$ at the point $x = \lambda e_1$.

By Lemma 4.2 we have

$$H_{f,\lambda e_1} = 2\sum_{k=1}^{r} \frac{|\lambda e_1 - w_k|^2 E_N - 2(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T}{|\lambda e_1 - w_k|^4}$$

For $1 < i \neq j$ we have

$$\begin{aligned}(H_{f,\lambda e_1})_{ij} &= 2\sum_{k=1}^{r} \frac{e_i^T|\lambda e_1 - w_k|^2 E_N e_j - 2e_i^T(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T e_j}{|\lambda e_1 - w_k|^4} \\ &= 2\sum_{k=1}^{r} \frac{-2e_i^T(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T e_j}{|\lambda e_1 - w_k|^4} = 0.\end{aligned}$$

For $1 < j$ we have

$$\begin{aligned}(H_{f,\lambda e_1})_{1j} &= 2\sum_{k=1}^{r} \frac{e_1^T|\lambda e_1 - w_k|^2 E_N e_j - 2e_1^T(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T e_j}{|\lambda e_1 - w_k|^4} \\ &= 2\sum_{k=1}^{r} \frac{-2e_1^T(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T e_j}{|\lambda e_1 - w_k|^4} \\ &= 2\sum_{k=1}^{r} \frac{2(\lambda - e_1^T w_k)w_k^T e_j}{|\lambda e_1 - w_k|^4} = 2\left(2\frac{\lambda}{|\lambda e_1 - e_j|^4} + 2\frac{\lambda(-1)}{|\lambda e_1 + e_j|^4}\right) = 0.\end{aligned}$$

In particular, we see that $H_{f,x}$ at the point $x = \lambda e_1$ is diagonal.

$$\begin{aligned}
(H_{f,\lambda e_1})_{11} &= 2\sum_{k=1}^{r}\frac{e_1^T|\lambda e_1 - w_k|^2 E_N e_1 - 2e_1^T(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T e_1}{|\lambda e_1 - w_k|^4} \\
&= 2\sum_{k=1}^{r}\frac{|\lambda e_1 - w_k|^2 - 2e_1^T(\lambda e_1 - w_k)(\lambda e_1 - w_k)^T e_1}{|\lambda e_1 - w_k|^4} \\
&= 2\left(\sum_{w_k\neq\pm e_1}\frac{(\lambda^2+1)-2\lambda^2}{(\lambda^2+1)^2} + \frac{(\lambda-1)^2-2(\lambda-1)^2}{(\lambda-1)^4} + \frac{(\lambda+1)^2-2(\lambda+1)^2}{(\lambda+1)^4}\right) \\
&= 2\left((2N-2)\frac{1-\lambda^2}{(1+\lambda^2)^2} - 2\frac{\lambda^2+1}{(\lambda^2-1)^2}\right) = -2N^2\frac{N-2}{N-1}.
\end{aligned}$$

For $1 < i$ a similar computation shows that

$$(H_{f,\lambda e_1})_{ii} = 2N^2\left(\frac{(N^2-2N-2)}{2N(N-2)^2} + \frac{N-1}{N}\right)$$

Summing up we get

$$H_{f,\lambda e_1} \sim \operatorname{diag}(-2N^2, 2N^2, \cdots, 2N^2)$$

as $N \to \infty$.

7.2. **Three elementary examples in $\mathbb{R}^3$.**
1) Consider the four vertices $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_3$, $w_4 = e_1 + e_2 + e_3$ of the regular symplex in $\mathbb{R}^3$.

Here

$$F(x) := \prod_{i=1}^{4}|x - w_i|^2.$$

has four absolute minima in w_1, w_2, w_3, w_4, a local minimum in the barycenter $B := \frac{1}{2}w_4$ and 4 non degenerate critical points (of negativity 1) in

$$y_i := \frac{1}{3}w_i + \frac{2}{3}B,\ 1 \leq i \leq 4.$$

2) Consider the following eight vertices of the regular cube in $\mathbb{R}^3$:

$$\{w_1, \ldots, w_8\} = \{0, e_1, e_2, e_3, e_1+e_2, e_1+e_3, e_2+e_3, e_1+e_2+e_3\}.$$

$$F(x) := \prod_{i=1}^{8}|x - w_i|^2.$$

Then F has eight absolute minima in $w_1, \ldots, w_8$, a local minimum in $\frac{1}{2}w_8$ and further 8 non degenerate critical points (of negativity 1).

3) Consider the following six points in $\mathbb{R}^3$:

$$\{w_1, \ldots, w_6\} = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3\}.$$

$$F(x) := \prod_{i=1}^{6} |x - w_i|^2.$$

Then F has eight absolute minima in $w_1, \ldots, w_6$, a local minimum in $\frac{1}{2}(e_1+e_2+e_3)$ and further 6 non degenerate critical points (of negativity 1).

7.3. **The regular triangular prism: an example with two local minima.**

Fix $a \in \mathbb{R}_+$ and consider the following six points: $w_j := (u_j, a),\ w'_j := (u_j, -a) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$, where for $1 \le j \le 3$ we set $u_j := e^{j\frac{2\pi i}{3}}$. Set

$$F_a(x) := \prod_{j=1}^{3} |x - w_j|^2 \cdot |x - w'_j|^2.$$

F_a has six absolute minima in the points w_j, w'_j.

We shall prove the following

Proposition 7.5.

(1) <u>$a = 1$:</u> *F_1 has a critical point in 0, whose Hessian has nullity 1 (and positivity two).*

(2) <u>$a < 1$:</u> *F_a has a local minimum in 0 with non degenerate Hessian and no further critical points of the form $(0, 0, x_3)$.*

(3) <u>$a > 1$:</u> *F_a has a non degenerate critical point in 0 with negativity 1, and two local minima in $(0, 0, \pm\sqrt{a^2 - 1})$.*

Then we have:

$$w_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, a),\ w_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, a),\ w_3 = (1, 0, a);$$

$$w'_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, -a),\ w'_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, -a),\ w'_3 = (1, 0, -a).$$

Moreover, write $F\colon = F_1 \cdot F_2 \cdot F_3$, where

$$F_j := |x - w_j|^2 \cdot |x - w'_j|^2.$$

Defining

$$g_1(x)\colon = (x_1 + \frac{1}{2})^2 + (x_2 - \frac{\sqrt{3}}{2})^2 + (x_3^2 - a^2),$$

$$g_2(x)\colon = (x_1 + \frac{1}{2})^2 + (x_2 + \frac{\sqrt{3}}{2})^2 + (x_3^2 - a^2),$$

$$g_3(x)\colon = (x_1 - 1)^2 + x_2^2 + (x_3^2 - a^2),$$

it is easy to verify that

$$(10)\quad \frac{\partial F_a}{\partial x_1}(x) = 4(x_1+\frac{1}{2})g_1(x)F_2(x)F_3(x)+ \\ +4(x_1+\frac{1}{2})g_2(x)F_1(x)F_3(x)+4(x_1-1)g_3(x)F_1(x)F_2(x),$$

$$(11)\quad \frac{\partial F_a}{\partial x_2}(x) = 4(x_2-\frac{\sqrt{3}}{2})g_1(x)F_2(x)F_3(x)+ \\ +4(x_2+\frac{\sqrt{3}}{2})g_2(x)F_1(x)F_3(x)+4x_2g_3(x)F_1(x)F_2(x),$$

$$(12)\quad \frac{\partial F_a}{\partial x_3}(x) \quad = \quad 4x_3\left(g_1(x)F_2(x)F_3(x)+g_2(x)F_1(x)F_3(x)+g_3(x)F_1(x)F_2(x)\right).$$

Remark 7.6. One sees immediately that $\frac{\partial F_a}{\partial x_3}(x)=0$ if and only if $x_3=0$ or

$$g_1(x)F_2(x)F_3(x)+g_2(x)F_1(x)F_3(x)+g_3(x)F_1(x)F_2(x)=0.$$

We assume now that $x=(0,0,x_3)$. Then

$$(13)\quad \frac{\partial F_a}{\partial x_1}(x) = 2(1+x_3^2-a^2)(1+(x_3-a)^2)^2(1+(x_3+a)^2)^2+ \\ +2(1+x_3^2-a^2)(1+(x_3-a)^2)^2(1+(x_3+a)^2)^2- \\ -4(1+x_3^2-a^2)(1+(x_3-a)^2)^2(1+(x_3+a)^2)^2=0,\ \forall x_3,\ \forall a.$$

Similarly, one verifies that $\frac{\partial F_a}{\partial x_2}(x)=0, \forall x_3, \forall a$. Moreover, we have that $\frac{\partial F_a}{\partial x_3}(0,0,x_3)=0$ if and only if $x_3=0$ or if

$$(14)\quad 0 = g_1(x)F_2(x)F_3(x)+g_2(x)F_1(x)F_3(x)+g_3(x)F_1(x)F_2(x)= \\ =3(1+x_3^2-a)(1+(x_3+a)^2)^2(1+(x_3-a)^2)^2 \iff x_3^2=a^2-1.$$

Therefore we have shown the following

Lemma 7.7.

(1) *If* $a\le 1$, *then* $\operatorname{grad} F_a(0,0,x_3)=0 \iff x_3=0.$
(2) *If* $a>1$, *then* $\operatorname{grad} F_a(0,0,x_3)=0 \iff x_3\in\{(0,,0,0),(0,0,\pm\sqrt{a^2-1})\}.$

We shall now calculate the Hessian of $f_a=\log F_a$ in $(0,0,0)$ and for $a>1$ in $(0,0,\pm\sqrt{a^2-1})$. In fact, we prove the following:

Lemma 7.8. *For $x = (0,0,t)$ we have*

$$H_{f_a,x} = \frac{6}{(1+(t-a)^2)^2}\begin{pmatrix}(t-a)^2 & 0 & 0\\ 0 & (t-a)^2 & 0\\ 0 & 0 & 1-(t-a)^2\end{pmatrix} + \\ + \frac{6}{(1+(t+a)^2)^2}\begin{pmatrix}(t+a)^2 & 0 & 0\\ 0 & (t+a)^2 & 0\\ 0 & 0 & 1-(t+a)^2\end{pmatrix}. \tag{15}$$

Proof. Note that $|x-w_k|^2 = 1+(t-a)^2$ and $|x-w'_k|^2 = 1+(t+a)^2$ for $1 \le k \le 3$. Using the formula for the Hesse matrix in Lemma 4.2 one can verify the claim. □

We can now give a proof for Proposition 7.5.

Proof. One sees immediately that

$$H_{f_a,0} = \frac{12}{(1+a^2)^2}\begin{pmatrix}a^2 & 0 & 0\\ 0 & a^2 & 0\\ 0 & 0 & 1-a^2\end{pmatrix}.$$

From this one sees immediately that the Hesse matrix iin 0 is degenerate for $a = 1$, positive definite for $a < 1$ and has negativity 1 for $a > 1$.
It remains to verify that for $a > 1$ and $x = (0,0,\pm\sqrt{a^2-1})$ the Hesse matrix $H_{f_a,x}$ is positive definite. From the formula above we see (since the first two entries of the diagonal are automatically positive) that it suffices to show that for $t^2 = a^2 - 1$

$$\frac{6}{(1+(t-a)^2)^2}(1-(t-a)^2) + \frac{6}{(1+(t+a)^2)^2}(1-(t+a)^2) > 0. \tag{16}$$

For this observe that setting $\lambda := (t-a)^2$, we have that $\frac{1}{\lambda} = (t+a)^2$. Therefore the left hand side of equation 16 becomes

$$\frac{6}{(1+(t-a)^2)^2}(1-(t-a)^2) + \frac{6}{(1+(t+a)^2)^2}(1-(t+a)^2) = \\ = \frac{1-\lambda}{(1+\lambda)^2} + \frac{1-\frac{1}{\lambda}}{(1+\frac{1}{\lambda})^2} = \frac{1-\lambda}{(1+\lambda)^2} + \frac{\lambda^2-\lambda}{(1+\lambda)^2} = \frac{(1-\lambda)^2}{(1+\lambda)^2} > 0. \tag{17}$$

This proves Proposition 7.5 □

8. $3h$ points on an equilateral triangular prism, with h preassigned local (non absolute) minima

Let $r_1 < r_2 < \cdots < r_h$ be arbitrary real numbers, $h > 1$. We are going to construct $3h$ points $w_1, w_2, \cdots, w_{3h} \in \mathbb{R}^3$ such that $F(x) = \prod_{j=1}^{3h} |x - w_j|^2$ has h local (non absolute and non degenerate) minima at the points $(0, 0, r_j)$, $j = 1, \cdots, h$.

Actually, F is going to have also $h-1$ saddles (non degenerate critical points of negativity 1) on the x_3-axis.

8.1. **The auxiliary polynomial P.** Given the r_j's take s_j such that $r_j < s_j < r_{j+1}$ for $j = 1, \cdots, h-1$. Let $P(X)$ be the polynomial

$$P'(X) = \left(\prod_{j=1}^{h-1} (X - r_j)(X - s_j) \right) (X - r_h) \, .$$

Then $\deg(P) = 2h$ and $P(X)$ is bounded from below. So we can assume w.l.o.g. that $P(X) > 0$ for all $X \in \mathbb{R}$.
By construction $P(X)$ has h local minima at the r_j's and $h-1$ local maxima at the s_j's.

Decompose $P(X)$ as

$$P(X) = \prod_{j=1}^{h} P_j(X)$$

where $P_j(X)$ are degree two monic real polynomials without real roots. Observe that $P_j \neq P_k$ for $j \neq k$ otherwise the derivative would have a double root, contradicting our construction of $P'(X)$.

We can also assume w.l.o.g. that $P_j(X) > 0$ for all $X \in \mathbb{R}$.

Hence there are real numbers $a_j, b_j \in \mathbb{R}$ such that

$$P_j(X) = (X - a_j)^2 + b_j^2$$

and we can assume that $b_j > 0$.

Summing up we have:

$$P(X) = \prod_{j=1}^{h} \left((X - a_j)^2 + b_j^2 \right) \tag{18}$$

8.2. **The $3h$ points $w_1, \ldots, w_{3h}$.**

We regard now $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$. For each $j \in \{1, \cdots, h\}$ we consider the following 3 points w_j^1, w_j^2, w_j^3 defined as follows:

$$w_j^i := (\xi^i b_j, a_j)$$

where $\xi = e^{\sqrt{-1}\frac{2\pi}{3}}$ and the a_j, b_j's are as in the factorization of $P(X)$ given in equation (18).

Proposition 8.1. *Let* $F(x) := \prod_{j=1}^{h} |x - w_j^1|^2 |x - w_j^2|^2 |x - w_j^3|^2$. *Then:*

(1) *F has h local (non absolute) nondegenerate minima in $(0, r_j) \in \mathbb{C} \times \mathbb{R}$, $1 \leq j \leq h$;*
(2) *F has $h-1$ saddle points (nondegenerate critical points of negativity 1) in the points $(0, s_j)$, $1 \leq j \leq h-1$;*
(3) *F has $3h$ absolute minima in the points w_j^i, $1 \leq j \leq h$, $1 \leq i \leq 3$;*

Proof.
Step I: *The $h + h - 1$ points $(0, r_j)$ and $(0, s_j)$ are critical points of F.*

To see this write $x = (z, t)$ so that

$$F(x) = F(z,t) = \prod_{j=1}^{h} \left(|z - \xi b_j|^2 + (t - a_j)^2\right) \left(|z - \xi^2 b_j|^2 + (t - a_j)^2\right) \left(|z - b_j|^2 + (t - a_j)^2\right)$$

hence

$$F(\xi z, t) = F(z, t).$$

I.e., F is invariant for the 120 degree rotation on the first factor $\mathbb{C}$.

This implies that the gradient ∇F at the points $(0, r_j)$ and $(0, s_j)$ has zero $\mathbb{C}$-component. The $\mathbb{R}$-component of ∇F is then $\frac{\mathrm{d}F(0,t)}{\mathrm{d}t}$. Now

$$F(0,t) = \prod_{j=1}^{h} \left(b_j^2 + (t - a_j)^2\right) \left(b_j^2 + (t - a_j)^2\right) \left(b_j^2 + (t - a_j)^2\right) = P(t)^3$$

where P is as in equation (18). So, as we claimed, $\frac{\mathrm{d}F(0,t)}{\mathrm{d}t}$ is zero at the r_j's and s_j's.

Step II: *At each critical point of the form $(0, r_j)$ and $(0, s_j)$ the Hessian matrix H_F has both factors $\mathbb{C}$ and $\mathbb{R}$ as invariant subspaces. Moreover $H_F|_{\mathbb{C}} = \lambda \mathrm{Id}_{\mathbb{C}}$, for $\lambda \in \mathbb{R}$.* Let in fact $R_{120} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of 120 degrees, induced by the multiplication by ξ on the first factor $\mathbb{C}$. Then, as observed above, $F \circ R_{120} = F$. The critical points $(0, r_j)$ and $(0, s_j)$ are fixed by R_{120} hence:

$$R_{120} H_{F,\mathbf{p}} = H_{F,\mathbf{p}} R_{120}$$

where $\mathbf{p} \in \{(0, r_j), (0, s_j)\}$. Since the $\mathbb{R}$ factor is the unique fixed line by R_{120} it follows that $H_{F,\mathbf{p}}$ preserves the $\mathbb{R}$ line hence $H_{F,\mathbf{p}}$ also preserves $\mathbb{C}$. The $\mathbb{Z}_3$-action generated by R_{120} is irreducible on $\mathbb{C}$. It follows that $H_F|_{\mathbb{C}} = \lambda \mathrm{Id}_{\mathbb{C}}$ as we claimed.

Step III: *The points $(0, r_j)$ are local (non absolute) minima of F whilst H_F has negativity 1 and is nondegenerate at the points $(0, s_j)$.* Since $h > 1$ the $3h$ points are not coplanar. So at each critical point $\mathbf{p} \in \{(0, r_j), (0, s_j)\}$ the constant λ in the first block of the Hessian matrix $H_{F,\mathbf{p}}$ must be positive, i.e. $\lambda > 0$. The constant in the $\mathbb{R}$ direction is given by

$$\frac{\mathrm{d}^2 F(0,t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 (P(t)^3)}{\mathrm{d}t^2}$$

By construction $P(X)$ has non degenerate local minima at the r_j's and non degenerate local maxima at the s_j's. Since $P > 0$ the cubic power does not change the sign of the derivatives and we get that also $F(0,t)$ has has non degenerate local minima at the r_j's and non degenerate local maxima at the s_j's. □

Remark 8.2. One can show that, for general choice of the numbers r_j, s_j, F has exactly $3h$ further saddle points (nondegenerate critical points of negativity 1).

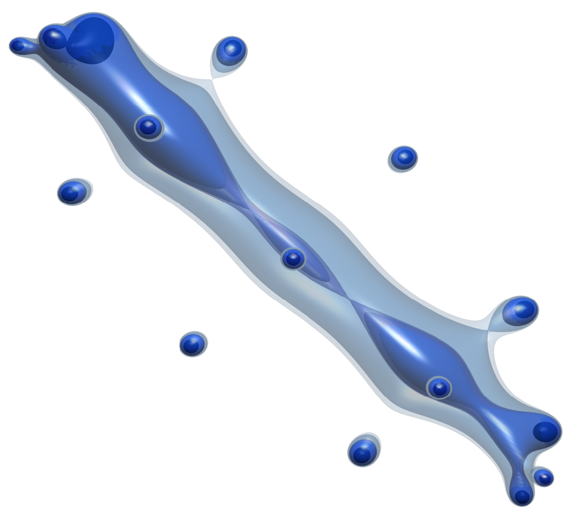

FIGURE 3. Several lemniscates of a perturbation of the configuration for $r = 15$ constructed with the method of this section.

9. Appendix

We give here a further concrete example.

9.1. **The vertices of the regular simplex in $\mathbb{R}^N$.**

Let $N \geq 4$ be a natural number and consider $w_1, w_2, \ldots, w_N \in \mathbb{R}^N$, where

$$w_i := e_i, \ 1 \leq i \leq N.$$

Here e_i is the i-th standard basis vector of $\mathbb{R}^N$. Let

$$F(x) := \prod_{i=1}^{N} |x - e_i|^2$$

and $f(x) := \log(F(x))$.

Proposition 9.1. *Then F has N absolute minima in the points $x = e_i$, $i = 1, \ldots, N$, a local minimum in $x = B := \frac{1}{N}\sum_{i=1}^{N} e_i$ and N non degenerate critical points of negativity 1 in $Q_i := \frac{2}{N-1}B + \frac{N-3}{N-1}e_i$, $i = 1, \ldots, N$.*

More precisely, $H_{f,B}$ has two eigenvalues:

- $\frac{2N^3}{N(N-1)}$ *with multiplicity 1 whose eigenvector is parallel to B and*
- $2(N-3)(\frac{N}{N-1})^2$ *with multiplicity $N-1$.*

The Hessian matrix H_{f,Q_i} has three eigenvalues:

- $-\frac{(N-3)N^2}{2(N-2)}$ *has multiplicity one with eigenvector $B - e_i$,*
- $\frac{(N-1)N^2}{2(N-2)}$ *with multiplicity one with eigenvector B, and*
- $\frac{8+(-3+N)N\left(4+N^2\right)}{2(-2+N)^2}$ *with multiplicity $(N-2)$.*

Proof. That the N points e_i, $i = 1, \ldots, N$ are absolute minima follows from the very definition of F. A critical point $x \neq e_i$ of F satisfies:

$$\sum_{i=1}^{N} \frac{x - e_i}{|x - e_i|^2} = 0$$

taking scalar product with e_j we get

$$\sum_{i=1}^{N} \frac{x_j - \delta_{ij}}{|x - e_i|^2} = 0\,.$$

Then

$$x_j \Sigma - \frac{1}{|x - e_j|^2} = 0$$

where $\Sigma := \sum_{i=1}^{N} \frac{1}{|x-e_i|^2}$. Hence the coordinates x_j (of the critical point x) are roots of the following polynomial $P(X)$:

$$P(X) = X^2 - \frac{(|x|^2 + 1)}{2}X + \frac{1}{2\Sigma} = (X - \alpha)(X - \beta)\ .$$

Due to the symmetry we can assume that

$$x = (\underbrace{\alpha, \cdots, \alpha}_{a-times}, \underbrace{\beta, \cdots, \beta}_{(N-a)-times})\,.$$

with $a > 1$. This implies

$$\begin{cases} \alpha + \beta = \frac{|x|^2+1}{2}, \\ a\alpha^2 + (N-a)\beta^2 = |x|^2, \\ a\alpha + (N-a)\beta = 1 \end{cases} \tag{19}$$

The last equality is due to the fact that x must be in the convex hull of the N points e_i, $i = 1, \ldots, N$.

If $a = N$ then $\alpha = \frac{1}{N}$ and $x = B = \frac{1}{N}(1, \cdots, 1)$.

Now assume that $1 < a \leq N-1$. From the equations (19) we get the following quadratic equation for β:

$$\frac{(1-(N-a)\beta)}{a} + \beta = \frac{a(\frac{(1-(N-a)\beta)}{a})^2 + (N-a)\beta^2 + 1}{2}$$

$$2 - 2(N-a)\beta + 2a\beta = (1-(N-a)\beta)^2 + a(N-a)\beta^2 + a$$

$$2 + 2a\beta = ((N-a)\beta)^2 + 1 + a(N-a)\beta^2 + a$$

so

$$2 + 2a\beta = (N-a)N\beta^2 + a + 1$$

or

$$0 = (N-a)N\beta^2 - 2a\beta + (a-1)$$

then

$$\beta = \frac{2a \pm \sqrt{4a^2 - 4N(N-a)(a-1)}}{2N(N-a)}.$$

Now for $N \geq 4$ and $1 < a < N-1$ the discriminant $a^2 - N(N-a)(a-1)$ is negative. It follows that $a = N-1$ hence

$$x = (\alpha, \alpha, \cdots, \alpha, \beta)$$

where

$$\beta = \frac{(N-1) \pm \sqrt{(N-1)^2 - N(N-2)}}{N} = \frac{(N-1) \pm 1}{N}$$

So there are two possibilities $\beta = 1, \alpha = 0$ or $\beta = \frac{N-2}{N}, \alpha = \frac{2}{N(N-1)}$. The first one corresponds to $x = e_N$ is excluded, since we assumed $x \neq e_i$, $i = 1, \cdots, N$. Then

$$x = (\frac{2}{N(N-1)}, \frac{2}{N(N-1)}, \cdots, \frac{2}{N(N-1)}, \frac{N-2}{N})$$

or

$$x = \frac{2}{N-1}B + \frac{N-3}{N-1}e_N.$$

This shows that the critical points of F are exactly: B, $Q_i := \frac{2}{N-1}B + \frac{N-3}{N-1}e_i$ and e_i, $i = 1, \ldots, N$.

The Hesse matrix of $f(x) = \log(F(x))$ at B is by Lemma 4.2:

$$H_{f,B} = 2\sum_{k=1}^{N} \frac{|B-e_k|^2 E_N - 2(B-e_k)(B-e_k)^T}{|B-e_k|^4},$$

Let $\xi_1, \xi_2, \cdots, \xi_{N-1}, \xi_N$ be an orthonormal base of $\mathbb{R}^N$ such that $\xi_N = \frac{B}{|B|}$. We are going to compute the Hesse matrix w.r.t. this base.

Then for $1 \le i \ne j < N$ we have

$$(H_{f,B})_{ij} = 2\sum_{k=1}^{N} -2\frac{\xi_i^T(B-e_k)(B-e_k)^T\xi_j}{|B-e_k|^4} = 2\sum_{k=1}^{N} -2\frac{\langle \xi_i, e_k\rangle\langle \xi_j, e_k\rangle}{\lambda^4} = \frac{-4}{\lambda^4}\langle \xi_i, \xi_j\rangle = 0$$

where $\lambda := |B-e_k| = \frac{\sqrt{N(N-1)}}{N}$. For $1 \le i < N$ we have

$$\begin{aligned}(H_{f,B})_{iN} &= 2\sum_{k=1}^{N} -2\frac{\langle \xi_i, (B-e_k)\rangle\langle \xi_N, (B-e_k)\rangle}{\lambda^4} = \frac{-4}{\lambda^4}\sum_{k=1}^{N}\langle \xi_i, (B-e_k)\rangle\langle \xi_N, (B-e_k)\rangle = \\ &= \frac{-4}{\lambda^4}\sum_{k=1}^{N}\langle \xi_i, -e_k\rangle\left(\langle \xi_N, B\rangle - \langle \xi_N, e_k\rangle\right) = \frac{-4}{\lambda^4}\left(\langle \xi_i, \xi_N\rangle + \sum_{k=1}^{N}\langle \xi_i, -e_k\rangle\langle \xi_N, B\rangle\right) = \\ &= \frac{-4}{\lambda^4}\left(\langle \xi_i, -\sum_{k=1}^{N} e_k\rangle\langle \xi_N, B\rangle\right) = \frac{-4}{\lambda^4}\langle \xi_i, -NB\rangle\langle \xi_N, B\rangle = 0\end{aligned}$$

This shows that w.r.t. the basis $\xi_1, \xi_2, \cdots, \xi_{N-1}, \xi_N$ the Hesse matrix $H_{f,B}$ is diagonal.

Now for $1 \le i < N$

$$\begin{aligned}(H_{f,B})_{ii} &= 2\sum_{k=1}^{N} \frac{\lambda^2 - 2\xi_i^T(B-e_k)(B-e_k)^T\xi_i}{\lambda^4} = 2\sum_{k=1}^{N} \frac{\lambda^2 - 2\langle \xi_i, e_k\rangle\langle \xi_i, e_k\rangle}{\lambda^4} = \\ &= 2\sum_{k=1}^{N} \frac{\lambda^2 - 2\langle \xi_i, e_k\rangle\langle \xi_i, e_k\rangle}{\lambda^4} = 2\frac{N}{\lambda^2} - 4\sum_{k=1}^{N}\frac{\langle \xi_i, e_k\rangle\langle \xi_i, e_k\rangle}{\lambda^4} \\ &= 2\frac{N}{\lambda^2} - \frac{4}{\lambda^4} = 2(N-3)(\frac{N}{N-1})^2.\end{aligned}$$

and

$$
\begin{aligned}
(H_{f,B})_{NN} &= 2\sum_{k=1}^{N}\frac{\lambda^2-2\langle(B-e_k),\xi_N\rangle^2}{\lambda^4} = 2\sum_{k=1}^{N}\frac{\lambda^2-2\left(|B|-\langle e_k,\xi_N\rangle\right)^2}{\lambda^4} = \\
&= \frac{2N}{\lambda^2}-\frac{4}{\lambda^4}\sum_{k=1}^{N}\left(|B|^2-2|B|\langle e_k,\xi_N\rangle+\langle e_k,\xi_N\rangle^2\right) = \\
&= \frac{2N}{\lambda^2}-\frac{4}{\lambda^4}\left(N|B|^2-2|B|\langle NB,\xi_N\rangle+1\right) \\
&= \frac{2N}{\lambda^2}-\frac{4}{\lambda^4}\left(N|B|^2-2N|B|^2+1\right) = \frac{2N}{\lambda^2} = \frac{2N^3}{N(N-1)}
\end{aligned}
$$

To compute H_{f,Q_1} we will use the standard orthogonal representation of the symmetric group $\rho:\mathfrak{S}_N\to O(N)$. It is well-known that $\mathbb{R}^N=\mathbb{R}B\oplus\mathbb{W}$ is an orthogonal $\mathfrak{S}_N$-invariant sum and $\mathfrak{S}_N$ acts irreducibly on $\mathbb{W}$. Let $(\mathfrak{S}_N)_{e_1}$ be the isotropy subgroup of e_1. Then $(\mathfrak{S}_N)_{e_1}$ fixes the critical point Q_1. The tangent space $T_{Q_1}\mathbb{R}^N$ splits as $T_{Q_1}\mathbb{R}^N=\mathrm{span}_{\mathbb{R}}\{B,e_1\}\oplus\mathbb{V}$ where the sum is orthogonal and $(\mathfrak{S}_N)_{e_1}$ acts irreducibly on $\mathbb{V}$. It follows that $H_{f,Q_1}(\mathbb{V})\subset\mathbb{V}$ and $H_{f,Q_1}(\mathrm{span}_{\mathbb{R}}\{B,e_1\})\subset\mathrm{span}_{\mathbb{R}}\{B,e_1\}$. Observe that the vector B is an eigenvector of H_{f,Q_1}. This follows by computing $\frac{\partial^2 f}{\partial\xi_i\xi_N}$ w.r.t. the coordinates given by the orthogonal base $\xi_1,\xi_2,\cdots,\xi_{N-1},\xi_N$ where $\xi_N=\frac{B}{|B|}$, since the set of points $e_1,\cdots,e_N$ is contained in the affine space $\sum_{k=1}^N x_k=1$. So from this direct computation we get:

$$
\begin{aligned}
H_{f,Q_1}(\xi_N) &= \left(\sum_{k=1}^{N}\frac{2}{|Q_1-e_k|^2}\right)\xi_N = \\
&= \left(\frac{2}{|Q_1-e_1|^2}+(N-1)\frac{2}{|Q_1-e_2|^2}\right)\xi_N = \frac{(N-1)N^2}{2(N-2)}\xi_N
\end{aligned}
$$

We define $\xi_1:=\frac{e_1-B}{|e_1-B|}$ so that $\mathrm{span}_{\mathbb{R}}\{B,e_1\}=\mathrm{span}_{\mathbb{R}}\{\xi_N=\frac{B}{|B|},\xi_1\}$. Then

$$
H_{f,Q_1}(\xi_1)=\mu\xi_1,
$$

where

$$\begin{aligned}\mu &= 2\frac{|Q_1-e_1|^2-2\langle\xi_1,Q_1-e_1\rangle^2}{|Q_1-e_1|^4}+2\sum_{k=2}^{N}\frac{|Q_1-e_k|^2-2\langle\xi_1,Q_1-e_k\rangle^2}{|Q_1-e_k|^4}\\ &= 2\frac{|Q_1-e_1|^2-2\langle\xi_1,Q_1-e_1\rangle^2}{|Q_1-e_1|^4}+2(N-1)\frac{|Q_1-e_2|^2-2\langle\xi_1,Q_1-e_2\rangle^2}{|Q_1-e_2|^4}\\ &= -\frac{(N-3)N^2}{2(N-2)}\end{aligned}$$

and

$$H_{f,Q_1}|_{\mathbb{V}} = \nu E_{\mathbb{V}},$$

where $E_{\mathbb{V}}$ is the identity of $E_{\mathbb{V}}$ due to the irreducibility of the action of $(\mathfrak{S}_N)_{e_1}$ on $\mathbb{V}$.

To compute ν we use the value $\mathrm{trace}(H_{f,Q_1}) = \frac{N-1}{2(4+N^2)}$ which is not difficult to compute w.r.t. the canonical basis $e_1, \cdots, e_N$. Then

$$\nu = \frac{\mathrm{trace}(H_{f,Q_1}) - \mu - \frac{(N-1)N^2}{2(N-2)}}{N-2} = \frac{8+(-3+N)N\,(4+N^2)}{2(-2+N)^2}$$

□

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References

[AMOP06] Gabriel Arcos, Guillermo Montilla, José Ortega, and Marco Paluszny, *Shape control of 3D lemniscates*, Math. Comput. Simulation **73** (2006), no. 1-4, 21–27, DOI 10.1016/j.matcom.2006.06.001. MR2307087 (2009e:65031) ↑2

[Arn92] V. I. Arnol'd, *Snake calculus and the combinatorics of the Bernoulli, Euler and Springer numbers of Coxeter groups*, Uspekhi Mat. Nauk **47** (1992), no. 1(283), 3–45, 240, DOI 10.1070/RM1992v047n01ABEH000861 (Russian); English transl., Russian Math. Surveys **47** (1992), no. 1, 1–51. MR1171862 (93h:20042) ↑

[Arn91] ———, *Bernoulli-Euler updown numbers associated with function singularities, their combinatorics and arithmetics*, Duke Math. J. **63** (1991), no. 2, 537–555, DOI 10.1215/S0012-7094-91-06323-4. MR1115120 (93b:58020) ↑

[BC97] Ingrid Bauer and Fabrizio Catanese, *Generic lemniscates of algebraic functions*, Math. Ann. **307** (1997), no. 3, 417–444, DOI 10.1007/s002080050042. MR1437047 (98f:57003) ↑2

[Cat99] Fabrizio Catanese, *Komplexe Mannigfaltigkeiten, Course Lecture Notes, taken by P. Frediani* (1999). ↑13

[CP91] Fabrizio Catanese and Marco Paluszny, *Polynomial-lemniscates, trees and braids*, Topology **30** (1991), no. 4, 623–640, DOI 10.1016/0040-9383(91)90043-4. MR1133876 (93b:57014) ↑2, 5, 6, 10, 13, 18

[CW91] Fabrizio Catanese and Bronislaw Wajnryb, *The fundamental group of generic polynomials*, Topology **30** (1991), no. 4, 641–651, DOI 10.1016/0040-9383(91)90044-5. MR1133877 (93b:57015) ↑

[CF93] Fabrizio Catanese and Paola Frediani, *Configurations of real and complex polynomials*, Astérisque **218** (1993), 61–93. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). MR1265309 (95c:57024) ↑

[Jac74] Nathan Jacobson, *Basic algebra. I*, W. H. Freeman and Co., San Francisco, Calif., 1974. MR0356989 (50 #9457) ↑17

[PMO05] Marco Paluszny, Guillermo Montilla, and José Rafael Ortega, *Lemniscates 3D: a CAGD primitive?*, Numer. Algorithms **39** (2005), no. 1-3, 317–327, DOI 10.1007/s11075-004-3645-6. MR2137759 (2006a:65020) ↑2

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