CANONICAL SURFACES OF HIGHER DEGREE

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Dedicated to Philippe Ellia on the occasion of his 60th birthday

Abstract. We consider a family of surfaces of general type S with K_S ample, having $K_S^2 = 24, p_g(S) = 6, q(S) = 0$. We prove that for these surfaces the canonical system is base point free and yields an embedding $\Phi_1 : S \to \mathbb{P}^5$.

This result answers a question posed by G. and M. Kapustka [Kap-Kap15].

We discuss some related open problems, concerning also the case $p_g(S) = 5$, where one requires the canonical map to be birational onto its image.

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Introduction

Among the many questions that one may ask about surfaces of general type the following one has not yet been sufficiently considered.

Question 0.1. Let S be a smooth surface with ample canonical divisor K_S, assume that $p_g(S) = 6$ and that the canonical map $\phi_1 : S \to \mathbb{P}^5$ is a biregular embedding.

Which values of K_S^2 can occur, in particular which is the maximal value that K_S^2 can reach?

Recall the Castelnuovo inequality, holding if ϕ_1 is birational (onto its image):

$$K_S^2 \geq 3p_g(S) - 7,$$

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and the Bogomolov-Miyaoka-Yau inequality

$$K_S^2 \leq 9\chi(S) = 9 + 9p_g(S) - 9q(S).$$

By virtue of these inequalities, under the assumptions of question 0.1 one must have:

$$11 \leq K_S^2 \leq 63.$$

In the article [Cat97] methods of homological algebra were used to construct such surfaces with low degree $11 \leq K_S^2 \leq 17$, and also to attempt a classification of them. Recently, M. and G. Kapustka constructed in [Kap-Kap15] such canonical surfaces of degree $K_S^2 = 18$, using the method of bilinkage. In a preliminary version of the article they even ventured to ask whether the answer to question 0.1 would be $K_S^2 \leq 18$.

Our main result consists in exhibiting such canonically embedded surfaces having degree $K_S^2 = 24$, and with $q(S) = 0$. The family of surfaces was indeed listed in the article [Cat99], dedicated to applications of the technique of bidouble covers; but it was not a priori clear that their canonical system would be an embedding (this was proven in [Cat84] for bidouble covers satisfying much stronger conditions).

Theorem 0.1. Assume that S is a bidouble cover of the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ branched on three curves D_1, D_2, D_3 of respective bidegrees $(2,3), (2,3)(4,1)$, which are smooth and intersect transversally. Assume moreover that the 12 intersection points $D_1 \cap D_2$ have pairwise different images via the second projection $p_v : Q \to \mathbb{P}^1$ (i.e., they have different coordinates $(v_0 : v_1)$).

Then S is a simply connected [1] surface with $K_S^2 = 24, p_g(S) = 6, q(S) = 0$, whose canonical map $\phi_1 : S \to \mathbb{P}^5$ is a biregular embedding. These surfaces form a (non empty) irreducible algebraic subset of dimension 25 of the moduli space.

We plan to try to describe the equations of the above surfaces $S \subset \mathbb{P}^5$; since, by a theorem of Walter [Wal96], each S is the Pfaffian locus of a twisted antisymmetric map $\alpha : \mathcal{E}(-t) \to \mathcal{E}$ of a vector bundle $\mathcal{E}$ on $\mathbb{P}^5$, a natural question is to describe the bundle $\mathcal{E}$.

Concerning question 0.1, we should point out that it is often easier to construct algebraic varieties as parametric images rather than as zero sets of ideals. Note that surfaces with $p_g(S) = 6, q(S) = 2, K_S^2 = 45$ (and with K_S ample since they are ball quotients) were constructed in [BC08]. For these, however, the canonical system has base points.

It would be interesting to see whether there do exist canonically embedded surfaces with $K_S^2 = 56$ which are regular surfaces isogenous to a product (see [Cat00]).

[1] This follows from theorem 3.8 of [Cat84].

We finish this introduction pointing out that a similar question is wide open also for $p_g(S) = 5$ (while for $p_g(S) = 4$ some work has been done, see for instance [Cat99]).

Question 0.2. Let S be a smooth minimal surface of general type, assume that $p_g(S) = 5$ and that the canonical map $\phi_1 : S \to \mathbb{P}^4$ is birational. What is the maximal value that K_S^2 can reach?

Again, Castelnuovo's inequalty gives $8 \leq K_S^2$, and Bogomolov-Miyaoka-Yau's inequality gives $K_S^2 \leq 54$.

The known cases where ϕ_1 is an embedding are just for $K_S^2 = 8, 9$, where S is a complete intersection of type $(4, 2)$ or $(3, 3)$.

Indeed these are the only cases by virtue of the following (folklore?) theorem which was stated and proven in [Cat97], propositions 6.1 and 6.2, corollary 6.3.

Theorem 0.2. Assume that S is the minimal model of a surface of general type with $p_g(S) = 5$, and assume that the canonical map ϕ_1 embeds S in $\mathbb{P}^4$.

Then S is a complete intersection with $\mathcal{O}_S(K_S) = \mathcal{O}_S(1)$, i.e., S is a complete intersection in $\mathbb{P}^4$ of type $(2, 4)$ or $(3, 3)$.

Moreover, if ϕ_1 is birational, and $K_S^2 = 8, 9$, then ϕ_1 yields an embedding of the canonical model of X as a complete intersection in $\mathbb{P}^4$ of type $(2, 4)$ or $(3, 3)$.

Recall that the first main ingredient of proof is the well known Severi's double point formula ([Sev01], see [Cat79] for the proof of some transversality claims made by Severi).

Write the double point formula in the form stated in [Hart77], Appendix A, 4.1.3. It gives, once we set $d := K_S^2$:

$$12\chi(S) = (17 - d)d.$$

Since $\chi(S) \geq 1$, we get $8 \leq d \leq 16$, and viewing the double point formula as an equation among integers, we see that it is only solvable for $d = 8, 9 \Rightarrow \chi(S) = 6 \Rightarrow q(S) = 0$, or for $d = 12, \Rightarrow \chi(S) = 5 \Rightarrow q(S) = 1$.

One sees that the last case cannot occur, since the Albanese map of S, $\alpha : S \to A$, has as image an elliptic curve A; if one denotes by g the genus of the Albanese fibres, the slope inequalities for fibred surfaces of Horikawa, or Xiao, or Konno ([Hor81], [Xiao87], [Kon93]), give $g = 2$: hence ϕ_1 cannot be birational.

One could replace in theorem 0.2 the hypothesis that ϕ_1 is an embedding of S in $\mathbb{P}^4$ by the weaker condition that ϕ_1 yields an embedding of the canonical model of X via an extension of the Severi double formula to the case of surfaces with rational double points as singularities [2].

[2]for which however we have not yet found a reference.

1. The construction of the family of surfaces

We consider the family of algebraic surfaces S, bidouble covers (Galois covers with group $(\mathbb{Z}/2)^2$) of the quadric $Q := \mathbb{P}^1 \times \mathbb{P}^1$, described in the fourth line of page 106 of [Cat99].

This means that we take three divisors

$$D_j = \{\delta_j = 0\}, \delta_1, \delta_2 \in H^0(\mathcal{O}_Q(2,3)),\ \delta_3 \in H^0(\mathcal{O}_Q(4,1)),$$

we consider divisor classes $L_1 = L_2, L_3$ with

$$\mathcal{O}_Q(L_1) = \mathcal{O}_Q(L_2) = \mathcal{O}_Q(3,2),\ \mathcal{O}_Q(L_3) = \mathcal{O}_Q(2,3)$$

and we define the surface S as

$$Spec(\mathcal{O}_Q \oplus w_1\mathcal{O}_Q(-L_1) \oplus w_2\mathcal{O}_Q(-L_2) \oplus w_3\mathcal{O}_Q(-L_3)),$$

where the ring structure is given by (here $\{i,j,k\}$ is any permutation of $\{1,2,3\}$)

$$w_i^2 = \delta_j\delta_k,\ w_iw_j = w_k\delta_k.$$

We refer to [Cat84] and [Cat99] for the basics on bidouble covers, which show that on the surface S the sections w_i can be written as a product of square roots of the sections $\delta_1, \delta_2, \delta_3$:

$$w_i = y_jy_k,\ y_i^2 = \delta_i.$$

As is in [Cat99], page 102, define

$$N := 2K_Q + \sum_1^3 D_i = 2K_Q + \sum_1^3 L_i,$$

obtaining a description of the canonical ring of S, which is a smooth surface with ample canonical divisor if the three curves D_1, D_2, D_3 are smooth and intersect transversally. The canonical ring is defined as usual by:

$$\mathcal{R} := \oplus_{m=0}^{\infty}\mathcal{R}_m,\ \mathcal{R}_m := H^0(\mathcal{O}_S(mK_S)),$$

and we have (ibidem)

$$\mathcal{R}_{2m+1} = y_1y_2y_3H^0(\mathcal{O}_Q(K_Q+mN)) \oplus (\oplus_{i=1}^3 y_iH^0(\mathcal{O}_Q(K_Q+mN+L_i)))$$

$$\mathcal{R}_{2m} = H^0(\mathcal{O}_Q(mN)) \oplus (\oplus_{i=1}^3 w_iH^0(\mathcal{O}_Q(mN-L_i))).$$

In particular, since in this case $\mathcal{O}_Q(N) = \mathcal{O}_Q(4,3)$,

$$H^0(\mathcal{O}_S(K_S)) = \oplus_{i=1}^3 y_iH^0(\mathcal{O}_Q(K_Q+L_i)) =< y_1u_0, y_1u_1, y_2u_0, y_2u_1, y_3v_0, y_3v_1 >,$$

where u_0, u_1 is a basis of $H^0(\mathcal{O}_Q(1,0))$, v_0, v_1 is a basis of $H^0(\mathcal{O}_Q(0,1))$.

In particular, $p_g(S) = 6$; moreover, $q(S) = 0$ since $H^1(\mathcal{O}_Q) = H^1(\mathcal{O}_Q(-L_i)) = 0, \forall i = 1,2,3$, as follows easily from the Künneth formula.

Finally, if $\pi : S \to Q$ is the bidouble cover, then $2K_S = \pi^*(N)$, hence K_S is ample and $K_S^2 = N^2 = 24$.

2. Proof that the canonical map is an embedding

Theorem 2.1. Assume that the three branch curves D_1, D_2, D_3 are smooth and intersect transversally, and moreover that the 12 intersection points $D_1 \cap D_2$ have pairwise different images via the second projection $p_v : Q \to \mathbb{P}^1$ (i.e., they have different coordinates $(v_0 : v_1)$).

Then the canonical map $\phi_1 : S \to \mathbb{P}^5$, $\phi_1(x) =$

$$= (y_1(x)u_0(x) : y_1(x)u_1(x) : y_2(x)u_0(x) : y_2(x)u_1(x) : y_3(x)v_0(x) : y_3(x)v_1(x)),$$

is a biregular embedding, i.e. we have an isomorphism

$$\phi_1 : S \to \Sigma := \phi_1(S).$$

Proof. Let $R_i \subset S, R_i := \{y_i = 0\}$. The curve R_i maps to D_i with degree 2, and $R_1 \cap R_2 \cap R_3 = \emptyset$ since by our assumption $D_1 \cap D_2 \cap D_3 = \emptyset$.

Claim 1: the canonical system is base-point free.

In fact, for each point x there is u_i, $i \in \{0, 1\}$, such that $u_i(x) \neq 0$, and similarly v_j, $j \in \{0, 1\}$, such that $v_j(x) \neq 0$. Hence x is a base point if $y_1 = y_2 = y_3 = 0$ at x, contradicting $R_1 \cap R_2 \cap R_3 = \emptyset$.

Claim 2: ϕ_1 is a local embedding.

2.1) At the points $x \in R_h \cap R_k$ the two sections y_h, y_k yield local coordinates, hence our assertion.

2.2) For the points $x \in S \setminus R$, where $R = R_1 \cup R_2 \cup R_3$ is the ramification divisor, let us consider the rational map $F_u : \Sigma \to \mathbb{P}^1$, induced by the linear projections

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \dashrightarrow (x_1 : x_2)$$

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \dashrightarrow (x_3 : x_4),$$

respectively the rational map $F_v : \Sigma \to \mathbb{P}^1$, induced by the linear projection

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \dashrightarrow (x_5 : x_6).$$

We have that $(u_0(x) : u_1(x)) = p_u \circ \pi = F_u \circ \phi_1, (v_0(x) : v_1(x)) = p_v \circ \pi = F_v \circ \phi_1$. Moreover, $F_u \circ \phi_1$ is a morphism outside the finite set $R_1 \cap R_2$, $F_v \circ \phi_1$ is a morphism outside R_3.

Hence $\pi = (F_u \circ \phi_1) \times (F_v \circ \phi_1) : S \setminus R$ is a morphism of maximal rank, so ϕ_1 is a local embedding outside of R.

2.3) Set for convenience $u := p_u \circ \pi, v := p_v \circ \pi$.

At the points of $R_1 \setminus (R_2 \cup R_3)$, then y_1 , u, v give local coordinates, so we are done; similarly for the points of $R_2 \setminus (R_1 \cup R_3)$.

At the points of $R_3 \setminus (R_1 \cup R_2)$, we need to show that y_3 and u give local coordinates. Here, we make the remark that D_3 is a divisor of bidegree $(4, 1)$, hence $p_u : D_3 \to \mathbb{P}^1$ is an isomorphism. Hence δ_3, u give local coordinates at the points of D_3, and we infer that y_3 , u, give local coordinates at the points of $R_3 \setminus (R_1 \cup R_2)$.

Claim 3: ϕ_1 is injective.

Assume that $\phi_1(x) = \phi_1(x')$. We observe preliminarily that this condition implies that, if $x \in R_i$, then also $x' \in R_i$, since for instance $x \in R_1 \Leftrightarrow x_1(\phi_1(x)) = x_2(\phi_1(x)) = 0$.

3.1) Assume that $x, x' \in S \setminus R$.

Then $\pi(x) = \pi(x')$, and since the elements of Galois group $G = (\mathbb{Z}/2)^2 = \{\pm 1\}^3/\{\pm 1\}$ are determined by $\epsilon \in \{\pm 1\}^3$, and each of them acts on $\phi_1(x) = (x_1 : x_2 : x_3 : x_4 : x_5 : x_6)$ by

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (\epsilon_1 x_1 : \epsilon_1 x_2 : \epsilon_2 x_3 : \epsilon_2 x_4 : \epsilon_3 x_5 : \epsilon_3 x_6),$$

the action of G on $\phi_1(x)$ has an orbit of cardinality 4, which is precisely the cardinality of $\pi^{-1}(\pi(x))$.

Hence $\phi_1(x) = \phi_1(x') \Rightarrow x = x'$.

3.2) Let $x, x' \in R_1 \setminus (R_2 \cup R_3)$. Again $\pi(x) = \pi(x')$, and we see that the orbit of $\phi_1(x)$ under G is the set of points

$$(0 : 0 : \epsilon_2 x_3 : \epsilon_2 x_4 : \epsilon_3 x_5 : \epsilon_3 x_6),$$

which has cardinality 2; this is precisely the cardinality of $\pi^{-1}(\pi(x))$, so again we are done.

The case $x, x' \in R_2 \setminus (R_1 \cup R_3)$ is completely analogous.

3.3) Let $x, x' \in R_3 \setminus (R_1 \cup R_2)$. Again, the orbit of $\phi_1(x)$ under G has cardinality 2, so we are done if we show that $\pi(x) = \pi(x')$. On the set R_3, however, π is not a morphism, so we argue differently.

We use instead that $u(x) = u(x')$, and that $p_u : D_3 \to \mathbb{P}^1$ is an isomorphism to conclude that $\pi(x) = \pi(x')$.

3.4) If $x, x' \in R_3 \cap R_2$ (for $x, x' \in R_3 \cap R_1$ the argument is entirely similar), we proceed as follows.

We obtain again $u(x) = u(x')$, and since $p_u : D_3 \to \mathbb{P}^1$ is an isomorphism, we get that $\pi(x) = \pi(x')$. However, the restriction of π to $R_3 \cap R_2$ is bijective, hence $x = x'$.

3.5) Assume finally that $x, x' \in R_1 \cap R_2$ and use again that the restriction of π to $R_1 \cap R_2$ π is bijective. It suffices therefore to show that $\pi(x) = \pi(x')$.

The condition $\phi_1(x) = \phi_1(x') = (0 : 0 : 0 : 0 : y_3 v_0 : y_3 v_1)$, since y_3 does not vanish on $R_1 \cap R_2$, implies that $v(x) = v(x')$, so the points $\pi(x), \pi(x') \in D_1 \cap D_2$ have the same v coordinate.

Hence, by our assumption, $\pi(x) = \pi(x')$, exactly as desired.

□

Proposition 2.2. The hypotheses of theorem 2.1 define a non empty family of dimension $11 + 11 + 9 = 31$.

Proof. D_1, D_2 vary in a linear system of dimension $3 \times 4 - 1 = 11$, D_3 varies in a linear system of dimension $5 \times 2 - 1 = 9$.

The condition that the divisors intersect transversally is a consequence of the fact that each of the three linear systems embeds Q in a projective space.

The final condition amounts to the following: write

$$\delta_j = u_0^2 A_j(v) + u_0 u_1 B_j(v) + u_1^2 C_j(v) = 0,\ \ j = 1, 2.$$

We can view the coefficients of δ_1, δ_2, six degree three homogeneous polynomials in $v = (v_0 : v_1)$, as giving a map ψ of $\mathbb{P}^1$ inside the $\mathbb{P}^5$ with coordinates

$$(A_1 : B_1 : C_1 : A_2 : B_2 : C_2)$$

parametrizing pairs of homogeneous polynomials of degree 2 in $u = (u_0 : u_1)$,

$$\mathcal{P}_j = u_0^2 A_j + u_0 u_1 B_j + u_1^2 C_j = 0,\ \ j = 1, 2.$$

Let Δ be the resultant

$$\Delta := Res_u(\mathcal{P}_1, \mathcal{P}_2) = (A_1 C_2 - A_2 C_1)^2 + (B_2 C_1 - C_2 B_1)(A_1 B_2 - A_2 B_1).$$

The resultant Δ defines a hypersurface of degree 4 in $\mathbb{P}^5$, which is reduced and irreducible, being a non degenerate quadric in the variables B_1, B_2.

We can therefore choose the six polynomials in a general way so that the twisted cubic curve $\psi(\mathbb{P}^1)$ intersects Δ transversally in 12 distinct points.

We conclude because the image of $D_1 \cap D_2$ via the projection p_v is given by the 12 zeros of

$$f(v_0 : v_1) := Res_u(\delta_1, \delta_2) = \Delta(A_1(v) : B_1(v) : C_1(v) : A_2(v) : B_2(v) : C_2(v)),$$

and by our general choice these are 12 distinct points.

□

Remark 2.3. i) Taking into account the group of automorphisms of $Q = \mathbb{P}^1 \times \mathbb{P}^1$, we see that the above family gives a locally closed subset of dimension 25 inside the moduli space of surfaces of general type. This dimension is more than the expected dimension $10\chi(S) - 2K_S^2 = 70 - 48 = 22$.

ii) since $H^0(\mathcal{O}_Q(D_i - L_i) = 0$ for each $i = 1, 2, 3$, there are no natural deformations ([Cat84], [Cat99]); it is not clear that our family yields an irreducible component of the moduli space, since the elementary method of cor. 2.20 and theorem 3.8 of [Cat84] do not apply.

3. The canonical ring as a module over the symmetric algebra $\mathcal{A} := Sym(\mathcal{R}_1)$

Let us first of all look at the homomorphism $m_2 : Sym^2(\mathcal{R}_1) \to \mathcal{R}_2$, keeping track of the eigenspace decompositions, and using the notation $H^0(\mathcal{O}_Q(a, b)) =: V(a, b)$. We have then

$$\mathcal{R}_1 = 0 \oplus y_1 V(1, 0) \oplus y_2 V(1, 0) \oplus y_3 V(0, 1),$$

$$\mathcal{R}_2 = V(4, 3) \oplus y_2 y_3 V(1, 1) \oplus y_1 y_3 V(1, 1) \oplus y_1 y_2 V(2, 0).$$

Since $V(1,0)\otimes V(0,1)\cong V(1,1)$, and we have a surjection $V(1,0)\otimes V(1,0)\to V(2,0)$, the non trivial character spaces of $\mathcal{R}_2$ are in the image of $m_2 : Sym^2(\mathcal{R}_1)\to\mathcal{R}_2$.

Moreover, the kernel of $y_1V(1,0)\otimes y_2V(1,0)\to y_1y_2V(2,0)$ is 1-dimensional, and provides a quadric $\{q(x)=0\}$ containing the canonical image Σ of S. To simplify our notation, we directly assume that $S\subset\mathbb{P}^5$, via the canonical embedding.

Then the quadric that we obtain is: $q(x) := x_2x_3 - x_1x_4$.

The trivial character space of $\mathcal{R}_2$ is isomorphic to $V(4,3)$ and contains the image of the subspace

$$W := y_1^2V(2,0)\oplus y_2^2V(2,0)\oplus y_3^2V(0,2)\subset Sym^2(\mathcal{R}_1),$$

which maps onto

$$W' := \delta_1V(2,0)+\delta_2V(2,0)+\delta_3V(0,2).$$

Lemma 3.1. *$W\to W'$ is an isomorphism.*

Proof. Assume that there is a kernel: then there are bihomogeneous polynomials P_1,P_2,P_3 such that $P_1\delta_1+\delta_2P_2=P_3\delta_3$.

Since δ_3 does not vanish at the 12 points where $\delta_1=\delta_2=0$, $P_3=P_3(v)$ should vanish on the projections of these 12 points under p_v. But this is a contradiction, since P_3 has degree 2, while the 12 projected points are distinct. Hence $P_3\equiv 0$, and $\delta_1|P_2$, a contradiction again since P_2 has bidegree $(2,0)$.

□

Corollary 3.1. S is contained in a unique quadric $\{q(x)=0\}$, and $m_2 : Sym^2(\mathcal{R}_1)\to\mathcal{R}_2$ has image of dimension $21-1=20$ and codimension 11.

Proof. $\mathcal{R}_m$ has dimension $dim(\mathcal{R}_m)=\chi(S)+\frac{1}{2}m(m-1)K_S^2 = 7+12m(m-1)$, which, for $m=2$, is equal to 31.

□

We shall now choose $z_1,\dots,z_{11}\in\mathcal{R}_2$ which, together with $Im(m_2)$, generate $\mathcal{R}_2$. The elements $z_1,\dots,z_{11}$ induce a basis of the quotient $\mathcal{R}_2/Im(m_2)$.

Theorem 3.2. Let $\mathcal{A} := Sym(\mathcal{R}_1)$ be the coordinate ring of $\mathbb{P}^5$, and consider $\mathcal{R}$ as an $\mathcal{A}$-module. Then $1,z_1,\dots,z_{11}$ is a minimal graded system of generators of $\mathcal{R}$ as an $\mathcal{A}$-module.

Proof. In view of the previous observations, it suffices to show that these elements generate $\mathcal{R}$.

Observe preliminarily that $V(a,b)\otimes V(c,d)\to V(a+c,b+d)$ is always surjective as soon as $a,b,c,d\geq 0$.

For $\mathcal{R}_3$, let us write:

$$\mathcal{R}_3 = y_1y_2y_3V(2,1)\oplus y_1V(5,3)\oplus y_2V(5,3)\oplus y_3V(4,4).$$

The last three eigenspaces are in the image of $V(4,3) \otimes \mathcal{R}_1 \subset \mathcal{R}_2 \otimes \mathcal{R}_1 \to \mathcal{R}_3$.

Also the first summand is in the image of $y_1 V(1,0) \otimes y_2 y_3 V(1,1)$.

The same argument works for $\mathcal{R}_{2m+1}$, while for $\mathcal{R}_{2m+2}$ we find surjections

$$H^0(\mathcal{O}_Q(N)) \otimes w_i H^0(\mathcal{O}_Q(mN - L_i)) \to w_i H^0(\mathcal{O}_Q((m+1)N - L_i)),$$

and

$$H^0(\mathcal{O}_Q(N)) \otimes H^0(\mathcal{O}_Q(mN)) \to H^0(\mathcal{O}_Q((m+1)N)).$$

Hence the claimed result follows by induction on m.

□

Remark 3.3. 1) Since $dim\ \mathcal{R}_3 = 79$, $dim\ \mathcal{A}_3 = 56$, we see that the 12 minimal generators of the module admit $56 + 6 \times 11 - 79 = 43$ relations in degree 3.

The module $\mathcal{R}$ is Cohen-Macaulay, hence it has a length 3 minimal graded resolution.

2) In general, $dim\ \mathcal{R}_m = 7 + 12m(m-1)$,

$$dim\ \mathcal{A}_m = \binom{m+5}{m},$$

hence the image of $\mathcal{A}_m \to \mathcal{R}_m$ has dimension less than or equal (because S is contained in a quadric) to $dim \mathcal{A}_m - dim \mathcal{A}_{m-2}$.

Hence the Hartshorne-Rao module $\oplus_m H^1(\mathcal{I}_S(m)$ is non zero in all degrees $m = 2,3,4,5,6$. This shows that the bundle $\mathcal{E}$ contructed by Walter using the Horrocks correspondence should be rather interesting.

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