# KODAIRA FIBRATIONS AND BEYOND: METHODS FOR MODULI THEORY

F. CATANESE

Contribution to the 16-th Takagi lectures 'In Celebration of the 100th Anniversary of Kodaira's Birth'

> ABSTRACT. Kodaira fibred surfaces are a remarkable example of projective classifying spaces, and there are still many intriguing open questions concerning them, especially the slope question. The topological characterization of Kodaira fibrations is emblematic of the use of topological methods in the study of moduli spaces of surfaces and higher dimensional complex algebraic varieties, and their compactifications. Our tour through algebraic surfaces and their moduli (with results valid also for higher dimensional varieties) deals with fibrations, questions on monodromy and factorizations in the mapping class group, old and new results on Variation of Hodge Structures, especially a recent answer given (in joint work with Dettweiler) to a long standing question posed by Fujita. In the landscape of our tour, Galois coverings, deformations and rigid manifolds (there are by the way rigid Kodaira fibrations), projective classifying spaces, the action of the absolute Galois group on moduli spaces, stand also in the forefront. These questions lead to interesting algebraic surfaces, for instance remarkable surfaces constructed from VHS, surfaces isogenous to a product with automorphisms acting trivially on cohomology, hypersurfaces in Bagnera-de Franchis varieties, Inoue-type surfaces.

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# 1. KODAIRA FIBRATIONS

It is well known that the topological Euler characteristic e is multiplicative for fibre bundles: this means that, if  $f: X \to B$  is a fibre bundle with fibre F, then

$$e(X) = e(B)e(F).$$

In 1957 Chern, Hirzebruch and Serre ([CHS57]) showed that the same holds true for the signature, also called index  $\sigma = b^+ - b^-$  (it is the index of the intersection form on the middle cohomology group) if

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the fundamental group of the base B acts trivially on the (rational) cohomology of the fibre F.

In 1967 Kodaira [Kod67] constructed examples of fibrations of a complex algebraic surface over a curve which are differentiable but not holomorphic fibre bundles for which <sup>1</sup>the multiplicativity of the signature does not hold true. In his honour such fibrations are nowadays called Kodaira fibrations. In fact, for a compact oriented two dimensional manifold the intersection form is antisymmetric, hence  $\sigma = 0$ , whereas Kodaira fibrations have necessarily  $\sigma > 0$ .

As I am now going to explain, there are many interesting properties and open questions concerning Kodaira fibrations.

1.1. Generalities on algebraic surfaces. The signature formula of Hirzebruch, Atiyah and Singer for a compact complex surface S is

$$\sigma(S) = \frac{1}{3}(K_S^2 - 2e(S)) = \frac{1}{3}(c_1(S)^2 - 2c_2(S)).$$

Here the Euler Number e(S) is the alternating sum of the Betti numbers

$$e(S) = 1 - b_1(S) + b_2(S) - b_3(S) + 1 = 2 - 2b_1(S) + b_2(S),$$

and it equals the second Chern class  $c_2(S)$  of the complex tangent bundle of S. Whereas  $K_S = -c_1(S)$  is, for an algebraic surface, the Cartier divisor of a rational section of the sheaf  $\Omega_S^2$  of holomorphic differential 2-forms.

In the Kähler case it was well known that the signature  $\sigma(S) = b^+(S) - b^-(S)$  is determined by the Hodge numbers, indeed  $b^+(S) = 2p_g(S) + 1, b^+(S) + b^-(S) = b_2(S)$ , where

- $p_g(S) := h^{2,0}(S) := h^0(\Omega_S^2)$  is the geometric genus of S,
- $q(S) := h^{0,1}(S) := h^1(\mathcal{O}_S)$  is the **irregularity** of S,
- $h^{1,0}(S) := h^0(\Omega_S^1)$  is the Albanese number (dimension of the Albanese variety).

Even if I will concentrate on algebraic or Kähler manifolds, I would like to point out how the beautiful series of papers by Kodaira 'On the structure of compact complex analytic surfaces' [Kod64-8] (extending the Enriques classification to non algebraic complex surfaces) begins with the following miracoulous consequence of the signature formula:

$$(b^+(S) - 2p_g(S)) + (2q(S) - b_1(S)) = 1.$$

Because both terms are easily shown to be non-negative, by the fact that the intersection form is positive definite on  $H^0(\Omega_S^2) \oplus \overline{H^0(\Omega_S^2)}$ , respectively that one has the exact sequence

 $0 \to H^0(d\mathcal{O}_S) \to H^1(S,\mathbb{C}) \to H^1(\mathcal{O}_S),$ 

and the Dolbeault inclusion  $\overline{H^0(d\mathcal{O}_S)} \subset H^1(\mathcal{O}_S)$ .

<sup>&</sup>lt;sup>1</sup>indeed, this is true for all such fibrations

The non Kähler case is just the case where the first Betti number  $b_1(S)$  is odd,  $b_1(S) = 2q(S) - 1, b^+(S) = 2p_g(S)$ ; this conjecture of Kodaira was proven through a long series of papers, culminating in [Siu83].

The so called 'surface geography' problem was raised by van de Ven ([vdV66]), and concerns the points of the plane with coordinates  $(\chi(S), K_S^2)$ ; here  $\chi(S) := \chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$  is the Euler Poincaré characteristic of the sheaf of holomorphic functions.

An important invariant, in the case  $\chi(S) \geq 1$  (recall Castelnuovo's theorem:  $\chi(S) < 0$  implies that S is ruled) is the so called slope  $\nu := K_S^2/\chi(S)$ , whose growth is equivalent to the one of the Chern slope, which is the ratio  $\nu_C(S) := c_1(S)^2/c_2(S)$  of the Chern numbers, because of the Noether formula

$$12\chi(S) = c_1(S)^2 + c_2(S) \Rightarrow c_1(S)^2 / c_2(S) = K_S^2 / (12\chi(S) - K_S^2) = \frac{\nu}{12 - \nu}$$

As conjectured by van de Ven, inspired by work of Bogomolov, and proven by Miyaoka and Yau, there is the slope inequality,

$$\nu(S) := K_S^2 / \chi(S) \le 9 \Leftrightarrow \nu_C(S) := c_1(S)^2 / c_2(S) \le 3$$

for surfaces with  $\chi(S) \ge 1$ .

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It is commonly called the Bogomolov-Miyaoka-Yau inequality.

Moreover, by the theorems of Aubin and Yau ([Aub78], [Yau77], [Yau78]) and Miyaoka's proof [Miy77], [Miy83] of ampleness of  $K_S$  in the case where equality holds, follows

**Theorem 1.** Is S a non ruled surface and  $K_S^2/\chi(S) = 9$ , then the universal cover  $\tilde{S}$  of S is biholomorphic to the ball, i.e. the unit disk  $\mathcal{D}_2 \subset \mathbb{C}^2$ .

Indeed, the argument of proof about the existence of the Kähler Einstein metric seemed to suggest that if the ratio  $K_S^2/\chi(S)$  would be rather close to 9, then the universal cover  $\tilde{S}$  of S would be diffeomorphic to Euclidean space. More than that, probably the imagination of many was struck by an impressive result by Mostow and Siu [M-S80]

**Theorem 2.** There exists an infinite series of rigid surfaces S with positive signature whose slope is very close to 9<sup>2</sup>, and which admit a metric of nonpositive sectional curvature. In particular, by the theorem of Cartan-Hadamard, their universal cover  $\tilde{S}$  is diffeomorphic to Euclidean space  $\mathbb{R}^4$ .

Recently, Roulleau and Urzua [RU15] disposed in the negative of a stronger form of this no name-conjecture

**Theorem 3.** The slopes of simply connected surfaces are dense in the interval [8,9].

 $<sup>^{2}</sup>$ The maximum achieved by Mostow and Siu is 8,8575.

They used a method introduced by Comessatti and, later, by Hirzebruch ([Hirz84]), to consider abelian coverings of the projective plane  $\mathbb{P}^2$  branched over special configurations of curves, especially configurations of lines; we shall discuss this method at a later moment.

Of course if the slope  $\nu(S) = 9$  then the universal cover S is the ball, and the fundamental group is countable; while the result of [RU15] does not exclude a priori the existence of a small region around the Miyaoka-Yau line  $K_S^2 = 9\chi(S)$  where the fundamental group is infinite and the universal cover is homeomorphic to a ball, it morally settles the question in the negative. A similar but even bolder conjecture, namely the so-called 'watershed conjecture', that every surface of positive index should have infinite fundamental group had been disproved thanks to Miyaoka's observation that the Galois closure S of a general projection of a surface  $X \to \mathbb{P}^2$  tends to have positive index (see [Miy83], [MT87], and [CZ87] for other examples).

1.2. Kodaira fibrations and Kodaira's construction. We consider in this subsection the following situation:  $f: S \to B$  is a holomorphic map of a compact complex surface S onto a curve B of genus b, which is a differentiable fibre bundle, with fibres F of genus g. In this case, the Euler number of S equals

$$e(S) = 4(g-1)(b-1).$$

This fibration clearly induces a morphism  $\Phi: B \to \mathfrak{M}_g$  into the moduli space of curves of genus g.

There are two cases: either  $\Phi$  is constant, or, as one says, we have a nonconstant moduli fibration. The second case is exactly the case of Kodaira fibrations. Following a generalization of Kodaira's method developed in our joint paper with Rollenske, we are going to show how this situation can be effectively constructed, and in such a way that we can calculate the slope  $\nu(S)$  explicitly.

On the other hand, the fact that Kodaira fibrations do in fact exist, can be shown non effectively in the following way.

1.2.1. General Kodaira fibrations. Let  $\overline{\mathfrak{M}_g}^*$  be the Satake compactification of the moduli space of curves of genus  $g \geq 3$ : this is the closure of  $\mathfrak{M}_g$ , embedded via the Torelli map into  $\mathcal{A}_g$ , inside the Satake compactification  $\overline{\mathcal{A}_g}^*$ .

From the moduli point of view, given a moduli-stable curve C of genus g, we associate to C the product of the Jacobian varieties of the components in the normalization  $\tilde{C}$  of C. It follows that the boundary  $\partial \overline{\mathfrak{M}_g}^*$  has dimension 3g - 5, hence codimension 2 inside  $\overline{\mathfrak{M}_g}^*$ . Similarly the singular set of  $\overline{\mathfrak{M}_g}^*$  corresponds to the locus  $\Sigma$  of curves with automorphisms for  $g \geq 4$ , and is contained in  $\Sigma$  for g = 3;  $\Sigma$  has codimension at least 2 for  $g \geq 4$ , and codimension equal to 1 for g = 3 (its divisorial part is the locus of hyperelliptic curves)

By the projectivity of  $\overline{\mathfrak{M}_g}^*$  we can find, for  $g \geq 4$ , a smooth linear section B of  $\overline{\mathfrak{M}_g}^*$  having dimension 1 and which avoids both  $\partial \overline{\mathfrak{M}_g}^*$  and the locus  $\Sigma$ .

So  $B \subset \mathfrak{M}_g$  and, since  $B \cap \Sigma = \emptyset$ , we have a family of curves over B with nonconstant moduli, and the fibre curves are all smooth.

We shall see in the sequel that a Kodaira fibration has always positive index, and indeed the Chern slope  $\nu_C(S)$  lies in the open interval (2,3).

One can show that, via Kodaira fibrations obtained as just described from a general complete intersection curve B (under the composition of the Torelli map with the Satake embedding of the moduli space  $\mathfrak{M}_g$ of curves of genus  $g \geq 4$ ), one obtains a slope which is rather small, at most around 2,18 (see [CatRol09]).

The above argument shows, via a non explicit construction, the existence of Kodaira fibrations for any fibre genus  $g \ge 4$ . The argument can also be adapted for the case g = 3, while for g = 2 there are no Kodaira fibrations; in fact  $\mathfrak{M}_2$  is affine, by a theorem of Igusa [Ig60]: hence if B is a complete curve, any morphism  $B \to \mathfrak{M}_2$  must be constant.

For g = 3 we take a smooth curve D which does not intersect  $\partial \mathfrak{M}_g^*$ and intersects  $\Sigma$  transversely in smooth points  $p_1, \ldots, p_k$  corresponding to general hyperelliptic curves  $C_1, \ldots, C_k$  (hence such that  $\operatorname{Aut}(C_i) = \mathbb{Z}/2$ ). In this case we do not have a family over D, but only over  $D \setminus \Sigma$ . Take now a double covering  $f : B \to D$  branched over the points  $p_1, \ldots, p_k$  and over other points. Let  $p'_j := f^{-1}(p_j)$ : then we observe that for each  $j = 1, \ldots, k$  the Kuranishi family of  $C_j$  has a map  $\psi$  to  $\mathfrak{M}_3$  which is a double covering ramified over the hyperelliptic locus. In local coordinates we may assume that the map is given by

$$\psi: (y_1, y_2, \dots, y_5, z) \mapsto (y_1, y_2, \dots, y_5, z^2), i.e., w = z^2$$

and the curve D is locally given by

$$y_2 = \dots = y_5 = 0, y_1 = kw,$$

hence

$$B = \{y_2 = \dots = y_5 = 0, y_1 = kz^2\}$$

and the family over  $B \setminus f^{-1}(\Sigma)$  extends to a family over B, with all the fibres smooth genus 3 curves.

As discussed with Terasoma after the Takagi lectures, the Kodaira construction does not work for g = 3. There is however an explicit construction for g = 3 due to Zaal [Zaal95], which uses Prym varieties, and other explicit constructions of curves B as above, by Gonzalez-Diez and Harvey [GD-H91].

We do not know the answer to the following question, which might turn out to be not too difficult

**Question 4.** Given an integer  $g \ge 3$ , which is the number b(g), the minimum value  $b \in \mathbb{N}$  such that there is a Kodaira fibration with fibres of genus g and base curve of genus b?

We just observe here that  $b \ge 2$ , and indeed  $b \ge 3$  for g = 3, 4. Because for Kodaira fibrations we have e(S) = 4(g-1)(b-1), the positive index inequality  $K_S^2 > 2e(S) = 8(g-1)(b-1)$  and the Kefeng Liu inequality [Liu96]  $K_S^2 < 9\chi(S)$  hold; combining with the Noether formula  $K_S^2 = 12\chi(S) - e(S)$  one gets then

$$3(g-1)(b-1) < 3\chi(S) < 4(g-1)(b-1),$$

by which our previous assertion follows immediately.

Observe moreover that once you have a Kodaira fibration  $f: S \to B$ with given fibre genus g, and base genus b, for each integer  $n \ge 1$  we can take the Kodaira fibration  $f_n: S(n) \to B$ , where the fibres of S(n) are the unramified coverings of the fibres F of f corresponding to the surjection  $\pi_1(F) \to H_1(F, \mathbb{Z}/n)$ . Then the fibre genus of  $f_n$  equals  $g_n := 1 + (g-1)n^{2g}$ , while the genus of the base curve remains b. This shows that

$$\operatorname{liminf}_{g \to \infty}(b_g) = \min(b_g).$$

1.2.2. *Kodaira's construction and its generalizations*. Kodaira constructed explicit examples of these Kodaira fibrations, and we are going now to describe how his method can be generalized.

The basic notion for the Kodaira type construction is the following:

**Definition 5.** A logarithmic Kodaira fibration is a quadruple (X, D, f, B) consisting of

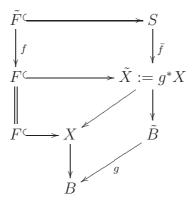
- (1) a smooth fibration  $\psi : X \to B$  of a surface to a curve, with fibres  $F_t$  and
- (2) a divisor  $D \subset X$  such that
  - (a) the projection  $D \to B$  is étale and
  - (b) the fibration of pointed curves  $(F_t, F_t \cap D)$  is not isotrivial, i.e., the fibres are not all isomorphic (as pointed curves).

Now, even if at first sight this does not seem to help, it really does: because now the fibration  $\psi$  might have constant moduli, but the points  $F_t \cap D$  may be moving.

The easiest case to consider is the case where X will be a product of curves  $X := B_1 \times B_2$  and D shall be a divisor such that the first projection  $D \to B_1$  is étale and the second projection  $D \to B_2$  is finite.

We shall now see that in order to construct Kodaira fibrations it suffices to construct log-Kodaira fibrations.

**Proposition 6.** Let  $(X, D) \to B$  be a log-Kodaira fibration and let  $f: \tilde{F} \to F$  be a Galois-covering of a fibre F, with Galois group G, and branched over  $D \cap F$ . Then we can extend f to a ramified covering of surfaces  $\bar{f}: S \to \tilde{X}$  obtaining a diagram



where

- $g: \tilde{B} \to B$  is an étale covering,
- $\tilde{X}$  is the pullback of X via g,
- $\bar{f}$  is a ramified covering with Galois group G branched over  $\tilde{D} := g^*D$  and such that  $\bar{f}|_{\tilde{r}} = f$ .

The idea of the proof (for which we refer to [CatRol09]) is that the covering  $\bar{f}|_{\tilde{E}} = f$  is determined by a monodromy homomorphism

$$\mu: \pi_1(F \setminus F \cap D) \to G,$$

which extends on neighbouring fibres by the differential local triviality of the logarithmic fibration: but then there will be an action of  $\pi_1(B)$ transforming the monodromy into another one. Since however there is only a finite set of possible such monodromy homomorphisms to the finite group G, we get a tautological finite étale covering  $g: \tilde{B} \to B$  associated to the 'monodromy of the monodromy', and then the ramified covering extends to the pull back of X.

The problem is thus reduced to finding disjoint étale correspondences between the two curves  $B_1, B_2$ , which give the connected components of the curve D we are looking for.

In turn, the easiest case of an étale correspondence is given by the following situation: D' is a curve with an automorphism group H, and there are two subgroups  $H_1, H_2 < H$  acting freely on D'. We then set  $B_j := D'/H_j$ . Then D' admits a morphism  $\psi : D' \to B_1 \times B_2$ , such that the composition of  $\psi$  with both projections is étale.

If the intersection of the two subgroups  $H_0 := H_1 \cap H_2$  is non trivial, the map  $\psi$  factors through the quotient  $D'/H_0$ ; in any case,  $\psi$  is injective if and only if there are no points  $x \neq y$  such that  $H_1x = H_1y$ and  $H_2x = H_2y$  (i.e., there do exist  $h_1 \in H_1 \setminus \{1_H\}, h_2 \in H_2$  such that  $y = h_1x, x = h_2h_1x$ ). Equivalently,

 $D' \to B_1 \times B_2$  embeds  $D' \Leftrightarrow H_2(H_1 \setminus \{1_H\}) \cap \mathcal{S} = \emptyset$ ,

where  $S = \{h \in H | \exists x \ s.t.hx = x\}$  is the set of stabilizers.

1.3. Slopes of Kodaira fibrations. An interesting and open question, raised by Le Brun, asks for the possible values of the Chern slope of a Kodaira fibred surface  $f: S \to B$ .

The Chern slope  $\nu_C(S) := c_1^2(S)/c_2(S) = K_S^2/e(S)$  of a Kodaira fibred surface lies in the interval (2,3), in view of the well known Arakelov inequality (that shall be discussed in a later section) and of the improvement by Kefeng Liu ([Liu96]) of the Bogomolov-Miyaoka-Yau inequality to  $K_S^2/e(S) < 3$ .

Le Brun raised the question whether the slopes of Kodaira fibred surfaces can be effectively bounded away from 3: is it true that there exists  $\epsilon > 0$  such that for a Kodaira fibred surface S we have  $\nu_C(S) \leq 3 - \epsilon$ ?

The examples by Atiyah, Hirzebruch and Kodaira have slope not greater than 2+1/3 = 2, 33... (see [BPHV], page 221) and, as observed already, if one considers Kodaira fibrations obtained from a general complete intersection curve under the composition of the Torelli map with the Satake embedding in the moduli space  $\mathfrak{M}_g$  of curves of genus  $g \geq 3$ , one obtains a smaller slope (around 2,18).

Now, given a Kodaira fibration  $f : S \to B$ , and any holomorphic map of curves  $\varphi : B' \to B$ , one can take the pull-back of f, namely the fibred product  $S' := B' \times_B S$ . The slope remains the same if  $\varphi$  is étale, but in case where we have a ramified map, then the slope decreases.

In fact, if we denote by d the degree of  $\varphi$  and by r the degree of the ramification divisor on B', then b' - 1 = d(b - 1) + r/2, hence  $e(S') = 4d(g-1)(b-1+\frac{r}{2d})$ , while  $K_{S'}^2 = dK_S^2 + 4r(g-1)$ . Hence

$$[\nu_C(S) - \nu_C(S')][4(g-1)(b-1)(b-1 + \frac{r}{2d})] = \frac{r}{2d}(K_S^2 - 8(b-1)(g-1)),$$

which is strictly positive as soon as r > 0.

We observe moreover that the slope

$$\nu_C(S') = \frac{K_S^2 + 4\frac{r}{d}(g-1)}{4(g-1)(b-1+\frac{r}{2d})}$$

tends to 2 as soon as  $\frac{r}{d}$  tends to infinity.

Therefore, once one has found a given slope, it looks more like a question of book-keeping to show that one can realize smaller slopes. While the hard question seems to be the one of finding higher slopes: for this reason we concentrate our attention on the problem of finding Kodaira fibrations with high slope.

The best known result in the direction of high slope is the following result of [CatRol09]:

**Theorem 7** (Catanese-Rollenske). There are Kodaira fibrations with slope equal to  $2 + 2/3 = 2, 66 \dots$ 

Our method of construction has been a variant of the one used by Kodaira, namely to consider *double Kodaira fibred surfaces*.

The first main point is that the slope of a Kodaira fibred surface  $f: S \to B'$  obtained as a Galois branched covering of a logarithmic Kodaira fibred surface  $\psi: (X, D) \to B$  is determined by the logarithmic structure of (X, D), namely, given the components  $D_1, \ldots, D_r$  of D, one associates to  $D_i$  the branching integer  $m_i$  of f along the divisor  $D_i$ , and for instance the canonical divisor  $K_S$  is the pull-back of the logarithmic divisor

$$K_X + \sum_i (1 - \frac{1}{m_i}) D_i.$$

As a consequence, if d = |G|, G being the Galois group, then, since the curves  $D_i$  are disjoint  $(D \to B$  being étale):

$$K_S^2 = d(K_X + \sum_i (1 - \frac{1}{m_i})D_i)^2 =$$
  
=  $d(K_X^2 + 2\sum_i (1 - \frac{1}{m_i})K_XD_i + \sum_i (1 - \frac{1}{m_i})^2D_i^2).$ 

Similarly, since in this case D is smooth,

$$e(S) = d[e(X) - \sum_{i} (1 - \frac{1}{m_i})e(D_i)] = d[e(X) + \sum_{i} (1 - \frac{1}{m_i})(K_X D_i + D_i^2)].$$

Since  $(1 - \frac{1}{m_i})^2 < (1 - \frac{1}{m_i})$ , we see that the slope  $\nu_C(S)$  has little chance to become larger unless  $D_i^2 < 0$ .

This is however often the case: write  $K_X = p^*K_B + K_{X|B}$ , and observe that  $K_{D_i} = p^*K_B$ . Then  $D_i^2 = K_BD_i - K_XD_i = -K_{X|B}D_i$ , which is negative if the relative canonical divisor  $K_{X|B}$  is nef (this fact is, under suitable assumptions, a consequence of Arakelov's theorem, and is used for the proof of the Mordell conjecture over function fields).

1.4. Double Kodaira fibrations and group theory. In [CatRol09] a logarithmic Kodaira fibration was defined to be very simple if  $X = B \times B$  and each  $D_i$  is the graph of an automorphism of B, where the genus b of B is at least two.

In this case  $D_i^2 = -2(b-1)$ ,  $K_X D_i = 4(b-1)$ ,  $K_X^2 = 8(b-1)^2$ ,  $e(X) = 4(b-1)^2$  and for the Chern slope we have

$$\nu_C(S) = 2 + \frac{2(b-1)\sum_i (1 - \frac{1}{m_i^2})}{4(b-1)^2 + \sum_i (1 - \frac{1}{m_i})2(b-1)} = 2 + \frac{\sum_i (1 - \frac{1}{m_i^2})}{2(b-1) + \sum_i (1 - \frac{1}{m_i})^2}$$

Now, as the indices  $m_i \to \infty$ , the slope tends to

$$\nu_C = 2 + \frac{r}{2(b-1)+r} = 2 + \frac{\alpha}{2+\alpha} = 3 - \frac{2}{2+\alpha}, \quad \alpha := \frac{r}{(b-1)}.$$

Hence one can maximize the slope if one can maximize the ratio  $\alpha := \frac{r}{(b-1)}$ .

One can indeed be more clever (as in loc. cit.) and take  $m_i = 3$ , so that

$$\nu_C(S) = 2 + \frac{8r}{18(b-1) + 6r} = 2 + \frac{4\alpha}{9+3\alpha}$$

Then , for  $\alpha = 3$ , we obtain

$$\nu_C(S) = 2 + \frac{12}{18} = \frac{8}{3}.$$

However, there are limits to maximizing  $\alpha$ : first of all, by Hurwitz' theorem,  $\alpha < 84$ , since  $|Aut(B)| \le 84(b-1)^{-3}$ .

This inequality is however much milder than the one given by the BMY-inequality, which implies  $\alpha \leq 9$  (indeed we can show that  $\alpha < 8$ ).

Now, the account for the drop from 84 to a much lower constant is due indeed to a cogent restriction, namely, that we want all the curves  $D_i$  to be disjoint!

We already showed that we would like to find an  $\alpha > 3$ , in particular the action of the group G := Aut(B) on B cannot be free (otherwise  $|G| \leq 2(b-1)!$ ).

Therefore, we denote as above by  $\mathcal{S}$  the subset of stabilizers in G,

$$\mathcal{S} := \{ g \in G | \exists x \in B, \ s.t. \ gx = x \}.$$

Let  $D_i = \{(x, g_i x)\}$ . Then

$$D_i \cap D_j \neq \emptyset \Leftrightarrow \exists x \in B, \ s.t.g_i x = g_j x \Leftrightarrow g_i^{-1}g_j \in \mathcal{S}.$$

The group theoretical question that we have therefore in mind is a sort of sphere packing problem for groups G acting on a curve. We define the sphere with center g to be the set gS.

Our problem reduces to find a maximal number r such that there are elements  $g_1, \ldots, g_r$  so that  $\forall i$  the sphere  $g_i \mathcal{S}$  contains only the element  $g_i$ , and no other  $g_j \neq g_i$ .

Moreover, by the proof of Hurwitz's theorem, and some easy arguments, one sees that the quotient  $B/G = \mathbb{P}^1$ , and then that G is a finite quotient of a polygonal group

$$T(n_1, n_2, \dots, n_k) = \langle \gamma_1, \dots, \gamma_k | \gamma_1 \dots \gamma_k = 1, \gamma_i^{n_i} = 1, \forall i = 1, \dots, k \rangle$$

which is hyperbolic, i.e.  $\sum_{i=1}^{k} (1 - \frac{1}{n_i}) > 2$ . We wonder whether Lubotzky' s theory of expander graphs might yield a possible method to approach the question of finding a large  $\alpha$ , and also of giving a better upper bound for  $\alpha$  ([Lub94]).

1.5. Moduli of Kodaira fibrations. The study of moduli of Kodaira fibrations was initiated by Jost and Yau [J-Y83] (after that Kas [Ks68] had proven that the small deformations of Kodaira fibred surfaces are unobstructed), showing that all deformations yield again a Kodaira fibred surface.

<sup>&</sup>lt;sup>3</sup>Observe that if equality holds there are several automorphisms with fixed points.

Kodaira surfaces are a typical issue of the case where topology determines the moduli space, and the best characterization was obtained by Kotschick [Kot99]

**Theorem 8.** Let S be a complex surface. A Kodaira fibration on S with fibres of genus g and base curve of genus b is equivalent to the datum of

(1) an exact sequence

$$1 \to \Pi_q \to \pi_1(S) \to \Pi_b \to 1$$

(here  $\Pi_g$  denotes the fundamental group of a compact curve of genus g) such that:

(2)

$$e(S) = 4(b-1)(g-1),$$

(3) the monodromy homomorphism  $m : \Pi_b \to Out(\Pi_g)$ , induced by conjugation in the previous exact sequence, has infinite image.

Kotschick used the methods that we shall review in section 3, especially the following facts:

i) a fibration of a Kähler manifold over a curve of genus  $b \ge 2$  without multiple fibres is determined by a surjection  $\pi_1(X) \to \Pi_b$  with finitely generated kernel [Cat03b], [Cat08]

2) The Zeuthen-Segre formula says that, in the case where X is a surface S,  $e(S) \ge 4(b-1)(g-1)$ , equality holding, when  $g \ge 2$ , iff we have a differentiable bundle

3) the fibration has constant moduli if and only if the image of the monodromy m is finite.

We see then that every surface with the same fundamental group and Euler number as S is again Kodaira fibred with the same fibre genus gand base genus b.

Of course, a major question which remains is:

**Question 9.** Let S be a Kodaira fibred surface: do then the surfaces with the same fundamental group and Euler number as S form a connected component of the moduli space, or the union of a connected component with its complex conjugate component?

One may moreover ask the following question.

**Question 10.** *How many Kodaira fibrations can a given algebraic surface possess?* 

Do there exist surfaces S with three distinct Kodaira fibrations?

Kodaira's original example, whose generalization was explained in the previous subsection, shows that S can have two distinct Kodaira fibrations, and one can indeed see that this is again a topological condition.

**Proposition 11.** Let S be a complex surface. A double Kodaira fibration on S is equivalent to the datum of two exact sequences

$$1 \longrightarrow \Pi_{g_i} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_i} \Pi_{b_i} \longrightarrow 1 \qquad i = 1, 2 ,$$

(here  $\Pi_g$  denotes as before the fundamental group of a compact curve of genus g) such that:

- (1) the monodromy homomorphisms  $m_i : \Pi_{b_i} \to Out(\Pi_{g_i})$  have infinite image;
- (2)  $b_i \ge 2, g_i \ge 3$ ,
- (3) the composition homomorphism

$$\Pi_{g_1} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_2} \Pi_{b_2}$$

is neither zero nor injective, and

(4) the Euler characteristic of S satisfies

$$e(S) = 4(b_1 - 1)(g_1 - 1) = 4(b_2 - 1)(g_2 - 1).$$

The above result shows that surfaces admitting a double Kodaira fibration form a closed and open subset in the moduli spaces of surfaces of general type; since for these one has a realization as a branched covering  $S \to B_1 \times B_2$ , branched over a divisor  $D \subset B_1 \times B_2$ , it makes sense to distinguish the étale case where D is smooth and the two projections  $D \to B_i$  are étale. It is not clear a priori that this property is also open and closed, but in [CatRol09] we were able to prove it.

**Theorem 12.** Double étale Kodaira fibrations form a closed and open subset in the moduli space of surfaces of general type.

Concerning the previous question 9 there are only partial results, Jost and Yau studied the deformations of the original example of Kodaira, while [CatRol09] considered a more general question for an important special class of double étale Kodaira fibrations. To describe the latter results we recall a definition from [CatRol09].

**Definition 13.** A double étale Kodaira fibration is said to be **standard** if the logarithmic double Kodaira fibred surface  $(B_1 \times B_2, D)$  reduces, after étale base changes for  $B_1$  and  $B_2$ , to the very simple case of a logarithmic double Kodaira fibred surface  $(B \times B, D)$  for which D is a union of graphs of automorphisms of B.

**Theorem 14.** The subset of the moduli space corresponding to standard double étale Kodaira fibred surfaces S with a fixed fundamental group consists of at most two irreducible connected components, exchanged by complex conjugation, which are isomorphic to the moduli space of pairs (B,G), where B is a curve of genus b at least two and G is a group of biholomorphisms of B of a given topological type.

An interesting by-product of the study of the moduli space of standard double étale Kodaira fibrations is the following result which contradicted a spread belief ([M-S80]), and shows that Kefeng Liu's theorem that a Kodaira fibration cannot exist on a (free) ball quotient cannot be shown invoking non-rigidity.

**Theorem 15.** There are double Kodaira fibred surfaces S which are rigid.

Using these explicit descriptions Rollenske went further in [Rol10] and showed that, in the case where the branched cover has a cyclic Galois group, then the closure of this irreducible component inside the Kollár-Shepherd Barron-Alexeev [K-SB88] compactification is again a connected component. One should observe that there are extremely few examples where the connected components of the KSBA compact moduli space have been investigated, apart from the obvious case of rigid surfaces (see [LW12] for the case of surfaces isogenous to a product).

Finally, concerning the existence problem for Kodaira fibrations, we have the following

Question 16. Given an exact sequence

$$1 \to \Pi_g \to \pi \to \Pi_b \to 1$$

such that the image of  $m : \Pi_b \to \mathcal{M}ap_g := Out^+(\Pi_g)$  is infinite, when does there exist a Kodaira fibred surface S with  $\pi_1(S) \cong \pi$ ?

An obvious necessary condition is that the Abelianization  $\pi^{ab} = \pi/[\pi,\pi]$  has even rank, by Hodge theory.

To the monodromy m is associated a continuous map  $f : B \to \mathcal{T}_g/Im(m)$ , where  $\mathcal{T}_g$  denotes Teichmüller space, equivalently, an m-equivariant map of the universal cover  $\tilde{B}, \tilde{f} : \tilde{B} \to \mathcal{T}_g$ . It is at present not clear to me if it is proven that f can be deformed to a harmonic map; the main difficulty seems however to be to show the holomorphicity of such a harmonic map, for which the condition on the Betti number being even is the first obstruction.

Recently Arapura <sup>4</sup> observed that there are other necessary conditions. Let  $m_H$  be the monodromy on  $V := \prod_g^{ab} \otimes \mathbb{Q}$ , i.e. the monodromy on cohomology, and let G be the connected component of the identity in the Zariski closure of  $Im(m_H)$ . Then we can first of all replace the above condition on the parity of the first Betti number as the condition that the space  $V_G$  of G-coinvariants (the largest quotient on which Gacts trivially) has even dimension.

Then there are necessary conditions in special cases: for instance, in the case where the space  $V_G$  of coinvariants is zero, G must be

 $<sup>^4\</sup>mathrm{Talk}$  at the Conference for Bob Friedman's 60-th birthday, in May 2015, and [Ara15].

semisimple of classical Hermitian type. In other words, if  $V_G = 0$ , then  $G(\mathbb{R})^0/K$  must be a Hermitian Symmetric Domain of classical type (here K is a maximal compact subgroup).

## 2. PROJECTIVE VARIETIES WHICH ARE CLASSIFYING SPACES

# 2.1. Generalities on projective varieties which are classifying spaces.

**Definition 17.** Define  $\mathcal{PC}$  as the class of projective varieties which are classifying spaces for their fundamental group  $\pi_1(Z)$ : equivalently,  $\mathcal{PC}$  is the class of projective varieties Z whose universal covering  $\tilde{Z}$  is contractible.

The class  $\mathcal{PC}$  is stable for Cartesian products, and for étale coverings, hence also for the relation of isogeny.

**Definition 18.** Two varieties X, Y are said to be isogenous if there exist a third variety Z, and étale finite morphisms  $f_X : Z \to X$ ,  $f_Y : Z \to Y$ .

The class  $\mathcal{PC}$  is however not stable for taking hyperplane sections: because for a compact manifold which is a classifying space its real dimension is read off by the top nonzero cohomology group  $H^m(\pi, \mathbb{Z}/2)$ , where  $\pi = \pi_1(Z)$ : since if Z is a classifying space, or a  $K(\pi, 1)$  as one also says in topology, then  $H^m(\pi, R) \cong H^m(Z, R)$ , for any ring of coefficients R.

Projective curves C of genus g are, by virtue of the uniformization theorem, in the class  $\mathcal{PC}$  if and only if  $g \geq 1$ . For  $g \geq 2$  they are the quotients  $C = \mathcal{H}/\Gamma$ , where  $\mathcal{H} := \{z \in \mathbb{C} | Im(z) > 0\}$  and  $\Gamma \subset \mathbb{P}SL(2,\mathbb{R})$  is a discrete subgroup isomorphic to  $\Pi_g$  (then necessarily the action is free and cocompact).

Primary examples of projective varieties which are  $K(\pi, 1)$ 's are curves and Abelian varieties, and the varieties which are isogenous to a product of these. Particularly interesting are the varieties isogenous to a product of curves of genera at least 2.

It is interesting to observe ([Cat15], corollary 82) that Abelian varieties are exactly the projective  $K(\pi, 1)$  varieties, for which  $\pi$  is an abelian group.

2.1.1. Locally symmetric manifolds of negative type. A very interesting class of projective varieties which are  $K(\pi, 1)$ 's are the **locally symmetric manifolds** Z with ample canonical divisor  $K_Z$ . These are in some sense a generalization in higher dimension of curves of genus  $g \ge 2$ : because the upper half plane  $\mathcal{H} := \{z \in \mathbb{C} | Im(z) > 0\}$  is biholomorphically equivalent to the unit disk  $\{z \in \mathbb{C} | |z| < 1\}$ .

Locally symmetric manifolds Z with ample canonical divisor  $K_Z$ (also called locally Hermitian symmetric manifolds of negative curvature) are the quotients of a **bounded symmetric domain**  $\mathcal{D}$  by a cocompact discrete subgroup  $\Gamma \subset Aut(\mathcal{D})$  acting freely.

Recall that a bounded symmetric domain  $\mathcal{D}$  is a bounded domain  $\mathcal{D} \subset \subset \mathbb{C}^n$  such that its group  $Aut(\mathcal{D})$  of biholomorphisms contains, for each point  $p \in \mathcal{D}$ , a holomorphic automorphism  $\sigma_p$  such that  $\sigma_p(p) = p$ , and such that the derivative of  $\sigma_p$  at p is equal to -Id. This property implies that  $\sigma$  is an involution (i.e., it has order 2), and that  $Aut(\mathcal{D})^0$  (the connected component of the identity) is transitive on  $\mathcal{D}$ ; therefore one can write  $\mathcal{D} = G/K$ , where G is a connected Lie group, and K is a maximal compact subgroup.

The classification of these bounded symmetric domains, done by Elie Cartan in [Car35], is based on the fact that such a  $\mathcal{D}$  splits uniquely as the product of irreducible bounded symmetric domains.

 $\mathcal{D}$  is a complete Riemannian manifold of negative sectional curvature, hence it is contractible, by the Cartan-Hadamard theorem, and  $Z = \mathcal{D}/\Gamma$  is a classifying space for the group  $\Gamma \cong \pi_1(X)$ .

There are four series of non sporadic bounded irreducible domains, in their bounded realization, plus two exceptional types:

- (i)  $I_{n,p}$  is the domain  $\mathcal{D} = \{ Z \in Mat(n, p, \mathbb{C}) : I_p {}^t Z \cdot \overline{Z} > 0 \}.$
- (ii)  $II_n$  is the intersection of the domain  $I_{n,n}$  with the subspace of skew symmetric matrices.
- (iii)  $III_n$  is instead the intersection of the domain  $I_{n,n}$  with the subspace of symmetric matrices.
- (iv) The Cartan Harish Chandra realization of a domain of type  $IV_n$  in  $\mathbb{C}^n$  is the subset  $\mathcal{D}$  defined by the inequalities (compare [Helga78], page 527)

$$|z_1^2 + z_2^2 + \dots + z_n^2| < 1,$$
  
 
$$1 + |z_1^2 + z_2^2 + \dots + z_n^2|^2 - 2(|z_1|^2 + |z_2|^2 + \dots + |z_n|^2) > 0.$$

- (v)  $\mathcal{D}_{16}$  is the exceptional domain of dimension d = 16.
- (vi)  $\mathcal{D}_{27}$  is the exceptional domain of dimension d = 27.

Each of these domains is contained in the so-called **compact dual**, which is a Hermitian symmetric spaces of compact type, the easiest example being, for type I, the Grassmann manifold.

Among the bounded symmetric domains are the so called **bounded symmetric domains of tube type**, those which are biholomorphic to a **tube domain**, a generalized Siegel upper half-space

$$T_{\mathcal{C}} = \mathbb{V} \oplus \sqrt{-1}\mathcal{C}$$

where  $\mathbb{V}$  is a real vector space and  $\mathcal{C} \subset \mathbb{V}$  is a *symmetric cone*, i.e., a self dual homogeneous convex cone containing no full lines.

In the case of type III domains, the tube domain is Siegel's upper half space:

$$\mathcal{H}_g := \{ \tau \in \operatorname{Mat}(g, g, \mathbb{C}) | \tau = {}^t \tau, \operatorname{Im}(\tau) > 0 \},\$$

a generalisation of the upper half-plane of Poincaré.

Borel proved in [Bore63] that for each bounded symmetric domain  $\mathcal{D}$  there exists a compact free quotient  $X = \mathcal{D}/\Gamma$ , called a compact Clifford-Klein form of the symmetric domain  $\mathcal{D}$ .

A classical result of J. Hano (see [Hano57] Theorem IV, page 886, and Lemma 6.2, page 317 of [Mil76]) asserts that a bounded homogeneous domain that is the universal cover of a compact complex manifold is symmetric.

2.1.2. Kodaira fibrations. The Kodaira fibrations  $f : S \to B$  are a remarkable example of surfaces in the class  $\mathcal{PC}$ .

Because S is a smooth projective surface and it is known that all the fibres of f are smooth curves of genus  $g \ge 3$ , whereas the base curve has genus  $b \ge 2$ .

By the fundamental group exact sequence

$$1 \to \Pi_g \to \pi_1(S) \to \Pi_b \to 1$$

the universal cover  $\tilde{S}$  is a differentiable fibre bundle over  $\tilde{B}$  with fibre  $\tilde{F}$ , hence it is diffeomorphic to a ball of real dimension 4.

By simultaneous uniformization ([Bers60]) the universal covering S of a Kodaira fibred surface S is biholomorphic to a bounded domain in  $\mathbb{C}^2$  (fibred over the unit disk  $\Delta := \{z \in \mathbb{C} | |z| < 1\}$  with fibres isomorphic to  $\Delta$ ), which is not homogeneous.

 $\tilde{S}$  is not homogeneous by the Hirzebruch proportionality principle; indeed there are only two bounded homogeneous domains in dimension 2: the bidisk and the 2-ball. The bidisk is biholomorphic to  $\mathcal{H} \times \mathcal{H}$ , and its group of biholomorphisms is a semidirect product of  $Aut(\mathcal{H}) \times$  $Aut(\mathcal{H})$  by  $\mathbb{Z}/2$ , in particular one has Chern forms for the tangent bundle, invariant by automorphisms, which can be written as

$$c_1 := \phi(z_1)dz_1 \wedge dz_1 \oplus \phi(z_2)dz_2 \wedge dz_2,$$
$$c_2 := \phi(z_1)\phi(z_2)dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} = \frac{1}{2}c_1^2.$$

Hence the Chern index  $\nu_C = 2$  if the universal cover of S is the bidisk, i.e.  $c_1(S)^2 = 2c_2(S)$ ; whereas, if the universal cover is the ball,  $c_1(S)^2 = 3c_2(S)$  by a similar argument ([Hirz58]).

2.1.3. Known projective classifying spaces in complex dimension two. In complex dimension 2, we have the following list of projective classifying spaces:

(1) Abelian surfaces

(2) Hyperelliptic surfaces: these are the quotients of a complex torus of dimension 2 by a finite group G acting freely, and in such a way that the quotient is not again a complex torus.

They have  $p_g = 0, q = 1$ .

These surfaces were classified by Bagnera and de Franchis ([BdF08], see also [ES09] and [BPHV]) and they are obtained as quotients  $(E_1 \times E_2)/G$  where  $E_1, E_2$  are two elliptic curves, and G is an abelian group acting on  $E_1$  by translations, and on  $E_2$  effectively and in such a way that  $E_2/G \cong \mathbb{P}^1$ . In other words, these are exactly the surfaces isogenous to a product of curves of genus 1 which are not Abelian surfaces.

(3) Surfaces isogenous to a product of curves  $C_1 \times C_2$ , where  $C_1$  has genus 1,  $C_2$  has genus  $g_2 \ge 2$ .

These are quotients  $(C_1 \times C_2)/G$ , where the finite group G acts freely, and where we can assume that G acts by a faithful diagonal action

$$g(x,y) = (g(x),g(y))$$

(we regard G as  $G \subset Aut(C_1), G \subset Aut(C_2)$ ). These surfaces have Kodaira dimension 1.

- (4) Ball quotients.
- (5) Bidisk quotients, divided into the **reducible case**, the case of surfaces  $(C_1 \times C_2)/G$ , isogenous to a product of curves of genera  $g_i \geq 2$ , and the **irreducible case** (for these rigidity holds, as proven by Jost and Yau [J-Y85]).
- (6) Kodaira fibred surfaces.
- (7) Mostow-Siu surfaces of negative curvature ([M-S80]): these are branched coverings of ball quotients, admitting a metric of negative scalar curvature. Their Chern slopes  $\nu_C(S)$  are very close to 3, but strictly smaller than 2, 96. These surfaces are rigid, in particular they are not the Kodaira examples of Kodaira fibrations.
- (8) More examples are gotten from coverings of the plane branched over configurations of lines, as we shall discuss in a later section (see [Zheng99], and [Pan09], [Pan11]).

**Proposition 19.** Every surface in  $\mathcal{PC}$  is necessarily minimal, and indeed it contains no rational curve.

*Proof.* In fact, if we have a rational curve  $\phi : \mathbb{P}^1 \to S$ , then the map  $\phi$  lifts to the universal cover, and is then null-homotopic because  $\tilde{S}$  is contractible. Hence also  $\phi$  is null-homotopic, and the image rational curve  $C := \phi(\mathbb{P}^1)$  is homologous to zero; this is a contradiction, as soon as S is a Kähler surface. A fortiori, if S is projective.

**Corollary 20.** Surfaces in  $\mathcal{PC}$  which are not of general type are exactly type 1), 2) and 3) above. Any surface in  $\mathcal{PC}$  which is of general type has ample canonical divisor.

*Proof.* The result follows by minimality and by Enriques' classification, if the Kodaira dimension is < 1.

If S has general type, then it contains no rational curves, in particular no (-2)-curves, hence  $K_S$  is ample.

If the Kodaira dimension is one, then S is minimal, properly elliptic, which means that a multiple of the canonical divisor yields a morphism  $f: S \to B$  with general fibre an elliptic curve. By Kodaira's classification of singular fibres of elliptic fibrations ([Kod60]) follows that all the fibres are either smooth elliptic, or multiple of a smooth elliptic curve. We use now the orbifold fundamental group exact sequence (see [CKO03], [Cat03b] or [Cat08])

$$\pi_1(F) \to \pi_1(S) \to \pi_1^{orb}(f) \to 1.$$

Here the orbifold fundamental group is defined as  $\pi_1(B^*)/K$ ,  $B^*$  being the set of non critical values of f, and K is normally generated by  $\gamma_i^{m_i}$ , for each geometric loop going around a point  $p_i \in B \setminus B^*$  for which the fibre  $f^{-1}(p_i)$  is multiple of multiplicity  $m_i$ . Hence there is an intermediate covering  $\hat{S} \to \hat{B}$ , possibly of infinite degree, such that all the fibres are smooth elliptic curves, and where  $\hat{B}$  is simply connected.

There is also a finite ramified covering  $B' \to B$  such that the pull back  $f': S' \to B'$  has all the fibres which are smooth. The *j*-invariant is constant on B' since B' is projective. Hence all the smooth fibres are isomorphic to a fixed elliptic curve E and therefore we obtain another finite cover  $B'' \to B'$  such that  $S'' = E \times B''$ . Since the Kodaira dimension of S is one, we obtain that B'' has genus at least two, and there exists another étale covering  $C \to B''$  such that  $C \to B$  is Galois hence  $S = (E \times C)/G$ .

2.2. Galois conjugate of projective classifying spaces. Let  $X \subset \mathbb{P}^n$  be a complex projective variety. Then, for each  $\sigma \in Aut(\mathbb{C})$ , the conjugate variety  $X^{\sigma}$  is the set  $\sigma(X)$ : it is the projective variety defined by the ideal  $I_X^{\sigma}$ , obtained from the ideal  $I_X$  of X applying the homomorphism  $\sigma$  to the coefficients of the polynomials in  $I_X$ .

If  $\sigma$  is complex conjugation, we get  $\overline{X}$ , which is diffeomorphic to X. Observe that, by the theorem of Steiniz, one has a surjection  $Aut(\mathbb{C}) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ , and that we have an action of the absolute Galois group  $GAL := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$  on the set of varieties X defined over  $\overline{\mathbb{Q}}$ .

X and the conjugate variety  $X^{\sigma}$  have the same Hodge numbers and Chern numbers. In particular, for curves, the genus is preserved.

It is immediate also that Galois conjugation by  $\sigma \in Aut(\mathbb{C})$  preserves products and the equivalence relation given by isogeny, indeed Galois conjugation does not change the algebraic fundamental group, as shown by Grothendieck [SGA1].

**Theorem 21.** Conjugate varieties  $X, X^{\sigma}$  have isomorphic algebraic fundamental groups

 $\pi_1(X)^{alg} \cong \pi_1(X^{\sigma})^{alg},$ 

 $(\pi_1(X)^{alg})$  is the profinite completion of the topological fundamental group  $G := \pi_1(X)$ , i.e.  $\pi_1(X)^{alg}$  is the inverse limit of the factor groups G/K, K being a normal subgroup of finite index in G).

It is easy to see ([Cat15], theorem 223) that

i) If X is an Abelian variety, the same holds for any Galois conjugate  $X^{\sigma}$ .

ii) If S is a Kodaira fibred surface, then any Galois conjugate  $S^{\sigma}$  is also Kodaira fibred.

The following attempt of conjecture is based mainly on the fact that it holds for all known examples.

**Conjecture 22.** Assume that X is a projective  $K(\pi, 1)$ , and assume  $\sigma \in Aut(\mathbb{C})$ .

Is then the conjugate variety  $X^{\sigma}$  still a classifying space  $K(\pi', 1)$ ?

We know since long, thanks to the result obtained by J.P. Serre [Ser64] in the 60's, that it is not true in general that  $\pi_1(X^{\sigma}) \cong \pi_1(X)$ . Serre showed that there exists a field automorphism  $\sigma$  in the absolute Galois group  $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ , and a variety X defined over a number field, such that X and the Galois conjugate variety  $X^{\sigma}$  have non isomorphic fundamental groups, in particular they are not homeomorphic.

This is also false for surfaces in the class  $\mathcal{PC}$ , for instance in a joint paper with I. Bauer and F. Grunewald [BCG14] we obtained the following theorem.

**Theorem 23.** If  $\sigma \in Gal(\mathbb{Q}/\mathbb{Q})$  is not in the conjugacy class of complex conjugation, then there exists a surface isogenous to a product Ssuch that S and the Galois conjugate surface  $S^{\sigma}$  have non-isomorphic fundamental groups.

I. Bauer and F. Grunewald and the author ([BCG06], [BCG14]) discovered also many explicit examples of algebraic surfaces isogenous to a product for which the same phenomenon holds (observe that the proof of the general theorem is, as it may be surmised, non constructive).

**Remark 24.** Gonzaléz-Diez and Jaikin-Zapirain [GD-JZ16] later extended theorem 23 to all automorphisms  $\sigma$  different from complex conjugation.

2.3. Some characterizations of locally symmetric varieties and Kazhdan's theorem in refined form. Proceeding with other projective  $K(\pi, 1)$ 's, the question becomes more subtle and we have to

appeal to a famous theorem by Kazhdan on arithmetic varieties (see [Kazh70], [Kazh83], [Milne01], [CaDS12], [C-DS14], [ViZu07]).

Here the result is in the end much stronger: not only the conjugate variety is again locally symmetric, but the universal cover is indeed the same bounded symmetric domain!

**Theorem 25.** Assume that X is a projective manifold with  $K_X$  ample, and that the universal covering  $\tilde{X}$  is a bounded symmetric domain. Let  $\tau \in \operatorname{Aut}(\mathbb{C})$  be an automorphism of  $\mathbb{C}$ . Then the conjugate variety  $X^{\tau}$  has universal covering  $\tilde{X}^{\tau} \cong \tilde{X}$ .

The above result rests in an essential way on the Aubin-Yau theorem ([Yau78], [Aub78]) about the existence of a Kähler Einstein metric for a projective manifold with ample canonical divisor  $K_X$ , and on the results of Berger [Ber53].

These results allow precise algebro-geometric characterizations of such locally symmetric varieties. These results were pioneered by Yau [Yau88] [Yau93], his treatment of the non tube case is however incorrect (he claims that (1) of theorem 26, which characterizes the tube case, holds also in the non tube case).

Simpler proofs follow from recent results obtained together with Antonio Di Scala. These results yield a simple and precise characterization of varieties possessing a bounded symmetric domain as universal cover, without having to resort to the existence of a finite étale covering where the holonomy splits.

For the tube case, we have the following theorem (see [CaDS12]), whose simple underlying idea is best illustrated in the case where the universal covering is a polydisk.

In this case one observes that, due to the nature of the automorphism group of  $\mathcal{H}^n$  as semidirect product of  $Aut(\mathcal{H})^n$  with the symmetric group in *n* letters, the following tensor

$$\Psi := \frac{dz_1 \cdots dz_n}{dz_1 \wedge \cdots \wedge dz_n}$$

is a semi-invariant for the group of automorphisms, it is multiplied by  $\pm 1$  according to the sign of the corresponding coordinates permutation.

It therefore descends to a section  $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$ , where  $\eta$  is the 2-torsion invertible sheaf associated to the sign character of the fundamental group  $\pi_1(X) \cong \Gamma < Aut(\mathcal{H}^n)$  (observe that, depending on the choice of  $\Gamma$ ,  $\eta$  may be trivial).

The existence of such a tensor is unfortunately, in complex dimension  $n \ge 4$ , not a characterizing property of polydisk quotients.

Indeed, in dimension 4, we have the bounded domain  $\Omega \subset \mathbb{C}^4$ ,

$$\Omega = \{ Z \in Mat(2, 2, \mathbb{C}) : I_2 - {}^t Z \cdot Z > 0 \},\$$

the bounded (Harish-Chandra) realization of the Hermitian symmetric space  $SU(2,2)/S(U(2) \times U(2))$ .

Here the holonomy action of  $(A, D) \in S(U(2) \times U(2))$  is given by  $Z \mapsto AZD^{-1}$ . Hence, the square of the determinant of Z yields a section  $\Psi$  which descends to a section  $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X))$ .

The main difference with respect to the polydisk quotient case is that the corresponding hypersurface in the projectivized tangent bundle of X is nonreduced, since we start from an invariant hypersurface of degree 4 which is twice a smooth quadric. With these examples in mind it should be easier to get the flavour of the following theorem.

**Theorem 26.** Let X be a compact complex manifold of dimension n with  $K_X$  ample.

Then the following two conditions (1) and (1'), resp. (2) and (2') are equivalent:

- (1) X admits a slope zero tensor  $0 \neq \psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$ , (for some positive integer m);
- (1')  $X \cong \Omega/\Gamma$ , where  $\Omega$  is a bounded symmetric domain of tube type and  $\Gamma$  is a cocompact discrete subgroup of  $Aut(\Omega)$  acting freely.
- (2) X admits a semi special tensor  $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$ , where  $\eta$  is a 2-torsion invertible sheaf, such that there is a point  $p \in X$  for which the corresponding hypersurface  $F_p := \{\phi_p = 0\} \subset \mathbb{P}(TX_p)$  is reduced.
- (2) The universal cover of X is a polydisk.

Moreover, in case (1), the degrees and the multiplicities of the irreducible factors of the polynomial  $\psi_p$  determine uniquely the universal covering  $\tilde{X} = \Omega$ .

The crucial underlying fact is the following discovery of Korányi-Vági.

Let  $D \subset \mathbb{C}^n$  be a homogeneous bounded symmetric domain in its circle realization around the origin  $0 \in \mathbb{C}^n$ .

Let K be the isotropy group of D at the origin  $0 \in \mathbb{C}^n$ , so that we have D = G/K.

A polynomial  $f \in \mathbb{C}[X_1, \ldots, X_n]$  is said to be K- semi-invariant if there is a character  $\chi \colon K \to \mathbb{C}$  such that, for all  $g \in K$ ,  $f(gX) = \chi(g)f(X)$ .

Observe that, since K is compact, we have:  $|\chi(g)| = 1$ .

Let  $D = D_1 \times D_2$  be the decomposition of D as a product of two domains where  $D_1$  is of tube type and  $D_2$  has no irreducible factor of tube type.

# Theorem 27. [KoVa79, Korányi-Vági]

Let  $D = D_1 \times D_2$  be the above decomposition and let moreover

$$D_1 = D_{1,1} \times D_{1,2} \times \cdots \times D_{1,p}$$

be the decomposition of  $D_1$  as a product of irreducible tube type domains  $D_{1,j}$ ,  $(j = 1, \dots, p)$ .

Then there exist, for each j = 1, ..., p, a unique  $K_j$ -invariant polynomial  $N_j(z_{1,j})$ , where  $K_j$  is the isotropy subgroup of  $D_{1,j}$ , such that:

for all K-invariant polynomial f there exist a constant  $c \in \mathbb{C}$  and exponents  $k_i$  with

(1)

$$f = c \prod_{j=1}^{p} N_j^{k_j},$$

hence in particular

(2)

$$f(z_1, z_2) = f(z_1) ,$$

where  $z_1$  denotes a vector in the domain  $D_1$  and  $z_2 \in D_2$ .

The above theorem follows almost directly from [KoVa79] by taking into account that a K-invariant polynomial is, up to a multiple, an *inner function*, i.e., a function such that |f(z)| = 1 on the Shilov boundary of D (on which K acts transitively).

Moreover the polynomials  $N_j$  have degree equal to the  $rank(D_j)$  of the irreducible domain  $D_j$ ,  $(rank(D_j)$  denotes the dimension r of the maximal totally geodesic embedded polydisc  $\mathcal{H}^r \subset D_j$ , or, equivalently, if D = G/K, with  $G = Aut(D)^0$ ,  $rank(D) = rank(G^{\mathbb{C}})$  = the dimension of a maximal algebraic torus contained in the complexification  $G^{\mathbb{C}}$ ).

The characterization is essentially a consequence of the unicity of these inner functions, and of the inequality  $rank(D_j) \leq dim(D_j)$ , where equality holds if and only if  $D_j = \mathcal{H}$ .

In the case where there are non tube domains in the irreducible decomposition, one has to use some ideas of Kobayashi and Ochiai [KobOchi81], developed by Mok [Mok89] who introduced and studied certain characteristic varieties which generalize the hypersurfaces defined by the Korányi-Vági - polynomials.

2.3.1. Algebraic curvature-type tensors and their First Mok characteristic varieties. Consider the situation where we are given a direct sum

$$T = T_1 \oplus \ldots \oplus T_k$$

of irreducible representations  $T_i$  of a group  $H_i$  (T shall be the tangent space to a projective manifold at one point, and  $H = H_1 \times \cdots \times H_k$  shall be the restricted holonomy group).

**Definition 28.** 1) An algebraic curvature-type tensor is a nonzero element

$$\sigma \in End(T \otimes T^{\vee}).$$

2) Its first Mok characteristic cone  $CS \subset T$  is defined as the projection on the first factor of the intersection of  $ker(\sigma)$  with the set of rank 1 tensors, plus the origin:

$$\mathcal{CS} := \{ t \in T | \exists t^{\vee} \in T^{\vee} \setminus \{0\}, (t \otimes t^{\vee}) \in ker(\sigma) \}.$$

3) Its first Mok characteristic variety is the subset  $S := \mathbb{P}(CS) \subset \mathbb{P}(T)$ .

4) More generally, for each integer h, consider

 $\{A \in T \otimes T^{\vee} | A \in ker(\sigma), \operatorname{Rank}(A) \le h\},\$ 

and consider the algebraic cone which is its projection on the first factor

 $\mathcal{CS}^h := \{ t \in T | \exists A \in ker(\sigma), \operatorname{Rank}(A) \le h, \exists t' \in T : t = At' \},\$ 

and define then  $S^h := \mathbb{P}(CS^h) \subset \mathbb{P}(T)$  to be the h-th Mok characteristic variety.

5) We define then the full characteristic sequence as the sequence

$$\mathcal{S} = \mathcal{S}^1 \subset \mathcal{S}^2 \subset \cdots \subset \mathcal{S}^{k-1} \subset \mathcal{S}^k = \mathbb{P}(T).$$

**Remark 29.** In the case where  $\sigma$  is the curvature tensor of an irreducible symmetric bounded domain  $\mathcal{D}$ , Mok ([Mok02]) proved that the difference sets  $S^h \setminus S^{h-1}$  are exactly all the orbits of the parabolic subgroup P associated to the compact dual  $\mathcal{D}^{\vee} = G/P$ .

The concept of an algebraic curvature type tensor  $\sigma$  can be then used to prove the following theorem.

**Theorem 30.** Let X be a compact complex manifold of dimension n with  $K_X$  ample.

Then the universal covering  $\tilde{X}$  is a bounded symmetric domain without factors isomorphic to higher dimensional balls if and only if there is a holomorphic tensor  $\sigma \in H^0(End(T_X \otimes T_X^{\vee}))$  enjoying the following properties:

1) there is a point  $p \in X$ , and a splitting of the tangent space  $T = T_{X,p}$ 

$$T = T'_1 \oplus \ldots \oplus T'_m$$

such that the first Mok characteristic cone CS is  $\neq T$  and moreover CS splits into m irreducible components CS'(j) with

2)  $\mathcal{CS}'(j) = T'_1 \times \ldots \times \mathcal{CS}'_j \times \ldots \times T'_m$ 

3)  $\mathcal{CS}'_j \subset T'_j$  is the cone over a smooth projective variety  $\mathcal{S}'_j$  unless  $\mathcal{CS}'_j = 0$  and dim  $(T'_j) = 1$ .

Moreover, we can recover the universal covering of X from the sequence of pairs  $(\dim(\mathcal{CS}'_i), \dim(T'_i))$ .

The case where there are ball factors is the case where the Yau inequality is used (see also [ViZu07]), and one can also in this case give a characterization which is given in terms of X and not of some

unspecified étale cover K' of X (Master Thesis of Daniel Mckenzie, 2013 [Mck13]).

A very interesting program, suggested by Yau in [Yau93] is to extend these characterizations to the case of quotients  $Y := \mathcal{D}/\Gamma$ , where  $\mathcal{D}$  is a bounded symmetric domain, but the quotient need not be compact, and the action may be non-free.

This should be done using logarithmic sheaves  $\Omega^1_X(logD)$ , where (X, D) is a normal crossing compactification and resolution of (Y, Sing(Y)). These sheaves can possibly only be defined in orbifold sense (similarly to what is done in the work of Campana et al. [CGP13]), otherwise it is not clear that the tensors which we considered above, and which descend to  $Y \setminus Sing(Y)$ , do indeed extend to X logarithmically.

2.4. Kodaira fibred surfaces and their conjugates. The bulk of the previous subsection was to show examples where the universal cover of a projective variety X and of its Galois conjugates  $X^{\sigma}$  are isomorphic.

Kodaira surfaces miracolously show us that we should not hope (by mere wishful thinking) that the same result should hold for all varieties in the class  $\mathcal{PC}$ .

In fact, it was proven by Shabat ([Shab77], [Shab83])

**Theorem 31.** Let  $f : S \to B$  be a Kodaira fibration, and let  $\hat{S}$  be the universal covering of S, a bounded domain in  $\mathbb{C}^2$ . Then the fundamental group  $\pi_1(S)$  has finite index inside  $Aut(\tilde{S})$ .

We have then the following consequence

**Theorem 32.** There exist families  $S_t, t \in T$  of Kodaira fibrations whose universal covers  $\tilde{S}_t$  are not isomorphic. In particular, there exist Kodaira fibred surfaces such that S and some Galois conjugate  $S^{\sigma}$  have non isomorphic universal covering.

Proof. It suffices to take a family  $S_t, t \in T$ ,  $dim(T) \geq 1$ , of Kodaira fibrations where the fibres of the map of T to the moduli space are finite. Then, by Shabat's theorem, it follows that, for  $t' \in T$ , the number of surfaces  $S_t$  whose universal cover is isomorphic to  $\tilde{S}'_t$  is countable (finite?), since these surfaces correspond to conjugacy classes of finite index subgroups of  $G' := Aut(\tilde{S}'_t)$  which are isomorphic to  $\pi_1(S)$ . And G' is finitely presented, as well as  $\pi_1(S)$ .

For the second assertion, let  $\mathfrak{M}(S_t)$  be the irreducible component of the moduli space of surfaces of general type (which is defined over  $\mathbb{Q}$ ) containing the image of T, and let  $S'_t$  be a surface whose moduli point is not algebraic. Then the set of isomorphism classes of Kodaira fibred surfaces with universal covering isomorphic to  $\tilde{S}'_t$  is countable, while the set of isomorphism classes of its conjugate surfaces  $S'_t^{\sigma}$  is uncountable (note that the corresponding moduli points do not need

a priori to belong to  $\mathfrak{M}(S_t)$ , but this is irrelevant and can be indeed arranged taking some explicit family of double Kodaira fibrations).

The result of Shabat was brought to attention by González-Diez and Reyes-Carocca [GD-RC15], who also discuss the 'arithmeticity' condition that a Kodaira fibration is defined over a number field. They show, as a consequence of Arakelov's finiteness theorem, that S is arithmetic if and only if the base curve is so. And from this they deduce that if two such have the same universal cover, and one is arithmetic, then also the other is arithmetic.

Their work suggests the following possible extension of theorem 32:

**Question 33.** Does there exist an arithmetic Kodaira surface S (i.e., S is defined over  $\overline{\mathbb{Q}}$ ) and an automorphism  $\sigma \in Aut(\mathbb{C})$  such that the universal coverings of S and  $S^{\sigma}$  are not isomorphic?

2.5. Surfaces fibred onto curves of positive genus which are classifying spaces. We consider now the following situation:  $f: S \rightarrow B$  is a fibration of an algebraic surface S onto a curve B and we want to find a criterion for the universal covering  $\tilde{S}$  to be contractible.

We already remarked in proposition 2.1.3 that if  $S \in \mathcal{PC}$  then S must be minimal, and indeed S contains no rational curves. More generally, the same argument shows that, for each curve  $C \subset S$ , the fundamental group of the normalization C' of C has infinite image in  $\pi_1(S)$ .

Use now the orbifold fundamental group exact sequence (see [CKO03], [Cat03b] or [Cat08])

$$\pi_1(F) \to \pi_1(S) \to \pi_1^{orb}(f) \to 1,$$

hence we obtain a fibration  $\tilde{f}: \tilde{S} \to \hat{B}$ , where  $\hat{B} \to B$  is the ramified covering of B corresponding to the surjection

$$\varphi: \pi_1(B \setminus \{p_1, \dots, p_r\}) \to \pi_1^{orb}(f)$$

(here  $p_1, \ldots, p_r$  are the points whose fibre is multiple, of multiplicity  $m_i \geq 2$ , and the kernel of  $\varphi$  is normally generated by  $\{\gamma_i^{m_i}\}$ , for  $\gamma_i$  a loop around  $p_i$ ).

Notice that  $\hat{B}$  is compact if and only if  $\pi_1^{orb}(f)$  is finite, and if and only if  $\hat{B} \cong \mathbb{P}^1$ , since  $\hat{B}$  is simply connected. But even when  $\pi_1^{orb}(f)$  is infinite there is a finite ramified covering B' of B such that the pull back of the fibration f has no multiple fibres (it suffices that the ramification index at each  $p_i$  is equal to the multiplicity  $m_i$  of the multiple fibre).

By passing therefore to a finite étale covering (of the surface S), we may assume that  $f: S \to B$  has no multiple fibres, and we shall assume in the sequel that the genus b of B is  $\geq 1$ .

**Question 34.** Are there examples of a fibration  $f : S \to B$  of an algebraic surface S onto a curve B, where S is a projective classifying

space, and  $\pi_1^{orb}(f)$  is finite? Equivalently, with  $B \cong \mathbb{P}^1$  and where f has no multiple fibres?

Assume now that f has no multiple fibres, so that  $\hat{B}$  is the universal cover  $\tilde{B}$  of B, and that the genus b of B is  $\geq 1$ .

Next, by the necessary condition that for each curve C contained in a fibre  $\pi_1(C') \to \pi_1(S)$  has infinite image, it follows that all the fibres of  $\tilde{f} : \tilde{S} \to \tilde{B}$  are homotopy equivalent to CW complexes of real dimension 1.

Let  $\mathcal{C} \subset B$  be the set of critical values of  $f, \mathcal{C} = \{p_1, \ldots, p_h\}$ . Choose a set  $\Sigma$  of non intersecting paths joining a fixed base point  $p_0$  with the points of  $\mathcal{C}$ , and similarly a set  $\Sigma'$  inside  $\tilde{B}$  of non intersecting paths joining a fixed base point  $y_0 \in \tilde{S}$  mapping to  $p_0$ , with the points of the inverse image of  $\mathcal{C}$ , which is a countable set  $\{y_n | n \in \mathbb{N}, n \geq 1\}$ .

**Lemma 35.** Let  $f: S \to B$  be a fibration without multiple fibres over a curve B of genus  $b \ge 1$ , and assume that, for each irreducible curve C contained in a fibre of f, C' denoting the normalization of C, the homomorphism  $\pi_1(C') \to \pi_1(S)$  has infinite image.

Since  $\Sigma'$  is deformation retract of  $\tilde{B}$ ,  $\tilde{S}$  is homotopically equivalent to a CW complex of dimension  $\leq 2$ , in particular  $\tilde{S}$  is contractible if and only if  $H_2(\tilde{S}, \mathbb{Q}) = 0$ .

*Proof.*  $\tilde{S}$  retracts onto  $K' := \tilde{f}^{-1}(\Sigma')$ , and this shows the first assertion.

The second follows since,  $\tilde{S}$  being simply connected and homotopically equivalent to a CW complex of dimension  $\leq 2$ , it is contractible if and only if  $H_2(\tilde{S}, \mathbb{Z}) = 0$ . In fact, by Hurewicz' theorem, the first non-zero homotopy group  $\pi_m(\tilde{S})$  is isomorphic to  $H_m(\tilde{S}, \mathbb{Z})$ , which is obviously zero for  $m \geq 3$ . Hence if  $H_2(\tilde{S}, \mathbb{Z}) = 0$  all homotopy groups  $\pi_i(\tilde{S}) = 0$  and  $\tilde{S}$  is contractible (the converse is obvious).

Finally, by the universal coefficient formula,  $H_2(\tilde{S}, \mathbb{Z})$  is torsion free, hence  $H_2(\tilde{S}, \mathbb{Z}) = 0$  if and only if  $H_2(\tilde{S}, \mathbb{Q}) = 0$ .

Let  $\tilde{F}_0$  be the fibre over  $y_0$ , and let  $\tilde{F}_n$  be the fibre over  $y_n$ .

We can write  $K' = \bigcup_{n \in \mathbb{N}} K_n$ , where  $K_n$  is the inverse image of the union  $\Sigma'_n$  of the segments joining  $y_0$  with  $y_i$ , for  $i \leq n$ .

Clearly

$$H_2(S, \mathbb{Q}) = 0 \Leftrightarrow H_2(K_n, \mathbb{Q}) = 0 \ \forall n \in \mathbb{N}.$$

We can now use the theorem of Mayer-Vietoris, using the following notation:  $K_n^*$  shall be the inverse image of  $\Sigma_n^*$ , the union of the open segments, together with the point  $y_0$ , whereas  $\partial K_n$  shall be the inverse image of  $\{y_1, \ldots, y_n\}$ .

Now,  $K_n$  is the union of two open subsets, the first is  $K_n^*$  which is homotopically equivalent to  $\tilde{F}_0$ , and the second which is homotopically

equivalent to  $\partial K_n$ , the disjoint union of the fibres  $\tilde{F}_i$ ,  $1 \leq i \leq n$ . Moreover, the intersection of the two open sets is homotopically equivalent to n disjoint copies of  $\tilde{F}_0$ .

Define  $H_i := H_1(F_i, \mathbb{Q})$ . Then we have the Mayer-Vietoris exact sequence

$$0 \to H_2(K_n, \mathbb{Q}) \to H_0^n \to H_0 \oplus H_1 \oplus \cdots \oplus H_n \to H_1(K_n, \mathbb{Q}) \to 0.$$

Here, we have a homomorphism  $r_i : H_0 \to H_i$  which is obtained by the fact that a neighbourhood of the fibre  $\tilde{F}_i$  retracts onto  $\tilde{F}_i$ :  $r_i$ is a surjection whose kernel is the group  $V_i$  of vanishing cycles. The homomorphism of the i-th summand  $H_0$  inside  $H_0 \oplus H_1 \oplus \cdots \oplus H_n$  has first component which is the identity, and all the other components equal to zero with the exception of one which is indeed  $r_i : H_0 \to H_i$ .

We obtain the following theorem.

**Theorem 36.** Let  $f : S \to B$  be a fibration without multiple fibres over a curve B of genus  $b \ge 1$ , and assume that, for each irreducible curve C contained in a fibre of f, C' denoting the normalization of C, the homomorphism  $\pi_1(C') \to \pi_1(S)$  has infinite image.

Then  $\tilde{S}$  is contractible if and only if, for each  $n \in \mathbb{N}$ , the subgroups of vanishing cycles form a direct sum  $V_1 \oplus \cdots \oplus V_n$  inside  $H_0$ .

*Proof.* The kernel of  $H_0^n \to H_0 \oplus H_1 \oplus \cdots \oplus H_n$  are the elements  $(x_1, \ldots, x_n)$  such that  $(x_1 + \cdots + x_n, r_1(x_1), \ldots, r_n(x_n)) = (0, 0, \ldots, 0)$ . Hence these are the elements of  $V_1 \oplus \cdots \oplus V_n$  which map to 0 inside  $H_0$ .

We observe that, even if the criterion is essentially a characterization of fibred algebraic surfaces which are projective classifying spaces, the condition on the vanishing cycles is not so easy to verify, and, up to now, good examples are still missing (cf. however later sections for candidates).

Observe also the following relation with the well known Shafarevich conjecture. Keep the assumption that  $f: S \to B$  is a fibration without multiple fibres over a curve B of genus  $b \ge 1$ , and assume that, for a general fibre F, the homomorphism  $\pi_1(F) \to \pi_1(S)$  has infinite image. If there is an irreducible curve C contained in a fibre of f such that, C'denoting the normalization of C, the homomorphism  $\pi_1(C') \to \pi_1(S)$ has finite image, while the homomorphism  $\pi_1(C) \to \pi_1(S)$  has infinite image, then S would be a counterexample to the Shafarevich conjecture that  $\tilde{S}$  is holomorphically convex (cf. [Bog-Katz-98]). Since then  $\tilde{S}$ would contain a connected infinite chain of compact curves.

#### KODAIRA AND MODULI THEORY

#### 3. Surfaces fibred onto curves

3.1. The Zeuthen-Segre formula. The Zeuthen Segre formula is a beautiful formula, valid for any smooth algebraic surface S:

**Theorem 37. (Zeuthen-Segre, classical)** Let S be a smooth projective surface, and let  $C_{\lambda}$ ,  $\lambda \in \mathbb{P}^1$ , be a linear pencil of curves of genus g which meet transversally in  $\delta$  distinct points. If  $\mu$  is the number of singular curves in the pencil (counted with multiplicity), then

$$\mu - \delta - 2(2g - 2) = I + 4,$$

where the integer I is an invariant of the algebraic surface, called Zeuthen-Segre invariant.

Here, the integer  $\delta$  equals the self-intersection number  $C^2$  of the curve C, while in modern terms the integer I + 4 is not only an algebraic invariant, but is indeed a topological invariant: the topological Euler-Poincaré characteristic e(S).

Observe that  $I + 4 + C^2$  is then the topological Euler-Poincaré characteristic of the surface blown up in the  $\delta = C^2$  base points of the pencil. In this way the Zeuthen-Segre formula generalizes for every fibration  $f: S \to B$  of an algebraic surface onto a curve B, and the formula measures the deviation from the case of a topological bundle, for which one would have e(S) = 4(b-1)(g-1), where g is the genus of a fibre.

The importance of the formula is that, as suggested by the interpretation of  $\mu$  as a number of points, the difference  $\mu := e(S) - 4(b-1)(g-1)$ is always non negative.

The formula is well known using topology (see [BPHV]), but it is very convenient to have an algebraic formula, which is proven using sheaves and exact sequences (see the lecture notes [CB]).

**Definition 38.** Let  $f : S \to B$  be a fibration of a smooth algebraic surface S onto a curve of genus b, and consider a singular fibre  $F_t = \sum n_i C_i$ , where the  $C_i$  are distinct irreducible curves.

Then the divisorial singular locus of the fibre is defined as the divisorial part of the critical scheme,  $D_t := \sum (n_i - 1)C_i$ , and the Segre number of the fibre is defined as

$$\mu_t := deg\mathcal{F} + D_t K_S - D_t^2,$$

where the sheaf  $\mathcal{F}$  is concentrated in the singular points of the reduction of the fibre, and is the quotient of  $\mathcal{O}_S$  by the ideal sheaf generated by the components of the vector  $d\tau/s$ , where s = 0 is the equation of  $D_t$ , and where  $\tau$  is the pull-back of a local parameter at the point  $t \in B$ .

More concretely,

$$\tau = \Pi_j f_j^{n_j}, s = \tau / (\Pi_j f_j),$$

and the logarithmic derivative yields

$$d\tau = s[\sum_{j} n_j (df_j \Pi_{h \neq j} f_h)].$$

The following is the refined Zeuthen-Segre formula

**Theorem 39. (Zeuthen-Segre, modern)** Let  $f : S \to B$  be a fibration of a smooth algebraic surface S onto a curve of genus b, and with fibres of genus g.

Then

$$c_2(S) = 4(g-1)(b-1) + \mu,$$

where  $\mu = \sum_{t \in B} \mu_t$ , and  $\mu_t \ge 0$  is defined as above. Moreover,  $\mu_t$  is strictly positive, except if the fibre is smooth or a multiple of a smooth curve of genus g = 1.

Most of the times, the formula is used in its non refined form: if g > 1, then either  $\mu > 0$ , or  $\mu = 0$  and we have a differentiable fibre bundle. In this case there are two alternatives: either we have a Kodaira fibration, or all the smooth fibres are isomorphic, and we have a holomorphic fibre bundle ([F-G65]).

## 3.2. The positivity results of Arakelov, Fujita and Kawamata.

### 3.2.1. Arakelov's theorem.

**Definition 40.** A fibration  $f: S \to B$  of a smooth algebraic surface S onto a curve of genus b is said to be (relatively) minimal, if there is no (-1)-curve contained in a fibre. Moreover, it is said to be **isotrivial**, or with constant moduli, if all the smooth fibres are isomorphic.

Isotriviality is equivalent to the condition that the moduli morphism  $\psi: B \to \mathfrak{M}_g$  is constant, and it implies that there exists a finite Galois base change  $B' \to B$  such that the pull back  $S' \to B'$  is birational to a product.

The theorem of Arakelov ([Ara71]) gives a numerical criterion for isotriviality.

**Theorem 41.** Let  $f : S \to B$  be a minimal fibration of a smooth algebraic surface S onto a curve of genus b, where the genus g of the fibres is strictly positive. Define  $K_{S|B}$ , the relative canonical divisor, as  $K_{S|B} := K_S - f^*(K_B)$ . Then  $K_{S|B}$  is nef, and big unless the fibration is isotrivial. In particular  $K_{S|B}^2 \ge 0$  and  $K_{S|B}^2 > 0$  if the fibration is not isotrivial.

*Proof.* (Idea) (I) The minimality of the fibration and  $g \ge 1$  ensure that  $K_{S|B} \cdot C = K_S \cdot C \ge 0$  for each curve C contained in a fibre.

(II) For the case where C is not contained on a fibre, we use that the line bundle  $\mathcal{O}_C(K_{S|B})$  is generically a quotient of the pull-back of  $V := f_*\mathcal{O}_S(K_{S|B})$ . Indeed, if  $p : C \to B$  is induced by f, there is a non zero morphism  $p^*V \to \mathcal{O}_C(K_{S|B})$  and one applies the theorem of

Fujita 43 (that we shall soon describe) stating that  $p^*V$  is nef, hence  $\mathcal{O}_C(K_{S|B})$  has non negative degree.

(III) Since the divisor  $K_{S|B}$  is nef, either  $K_{S|B}$  is big or the graded ring associated to it has Iitaka dimension 1, and yields a map to a curve  $C, \varphi : S \to C$ . We consider  $F := f \times \varphi : S \to B \times C$ , and we consider the Hurwitz formula:  $K_S = F^*(K_B + K_C) + R$ , which proves that  $K_{S|B} = F^*(K_C) + R$ . Since all the sections of multiples of  $K_{S|B}$ pull back from C, it follows that R is horizontal, hence all the fibres are ramified covers of the same curve C, and branched in the same set of points. From this it follows that all the smooth fibres are isomorphic.

The following corollary contains an observation by Beauville ??and the fact, already mentioned several times, that a Kodaira fibred surface has positive index.

**Corollary 42.** Under the same assumptions as in Arakelov's theorem 41, but assuming  $g \ge 2$ : then  $\chi(S) \ge (g-1)(b-1)$ , equality if and only if we have a holomorphic bundle.

In particular, a Kodaira fibred surface S has a strictly positive index, i.e. S has  $c_1^2(S) = K_S^2 > 2e(S) = 2c_2(S)$ .

*Proof.* The Arakelov inequality  $K_{S|B}^2 = K_S^2 - 8(b-1)(g-1) \ge 0$  and the Zeuthen-Segre inequality  $e(S) \ge 4(b-1)(g-1)$  add up, in view of Noether's theorem (the first equality in the next formula) to yield the new inequality

$$12\chi(S) = K_S^2 + e(S) \ge 12(b-1)(g-1).$$

Moreover, equality holds if both Arakelov's and Zeuthen-Segre's inequality are equalities, implying that we have an isotrivial fibration and a differentiable bundle: hence, as we already observed, f is a holomorphic bundle.

If we have a Kodaira fibration, then e(S) = 4(b-1)(g-1) and the fibration is not isotrivial, hence  $K_S^2 > 8(b-1)(g-1) = 2e(S)$ .

3.2.2. *Fujita's direct image theorems*. An important progress in classification theory was stimulated by a theorem of Fujita, who showed ([Fujita78a]) that the direct image of the relative dualizing sheaf

$$\omega_{X|B} = \mathcal{O}_X(K_{X|B}) := \mathcal{O}_X(K_X - f^*K_B)$$

is numerically positive, i.e. every quotient bundle has non negative degree (a fact that was used in the previous subsection to give a different proof of Arakelov's theorem).

**Theorem 43. Fujita's first theorem** If X is a compact Kähler manifold and  $f : X \to B$  is a fibration onto a projective curve B (i.e., f has connected fibres), then the direct image sheaf

$$V := f_* \omega_{X|B}$$

is a nef vector bundle on B, equivalently V is 'numerically semipositive', meaning that each quotient bundle Q of V has degree  $\deg(Q) \ge 0$ .

In particular, if X is an algebraic surface S, then  $deg(V) \ge 0$ , which is the inequality observed previously

$$deg(V) = \chi(S) - (g-1)(b-1) \ge 0$$

(except that the characterization of the case of equality does not follow right away from theorem 43, one needs theorem 44)

In the note [Fujita78b] Fujita announced the following quite stronger result:

## Theorem 44. (Fujita's second theorem, [Fujita78b])

Let  $f : X \to B$  be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf

$$V := f_*\omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^*K_B)).$$

Then V splits as a direct sum  $V = A \oplus Q$ , where A is an ample vector bundle and Q is a unitary flat bundle.

Fujita sketched the proof, but referred to a forthcoming article concerning the positivity of the so-called local exponents; the result was used in the meantime, for instance I used it in my joint work with Pignatelli, but since Fujita's detailed article was never written, there was some objection (see [Barja98]) to the use of this beautiful result: this was a motivation for Dettweiler and myself to write down a complete proof, which is going to appear in the article contributed to Kawamata's 60-th birthday volume ([CatDet13]).

3.2.3. Kawamata's positivity theorems. In the meantime the idea of the proof had become more transparent, through Kawamata's use of Griffihts' results on Variation of Hodge Structures ( the relation being that the fibre of  $V := f_*\omega_{X|B}$  over a point  $b \in B$ , such that  $X_b := f^{-1}(b)$ is smooth, is the vector space  $V_b = H^0(X_b, \Omega_{X_b}^{n-1})$ .

Kawamata ([Kaw81] [Kaw82]) improved on Fujita's result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

$$Kod(X) \ge Kod(B) + Kod(F),$$

(here F is a general fibre) showing the semipositivity also for the direct image of higher powers of the relative dualizing sheaf

$$W_m := f_*(\omega_{X|B}^{\otimes m}) = f_*(\mathcal{O}_X(m(K_X - f^*K_B))).$$

Kawamata also extended his result to the case where the dimension of the base variety B is > 1 in [Kaw81], giving later a simpler proof of semipositivity in [Kaw02]. There has been a lot of literature on the subject ever since, (see the references we cited in [CatDet13], see [E-V90] for the ampleness of  $W_m$  when the fibration is not birationally isotrivial, and see [FF14] and [FFS14]).

Kawamata also introduced a simple lemma, concerning the degree of line bundles on a curve whose metric grows at most logarithmically around a finite number of singular points, which played a crucial role for the proof of Fujita's second theorem.

Indeed the missing details concerning the proof of the second theorem of Fujita, using Kawamata's lemma and some crucial estimates given by Zucker ([Zuc79]) for the growth of the norm of sections of the  $L^2$ -extension of Hodge bundles, were provided in [CatDet13], whose main contribution was a negative answer to a question posed by Fujita in 1982 (Problem 5, page 600 of [Katata83], Proceedings of the 1982 Taniguchi Conference).

3.3. Fujita's semiampleness question. To understand the question posed by Fujita it is not only important to have in mind Fujita's second theorem, but it is also very convenient to recall the following classical definition used by Fujita in [Fujita78a], [Fujita78b].

Let V be a holomorphic vector bundle over a projective curve B.

**Definition 45.** Let  $p : \mathbb{P} := Proj(V) = \mathbb{P}(V^{\vee}) \to B$  be the associated projective bundle, and let H be a hyperplane divisor (s.t.  $p_*(\mathcal{O}_{\mathbb{P}}(H)) = V$ ).

Then V is said to be:

(NP) numerically semi-positive if and only if every quotient bundle Q of V has degree  $deg(Q) \ge 0$ ,

(NEF) nef if and only if H is nef on  $\mathbb{P}$ ,

(A) ample if and only if H is ample on  $\mathbb{P}$ 

(SA) semi-ample if and only H is semi-ample on  $\mathbb{P}$  (there is a positive multiple mH such that the linear system |mH| is base-point free).

**Remark 46.** Recall that  $(A) \Rightarrow (SA) \Rightarrow (NEF) \Leftrightarrow (NP)$ , the last follows from the following result due to Hartshorne.

**Proposition 47.** A vector bundle V on a curve is nef if and only it is numerically semi-positive, i.e., if and only if every quotient bundle Q of V has degree  $deg(Q) \ge 0$ , and V is ample if and only if every quotient bundle Q of V has degree deg(Q) > 0.

Recall the following standard definition, used in the statement of Fujita's second theorem.

**Definition 48.** A flat holomorphic vector bundle on a complex manifold M is a holomorphic vector bundle  $\mathcal{H} := \mathcal{O}_M \otimes_{\mathbb{C}} \mathbb{H}$ , where  $\mathbb{H}$  is a local system of complex vector spaces associated to a representation  $\rho : \pi_1(M) \to GL(r, \mathbb{C}),$ 

$$\mathbb{H} := (M \times \mathbb{C}^r) / \pi_1(M),$$

 $\tilde{M}$  being the universal cover of M (so that  $M = \tilde{M}/\pi_1(M)$ ). We say that  $\mathcal{H}$  is unitary flat if it is associated to a representation  $\rho: \pi_1(M) \to U(r, \mathbb{C}).$ 

We come now to Fujita's question:

Question 49. (Fujita) Is the direct image  $V := f_* \omega_{X|B}$  semi-ample?

In [CatDet13] we established a technical result which clarifies how Fujita's question is very closely related to Fujita's II theorem

**Theorem 50.** Let  $\mathcal{H}$  be a unitary flat vector bundle on a projective manifold M, associated to a representation  $\rho : \pi_1(M) \to U(r, \mathbb{C})$ . Then  $\mathcal{H}$  is nef; moreover  $\mathcal{H}$  is semi-ample if and only if  $Im(\rho)$  is finite.

The idea of the proof is to reduce to the case where we have such a bundle over a curve (obtained by taking successive hyperplane sections of M), decompose the representation as a direct sum of irreducible unitary representations, and then use the theorem of Narasimhan and Seshadri [NS65] implying that a unitary flat holomorphic bundle over a curve is holomorphically trivial if and only if the representation is trivial.

Hence in our particular case, where  $V = A \oplus Q$  with A ample and Q unitary flat, the semiampleness of V simply means that the flat bundle has finite monodromy (this is another way to wording the fact that the representation of the fundamental group  $\rho : \pi_1(B) \to U(r, \mathbb{C})$  associated to the flat unitary rank-r bundle Q has finite image).

The main new result in our joint work [CatDet13] was, as already said, to provide a negative answer to Fujita's question in general:

**Theorem 51.** There exist surfaces X of general type endowed with a fibration  $f : X \to B$  onto a curve B of genus  $\geq 3$ , and with fibres of genus 6, such that  $V := f_*\omega_{X|B}$  splits as a direct sum  $V = A \oplus Q_1 \oplus Q_2$ , where A is an ample rank-2 vector bundle, and the flat unitary rank-2 summands  $Q_1, Q_2$  have infinite monodromy group (i.e., the image of  $\rho_j$  is infinite). In particular, V is not semi-ample.

Recently ([CatDet15]) we have found an infinite series of counterexamples, which are based on the same ideas, but are quite simpler; we shall report on them in a later subsection (they shall be called here BCDH-surfaces).

Notice that Fujita's second theorem follows right away from the first in the case where the base curve is  $\mathbb{P}^1$ , since then every vector bundle splits as a direct sum of line bundles, and we can separate the summands with strictly positive degree from the trivial summands.

Also, one can say something more precise in the case where the base curve B has genus 1, or under other assumptions, which imply that indeed V is semiample.

**Corollary 52.** Let  $f : X \to B$  be a fibration of a compact Kähler manifold X over a projective curve B. Then  $V := f_* \omega_{X|B}$  is a direct sum  $V = A \bigoplus (\bigoplus_{i=1}^h Q_i)$ , with A ample and each  $Q_i$  unitary flat without any nontrivial degree zero quotient. Moreover,

(I) if  $Q_i$  has rank equal to 1, then it is a torsion bundle ( $\exists m \text{ such that } Q_i^{\otimes m}$  is trivial) (Deligne)

(II) if the curve B has genus 1, then rank  $(Q_i) = 1, \forall i$ .

(III) In particular, if B has genus at most 1, then V is semi-ample. Idea of proof:

(I) was proven by Deligne [Del71] (and by Simpson [Simp93] using the theorem of Gelfond-Schneider), while

(II) Follows since  $\pi_1(B)$  is abelian, if B has genus 1: hence every representation splits as a direct sum of 1-dimensional ones.

3.3.1. How do the flat bundles appear. In order to get a fibration  $f : S \to B$  where we have a splitting  $V = f_* \omega_{S|B} \bigoplus Q$  our idea is to use symmetry, for instance the fibres are cyclic coverings C of  $\mathbb{P}^1$  with group  $\mathbb{Z}/n$  and branched in 4 points.

Then we get a curve  $C = C_x$  birational to the curve described by an equation of the form:

(53) 
$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - xy_0)^{m_3}, x \in \mathbb{C} \setminus \{0, 1\}.$$

Here, we shall make the restrictive assumption that

$$0 < m_j \le n - 3$$
, and  $m_0 + m_1 + m_2 + m_3 = n$ .

Then C admits a Galois cover  $\phi : C \to \mathbb{P}^1$  with cyclic Galois group equal to the group of n-th roots of unity in  $\mathbb{C}$ ,

$$G = \{ \zeta \in \mathbb{C}^* | \zeta^n = 1 \},\$$

acting by scalar multiplication on  $z_1$ . The choice of a generator in G yields an isomorphism  $G \cong \mathbb{Z}/n$ .

Now, the vector space  $V_x := H^0(\Omega^1_{C_x})$  splits according to the characters  $\chi : G \to \mathbb{C}^*$ ,

$$V_x = \bigoplus_{\chi} V^{\chi}$$

and each character can be written as  $\zeta \mapsto \zeta^i$ .

There is an easy formula (see [D-M86]) for  $dim(V^{\chi})$ :

$$dim(V^{\chi}) = \frac{1}{n}([im_0] + [im_1] + [im_2] + [im_3] - n), \ i \neq 0$$

where  $0 \leq [m] \leq n-1$  denotes the remainder of division by n (a standard representative of the residue class).

For the dual character then  $dim(V^{\bar{\chi}}) = \frac{1}{n}(n - [im_0] + n - [im_1] + n - [im_2] + n - [im_3] - n)$  hence

$$\dim(V^{\chi}) + \dim(V^{\bar{\chi}}) = 2.$$

Hence, by Hodge Theory, if  $H^{\chi} = V^{\chi} \oplus V^{\bar{\chi}}$  is the corresponding eigenspace for the action of G on  $H^1(C_x, \mathbb{C})$ , its dimension equals 2.

By our choice for the integers  $m_i$ , for i = 1 we get a character  $\chi_1$  such that  $V^{\chi_1} = 0$ , hence

$$H^{\chi_1} = V^{\chi_{n-1}},$$

hence we get a flat summand for V.

The next question is: how can we assert that the monodromy group for the flat summand is infinite?

The idea can be explained in simple terms like this: if it were finite, then there would be a monodromy invariant positive definite scalar product. This would also hold when we conjugate the characters via the action of the absolute Galois group (for *n* prime, they are all conjugate). If the monodromy is irreducibile, there is only one monodromy invariant scalar product, hence for all conjugate characters we would have a positive definite scalar product, which amounts to the condition that one never has  $dim(V^{\chi}) = dim(V^{\bar{\chi}}) = 1$ .

On the other hand, if we assume that  $m_0 + m_3$  is invertible in  $\mathbb{Z}/n\mathbb{Z}$ , there is a j such that  $j(m_0 + m_3) \equiv -1 \pmod{n}$ . For instance  $m_0 + m_3$ is invertible in  $\mathbb{Z}/n\mathbb{Z}$  if we take  $m_0 = 1, m_1 = 1, m_2 = 1, m_3 = n - 3$ and we assume that n is an odd number.

Define now  $m'_i := [m_i j]$ , where [a] denotes the remainder of division by n. Hence  $m'_0 + m'_3 = n - 1$ .

We have the obvious inequalities  $2 \le m'_1 + m'_2 \le 2n - 2$ . Hence

$$n+1 \le m'_0 + \dots + m'_3 \le 3n-3$$

and therefore

$$m_0' + \dots + m_3' = 2n.$$

Hence the underlying unitary form is indefinite for the character  $\chi_j$ , whereas it is definite for the conjugate character  $\chi_{n-j}$ .

Moreover, (see [CatDet15]) once the rank one summand  $V^{\chi_j}$  is shown to be ample, the irreducibility of the flat bundle associated to  $H^{\chi_j}$ (hence of the monodromy) follows for the following chain of arguments:

• If the flat vector bundle were reducible there would be an exact sequence of flat vector bundles

$$0 \to \mathcal{H}' \to H^{\chi_j} \to \mathcal{H}'' \to 0$$

where both  $\mathcal{H}'$ ,  $\mathcal{H}''$  have rank 1.

- $V^{\chi_j}$  is a holomorphic subbundle of  $H^{\chi_j}$ , hence it has a nontrivial homomorphism to it
- every homomorphism of  $V^{\chi_j}$  to a flat rank one bundle  $\mathcal{H}', \mathcal{H}''$  must be zero
- contradiction!

The final conclusion is that the monodromy of the flat factor  $V^{\chi_{n-1}}$  is infinite, and that V is not semiample; the concrete examples of this situation shall be described in section 5.

3.4. Castelnuovo-de Franchis and morphisms onto curves. In order to address the study of moduli spaces of Kodaira fibred surfaces, or of surfaces which are isotrivially fibred, it is convenient to review some simple results, pioneered by Siu, and which concern topological conditions implying the existence of holomorphic maps of a compact Kähler manifold X onto an algebraic curve.

Siu used harmonic theory in order to construct holomorphic maps from Kähler manifolds to projective curves. The first result in this direction was the theorem of [Siu87], also obtained by Jost and Yau (see [J-Y93] and also [J-Y83] for other results).

**Theorem 54.** (Siu) Assume that a compact Kähler manifold X is such that there is a surjection  $\phi : \pi_1(X) \to \pi_g$ , where  $g \ge 2$  and, as usual,  $\pi_g$  is the fundamental group of a projective curve of genus g. Then there is a projective curve C of genus  $g' \ge g$  and a fibration  $f : X \to C$  (i.e., the fibres of f are connected) such that  $\phi$  factors through  $\pi_1(f)$ .

In this case the homomorphism leads to a harmonic map to a curve, and one has to show that the Stein factorization yields a map to some Riemann surface which is holomorphic for some complex structure on the target.

It can be seen more directly how the Kähler assumption is used, because this assumption guarantees that holomorphic forms are closed, i.e.,  $\eta \in H^0(X, \Omega_X^p) \Rightarrow d\eta = 0.$ 

At the turn of last century this fact was used by Castelnuovo and de Franchis ([Cast05], [dF05]):

**Theorem 55.** (Castelnuovo-de Franchis) Assume that X is a compact Kähler manifold,  $\eta_1, \eta_2 \in H^0(X, \Omega^1_X)$  are  $\mathbb{C}$ -linearly independent, and the wedge product  $\eta_1 \wedge \eta_2$  is d-exact. Then  $\eta_1 \wedge \eta_2 \equiv 0$  and there exists a fibration  $f: X \to C$  such that  $\eta_1, \eta_2 \in f^*H^0(C, \Omega^1_C)$ . In particular, C has genus  $g \geq 2$ .

Even if the proof is well known, let us point out that the first assertion follows from the Hodge-Kähler decomposition, while  $\eta_1 \wedge \eta_2 \equiv 0$ implies the existence of a non constant rational function  $\varphi$  such that  $\eta_2 = \varphi \eta_1$ . This shows that the foliation defined by the two holomorphic forms has Zariski closed leaves, and the rest follows then rather directly taking the Stein factorization of the rational map  $\varphi : X \to \mathbb{P}^1$ .

Now, the above result, which is holomorphic in nature, combined with the Hodge decomposition, produces results which are topological in nature (they actually only depend on the cohomology algebra structure of  $H^*(X, \mathbb{C})$ ).

To explain this in the most elementary case, we start from the following simple observation. If two linear independent vectors in the first cohomology group  $H^1(X, \mathbb{C})$  of a Kähler manifold have wedge product which is trivial in cohomology, and we represent them as  $\eta_1 + \overline{\omega_1}, \eta_2 + \overline{\omega_2}$ , for  $\eta_1, \eta_2, \omega_1, \omega_2 \in H^0(X, \Omega^1_X)$ , then by the Hodge decomposition and the first assertion of the theorem of Castelnuovo-de Franchis

$$(\eta_1 + \overline{\omega_1}) \wedge (\eta_2 + \overline{\omega_2}) = 0 \in H^2(X, \mathbb{C})$$

implies

$$\eta_1 \wedge \eta_2 \equiv 0, \ \omega_1 \wedge \omega_2 \equiv 0.$$

We can apply Castelnuovo-de Franchis unless  $\eta_1, \eta_2$  are  $\mathbb{C}$ -linearly dependent, and similarly  $\omega_1, \omega_2$ . W.l.o.g. we may assume then  $\eta_2 \equiv 0$  and  $\omega_1 \equiv 0$ . But then  $\eta_1 \wedge \overline{\omega_2} = 0$  implies that the Hodge norm

$$\int_X (\eta_1 \wedge \overline{\omega_2}) \wedge \overline{(\eta_1 \wedge \overline{\omega_2})} \wedge \xi^{n-2} = 0,$$

where  $\xi$  is here the Kähler form. A simple trick is to observe that

$$0 = \int_X (\eta_1 \wedge \overline{\omega_2}) \wedge \overline{(\eta_1 \wedge \overline{\omega_2})} \wedge \xi^{n-2} = -\int_X (\eta_1 \wedge \omega_2) \wedge \overline{(\eta_1 \wedge \omega_2)} \wedge \xi^{n-2},$$

therefore the same integral yields that the Hodge norm of  $\eta_1 \wedge \omega_2$  is zero, hence  $\eta_1 \wedge \omega_2 \equiv 0$ ; the final conclusion is that we can in any case apply Castelnuovo-de Franchis and find a map to a projective curve Cof genus  $g \geq 2$ .

More precisely, one gets the following theorem ([Cat91]):

**Theorem 56.** (Isotropic subspace theorem) On a compact Kähler manifold X there is a bijection between isomorphism classes of fibrations  $f: X \to C$  to a projective curve of genus  $g \ge 2$ , and real subspaces  $V \subset H^1(X, \mathbb{C})$  ('real' means that V is self conjugate,  $\overline{V} = V$ ) which have dimension 2g and are of the form  $V = U \oplus \overline{U}$ , where U is a maximal isotropic subspace for the wedge product

$$H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C}).$$

The above result, as simple as it may be, implies the few relations theorem of Gromov ([Grom89]), which in turn implies theorem 54 of Siu (see e.g. [Cat15] for an ampler discussion).

There is another result ([Cat08]) which again, like the isotropic subspace theorem, determines explicitly the genus of the target curve (a result which is clearly useful for classification and moduli problems).

**Theorem 57.** Let X be a compact Kähler manifold, and let  $f: X \to C$  be a fibration onto a projective curve C, of genus g, and assume that there are exactly r fibres which are multiple with multiplicities  $m_1, \ldots m_r \geq 2$ . Then f induces an orbifold fundamental group exact sequence

$$\pi_1(F) \to \pi_1(X) \to \pi_1(g; m_1, \dots m_r) \to 0,$$

where F is a smooth fibre of f, and where the orbifold fundamental group

$$\pi_1(g;m_1,\ldots m_r)$$

is defined as

 $\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_r | \Pi_1^g [\alpha_j, \beta_j] \Pi_1^r \gamma_i = \gamma_1^{m_1} = \dots = \gamma_r^{m_r} = 1 \rangle.$ Conversely, let X be a compact Kähler manifold and let  $(g, m_1, \dots, m_r)$ 

be a hyperbolic type, i.e., assume that  $2g - 2 + \sum_i (1 - \frac{1}{m_i}) > 0$ . Then each epimorphism  $\phi : \pi_1(X) \to \pi_1(g; m_1, \dots, m_r)$  with finitely generated kernel is obtained from a fibration  $f : X \to C$  of type  $(g; m_1, \dots, m_r)$ .

With these results and the Zeuthen-Segre formula, it is easy to explain on the one hand the characterization of surfaces isogenous to a product, and on the other hand Kotschick's result on moduli of Kodaira surfaces.

**Theorem 58.** a) A projective smooth surface S is isogenous to a product of two curves of respective genera  $g_1, g_2 \ge 2$ , if and only if the following two conditions are satisfied:

1) there is an exact sequence

$$1 \to \pi_{q_1} \times \pi_{q_2} \to \pi = \pi_1(S) \to G \to 1,$$

where G is a finite group and where  $\pi_{g_i}$  denotes the fundamental group of a projective curve of genus  $g_i \geq 2$ ;

2)  $e(S)(=c_2(S)) = \frac{4}{|G|}(g_1-1)(g_2-1).$ 

b) Write  $S = (C_1 \times C_2)/G$ . Any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S and is also isogenous to a product.

c) The corresponding subset of the moduli space of surfaces of general type  $\mathfrak{M}_S^{top} = \mathfrak{M}_S^{diff}$ , corresponding to surfaces orientedly homeomorphic, resp. orientedly diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.

Idea of the proof of b)  $\Gamma := \pi_1(S)$  admits a subgroup  $\Gamma'$  of index d such that  $\Gamma' \cong (\pi_{g_1} \times \pi_{g_2})$ . Let S' be the associated unramified covering of S. Then application of the isotropic subspace theorem or of theorem 57 yields a pair of holomorphic maps  $f_j : S' \to C_j$ , hence a holomorphic map

$$F := f_1 \times f_2 : S' \to C'_1 \times C'_2.$$

Then the fibres of  $f_1$  have genus  $h_2 \ge g_2$ , hence by the Zeuthen Segre formula  $e(S') \ge 4(h_2 - 1)(g_1 - 1)$ , equality holding if and only if all the fibres are smooth.

But  $e(S') = 4(g_1 - 1)(g_2 - 1) \leq 4(h_2 - 1)(g_1 - 1)$ , so  $h_2 = g_2$ , all the fibres are smooth hence isomorphic to  $C'_2$ ; therefore F is an isomorphism.

The previous theorem generalizes also to varieties isogenous to a product of curves of dimension  $n \ge 3([Cat00], [Cat15])$ .

We try now to compare the cited theorem of Kotschick (theorem 8) with the previous one.

**Theorem 59.** Assume that S is a compact Kähler surface, and that (i) if f is a compact Kähler surface in f is a compact Kähler surface.

(i) its fundamental group sits into an exact sequence, where  $g, b \ge 2$ :

$$1 \to \pi_g \to \pi_1(S) \to \pi_b \to 1$$

(*ii*) e(S) = 4(b-1)(g-1).

Then S has a smooth holomorphic fibration  $f : S \to B$ , where B is a projective curve of genus b, and where all the fibres are smooth projective curves of genus g. f is a Kodaira fibration if and only if the associated homomorphism  $\rho : \pi_b \to \mathcal{M}ap_g$  has image of infinite order, else it is a surface isogenous to a product of unmixed type and where the action on the first curve is free.

Proof. By theorem 57 the above exact sequence yields a fibration  $f: S \to B$  such that there is a surjection  $\pi_1(F) \to \pi_g$ , where F is a smooth fibre. Hence, denoting by h the genus of F, we conclude that  $h \geq g$ , and again we can use the Zeuthen-Segre formula to conclude that h = g and that all fibres are smooth. So F is a smooth fibration. Let  $C' \to C$  be the unramified covering associated to  $\ker(\rho)$ : then the pull back family  $S' \to C'$  has a topological trivialization, hence is a pull back of the universal family  $\mathcal{C}_g \to \mathcal{T}_g$  for an appropriate holomorphic map  $\varphi: C' \to \mathcal{T}_g$ .

If ker( $\rho$ ) has finite index, then C' is compact and, since Teichmüller space is a bounded domain in  $\mathbb{C}^{3g-3}$ , the holomorphic map is constant. Therefore S' is a product  $C' \times C_2$  and, denoting by  $G := \text{Im}(\rho)$ ,  $S = (C' \times C_2)$ , and we get exactly the surfaces isogenous to a product such that the action of G on the curve C' is free.

If instead  $G := \text{Im}(\rho)$  is infinite, then the map of C' into Teichmüller space is not constant, since the isotropy group of a point corresponding to a curve F is, as we saw, equal to the group of automorphisms of F(which is finite). Therefore, in this case, we have a Kodaira fibration.

3.5. Singular fibres and mapping class group monodromy. Let us consider again the situation where we have a fibration  $f: S \to B$ of an algebraic surface S onto a curve of genus b, and where the fibres have genus  $g \ge 2$ . We assume throughout that the fibration is relatively minimal and not a product.

We have devoted most of our attention to the consideration of the case where all the fibres F of f are smooth curves. This can occur whenever the genus  $b \ge 1$ : just take a surface isogenous to a product

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of unmixed type, i.e.,  $S = (C_1 \times C_2)/G$ , where G acts diagonally and freely on  $C_1$ , with quotient  $C_1/G = B$ .

Then  $f: S = (C_1 \times C_2)/G \to C_1/G = B$  is the desired holomorphic bundle.

For Kodaira fibrations the situation is slightly more complicated, it is possible that the genus b = 2, as we saw, but the fibre genus g should be at least 3.

At any rate, if we have a differentiable fibre bundle over a curve B of genus 1, then,  $\tilde{B}$  being the universal covering of B,  $\tilde{B} \cong \mathbb{C}$ , the Torelli map  $\tilde{B} \to \mathcal{H}_g$  into Siegel's half space must be constant, therefore we have a holomorphic bundle, which is then isogenous to a product, as described above.

The conclusion is that there must be at least one singular fibre if b = 1 and we have nonconstant moduli.

On the other hand, when b = 0, and the fibration is not a product, the number of singular fibres must be at least three.

Indeed, if  $B^* = B \setminus C$  is the set of noncritical values of f, we want that there is a nonconstant map  $B^* \to \mathcal{H}_g$ ; if  $B = \mathbb{P}^1$ , therefore the number of singular fibres, i.e.  $|\mathcal{C}|$ , must at least three.

If we assume however that all the fibres are moduli semi-stable curves, then the number of singular fibres must be at least 4 for  $g \ge 1$  and at least 5 for  $g \ge 2$ , as proven by Beauville and Tan [Bea81] and [Tan95] ([Za04] gave an improvement for  $g \ge 3$ , that the number must be at least 6).

Tan's inequality is

$$\frac{g}{2}(2b-2+s) > deg(f_*(\omega_{S|B})) = \chi(S) - (b-1)(g-1),$$

so that for b = 1 the number s of singular fibres is  $> \frac{2}{g}$ , which gives  $s \ge 2$  only if g = 2.

Can one obtain better estimates on the number of singular fibres also when the genus of the base curve is b = 1?

A very interesting result was obtained by Ishida [Ishi06] who found, in the moduli space of surfaces with  $q = p_g = 1, K^2 = 3$ , described in joint work with Ciliberto ([CaCi93]) some surfaces whose Albanese fibration (onto a curve of genus b = 1) has a unique singular fibre, and fibre genus g = 3; but the singular fibre is not reduced.

As far as I know, there are no known examples of a non isotrivial fibration over a curve B of genus b = 1, and possessing only one singular fibre, irreducible and with only nodes as singularities.

If such a fibration were to exist, the local monodromy around the singular fibres, a product of commuting Dehn twists, would be a commutator in the mapping class group. This raises a general question

**Question 60.** Which products of commuting Dehn twists are a commutator in the mapping class group  $Map_g$ , for  $g \ge 2$ ?

The question in genus g = 1 is clear, since then the mapping class group is  $SL(2,\mathbb{Z})$ , and there the transvection (Picard-Lefschetz transformation)

$$T_1: e_1 \mapsto e_1, e_2 \mapsto e_1 + e_2$$

is not a commutator, and indeed no parabolic transformation in

$$\mathbb{P}SL(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3 = \langle A, B | A^2 = B^3 = 1 \rangle$$

is a commutator. The easiest way to see this is to express  $T_1$  as a product of the two generators,  $T_1^{-1} = AB$ , so that the image of  $T_1$  in the Abelianization  $\mathbb{Z}/2 \times \mathbb{Z}/3$  is the element (1, -1). In fact, no power of  $T_1$  is a commutator in  $SL(2, \mathbb{Z})$ .

In higher genus there is a surjection  $Map_g \to Sp(2g, \mathbb{Z})$  and there the obstructions seem to cease to exist, as shown by Corvaja and Zannier (personal communication). Writing  $T_2$  for the transvection such that

$$T_2: e_1 \mapsto e_1, e_2 \mapsto e_1 + e_2, e_3 \mapsto e_3, e_4 \mapsto e_3 + e_4,$$

they show that  $T_2$  is a commutator in  $Sp(4,\mathbb{Z})$ , and that  $T_1$  and the similarly defined  $T_3$  are commutators in  $Sp(6,\mathbb{Z})$ .

In the mapping class group the question becomes more subtle: first of all, there are Dehn twists on nonseparating and on separating curves; in the latter case the homology class of the curve is trivial, and the image of the Dehn twist in  $Sp(2g, \mathbb{Z})$  is trivial.

Endo and Kotschick [E-K01] showed that in the separating case the Dehn twist cannot be a commutator: the idea is that otherwise one would have a symplectic fibration over a torus with only one singular fibre with the Dehn twist as local monodromy.

Cartwright and Steger (see [CKY14]) recently constructed via computer calculations a surface of general type with  $q = p_g = 1$  and  $K^2 = 9$ (hence a ball quotient). In this case the Albanese map [CK14] has either three irreducible singular fibres with one node, or one irreducible singular fibre having either three nodes, or a tacnode.

The existence of this surface shows that the product of three Dehn twists can be a commutator in  $Map_g$ , and contradicts [Kot04]. Recently Stipsicz and Yun [S-Y16] announced the following result: a product of at most two Dehn twists cannot be a commutator in  $Map_g$  for  $g \ge 1$ .

# 4. Covers branched over line configurations

4.1. Generalities on Abelian coverings of the plane. Assume that in the projective plane  $\mathbb{P} := \mathbb{P}^2$  are given distinct irreducible curves  $C_j$  of respective degrees  $m_j$ ; we shall denote by C the union of the curves  $C_j$ .

Then an irreducible Abelian cover branched over the union C of these curves and with group G is determined by:

• elements  $g_j \in G$  such that they generate G and such that

• the sum of the  $m_j g_j$  is equal to zero in G:

$$\sum_{j} m_j g_j = 0 \in G.$$

In fact, one has an exact sequence

$$H^2(\mathbb{P},\mathbb{Z}) \to H^2(C,\mathbb{Z}) \to H^3(\mathbb{P},C,\mathbb{Z}) \to 0$$

and the last group is isomorphic to  $H_1(\mathbb{P} \setminus C, \mathbb{Z})$  via Lefschetz duality, while  $H^2(C, \mathbb{Z}) \cong \bigoplus_j \mathbb{Z}C_j$ . This means that  $H_1(\mathbb{P} \setminus C, \mathbb{Z})$  is generated by loops around the curves  $C_j$ , but these satisfy one relation.

The relation has the above form because, denoting  $r : H^2(\mathbb{P}, \mathbb{Z}) \to \bigoplus_j \mathbb{Z}C_j$ , the image of a line L in  $coker(r) = H_1(\mathbb{P} \setminus C, \mathbb{Z})$  is zero.

If we blow up some points  $P_i$  which are singular for C, obtaining a birational morphism  $\pi: Y \to \mathbb{P}$ , then the homology of the complement of the preimage of C,  $H_1(Y \setminus \pi^{-1}C, \mathbb{Z})$ , is generated by the loops around the curves  $D_j$  which are strict transform of the curves  $C_j$ , and by the loops around the exceptional curves  $E_i$ .

The monodromy  $\mu$  of the Abelian cover takes these generators to respective elements  $g_j$  and  $\epsilon_i$  of the group.

Recall the relation  $\sum_{j} m_j g_j = 0$ .

If we write

$$D_j = m_j L - \sum_i a_{j,i} E_i = 0$$

we can repeat the same argument with Lefschetz duality, and the relation saying that the image of  $E_i$  in  $coker(r_Y)$  is zero (here  $r_Y :$  $H^2(Y,\mathbb{Z}) \to \bigoplus_j \mathbb{Z} D_j \bigoplus \bigoplus_i \mathbb{Z} E_i$ ) yields

$$\sum_{j} a_{j,i} g_j = \epsilon_i.$$

This third formula determines the image  $\epsilon_i$  of a loop around  $E_i$  under the monodromy homomorphism  $\mu$ .

# Definition 61. (DEFINITION OF A MAXIMAL COVER)

Let  $d_j$  be the order of the element  $g_j$  and consider the following group G'' (relating to the terminology used e.g. in the lecture notes [Cat08], G'' is the abelianization of the orbifold fundamental group of the cover), defined as:

G'' is the quotient of the direct sum of  $\mathbb{Z}/d_i\mathbb{Z}$  by the relation

$$\sum_{j} m_j g_j'' = 0$$

where  $g''_j$  is the standard generator of the summand  $G''_j = \mathbb{Z}/d_j\mathbb{Z}$ . Clearly the monodromy  $\mu$  factors through  $\mu''$ , which sends the loop around  $C_j$  (resp.  $D_j$ ) to  $g''_j$ , and the obvious surjection of  $G'' \to G$ . We get corresponding (irreducible) normal coverings Z'', Z of the plane  $\mathbb{P}$  such that

$$\mathbb{P} = Z''/G'' = Z/G , \ Z = Z''/H,$$

where H is the kernel of the surjection  $G'' \to G$ . A cover is said to be **maximal** if G'' = G, i.e., Z = Z''.

**Remark 62.** The quotient Z = Z''/H is only ramified in a finite set.

The importance of the concept of maximal covering is the following: let Y be the surface obtained by blowing up the points  $P_i$  where C is not a normal crossing divisor, and assume that the divisor D in Y, union of the  $E_i$ 's and the  $D_j$ 's, is a normal crossing divisor (this happens if and only if C has only ordinary singularities).

In this case the local monodromies are the elements  $g''_i$ , respectively

$$\epsilon_i'' = \sum_j a_{j,i} g_j''.$$

Now, we can write our covering  $X \to Y$  (the normalized fibre product  $Z \times_{\mathbb{P}^2} Y$ ) as a GLOBAL QUOTIENT X = X''/H, and since the covering  $X'' \to X$  is only ramified in a finite set, we get that, if X'' is smooth, then X has only cyclic quotient singularities (this approach has the advantage of making the description of the singularities shorter).

It is therefore important to see general conditions which ensure the smoothness of X'': this is however technical, so we skip this analysis here.

As done by Hirzebruch ([Hirz83], [BHH87]) formulae simplify drastically if we require all the curves  $C_j$  to be lines  $L_j$ , and we let all the orders  $d_j$  of the elements  $g_j$  to be equal to the same integer n.

In this case the maximal cover is called the Kummer cover of exponent n of the plane branched on the r lines  $L_1, \ldots, L_r$  and its Galois group is the group

$$(\mathbb{Z}/n)^{r-1} = (\mathbb{Z}/n)^r / \mathbb{Z}e,$$

where  $e := \sum_{i} e_i$ .

**Definition 63.** Let C be a configuration of distinct lines

$$L_1 = \{l_1(x_1, x_2, x_3) = 0\}, \dots, L_r = \{l_r(x_1, x_2, x_3) = 0\} \subset \mathbb{P}^2,$$

where we assume that there is no point in the plane P belonging to each line  $L_i$ . Hence, without loss of generality, we can assume, after changing the numbering of the lines, and after a projective change of coordinates, that  $L_i = \{x_i = 0\}$  for i = 1, 2, 3.

The linear forms  $(l_1, \ldots, l_r)$  yield an embedding  $l : \mathbb{P}^2 \to \mathbb{P}^{r-1}$ , and let  $(y_1, \ldots, y_r)$  be coordinates in  $\mathbb{P}^{r-1}$ . Consider next the Galois covering

$$\psi_n: \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}, \quad \psi_n((z_1, \dots, z_r)) = (z_1^n, \dots, z_r^n),$$

with Galois group  $(\mathbb{Z}/n)^{r-1}$ .

Let Y be the fibre product of L and  $\psi$ :

$$Y = \{(x, z) \in \mathbb{P}^2 \times \mathbb{P}^{r-1} | l(x) = \psi(z)\}.$$

Under our assumption on the linear forms, Y indeed embeds in  $\mathbb{P}^{r-1}$ as the complete intersection of r-3 hypersurfaces:

$$Y = \{(z) | z_j^n = l_j(z_1^n, z_2^n, z_3^n), \ j = 4, \dots, r\}.$$

The minimal resolution of singularities X of Y is called the **Hirzebruch-Kummer covering** of  $\mathbb{P}^2$  of exponent n branched on the configuration  $\mathcal{C}$  of lines, and denoted  $HK_{\mathcal{C}}(n)$ .

Quite similarly one defines the Hirzebruch-Kummer covering of  $\mathbb{P}^m$  of exponent n branched on a configuration  $\mathcal{C}$  of hyperplanes.

**Remark 64.** a) When r = m + 2 one gets a hypersurface, the Fermat hypersurface.

b) In the case of  $\mathbb{P}^1$  one gets curves which are also called **generalized** Fermat curves.

c) If the configuration is a normal crossing configuration, Y is a smooth complete intersection, so it has a lot of deformations.

4.2. **Invariants.** Assume that, as in the previous subsection, we have an Abelian cover  $S \to Y$  branched on a normal crossing divisor  $D' := \sum_j D_j + \sum_i E_i$ . To each divisor is associated the cyclic subgroup (inertia subgroup) generated by the local monodromy:  $\langle g_j \rangle$  in the case of  $D_j$ , and  $\langle \epsilon_i \rangle$  in the case of  $E_i$ . Let  $d_j$  be the order of  $g_j$  and  $d'_i$ the order of  $\langle \epsilon_i \rangle$ .

The covering S is smooth if and only if, at each intersection point of two branch curves, the corresponding two inertia subgroups yield a direct sum. In this case also the maximal cover S'' is smooth, at least provided that the order of  $\epsilon''_i$  in G'' equals  $d'_i$ ; and S'' is an étale covering of S with group H, such that  $G''/H \cong G$ .

The Chern numbers of S can be easily calculated, since  $K_S$  is the pull back of a divisor with rational coefficients

$$K_S = p^* (K_Y + \sum_j (1 - \frac{1}{d_j})D_j + \sum_i (1 - \frac{1}{d'_i})E_i) =$$
$$= p^* (-3L + \sum_j (1 - \frac{1}{d_j})(m_j L - \sum_i a_{j,i}E_i) + \sum_i (2 - \frac{1}{d'_i})E_i).$$

Whereas, for the Euler number one uses the fact that it is additive for a stratification with strata which are orientable (non compact) manifolds (of different dimensions).

The simplest case is the case of a Kummer covering <sup>5</sup> of exponent n branched on r lines, where  $m_i = 1$ ,  $d_i = n$ , and where we assume that

<sup>&</sup>lt;sup>5</sup> One can define in the utmost generality the Kummer covering of exponent n of a normal variety Y branched on B as the normal finite covering associated to the epimorphism  $\pi_1(Y \setminus B) \to H_1(Y \setminus B, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/n$ 

also  $d'_i = n$ , for all i = 1, ..., k. Then, observing that  $a_{j,i} \in \{0, 1\}$  and writing  $v_i := \sum_j a_{j,i}$  for the valency of the point  $p_i$ , we get:

$$K_{S} = p^{*}(-3L + \sum_{j}(1 - \frac{1}{n})(L - \sum_{i}a_{j,i}E_{i}) + \sum_{i}(2 - \frac{1}{n})E_{i}) =$$
$$p^{*}((-3 + r(1 - \frac{1}{n}))L + \sum_{i}(1 + (1 - \frac{1}{n})(1 - v_{i}))E_{i}).$$

Whence

$$K_S^2 = n^{r-1} \left[ \left( -3 + r(1 - \frac{1}{n}) \right)^2 - \sum_i \left( 1 + \left( 1 - \frac{1}{n} \right) (1 - v_i) \right)^2 \right]$$

Observe that we only needed to blow up the points of valency  $v_i \geq 3$ .

Whereas, to calculate the Euler number we write

$$k + 3 = e(Y) = e(Y - D') + \sum_{i} e(E_i^*) + \sum_{j} e(D_j^*) + N,$$

where \* denotes the intersection of a component of D' with the smooth locus of D', and N is the number of singular points of D'.

Let  $\delta$  be the number of double points of  $C = \sum_j L_j$ ,

$$\delta = \frac{1}{2}r(r-1) - \frac{1}{2}\sum_{i}v_i(v_i-1).$$

Then

$$k+3 = e(Y) = e(Y - D') + (2k - \sum_{i} v_i + 2r - \sum_{i} v_i - 2\delta) + (\sum_{i} v_i + \delta).$$

hence, writing  $v := \sum_i v_i$ 

$$k + 3 = e(Y) = e(Y - D') + (2k - 2v + 2r - 2\delta) + (v + \delta).$$

the Euler number of S is then just equal to

$$e(S) = n^{r-1}\left[k + 3 - \left(1 - \frac{1}{n}\right)\left(2k - 2v + 2r - 2\delta\right) - \left(1 - \frac{1}{n^2}\right)\left(v + \delta\right)\right].$$

In order to calculate the irregularity of S, and in general also  $q = h^1(\mathcal{O}_S), p_g = h^2(\mathcal{O}_S)$ , the best method is (see [BC08]) to calculate explicitly the decomposition of  $p_*\mathcal{O}_S$  into eigensheaves,

$$p_*\mathcal{O}_S = \mathcal{O}_Y \oplus (\bigoplus_{\chi} \mathcal{O}_Y(-L_{\chi})).$$

4.3. Covers branched on lines in general position. Assume that we have an Abelian covering with  $d_j = n \ \forall j = 1, \ldots, r$ , branched on the *r* lines  $L_1, \ldots, L_r$ . We assume that the *r* lines are in general position, this means that *C* has only double points, hence we get exactly  $\delta = \frac{1}{2}r(r-1)$  intersection points. In this case the fundamental group of  $\mathbb{P}^2 \setminus C$  is Abelian, free of rank (r-1).

Then any such covering is a quotient of the Hirzebruch-Kummer covering of exponent n, and we can assume that  $r \ge 4$ , otherwise our surface is either singular or equal to  $\mathbb{P}^2$ .

We saw in remark 64 that for r = 4 the Hirzebruch-Kummer covering yields the Fermat surface of degree n in  $\mathbb{P}^3$ .

The simplest case is n = 2 and r = 4: the Hirzebruch-Kummer cover is the smooth quadric

$$\{u^2 = x^2 + y^2 + z^2\} \subset \mathbb{P}^3,$$

hence the Del Pezzo surface  $\mathbb{P}^1 \times \mathbb{P}^1$ .

The next case of five lines in general position yields as Hirzebruch-Kummer cover a complete intersection Y of type (2, 2) in  $\mathbb{P}^4$ , hence again a Del Pezzo surface, of degree 4. There is only one intermediate covering branched on the five lines: it is a singular Del Pezzo surface of degree 2.

The next case of six lines in general position is interesting, the Kummer covering is a smooth K3 surface, with a group  $(\mathbb{Z}/2)^5$  of covering automorphisms, whereas the double cover branched on the six lines is a K3 surface with 15 nodes coming from the  $\delta = 15$  double points; in the case where the six lines are tangent to a conic Q, it is a 16-nodal Kummer surface blown up in one node (the blown up node corresponds to one of the two components in the inverse image of Q).

In general, in order to obtain a surface of general type, we must have at least 4 lines, and moreover we need, in the case where  $d_j = n \forall j$ , that  $r(n-1) > 3n \Leftrightarrow (r-3)n > r$ .

Hence the smallest such case is for r = 4, n = 5. If the 4 lines are in general position, as we already mentioned, the Hirzebruch-Kummer covering is the Fermat quintic

$$\{u^5 = x^5 + y^5 + z^5\} \subset \mathbb{P}^3,$$

and any intermediate free  $\mathbb{Z}/5$  quotient yields a classical Godeaux surface.

4.3.1. Pardini's surfaces. Even if the surfaces we have been playing with right now may a priori look rather uninteresting, and more fun than serious mathematics, a remarkable example in the class of abelian coverings branched on lines in general position was found by Pardini ([Par91b]); it belongs to the next case, with r = 5 lines, n = 5, and group  $G = (\mathbb{Z}/5)^2$ .

The covering surfaces have, by the above formulae,  $K_S^2 = 25, e(S) = 35$  hence, by Noether's formula  $\chi(S) = \frac{1}{12}(25+35) = 5$ .

Since the fundamental group of the surface is  $(\mathbb{Z}/5)^2$  (the Kummer cover being simply connected, see 63), the conclusion is that q(S) = 0,  $p_g(S) = 4$ , and the canonical map maps to  $\mathbb{P}^3$ .

To make a long story short, recall that, when the canonical map of a surface of general type has degree  $\geq 2$ , then only two alternatives are possible (see [Bea79]):

(1) the canonical image of  $S, \Sigma := \Phi_{K_S}(S)$  has  $p_g(S) = 0$ , or

(2) the canonical image of  $S, \Sigma := \Phi_{K_S}(S)$  is canonically embedded.

Babbage claimed that only the first case should occur, then three essentially equivalent examples of case 2) were found by Beauville, van der Geer-Zagier, and the present author, ([Cat81], [vdGZ77], [Bea79]), where the degree of the canonical map  $\Phi_{K_S}$  equals 2.

It is still an open question which is the highest possible value for  $deg(\Phi_{K_S})$  in case 2), but the world record is 5, obtained by Pardini (and later, independently, by Tan [Tan92]).

To give a smooth covering with group  $G = (\mathbb{Z}/5)^2$ , branched over five lines in general position, it is equivalent to give the 5 monodromy vectors  $g_1, \ldots, g_5 \in (\mathbb{Z}/5)^2$ , with the property that:

i)  $\sum_{j} g_j = 0$ , and

ii) two distinct vectors  $g_i$  and  $g_j$  are linearly independent (this is the condition for the smoothness of S).

Pardini's choice can best be explained in the following abstract way: we take the five vectors to be the five points of an affine line in  $(\mathbb{Z}/5)^2$ not passing through the origin!

Now, conditions i) and ii) are satisfied trivially, and there remains to see the advantage of this clever choice.

This rests on the fact that there is a homomorphism  $\psi : G \to \mathbb{Z}/5$ mapping all the five elements  $g_j$  to  $1 \in \mathbb{Z}/5$ . To the homomorphism  $\psi$ there corresponds an intermediate  $\mathbb{Z}/5$ -covering Z of  $\mathbb{P}^2$ , i.e. we have a factorization  $S \to Z \to \mathbb{P}^2$  of the original covering.

If  $l_j(x_0, x_1, x_2) = 0$  is the equation of the line  $L_j$ , then we see right away that

$$Z = \{x_3^5 = \prod_{j=1}^5 l_j(x_0, x_1, x_2)\} \subset \mathbb{P}^3.$$

Now, Z is a quintic in  $\mathbb{P}^3$ , whose only singularities are the 10 points  $x_3 = l_i(x_0, x_1, x_2) = l_j(x_0, x_1, x_2) = 0$ . These singularities are rational double points of type  $A_4$ , hence Z is a canonical model with  $p_g(Z) = 4$ . Since  $p_g(S) = p_g(Z) = 4$ , the conclusion is:

**Theorem 65. (Pardini)** There exist surfaces S with  $K_S^2 = 25$ ,  $K_S$  ample,  $p_g(S) = 4$ , q(S) = 0, such that the canonical map of S maps with degree 5 onto a canonically embedded surface Z.

One can ask what happens if the five monodromy vectors  $g_1, \ldots, g_5$  are affinely independent, yet satisfy i) and ii). This is interesting, since it shows the usefulness of calculating the character sheaves.

We write a character  $\chi : G \to \mathbb{Z}/5$  as a pair  $(a, b), 0 \le a, b, < 5$ , so that, for  $(x, y) \in (\mathbb{Z}/5)^2$ ,  $\chi(x, y) = ax + by \in \mathbb{Z}/5$ .

The character sheaves are of the form  $\mathcal{O}_{\mathbb{P}^2}(-L_{\chi})$ , where the  $L_{\chi}$  are calculated applying  $\chi$  to the monodromy vectors, taking the remainder for division by 5 ( we denote by [d] the remainder for division of d by 5, hence  $0 \leq [d] < 5$ ), and then summing all these remainders.

Just in order to have a concrete example, take the five vectors

$$(1,0), (0,1), (1,1), (2,-1), (1,-1):$$

then

$$5L_{\chi} = a + b + [a + b] + [2a - b] + [a - b].$$

This is important, because one can write

$$H^0(S, \mathcal{O}_S(K_S) = \bigoplus_{\chi} z_{\chi} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + L_{\chi}),$$

where  $z_j = 0$  is the equation of the ramification divisor  $R_j$  corresponding to the j-th line  $L_j$ , and  $z_{\chi} := \prod_{\chi(j)=0} z_j^{5-1-[\chi(g_j)]}$ .

In our example,  $p_g = 4$  and there are exactly 4 characters such that  $L_{\chi} = 3$ , namely, a = b = 4, a = 4, b = 0, a = 3, b = 4, a = 1, b = 3.

One sees therefore that the base locus of the canonical system  $K_S$  equals the subscheme intersection of the four divisors

$$R_3 + 4R_5, R_1 + 2R_3 + 2R_4, 4R_2 + R_4, 3R_1 + R_2 + R_5,$$

that is, we get the points  $R_2 \cap R_3$  and  $R_4 \cap R_5$  with multiplicity 1. Hence

**Proposition 66.** A  $(\mathbb{Z}/5)^2$ -covering of the plane branched on 5 lines in general position, and with monodromy vectors

$$(1,0), (0,1), (1,1), (2,-1), (1,-1)$$

is smooth with  $K_S^2 = 25$ ,  $p_g(S) = 4$ , q(S) = 0, has a canonical system with 2 simple base points, and its canonical map is birational onto a surface  $\Sigma$  of degree 23 in  $\mathbb{P}^3$ .

We remark that there are other Abelian covers branched on 5 lines in general position, which are **uniform** (i.e. with  $d_j = n \forall j = 1, ..., r$ ) and with exponent n = 5, but we do not pursue their classification here.

4.4. Hirzebruch's and other ball quotients. One obtains an infinite fundamental group whenever we take a **uniform** (i.e. with  $d_j = n \ \forall j = 1, ..., r$ ) Abelian cover branched on a union of lines, such that there exists a point  $p_i$  of valency  $v_i =: w \ge 3$ .

This is because projection with centre the point  $p_i$  leads to a fibration  $S'' \to B$ , where B is an Abelian covering of  $\mathbb{P}^1$  branched on w points, and with group  $(\mathbb{Z}/n)^{w-1}$ .

The genus of B satisfies  $2(b-1) = n^{w-1}(-2 + w(1-\frac{1}{n}))$ , hence b > 1 as soon as (w-2)(n-1) > 2, which we shall now assume, and which holds for  $n \ge 4$ .

Since the maximal covering S'' maps onto a curve with genus  $b \ge 2$ , the fundamental group of S'' is infinite, as well as the one of S, of which S'' is a finite étale covering.

Yet, it may still happen that the first Betti number of S is zero, as we shall see.

Hirzebruch [Hirz83] found explicit examples of ball quotients by taking Kummer coverings branched on a union of lines.

The one on which we shall concentrate mostly is the one obtained for n = 5 taking 6 lines which are the sides of a complete quadrangle (which can be also visualized as the sides of a triangle plus its three medians, which meet in the barycentre).

Other important examples were the Hirzebruch-Kummer coverings (cf. 63)  $HK_{\mathcal{H}}(5)$  and  $HK_{\mathcal{DH}}(3)$  associated to the Hesse configuration of lines  $\mathcal{H}$ , a configuration of type  $(9_4, 12_3)$  formed by the 9 flexes of a plane cubic curve, and the twelve lines joining pairs of flexpoints, respectively to the dual Hesse configuration  $\mathcal{DH}$  of type  $(12_3, 9_4)$  of the nine lines dual to the flexpoints.

Recall that every smooth plane cubic curve is isomorphic to one in the Hesse pencil:

$$C_{\lambda} := \{x^3 + y^3 + z^3 + 6\lambda xyz = 0\}.$$

 $C_{\lambda}$  is smooth, except for  $\lambda = \infty$ , or  $8\lambda^3 = -1$ ; its flexes are the intersection of  $C_{\lambda}$  with its Hessian cubic curve, which is precisely

$$H_{\lambda} := \{\lambda^2(x^3 + y^3 + z^3) - (1 + 2\lambda^3)xyz = 0\} = C_{\mu}, \ \mu = -\frac{1 + 2\lambda^3}{6\lambda^2}.$$

Hence the nine points are the base points of the pencil,

$$\{xyz = x^3 + y^3 + z^3 = 0\},\$$

while  $C_{\lambda} = H_{\lambda}$  exactly when  $8\lambda^3 = -1$ : hence the 4 singular curves of the pencil are 4 triangles (every point is a flexpoint!), and these 4 triangles produce the 12 lines.

More examples of ball quotients can be found in [Hirz85] and [BHH87] (they are also related to hypergeometric integrals, see [D-M86]).

4.5. Symmetries of the del Pezzo surface of degree 5. The blowup Y of 4 points  $P_1, \ldots, P_4 \in \mathbb{P}^2$  in general position is the del Pezzo surface of degree 5.

This surface is the moduli space of ordered quintuples of points in  $\mathbb{P}^1$ , as we shall now see. The six lines can be labelled  $L_{i,j}$ , with  $i \neq j \in \{1, 2, 3, 4\}$  ( $L_{i,j}$  is the line  $\overline{P_i P_j}$ ).

The following is well known

**Theorem 67.** The automorphism group of the Del Pezzo surface Y of degree 5 is isomorphic to  $\mathfrak{S}_5$ .

*Proof.* There is an obvious action of the symmetric group  $\mathfrak{S}_4$  permuting the 4 points, but indeed there is more (hidden) symmetry, by the symmetric group  $\mathfrak{S}_5$ . This can be seen denoting by  $E_{i,5}$  the exceptional curve lying over the point  $p_i$ , and denoting, for  $i \neq j \in \{1, 2, 3, 4\}$ , by  $E_{i,j}$  the line  $L_{h,k}$ , if  $\{1, 2, 3, 4\} = \{i, j, h, k\}$ .

For each choice of 3 of the four points,  $\{1, 2, 3, 4\} \setminus \{h\}$ , consider the standard Cremona transformation  $\sigma_h$  based on these three points. To  $\sigma_h$  we associate the transposition  $(h, 5) \in \mathfrak{S}_5$ , and the upshot is that  $\sigma_h$  transforms the 10 (-1) curves  $E_{i,j}$  via the action of (h, 5) on pairs of elements in  $\{1, 2, 3, 4, 5\}$ .

Indeed there are five geometric objects permuted by  $\mathfrak{S}_5$ : namely, 5 fibrations  $\varphi_i : Y \to \mathbb{P}^1$ , induced, for  $1 \leq i \leq 4$ , by the projection with centre  $P_i$ , and, for i = 5, by the pencil of conics through the 4 points. Each fibration is a conic bundle, with exactly three singular fibres, corresponding to the possible partitions of type (2, 2) of the set  $\{1, 2, 3, 4, 5\} \setminus \{i\}$ .

To conclude that  $\mathfrak{S}_5 = \operatorname{Aut}(Y)$ , we observe that Y contains exactly ten lines, i.e. irreducible curves E with  $E^2 = EK_Y = -1$ . We have the following easy result:

**Lemma 68.** The curves  $E_{i,j}$ , which generate the Picard group, have an intersection behaviour which is dictated by the simple rule (recall that  $E_{i,j}^2 = -1, \forall i \neq j$ )

 $E_{i,j} \cdot E_{h,k} = 1 \iff \{i, j\} \cap \{h, k\} = \emptyset, \quad E_{i,j} \cdot E_{h,k} = 0 \iff \{i, j\} \cap \{h, k\} \neq \emptyset.$ In this picture the three singular fibres of  $\varphi_1$  are

 $E_{3,4} + E_{2,5}, E_{2,4} + E_{3,5}, E_{2,3} + E_{4,5}.$ 

The relations among the  $E_{i,j}$ 's in the Picard group come from the linear equivalences  $E_{3,4} + E_{2,5} \equiv E_{2,4} + E_{3,5} \equiv E_{2,3} + E_{4,5}$  and their  $\mathfrak{S}_5$ -orbits.

Therefore each automorphism  $\psi$  of Y permutes the 10 lines, preserving the incidence relation. Up to multiplying  $\psi$  with an element of the subgroup  $\mathfrak{S}_5$ , we may assume that  $\psi$  fixes  $E_{1,2}$ , hence that  $\psi$  permutes the three curves  $E_{3,4}, E_{3,5}, E_{4,5}$ . By the same trick we may assume that  $\psi$  fixes  $E_{1,2}, E_{3,4}, E_{3,5}, E_{4,5}$ . Hence  $\psi(E_{1,j}), 3 \leq j \leq 5$  is either  $E_{1,j}$  or  $E_{2,j}$ . Multiplying  $\psi$  by the transposition (1, 2) we may assume that also  $\psi(E_{1,3}) = E_{1,3}$ . The incidence relation now says that  $\psi$  fixes all the ten lines, and there remains to show that  $\psi$  is the identity. But blowing

down the four curve  $E_{i,5}$  we see that  $\psi$  induces an automorphism of  $\mathbb{P}^2$  fixing each of the points  $P_1, P_2, P_3, P_4$ . Hence  $\psi$  is the identity and we are done.

**Remark 69.** We have the following correspondences:

- The lines  $E_{i,j}$  correspond to the transpositions in  $\mathfrak{S}_5$ .
- The 15 intersection points  $E_{i,j} \cdot E_{h,k}$ , for  $|\{i.j.h.k\}| = 4$  correspond to isomorphisms of  $(\mathbb{Z}/2)^2$  with a subgroup of  $\mathfrak{S}_5$ .
- The five 2-Sylow subgroups correspond to the five conic bundles  $\varphi_i, i = 1, ..., 5$ , and the triples of singular fibres correspond to the three different embeddings  $(\mathbb{Z}/2)^2 \to \mathfrak{S}_5$  with the same image.
- The six 5-Sylow subgroups correspond combinatorially to pairs of opposite **pentagons**. Here a pentagon is the equivalence class of a bijection  $\mathcal{P} : \mathbb{Z}/5 \to \{1, 2, 3, 4, 5\}$  for the action of the dihedral group  $D_5$  on the source ( $n \in \mathbb{Z}/5 \mapsto \pm n + b, b \in \mathbb{Z}/5$ ). Whereas a pair of opposite pentagons is the equivalence class for the action of the affine group  $A(1, \mathbb{Z}/5)$  on the source.
- To a combinatorial pentagon corresponds a **geometric pentagon**, i.e. a union of lines  $E_{\mathcal{P}(i),\mathcal{P}(i+1)}, i \in \mathbb{Z}/5$  each meeting the following line  $E_{\mathcal{P}(i+2),\mathcal{P}(i+3)}$ . The divisor of a geometric pentagon is an anticanonical divisor  $D_{\mathcal{P}}$ , and the sum of two opposite geometric pentagons  $D_{\mathcal{P}} + D_{\mathcal{P}^o}$  is just the sum of the 12 lines  $E_{i,j}$ .
- If  $D_{\mathcal{P}} = div(s_{\mathcal{P}}), s_{\mathcal{P}} \in H^0(\mathcal{O}_Y(-K_Y))$ , we obtain five independent quadratic equations for the anticanonical embedding

$$Y \to \mathbb{P}(H^0(\mathcal{O}_Y(-K_Y))^{\vee}),$$

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from the six equations  $s_{\mathcal{P}}s_{\mathcal{P}^o} = \delta$ , where  $div(\delta) = \sum E_{i,j}$ .

• The above symmetry is the projective icosahedral symmetry, i.e., the symmetry of the image of the icosahedron in  $\mathbb{P}^2(\mathbb{R})$ : the 10 lines correspond to pairs of opposite faces, the 15 points to pairs of opposite edges, the 6 pairs of opposite pentagons correspond to pairs of opposite vertices.

A basis for  $H^0(\mathcal{O}_Y(-K_Y))$  is given by the six sections corresponding to the pentagons where 4, 5 are never neighbours.

Written as sections of  $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$  vanishing at the points  $P_1, \ldots, P_4$  which we assume to be the coordinate points and the point (1, 1, 1), they are:

$$s_{i,j} = x_i x_j (x_j - x_k), \ \{i, j, k\} = \{1, 2, 3\}.$$

All this leads to beautiful Pfaffian equations for  $Y \subset \mathbb{P}^5$ , which are  $\mathfrak{S}_5$  equivariant. Indeed  $H^0(\mathcal{O}_Y(-K_Y))$  is the unique irreducible representation  $\chi_6$  of  $\mathfrak{S}_5$  of dimension 6, whereas the representation

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 $\rho$  on the set of pairs of opposite pentagons splits as the direct sum of the trivial representation with an irreducible representation  $\chi_5$  of dimension 5, the representation  $\tilde{\rho}$  on the pentagons is the direct sum of  $\rho$  with the tensor product of  $\rho$  with the signature character  $\chi_1$ . The permutation representation of  $\mathfrak{S}_5$  on  $\{1, 2, 3, 4, 5\}$  splits as the trivial representation direct sum with an irreducible representation  $\chi_4$ of dimension 4.  $\chi_6$  is the only irreducible representation such that  $\chi_6 \cong \chi_6 \otimes \chi_1$ .

In this way one obtains all the irreducible representations of  $\mathfrak{S}_5$ , see [J-L], page 201. We shall not go further here with the equations of Y, for the anticanonical embedding  $Y \subset \mathbb{P}^5$ , and for the embedding  $Y \subset (\mathbb{P}^1)^5$  via  $\varphi_1 \times \ldots \varphi_5$ .

But we shall now prove the classical:

**Theorem 70.** The Del Pezzo surface Y of degree 5 is the moduli space for ordered 5-uples of points in  $\mathbb{P}^1$ , i.e., the GIT quotient of  $(\mathbb{P}^1)^5$  by  $\mathbb{P}GL(2,\mathbb{C})$ .

# Proof.

Another model for Y is the blow up of  $\mathbb{P}^1 \times \mathbb{P}^1$  in the three diagonal points  $(\infty, \infty)$ , (0, 0), (1, 1), and the ten lines come from the three blown up points, plus the strict transforms of the diagonal and of the vertical and horizontal lines  $x = 0, x = 1, x = \infty, y = 0, y = 1, y = \infty$ .

Removing these seven lines in  $\mathbb{P}^1 \times \mathbb{P}^1$  we obtain a point (u, v) such that the five points  $\infty, 0, 1, u, v$  are all distinct.

If we approach a smooth point in the diagonal line, say (u, u) we obtain the 5-uple  $\infty$ , 0, 1, u, u where the fourth and the fifth points are equal  $P_4 = P_5$ , and the other three are different (so that set theoretically we have four distinct points). By  $\mathfrak{S}_5$ -symmetry, the same occurs whenever we get a smooth point of the divisor  $\sum E_{i,j} \subset Y$ :  $P_i = P_j$  and the other three are different from  $P_i$  and pairwise different.

If we tend to the point (0, 1), we get the 5-uple  $\infty, 0, 1, 0, 1$ , where  $P_2 = P_4, P_3 = P_5$ ; again by symmetry, to the point  $E_{i,j} \cap E_{h,k}$  corresponds a quintuple with  $P_i = P_j, P_h = P_k$ , and where set theoretically we have three distinct points.

Now, as shown in Mumford's book [Mum65], especially proposition 3.4, page 73, in this case the semistable 5-uples are stable, and a quintuple is unstable if and only if three points are equal. One can easily conclude that we have an isomorphism of Y with the GIT-quotient  $(\mathbb{P}^1)^5 // \mathbb{P}GL(2,\mathbb{C}).$ 

4.5.1. Hirzebruch-Kummer covers of a del Pezzo surface of degree 5. Let us now take as branch locus  $D := \sum_{i,j} E_{i,j}$ , and let us notice that D is linearly equivalent to twice the anticanonical divisor  $-K_Y$ .

Therefore, for any n- uniform covering S of Y branched on the 10 lines, and which is smooth,

$$K_S = p^* (K_Y - 2(1 - \frac{1}{n})K_Y) = p^* (-(\frac{n-2}{n})K_Y).$$

In particular, for the Kummer covering, one has

$$K_{S''}^2 = 5(n-2)^2 n^3.$$

Whereas the Euler number can be calculated (in [CatDet15] it is calculated in an elegant way) as:

$$e(S'') = n^3(3 + 2(n-2)(n-3)) = n^3(2n^2 - 10n + 15).$$

The Chern slope equals then

$$\nu_C(S'') = \frac{5(n-2)^2}{3+2(n-2)(n-3)} = \frac{5}{2} \frac{n-2}{n-3+\frac{3}{2(n-2)}} > \frac{5}{2}.$$

The same formula shows that the slope is a decreasing function of n for  $n \geq 5$ , and for n = 5 we obtain  $\nu_C(S'') = \frac{45}{15} = 3$ . Hence the theorem of Hirzebruch [Hirz83]:

**Theorem 71.** A smooth n-uniform Abelian cover S with  $n \ge 5$  branched on the 10 lines of a del Pezzo surface of degree 5 is a surface with ample canonical divisor, with positive index, and indeed a ball quotient if and only if n = 5.

We observe, as a side remark, that the Kummer coverings above embed into  $C_n^5$ , where  $C_n$  is the Fermat curve of degree n,

$$C_n = \{x^n + y^n + z^n = 0\} \subset \mathbb{P}^2,$$

via the Cartesion product of the Stein factorizations of the maps induced by the  $\varphi_i$ ,  $i = 1, \ldots, 5$ .

We shall next discuss some of the surfaces S mentioned in the previous theorem, addressing the question of their irregularity q(S).

4.5.2. A problem posed by Enriques, and its partial solution. Enriques posed in his book [Enr49] the following problems:

Question 72. Given a surface S with  $p_g(S) = 4$  and with birational canonical map onto its image  $\Sigma \subset \mathbb{P}^3$ ,

I) what is the maximum value for its canonical degree  $K_S^2$ ?

II) what is the maximum value for  $deg(\Sigma)$ ?

He indeed suspected that there should be an upper bound equal to 24, for which counterexamples were given in [Cat99].

Now, without loss of generality, we may assume that S is minimal, because otherwise  $K_S^2$  decreases, and we observe that, since  $K_S$  is nef, we have

$$45 \ge K_S^2 \ge deg(\Phi_{K_S}) \cdot deg(\Sigma),$$

where the first inequality is a consequence of the Miyaoka-Yau inequality  $K_S^2 \leq 9(1 - q(S) + p_g(S)) = 9(5 - q(S)).$ 

Hence, in order to achieve the equality  $K_S^2 = 45$ , when  $p_g(S) = 4$ , we must have a ball quotient which has q(S) = 0.

With I. Bauer [BC08] we showed that there exists such a surface.

**Theorem 73.** There exists an Abelian cover of the del Pezzo surface Y of degree 5, with group  $(\mathbb{Z}/5)^2$ , and branched on the 10 lines, which is regular, i.e. q(S) = 0, has  $p_g(S) = 4$ ,  $K_S^2 = 45$ , and has a birational canonical map onto a surface  $\Sigma$  of degree 19.

Indeed in the course of the search we classified all such coverings of the del Pezzo surface Y of degree 5, with group  $(\mathbb{Z}/5)^2$ , and branched on the 10 lines.

We considered the group  $\mathcal{G}$ , generated by  $\mathfrak{S}_5$  and  $Gl(2, \mathbb{Z}/5\mathbb{Z})$ , acting on the set of admissible monodromy vectors.

A MAGMA computation showed that  $\mathcal{G}$  has four orbits, and representatives for these orbits could be taken as:

$$\begin{aligned} \mathfrak{U}_{1} &= \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 2\\1 \end{pmatrix}, \begin{pmatrix} 2\\1 \end{pmatrix}, \begin{pmatrix} 4\\2 \end{pmatrix}); \\ \mathfrak{U}_{2} &= \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 2\\1 \end{pmatrix}, \begin{pmatrix} 4\\2 \end{pmatrix}, \begin{pmatrix} 2\\1 \end{pmatrix}); \\ \mathfrak{U}_{3} &= \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 4\\1 \end{pmatrix}, \begin{pmatrix} 3\\2 \end{pmatrix}, \begin{pmatrix} 1\\1 \end{pmatrix}); \\ \mathfrak{U}_{4} &= \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1 \end{pmatrix}, \begin{pmatrix} 0\\3 \end{pmatrix}, \begin{pmatrix} 2\\0 \end{pmatrix}). \end{aligned}$$

In particular we saw that  $\mathcal{G} \cong Gl(2, \mathbb{Z}/5\mathbb{Z}) \times \mathfrak{S}_5$ , and concluded the classification with the following result:

**Theorem 74.** Let  $S_i$  be the minimal smooth surface of general type with  $K^2 = 45$  and  $\chi = 5$  obtained from the covering induced by the admissible six - tuple  $\mathfrak{U}_i$ , where  $i \in \{1, 2, 3, 4\}$ . Then we have that  $S_3$ is regular (i.e.,  $q(S_3) = 0$ ), whereas  $q(S_i) = 2$  for  $i \neq 3$ .

The story concerning Enriques' question is not yet completely finished, because it is not clear whether one can achieve  $deg(\Sigma) = 45$  (the current published record <sup>6</sup> in this direction is  $deg(\Sigma) = 28$ , [Cat99]).

<sup>&</sup>lt;sup>6</sup>Recently we achieved  $deg(\Sigma) = 32$ .

4.6. Bogomolov-Miyaoka-Yau fails in positive characteristic. Even if the BMY inequality was proven by Miyaoka with purely algebraic methods, still the proof uses characteristic 0 arguments in an essential way.

Robert Easton [Easton08] gave easy examples showing that indeed the BMY inequality does not hold in positive characteristic. These examples are related to the Hirzebruch-Kummer coverings of the plane, and the main idea is that in characteristic p > 0 there are configurations of lines which cannot exist in characteristic zero.

These configurations are just the projective planes  $\mathbb{P}^2_{\mathbb{Z}/p} \subset \mathbb{P}^2_K$ , for each algebraically closed field K of characteristic p > 0.

The easiest case is the so called **Fano plane**  $\mathbb{P}^2_{\mathbb{Z}/2} \subset \mathbb{P}^2_K =: \mathbb{P}^2$ . We have a configuration  $\mathcal{C}$  of type  $7_37_3$ , seven lines passing through seven points, each triple for the configuration.

For each odd number  $n \geq 5$  we consider the Hirzebruch-Kummer covering of exponent  $n, S_n$ , a finite Galois cover of the blow up Z of  $\mathbb{P}^2$  in the seven points, with Galois group  $(\mathbb{Z}/n)^6$ .

The canonical divisor of Z is  $K_Z = -3H + \sum_{i=1}^{7} E_i$ , where  $E_i$  is the inverse image of the point  $P_i$ . Denoting by  $L_i$  the proper transform of the line  $L_i$ , since  $\sum_{i=1}^{7} L_i = 7H - 3\sum_{i=1}^{7} E_i$ , the canonical divisor of  $S_n$  is the pull back of

$$K_Z + (1 - \frac{1}{n}) \left(\sum_{1}^{7} L_i + \sum_{1}^{7} E_i\right) = \left(-3 + 7\left(1 - \frac{1}{n}\right)\right) H + \left(1 - 2\left(1 - \frac{1}{n}\right)\right) \sum_{1}^{7} E_i$$

Hence

$$K_{S_n}^2 = n^6 \left[ \left(4 - \frac{7}{n}\right)^2 - 7\left(\left(1 - \frac{2}{n}\right)^2\right) \right] = n^6 \left[9 - \frac{28}{n} + \frac{21}{n^2}\right].$$

The formulae we illustrated earlier yield

$$c_2(S_n) = n^6 [10 + 14(1 - \frac{1}{n}) - 21(1 - \frac{1}{n^2})] = n^6 [3 - 14\frac{1}{n} + 21\frac{1}{n^2}].$$

Hence

$$K_{S_n}^2 = 3c_2(S_n) + n^6(14\frac{1}{n} - 42\frac{1}{n^2}) = 3c_2(S_n) + n^4(14n - 42).$$

Hence a particular case of Easton 's theorem

**Theorem 75.** In characteristic equal to 2 the Hirzebruch-Kummer coverings of the plane branched on the Fano of configuration of lines have Chern slope

$$\nu(S_n) = \frac{K_{S_n}^2}{c_2(S_n)} = 3\left(1 + \frac{14n - 42}{9n^2 - 42n + 63}\right) > 3,$$

violating the Bogomolov Miyaoka Yau inequality. The maximum of the slope is attained for n = 5,  $\nu(S_5) = 4 + \frac{3}{39}$ .

## KODAIRA AND MODULI THEORY

# 5. Counterexamples to Fujita's semiampleness question, rigid manifolds

5.1. BCDH surfaces, counterexamples to Fujita's question. Recently, we found new counterexamples to Fujita's question 49(we say counterexamples since we heard through the grapevine that experts were expecting a positive answer; however this counterexample does not inficiate the abundance conjecture).

**Theorem 76.** There exists an infinite series of surfaces with ample canonical bundle, whose Albanese map is a fibration  $f: S \to B$  onto a curve B of genus  $b = \frac{1}{2}(n-1)$ , and with fibres of genus g = 2b = n-1, where n is any integer relatively prime with 6.

These Albanese fibrations yield negative answers to Fujita's question about the semiampleness of  $V := f_*\omega_{S|B}$ , since here  $V := f_*\omega_{S|B}$  splits as a direct sum  $V = A \oplus Q$ , where A is an ample vector bundle, and Q is a unitary flat bundle with infinite monodromy group.

The fibration f is semistable: indeed all the fibres are smooth, with the exception of three fibres which are the union of two smooth curves of genus b which meet transversally in one point.

For n = 5 we get three surfaces which are rigid, and are quotient of the unit ball in  $\mathbb{C}^2$  by a torsion free cocompact lattice  $\Gamma$ . We shall call them BCD-surfaces (cf. theorem 74). The rank of A, respectively Q, is in this case equal to 2.

The easiest way to describe these surfaces, which are Abelian covers of the del Pezzo surface Y of degree 5 with group  $(\mathbb{Z}/n)^2$ , branched over the 10 lines of Y, is to look at a birational model which is an Abelian covering of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over the diagonal and of the vertical and horizontal lines  $x = 0, x = 1, x = \infty, y = 0, y = 1, y = \infty$ .

We consider again the equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - xy_0)^{m_3}, \ x \in \mathbb{C} \setminus \{0, 1\}$$

but we homogenize it to obtain the equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (x_0 y_1 - x_1 y_0)^{m_3} x_0^{n - m_3}.$$

The above equation describes a singular surface  $\Sigma'$  which is a cyclic covering of  $\mathbb{P}^1 \times \mathbb{P}^1$  with group  $G := \mathbb{Z}/n$ ;  $\Sigma'$  is contained inside the line bundle  $\mathbb{L}_1$  over  $\mathbb{P}^1 \times \mathbb{P}^1$  whose sheaf of holomorphic sections  $\mathcal{L}_1$  equals  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ .

The first projection  $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$  induces a morphism  $p: \Sigma' \to \mathbb{P}^1$ and we consider the curve B, normalization of the covering of  $\mathbb{P}^1$  given by

$$w_1^n = x_0^{n_0} x_1^{n_1} (x_1 - x_0)^{n_2}.$$

We consider the normalization  $\Sigma$  of the fibre product  $\Sigma' \times_{\mathbb{P}^1} B$ .

 $\Sigma$  is an abelian covering of  $\mathbb{P}^1 \times \mathbb{P}^1$  with group  $(\mathbb{Z}/n)^2$ , and, if GCD(n, 6) = 1, for a convenient choice of the integers  $m_j, n_i$ , for instance for

$$m_0 = m_1 = m_2 = 1, \ m_3 = n - 3, n_0 = n_1 = 1, n_2 = n - 2$$

we obtain a smooth  $(\mathbb{Z}/n)^2$ - Abelian covering of the blow up of  $\mathbb{P}^1 \times \mathbb{P}^1$ in the three diagonal points  $(\infty, \infty), (0, 0), (1, 1)$ , which is the del Pezzo surface Y of degree 5.

The corresponding singular fibres are only 3, and come from one of the 5 conic bundle structures on Y, here given by the first projection  $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ : hence one sees right away that the singular fibres are reducible, and that they are the union of two smooth curves of genus b intersecting transversally in one point.

The surfaces in theorem 76, which we shall call BCDH-surfaces, have an étale unramified covering given by the Hirzebruch Kummer coverings of the del Pezzo surface Y of degree 5 with group  $(\mathbb{Z}/n)^2$ , branched over the 10 lines of Y, which we shall denote HK-surfaces.

In the case n = 5 we get ball quotients, and, for 5|n, we get therefore that the universal covering is a branched covering of the ball. This motivates the following questions, which are clearly satisfied for n = 5:

# Question 77. (1) Are the BCDH-surfaces rigid?

- (2) Is their universal covering  $\tilde{S}$ , or more generally the universal covering of the HK-surfaces, a Stein manifold?
- (3) Is the universal covering  $\tilde{S}$  of the BCDH- and HK-surfaces contractible?
- (4) Do the BCDH-surfaces admit a metric of negative curvature?

The last question is motivated by analogy with the examples considered by Mostow-Siu: namely, one has a Kähler-Einstein metric on Y deprived of the 10 lines, and on the covering one should interpolate with another metric localized on the ramification divisor.

Of course, an interesting question is whether the Kähler-Einstein metric  $\omega_n$  on  $S_n$  is negatively curved. To analyse the question one can observe that  $\omega_n$  is the pull-back of a metric on Y with given cone angles along the branch divisors.

Now, a positive answer to (4) would imply (3); however, for (3), the vanishing cycles criterion which was given earlier may be applied.

Question (2) should have a positive answer, while for the answer to question (1), it is positive, as proven in joint work with I. Bauer [B-C16]:

**Theorem 78.** The HK-surfaces are rigid for  $n \ge 4$ , hence also the BCDH-surfaces are rigid.

Observe that, since the BCDH-surfaces have the HK surfaces as étale covers, the non rigidity of the former would imply non rigidity of the

latter (see 85). Moreover, for n = 3 rigidity does not hold, this surface was studied by Roulleau in [Rou11].

The proof for HK surfaces uses many tools, first of all the special geometry of the del Pezzo surface of degree 5, and rather complicated arguments using logarithmic sheaves in order to control the deformations of the Abelian coverings.

**Corollary 79.** The Albanese fibrations of BCDH-surfaces yield rigid curves in the moduli spaces  $\overline{\mathfrak{M}_{n-1}}$ .

We noticed that all the fibres of the Albanese map have compact Jacobian, hence the following:

**Question 80.** Do the Albanese fibrations of BCDH-surfaces yield rigid curves in the moduli spaces  $\mathfrak{A}_{n-1}$ ?

Theorem 78 raises several questions: here, given a configuration of lines C, we denote by  $HK_{\mathcal{C}}(n)$  the Hirzebruch-Kummer covering of exponent n ramified on the lines of the configuration C.

Natural questions are (see [B-C16]:

**Question 81. I)** For which rigid configuration C of lines in  $\mathbb{P}^2$  is the associated Hirzebruch Kummer covering  $HK_{\mathcal{C}}(n)$  rigid for n >> 0?

**II)** For which rigid configuration C of lines in  $\mathbb{P}^2$  is the associated Hirzebruch Kummer covering  $HK_{\mathcal{C}}(n)$  a  $K(\pi, 1)$  for n >> 0?

**III)** For which rigid configuration C of lines in  $\mathbb{P}^2$  does the associated Hirzebruch Kummer covering  $HK_C(n)$  possess a Kähler metric of negative sectional curvature for  $n \gg 0$ ?

Observe that if III) has a positive answer, then also II), by the Cartan-Hadamard theorem.

Abelian coverings branched over configurations were also used by Vakil in [Va06], who showed that for n >> 0 the local deformations of  $HK_{\mathcal{C}}(n)$  correspond to the product of the deformations of the configuration  $\mathcal{C}$  with a smooth manifold. Vakil used a result of Mnev [Mnev88] in order to show that, up to a product with a smooth manifold, one obtains all possible singularity types.

Let me end this subsection by commenting that not only BCD- and BCDH-surfaces are quite interesting from many algebro-geometric and complex analytic points of view, but that the features of their Albanese fibration have also found very interesting applications for the construction of remarkable symplectic manifolds, in the work of Akhmedov and coworkers (see [AkSa15] and literature cited therein).

5.2. Rigid compact complex manifolds. Recall the following notions of rigidity (see [B-C16] for more details)

Definition 82.

(1) Two compact complex manifolds X and X' are said to be *de-formation equivalent* if and only if there is a proper smooth holomorphic map

 $f:\mathfrak{X}\to\mathcal{B}$ 

where  $\mathcal{B}$  is a connected (possibly not reduced) complex space and there are points  $b_0, b'_0 \in \mathcal{B}$  such that the fibres  $X_{b_0} := f^{-1}(b_0), X_{b'_0} := f^{-1}(b'_0)$  are respectively isomorphic to X, X' $(X_{b_0} \cong X, X_{b'_0} \cong X').$ 

- (2) A compact complex manifold X is said to be globally rigid if for any compact complex manifold X', which is deformation equivalent to X, we have an isomorphism  $X \cong X'$ .
- (3) A compact complex manifold X is instead said to be *(locally)* rigid (or just rigid) if for each deformation of X,

$$f: (\mathfrak{X}, X) \to (\mathcal{B}, b_0)$$

there is an open neighbourhood  $U \subset \mathcal{B}$  of  $b_0$  such that  $X_t := f^{-1}(t) \cong X$  for all  $t \in U$ .

(4) A compact complex manifold X is said to be *infinitesimally* rigid if

$$H^1(X, \Theta_X) = 0,$$

where  $\Theta_X$  is the sheaf of holomorphic vector fields on X.

- (5) X is said to be strongly rigid if the set of compact complex manifolds Y which are homotopically equivalent to X,  $\{Y|Y \sim_{h.e.} X\}$  consists of a finite set of isomorphism classes of globally rigid varieties.
- (6) X is said to be *étale rigid* if every étale (finite unramified) cover Y of X is rigid.

**Remark 83.** 1) If X is infinitesimally rigid, then X is also locally rigid. This follows by the Kodaira-Spencer-Kuranishi theory, since  $H^1(X, \Theta_X)$  is the Zariski tangent space of the germ of analytic space which is the base Def(X) of the Kuranishi semiuniversal deformation of X. If  $H^1(X, \Theta_X) = 0$ , Def(X) is a reduced point and all deformations are locally trivial.

2) Obviously strong rigidity implies global rigidity; both global rigidity and étale rigidity imply local rigidity.

6) The simplest example illustrating the difference between global and infinitesimal rigidity is the Del Pezzo surface  $Z_6$  of degree 6, blow up of the plane  $\mathbb{P}^2$  in three non collinear points. It is infinitesimally rigid, but it deforms to the weak del Pezzo surface of degree 6, the blow up  $Z'_6$  of the plane  $\mathbb{P}^2$  in three collinear points.  $Z'_6$  is not isomorphic to  $Z_6$ because for the second surface the anticanonical divisor is not ample.

The following useful general result is established in [B-C16] using many earlier results (and the Riemann Roch theorem for the second statement):

**Theorem 84.** A compact complex manifold X is rigid if and only if the Kuranishi space Def(X) (base of the Kuranishi family of deformations) is 0-dimensional.

In particular, if X = S is a smooth compact complex surface and

$$10\chi(\mathcal{O}_S) - 2K_S^2 + h^0(X,\Theta_S) > 0,$$

then S is not rigid.

We have moreover (ibidem)

**Proposition 85.** If  $p: Z \to X$  is étale, i.e. a finite unramified holomorphic map between compact complex manifolds, then the infinitesimal rigidity of Z implies the infinitesimal rigidity of X. Moreover, if Z is rigid, then also X is rigid.

Idea of proof. For the first assertion, one observes that  $H^1(Z, \Theta_Z) = H^1(X, p_*(\Theta_Z)) = 0$ , and that  $p_*(\Theta_Z) = p_*(p^*\Theta_X) = \Theta_X \otimes (p_*\mathcal{O}_Z)$  has  $\Theta_X$  as a direct summand.

For the second assertion one reduces to the Galois case X = Z/G, where, as shown in [Cat88]:

$$\operatorname{Def}(X) = \operatorname{Def}(Z)^G \subset \operatorname{Def}(Z).$$

Hence, if Def(Z) has dimension 0, a fortiori also Def(X).

While the only rigid curve is  $\mathbb{P}^1$ , in the case of surfaces the list is only known for surfaces which are not of general type. Indeed, using surface classification, it is shown in [B-C16]:

**Theorem 86.** Let S be a smooth compact complex surface, which is (locally) rigid. Then either

- (1) S is a minimal surface of general type, or
- (2) S is a Del Pezzo surface of degree  $d \ge 5$  (i.e.,  $\mathbb{P}^2$ ,  $\mathbb{P}^1 \times \mathbb{P}^1$ , S<sub>8</sub>, S<sub>7</sub>, S<sub>6</sub>, S<sub>5</sub>, where S<sub>9-r</sub> is the blow-up of  $\mathbb{P}^2$  in r points which are in general linear position.)
- (3) S is an Inoue surface of type  $S_M$  or  $S_{N,p,q,r}^{(-)}$  (cf. [In94]).
- (4) Rigid surfaces in class (1) are also globally rigid, surfaces in class (3) are infinitesimally and globally rigid, surfaces in class (2) are infinitesimally rigid, but the only rigid surface in class (2) is the projective plane P<sup>2</sup>.

In particular, rigid surfaces have Kodaira dimension either 2 (general type), or  $-\infty$ . In higher dimension  $n \ge 3$  it shown in [B-C16] that there are rigid compact complex manifolds for each possible Kodaira dimension, except possibly Kodaira dimension = 1. Probably this exception does not really occur, at least for large n.

**Remark 87.** For surfaces of general type it is expected to find examples which are rigid, but not infinitesimally rigid: such an example would

be the one of a minimal surface S such that its canonical model X(S) is infinitesimally rigid and singular (see [B-W74]).

The intriguing part of the story is that all the old examples of globally rigid surfaces, except  $\mathbb{P}^2$ , are projective classifying spaces. Indeed, before the examples which we denote here by HK(n)-surfaces, all known examples of rigid surfaces of general type were the following:

- (1) the *ball quotients*, which are infinitesimally rigid, strongly rigid and étale rigid ([Siu80], [Most73]).
- (2) irreducible bi-disk quotients, i.e. those surfaces whose universal covering of S is  $\mathbb{B}_1 \times \mathbb{B}_1 \cong \mathbb{H} \times \mathbb{H}$ , where  $\mathbb{H}$  is the upper half plane, and moreover if we write  $S = \mathbb{H} \times \mathbb{H}/\Gamma$  the fundamental group  $\Gamma$  has dense image for any of the two projections  $\Gamma \to PSL(2,\mathbb{R})$ ; they are infinitesimally rigid, strongly rigid and étale rigid ([J-Y85], [Mok88]).
- (3) *Beauville surfaces*; they are infinitesimally rigid, strongly rigid but not étale rigid ([Cat00]).
- (4) *Mostow-Siu surfaces*, [M-S80]; these are infinitesimally rigid, strongly rigid and étale rigid.
- (5) the rigid Kodaira fibrations constructed by the author and Rollenske, [CatRol09], and mentioned in theorem 15; these are rigid and strongly rigid, infinitesimal rigidity and étale rigidity is not proven in [CatRol09] but could be true.

**Remark 88.** Examples (1)-(3), and (5) are strongly rigid.

A natural question is therefore:

**Question 89.** Do there exist infinitesimally rigid surfaces of general type which are not projective classifying spaces ?

We believe that the answer should be yes, not only because wishful thinking in the case of surfaces almost invariably turns out to be contradicted, but for the following reason, which relates to the later section on Inoue -type varieties.

**Remark 90.** (i) Assume that Z is a projective classifying space of dimension  $n \ge 3$ , and let Y be a smooth hyperplane section of Z: then Y is not a projective classifying space.

This is a consequence of the Lefschetz hyperplane theorem:  $\pi_1(Y) \cong \pi_1(Z)$ .

If Y were a classifying space for  $\pi_1(Y) \cong \pi_1(Z)$ , then

 $H^*(Y,\mathbb{Z}) \cong H^*(\pi_1(Y),\mathbb{Z}) \cong H^*(\pi_1(Z),\mathbb{Z}) \cong H^*(Z,\mathbb{Z}),$ 

in particular  $H^{2n}(Y,\mathbb{Z}) = \mathbb{Z}$ , against the fact that the real dimension of Y is 2n - 2, which implies that  $H^{2n}(Y,\mathbb{Z}) = 0$ .

(ii) The same occurs if Y is an iterated hyperplane section of Z, with  $dim(Y) \ge 2$ .

(iii) In particular, even if Z admits a metric of negative curvature, Y cannot have one such negative metric, by the Cartan-Hadamard theorem. In fact, concerning the metric inherited from Z, observe that only the Hermitian curvature decreases in subbundles.

Concerning question 89, the case of BCDH-surfaces, and HK(n)-surfaces is not completely settled.

For the surfaces HK(n) the answer is positive, in case that 5 divides n, due to the work of Fangyang Zheng [Zheng99] who extended the Mostow-Siu technique to the case of normal crossings; from this also strong and étale rigidity follow in this case.

For other values of  $n \ge 4$  the work of Panov [Pan09] gives a positive answer for n >> 0 (but unspecified): his method consists in finding polyhedral metrics of negative curvature.

# 6. Surfaces isogenous to a product and their use

Even if the topic of surfaces isogenous to a product does appear at first sight skew to the topic of surfaces which are ramified coverings branched on a union of lines, this idea is deceptive.

In fact, a simple way to construct a curve  $C_i$  with G-symmetry is to construct a Galois branched covering of  $\mathbb{P}^1 \setminus \mathcal{B}_i$ , with group G, and where we assume that the branch locus is exactly the finite set  $\mathcal{B}_i$ .

Hence, the surface isogenous to a product  $S := (C_1 \times C_2)/G$  is a ramified covering of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched on the horizontal and vertical lines

$$(\mathbb{P}^1 \times \mathcal{B}_2) \cup (\mathcal{B}_1 \times \mathbb{P}^1).$$

Note that we have  $\Delta_G \subset G \times G$ , the diagonal subgroup, and a Galois diagram

$$C_1 \times C_2 \to S := (C_1 \times C_2) / \Delta_G \to \mathbb{P}^1 \times \mathbb{P}^1 = (C_1 \times C_2) / (G \times G).$$

The covering  $S \to \mathbb{P}^1 \times \mathbb{P}^1$  is Galois if and only if G is Abelian.

By the main theorem on surfaces which are isogenous to a product, such a surface S is (strongly) rigid if  $|\mathcal{B}_1| = |\mathcal{B}_2| = 3$ , and in this case in [Cat00] we called these surfaces 'Beauville surfaces'.

The reason for this is that Beauville surfaces with Abelian group occur only when  $G = (\mathbb{Z}/n)^2$  with GCD(n, 6) = 1, as shown in [Cat00], and the original example by Beauville in [Bea78] was exactly the case of  $G = (\mathbb{Z}/5)^2$ . For these surfaces  $C_1 = C_2$  is the Fermat curve of degree n, and the only difficulty consists in finding actions such that  $\Delta_G$  acts freely on the product.

In the article [BCG05] the existence problem for Beauville surfaces was translated into group theoretical terms: because each covering  $C_1 \rightarrow \mathbb{P}^1$  is determined, in view of the Riemann existence theorem, by its branch locus (here fixed!) and its monodromy; and in this case the monodromy means the datum of three elements  $a, b, c \in G$  which generate G and satisfy  $abc = 1_G$ .

Then one gets two triples (a, b, c)(a', b', c') and the condition that the action on  $C_1 \times C_2$  is free amounts to the disjointness of the stabilizers,  $S_1 \cap S_2 = \{1\}$ , here the stabilizer set  $S_1$  is the union of the conjugates of the powers of the respective elements a, b, c, and similarly for  $S_2$ .

In particular the question: which (nonabelian) simple groups except  $\mathfrak{A}_5$  occur as G for a Beauville surface? It was solved for many groups in loc. cit., has then attracted the attention of group theorists and was solved in the affirmative, even if there are still open questions concerning whether this can also be achieved via non real structures, via real structures, via strongly real structures (see [GLL12], [GM12], [FMP13], [BBPV15], see also the book [BGV15] and references therein for a partial account).

Now, our contention here is that Beauville surfaces not only create links between algebraic geometry and group theory, but that they continue to yield very interesting algebro-geometric examples.

For instance, Beauville surfaces with Abelian group  $G = (\mathbb{Z}/n)^2$  were used in [Cat14] to answer a question posed by Jonathan Wahl, hence giving rise to new examples of threefolds Z (obtained as cones over such a surface S) which fulfill the following properties:

1) Z is Cohen-Macaulay,

2) the dualizing sheaf  $\omega_Z$  is torsion,

3) the index 1 -cover Z' is not Cohen-Macaulay, in particular Z is not  $\mathbb Q\text{-}\mathrm{Gorenstein}.$ 

This is the technical result answering the question by Wahl, showing in particular the existence of regular surfaces with subcanonical ring which is not Cohen-Macaulay.

**Theorem 91.** For each r = n-3, where  $n \ge 7$  is relatively prime to 30, and for each  $m, 1 \le m \le r-1$ , there are Beauville type surfaces S with q(S) = 0 ( $q(S) := \dim H^1(S, \mathcal{O}_S)$ ) s.t.  $K_S = rL$ , and  $H^1(mL) \ne 0$ .

6.1. Automorphisms acting trivially on cohomology. Another application is in the direction of giving examples of surfaces admitting automorphisms which act trivially on integral cohomology (but are not isotopic to the identity).

In this context, Cai, Liu and Zhang [C-L-Z13] have proven the following theorem

**Theorem 92.** Let S be a minimal smooth surface of general type with  $q(S) \ge 2$ . Then either S is rational cohomologically rigidified, i.e. every automorphism acts trivially on  $H^*(S, \mathbb{Q})$ , or the subgroup  $Aut(S)_{\mathbb{Q}}$  of automorphisms acting trivially on the rational cohomology algebra is isomorphic to  $\mathbb{Z}/2$ , and S satisfies

I)  $K_S^2 = 8\chi, q(S) = 2$ ,

II) the Albanese map is surjective, and

III) S has a pencil of genus b = 1.

To show that the estimate is effective, they classified the surfaces isogenous to a product such that  $q(S) \ge 2$  and  $Aut(S)_{\mathbb{Q}} \ne 0$ .

In joint work with Gromadzki we have shown that indeed, for one of these examples (and probably for both), the action is even trivial on  $H^*(S,\mathbb{Z})$ ; but it is not trivial on the fundamental group. To show triviality of the action on  $H^*(S,\mathbb{Z})$  it suffices to show that the action is trivial on the torsion group  $H_1(S,\mathbb{Z})$ ; because then the result follows from the universal coefficients theorem and by Poincaré duality.

The following is still an open question

**Question 93.** Do there exist surfaces of general type with nontrivial automorphisms which are isotopic to the identity?

The question (see [Cat13]) is crucial in order to compare the Kuranishi and the Teichmüller space of surfaces (and higher dimensional varieties).

## 7. TOPOLOGICAL METHODS FOR MODULI

In this very short section, devoted to concrete moduli theory in the tradition of Kodaira and Horikawa, i.e. the fine classification of complex projective varieties (see e.g. [Hor75]), we shall try to show how in some lucky cases, with big fundamental group, topology helps to achieve the fine classification, allowing explicit descriptions of the structure of moduli spaces.

This was done quite effectively in several papers ([BC09b], [BC09a], [BC10], [BC10-b], [B-C12], [B-C13], [BCF15a]), and in the article [Cat15] we already amply reported on this direction of research.

For this reason the exposition in this section shall be rather brief, we refer to [Cat15] for several preparatory results, and for other related topics, such as orbifold fundamental groups, Teichmüller spaces, moduli spaces of curves with symmetry, and also for an account of the results on the regularity of classifying maps, such as harmonicity, addressed by Eells and Sampson, and their complex analyticity, addressed by Siu, which are key ingredients for the study of moduli through topological methods (see especially [ABCKT96] on this topic); and which lead to rigidity and quasi-rigidity properties of projective varieties which are classifying spaces (meaning that their moduli spaces are completely determined by their topology).

We shall focus here instead on a few concrete problems in moduli theory, in particular new constructions of surfaces with  $p_g = q = 0$  or  $p_q = q = 1$ .

7.1. Burniat surfaces and Inoue type varieties. Among the algebraic surfaces obtained as Abelian coverings of the plane branched

on interesting configurations of lines, the oldest examples were the socalled Burniat surfaces, some surfaces with  $q = p_g = 0$  and  $K_S^2 = 6, 5, 4, 3, 2$ .

Again, I shall skip their description, especially since I already reported on them at the Kinosaki Conference in the Fall of 2011; even if there are still interesting open questions concerning the connected component of the moduli space containing Burniat surfaces with  $K_S^2 = 3$ .

Following a suggestion of Miles Reid, Masahisa Inoue [In94] gave another description of the Burniat surfaces, as quotients of a hypersurface in the product of three elliptic curves. Using this method, he went further with his construction, and obtained new (minimal) surfaces of general type with  $q = p_g = 0$  and  $K_S^2 = 7$ , which are now called (algebraic) Inoue surfaces.

These were given as quotients of a complete intersection of two surfaces inside the product of four elliptic curves, but a closer inspection showed that indeed they are quotients of a surface inside the product of a curve of genus 5 with two elliptic curves. More precisely, an Inoue surface S admits an unramified  $(\mathbb{Z}/2\mathbb{Z})^5$  - Galois covering  $\hat{S}$  which is an ample divisor in  $E_1 \times E_2 \times D$ , where  $E_1, E_2$  are elliptic curves and D is a projective curve of genus 5.

Hence the fundamental group of an Inoue surface with  $p_g = 0$  and  $K_S^2 = 7$  sits in an extension ( $\Pi_g$  being as usual the fundamental group of a projective curve of genus g):

$$1 \to \Pi_5 \times \mathbb{Z}^4 \to \pi_1(S) \to (\mathbb{Z}/2\mathbb{Z})^5 \to 1.$$

It turned out that the ideas needed to treat the moduli space of this special family of Inoue surfaces could be put in a rather general framework, valid in all dimensions, and together with I. Bauer we proposed the study, obtaining several results, of what we called Inoue-type varieties.

**Definition 94.** ([B-C12]) Define a complex projective manifold X to be an **Inoue-type manifold** if

- (1)  $dim(X) \ge 2;$
- (2) there is a finite group G and an unramified G-covering  $\hat{X} \to X$ , (hence  $X = \hat{X}/G$ ) such that
- (3)  $\hat{X}$  is an ample divisor inside a  $K(\Gamma, 1)$ -projective manifold Z, (hence by the theorem of Lefschetz  $\pi_1(\hat{X}) \cong \pi_1(Z) \cong \Gamma$ ) and moreover
- (4) the action of G on  $\hat{X}$  yields a faithful action on  $\pi_1(\hat{X}) \cong \Gamma$ : in other words the exact sequence

$$1 \to \Gamma \cong \pi_1(X) \to \pi_1(X) \to G \to 1$$

gives an injection  $G \to \text{Out}(\Gamma)$ , defined by conjugation by lifts of elements of G; (5) the action of G on  $\hat{X}$  is induced by an action of G on Z.

We say that an Inoue-type manifold X is a special Inoue type manifold if moreover

$$Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h) \times (M_1 \times \cdots \times M_s)$$

where each  $A_i$  is an Abelian variety, each  $C_j$  is a curve of genus  $g_j \ge 2$ , and  $M_i$  is a compact quotient of an irreducible bounded symmetric domain of dimension at least 2 by a torsion free subgroup; and a **classical Inoue type manifold** if instead  $Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h)$ where as above each  $A_i$  is an Abelian variety, each  $C_j$  is a curve of genus  $g_j \ge 2$ .

The main idea underlying the importance of this notion is the fact that, under suitable assumptions, one can say that if X is an Inoue type manifold, and Y is homotopically equivalent to X (indeed even some weaker cohomological conditions are sufficient), then also Y is an Inoue type manifold, and in special cases one can conclude that Y and X belong to the same irreducible connected component of the moduli space. We omit to state the general results, referring to [BC08] and [Cat15], and here we shall just treat a concrete case, in the next subsection.

7.2. Bagnera-de Franchis varieties and applications to moduli. In our article [BCF15b], appeared in the volume dedicated to Kodaira, we treated a special case of the theory of Inoue type varieties, the one where the action of G happens to be free also on Z, and Z is the simplest projective classifying space, an Abelian variety.

Define the **Generalized Hyperelliptic Varieties** (GHV) as the quotients A/G of an Abelian Variety A by a finite group G acting freely, and with the property that G is not a subgroup of the group of translations. Without loss of generality one can then assume that G contains no translations, since the subgroup  $G_T$  of translations in G would be a normal subgroup, and if we denote  $G' = G/G_T$ , then A/G = A'/G', where A' is the Abelian variety  $A' := A/G_T$ .

A smaller class is the class of **Bagnera-de Franchis (BdF) Va**rieties: these are the quotients X = A/G were G contains no translations, and G is a cyclic group of order m, with generator g (observe that, when A has dimension n = 2, the two notions coincide, thanks to the classification result of Bagnera-de Franchis in [BdF08]).

Bagnera-de Franchis varieties have a simple description as quotients of Bagnera-de Franchis varieties of product type, according to the following definition:

**Definition 95.** A Bagnera-de Franchis manifold (resp.: variety) of product type is a quotient X = A/G where  $A = A_1 \times A_2$ ,  $A_1, A_2$ are complex tori (resp.: Abelian Varieties), and  $G \cong \mathbb{Z}/m$  is a cyclic

group operating freely on A, generated by an automorphism of the form

$$g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2))_{\alpha_1}$$

where  $\beta_1 \in A_1[m]$  is an element of order exactly m, and similarly  $\alpha_2 : A_2 \to A_2$  is a linear automorphism of order exactly m without 1 as eigenvalue (these conditions guarantee that the action is free).

This is then the characterization of general Bagnera- de Franchis varieties.

**Proposition 96.** Every Bagnera-de Franchis variety X = A/G, where  $G \cong \mathbb{Z}/m$  contains no translations, is the quotient of a Bagnera-de Franchis variety of product type,  $(A_1 \times A_2)/G$  by any finite subgroup T of  $A_1 \times A_2$  which satisfies the following properties:

1) T is the graph of an isomorphism between two respective subgroups  $T_1 \subset A_1, T_2 \subset A_2$ ,

 $2) (\alpha_2 - \mathrm{Id})T_2 = 0$ 

3) if  $g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2))$ , then the subgroup of order m generated by  $\beta_1$  intersects  $T_1$  only in  $\{0\}$ .

In particular, we may write X as the quotient  $X = (A_1 \times A_2)/(G \times T)$ by the abelian group  $G \times T$ .

This notion was then used in [BCF15a] to make a construction that we briefly describe.

Let  $A_1$  be an elliptic curve, and let  $A_2$  be an Abelian surface with a line bundle  $L_2$  yielding a polarization of type (1, 2). Take as  $L_1$  the line bundle  $\mathcal{O}_{A_1}(2O)$ , and let L be the line bundle on  $A' := A_1 \times A_2$ obtained as the exterior tensor product of  $L_1$  and  $L_2$ , so that

$$H^{0}(A', L) = H^{0}(A_{1}, L_{1}) \otimes H^{0}(A_{2}, L_{2}).$$

Moreover, choose the origin in  $A_2$  so that the space of sections  $H^0(A_2, L_2)$  consists only of *even* sections.

We take then, using properties of the Stone-von Neumann representation of the Heisenberg group, a Bagnera-de Franchis threefold X := A/G, where  $A = (A_1 \times A_2)/T$ , and  $G \cong T \cong \mathbb{Z}/2$ , and a surface  $S \subset X$  which is the quotient of a  $(G \times T)$ -invariant  $D \in |L|$ , so that  $S^2 = \frac{1}{4}D^2 = 6$ .

We could then prove the following.

**Theorem 97.** Let S be a surface of general type with invariants  $K_S^2 = 6$ ,  $p_g = q = 1$  such that there exists an unramified double cover  $\hat{S} \to S$  with  $q(\hat{S}) = 3$ , and such that the Albanese morphism  $\hat{\alpha} : \hat{S} \to A$  is birational onto its image Z, a divisor in A with  $Z^3 = 12$ .

Then the canonical model of  $\hat{S}$  is isomorphic to Z, and the canonical model of S is isomorphic to  $Y = Z/(\mathbb{Z}/2)$ , which a divisor in a Bagnera-De Franchis threefold X := A/G, where  $A = (A_1 \times A_2)/T$ ,  $G \cong T \cong \mathbb{Z}/2$ , and where the action is given by

(98) 
$$G := \{Id, g\}, g(a_1+a_2) := a_1 + \tau/2 - a_2 + \lambda_2/2, \forall a_1 \in A_1, a_2 \in A_2$$

(99) 
$$T := (\mathbb{Z}/2)(1/2 + \lambda_4/2) \subset A = (A_1 \times A_2).$$

These surfaces exist, have an irreducible four dimensional moduli space, and their Albanese map  $\alpha : S \to A_1 = A_1/A_1[2]$  has general fibre a non hyperelliptic curve of genus g = 3.

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LEHRSTUHL MATHEMATIK VIII, MATHEMATISCHES INSTITUT DER UNIVER-SITÄT BAYREUTH, NW II, UNIVERSITÄTSSTR. 30, 95447 BAYREUTH

E-mail address: fabrizio.catanese@uni-bayreuth.de