

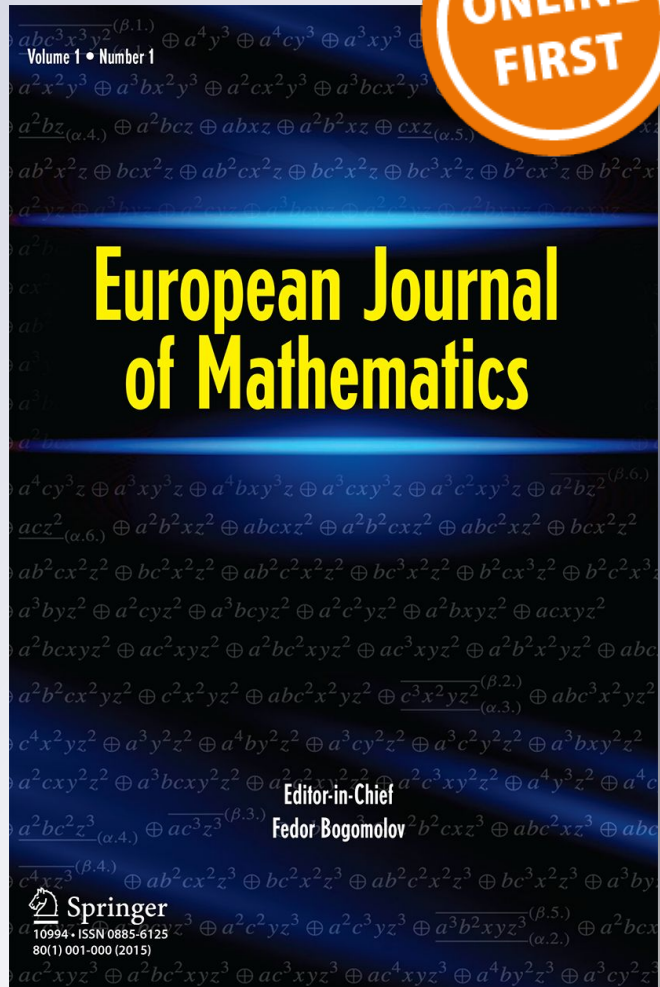
Fujita decomposition over higher dimensional base

Fabrizio Catanese & Yujiro Kawamata

European Journal of Mathematics

ISSN 2199-675X

European Journal of Mathematics
DOI 10.1007/s40879-018-0287-0



Your article is protected by copyright and all rights are held exclusively by Springer Nature Switzerland AG. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".



Fujita decomposition over higher dimensional base

Fabrizio Catanese¹ · Yujiro Kawamata²

Received: 30 September 2017 / Accepted: 3 September 2018
© Springer Nature Switzerland AG 2018

Abstract

We generalize a result of Fujita, on the decomposition of Hodge bundles over curves, to the case of a higher dimensional base.

Keywords Hodge bundle · Hermitian metric · Canonical divisor · Semi-positive · Nef · Unitary flat

Mathematics Subject Classification 14D07 · 14C30 · 32G20 · 33C60

1 Introduction

The purpose of this paper is to generalize a result of Fujita, on the decomposition of Hodge bundles over curves, to the case of a higher dimensional base. We first consider the direct image sheaf of the relative canonical sheaf for a fibration whose degeneracy locus (set of critical values) is contained in a normal crossing divisor and with unipotent local monodromies. It is known that such a sheaf is locally free and numerically semipositive (or *nef*), see Fujita [5] and Kawamata [10]. Moreover if the base space of the fibration is a curve, then Fujita proved that the direct image sheaf is a direct sum of an ample vector bundle and a unitary flat bundle with respect to

This paper grew out of a discussion when the second author visited the first author at the University of Bayreuth. The present work took place in the framework of the ERC Advanced grant no. 340258, “TADMICAMT”, and of the JSPS Grant-in-Aid (A) 16H02141. The second author would also like to thank the National Center for Theoretical Sciences at Taiwan University and Professor Jungkai Chen for the excellent working conditions.

✉ Yujiro Kawamata
kawamata@ms.u-tokyo.ac.jp

Fabrizio Catanese
fabrizio.catanese@uni-bayreuth.de

¹ Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth, Germany

² Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan

the natural Hermitian metric [6]. We shall generalize the latter result to the case of a higher dimensional base.

The algebraic statement on the nefness of the Hodge bundle was the starting point of many positivity results concerning canonical and pluricanonical sheaves. These results were used in the minimal model program. But the original theorem's statement is more analytic saying that the direct image sheaf carries a semi-positive Hermitian metric with mild but not algebraic singularities. We believe that this analytic statement should have more consequences (see e.g. Theorem 2.1).

The following is our first main result:

Theorem 1.1 (Unipotent monodromies case) *Let $f : X \rightarrow Y$ be a proper surjective morphism from a compact connected Kähler manifold to a smooth projective variety. Assume that there are respective simple normal crossing divisors B on X and D on Y such that f is smooth over $Y \setminus D$ and that $f^{-1}(D) = B$ set-theoretically. Set $X^o := X \setminus B$, $Y^o := Y \setminus D$, $f^o := f|_{X^o}$ and $n := \dim X - \dim Y$. Assume that the local monodromies of the local system $R^n f_*^o \mathbf{C}_{X^o}$ around the branches of D are unipotent. Let $V := f_* \mathcal{O}_X(K_X - f^* K_Y)$ be the direct image sheaf of the relative canonical sheaf. Then*

- (1) V is a locally free sheaf which is numerically semi-positive.

Moreover,

- (2) there is a decomposition $V = U \oplus W$, an orthogonal direct sum with respect to the natural Hermitian metric, where U is a unitary flat bundle and W is generically ample, i.e., $W|_C$ is an ample vector bundle for any general curve section C of Y .

The first assertion is proved in [5] for the case $\dim Y = 1$ and in [10] generally. See also [12]. See [11] for a generalization to pluricanonical forms. The second assertion when $\dim Y = 1$ is the essential part of Fujita's theorem [6]. See [1,2] for details of proof of Fujita's more general statement. We recall that one or both of the two summands U, W can be $= 0$; in [2,3] examples are given where U corresponds to a representation of $\pi_1(Y)$ of infinite order, and $W \neq 0$, in particular V is not semi-ample.

Next we prove the following full fledged analogue of Fujita's decomposition theorem over a higher dimensional base:

Theorem 1.2 (General case) *Let $f : X \rightarrow Y$ be a proper surjective morphism from a compact connected Kähler manifold to a smooth projective variety. Let $V := f_* \mathcal{O}_X(K_X - f^* K_Y)$ be the direct image sheaf of the relative canonical sheaf. Then*

- (1) V is a torsion free sheaf which is weakly-positive.
- (1') V is a locally free sheaf if there is a normal crossing divisor D such that f is smooth over $Y \setminus D$.

Moreover,

- (2) there is a decomposition $V = U \oplus W$, an orthogonal direct sum with respect to the natural Hermitian metric, where U is a locally free sheaf which is unitary flat with respect to the natural Hermitian connection and W is generically ample, i.e., $W|_C$ is an ample vector bundle for any general curve section C of Y .

(1) and (1') are proved in [14] using [10]. There are two ways to prove the local freeness of V :

- (i) reducing to the unipotent monodromy case by using the flatness of the covering of the base as in [14];
- (ii) as a consequence of the extension of the Hodge filtration to the canonical extension, in the same way as in the unipotent monodromy case (cf. [12]).

Recall that ([5, 10], resp. [14]) a locally free sheaf V on a projective variety Y is said to be *numerically semi-positive*, or *nef*, if given any curve C and any quotient bundle Q of $V|_C$, $\text{deg}(Q) \geq 0$; whereas a torsion free sheaf V on a projective variety Y is said to be *weakly positive* if there are a Zariski open subset U and an ample divisor H on Y such that, for each positive integer n , the natural homomorphism

$$H^0(Y, S^{mn}(V)^{\vee\vee} \otimes \mathcal{O}_Y(mH)) \otimes \mathcal{O}_Y \rightarrow S^{mn}(V)^{\vee\vee} \otimes \mathcal{O}_Y(mH)$$

is surjective on U for some positive integer m (here $V^{\vee\vee}$ is the double dual of V).

It makes sense to separate Theorems 1.1 and 1.2, since from the former follows an orbifold version of Fujita's decomposition theorem in the general case.

This orbifold version is proved in the final section, where we shall compare, through an appropriate diagram, the direct image sheaves in the different situations which occur in practice, when one takes first a normal crossing resolution of the degeneracy locus, and then, as in [10], a finite cover of the base in order to reduce to the case of unipotent local monodromies.

We define a flat unitary orbifold sheaf on a smooth variety Y in the following way. Let D be a divisor on Y , with irreducible components D_1, \dots, D_k . An orbifold unitary representation $\rho: \pi_1(Y \setminus D) \rightarrow U(r)$, with orders $m_1, \dots, m_k \in \mathbb{N}_{\geq 1}$ is defined to be one such that, for a small loop γ_j around D_j , the image $\rho(\gamma_j)$ has order exactly m_j . Let Y^{**} be the complement of the locus where D is not normal crossing. We define a locally free sheaf U^{**} on Y^{**} by taking the upper canonical extension of the flat unitary bundle \mathcal{U} on $Y^0 = Y \setminus D$ associated to the representation ρ . Then we define a *flat unitary orbifold sheaf* $U^{\text{orb}}(\rho)$ associated to the orbifold unitary representation ρ as the direct image sheaf $i_*(U^{**})$, where $i: Y^{**} \rightarrow Y$ is the inclusion.

Proposition 1.3 *In the general case we have*

- an orthogonal splitting $V = U^{\text{orb}} \oplus W$, where W is generically ample but where this time U^{orb} is an orbifold flat unitary sheaf, having vanishing curvature outside the locus D over which f is not smooth.
- U^{orb} is a locally free sheaf on a big open subset Y^{oo} of Y on which D is a normal crossing divisor. On Y^{oo} , U^{orb} coincides with the upper canonical extension (see the next section) corresponding to a unitary representation $\rho: \pi_1(Y \setminus D) \rightarrow U(r)$, with $r = \text{rank}(U^{\text{orb}})$, such that, for small loops γ_j around the divisorial components D_j of D , the $\rho(\gamma_j)$'s have finite order.
- U is a locally free summand of U^{orb} , corresponding to the irreducible summands ρ_h of ρ such that $\rho_h(\gamma_j) = 1$ for all γ_j .

2 Proof of Theorem 1.1

We use the notation of Theorem 1.1. The first assertion (1), the nefness of V , is proved in the following way. This is not new (see the papers cited in the bibliography, also for more references regarding several assertions made here), but we briefly recall the proof because we need the notation used in the argument.

We assume that the degeneracy locus D of f (the set of critical values of f) is contained in a normal crossing divisor, but do not assume that the local monodromies around D are unipotent, so that the following construction can be used also for the proof of Theorem 1.2.

The primitive part $H_{\mathbb{Z}}^o$ of the local system $R^n f_* \mathbb{C}_{X^o}$ underlies a polarized variation of Hodge structures. Let $H^o := H_{\mathbb{Z}}^o \otimes_{\mathbb{C}} \mathcal{O}_{Y^o}$. It is a locally free sheaf on Y^o with a flat connection, called the *Gauss–Manin connection*, whose sheaf of flat sections is $H_{\mathbb{C}}^o$. The direct image sheaf $V^o = f_*^o \mathcal{O}_X(K_X - f^* K_Y)$ is identified as the subsheaf $F^n(H^o)$, where F^\cdot denotes the Hodge filtration. The cup product along the fibres of f^o determines a polarization of the Hodge structures, a non-degenerate but not definite Hermitian metric on H^o which is flat with respect to the Gauss–Manin connection, but with the property that the restriction of this metric to V^o is positive definite.

H^o has an extension H as a locally free sheaf on Y called the *canonical extension* constructed in the following way [4]. Let x_j be local coordinates on Y such that D is expressed as $\prod_{j=1}^r x_j = 0$, and let T_j be the corresponding local monodromy transformations for $1 \leq j \leq r$. Then H is locally generated by the following holomorphic sections

$$\exp\left(-\frac{1}{2\pi i} \sum_{j=1}^r \log T_j \log x_j\right) v$$

where the v are multi-valued flat sections of H^o , and the matrices $\log T_j$ are defined to be $\log U_j + \log S_j$ for the decomposition $T_j = U_j S_j$ into unipotent and semisimple part. We take the branches of the logarithm function such that the eigenvalues of the matrices $\log S_j$ belong to the interval $[0, 2\pi i)$. We note that $\log U_j$ is well defined because $U_j - I$ is nilpotent, and that the expression is single-valued even if v is multi-valued.

If the local monodromies are unipotent, then the local holomorphic sections of H are characterized, among local holomorphic sections of H^o , by the condition that the norm of their restrictions to H^o grows at most logarithmically along the boundary D . In general, they grow asymptotically as $\prod_{j=1}^r |x_j|^{-p_j} |\log x_j|^{q_j}$ for some p_j, q_j such that $0 \leq p_j < 1$. We note that the failure of the unipotency of the T_j is reflected in the eigenvalues of the S_j , whence the exponents p_j above. We call these exponents p_j *boundary contributions*.

The Hodge filtration on H^o extends to H in such a way that the $F^p(H)$'s are locally free subbundles. Moreover we have $V = f_* \mathcal{O}_X(K_X - f^* K_Y) = F^n(H)$. This fact holds even if the local monodromies are not unipotent as long as f is smooth outside a normal crossing divisor. In particular V is locally free (see also [12]).

The Hodge–Riemann bilinear relations and Griffiths’ transversality [7] imply that the curvature of the connection of V^o corresponding to the restricted metric on V^o is semi-positive. The nefness of V follows from this semi-positivity together with the observation that the boundary contributions along D vanish because the metric grows at most logarithmically.

Now we prove the latter assertion (2). We fix an ample line bundle L on Y . Let $0 \subset N_1 \subset \dots \subset N_m = V$ be a Harder–Narasimhan filtration with respect to L ; the N_i are reflexive subsheaves of V such that the subquotients N_i/N_{i-1} are semistable torsion free sheaves with respect to L , and the inequalities $\mu(N_i/N_{i-1}) > \mu(N_{i+1}/N_i)$ hold for all i , where the slope μ of a torsion free sheaf is defined by $\mu(\bullet) = \text{deg}(\bullet)/\text{rank}(\bullet)$ with $\text{deg}(\bullet) = c_1(\bullet)L^{\dim Y-1}$.

By a theorem of Mehta–Ramanathan [13], if we take a general curve section C (a complete intersection of general hypersurfaces of sufficiently high degrees), then the restriction $0 \subset N_{1,C} \subset \dots \subset N_{m,C} = V_C$ for $N_{i,C} = N_i \otimes \mathcal{O}_C$ is a Harder–Narasimhan filtration of $V_C = V \otimes \mathcal{O}_C$.

Let $U' = V/N_{m-1}$ be the last quotient. Since V is nef, we have $\text{deg}(U') \geq 0$. If $\text{deg}(U') > 0$, then V_C is an ample vector bundle on C by Hartshorne [8]. In this case, we have $V = W$.

Hence we may assume in the following that $\text{deg}(U') = 0$. In this case, we set $W := N_{m-1}$. Then $W_C = N_{m-1,C}$ is an ample vector bundle on C . We shall prove that the quotient bundle U' splits and there is a subbundle $U \cong U'$ which is a unitary flat bundle using the following general result (a similar argument is given on page 86 of [2]). Shigeharu Takayama communicated to the authors that the following result was independently proved by Genki Hosono (cf. [9]):

Theorem 2.1 *Let X be a complex manifold which is not necessarily compact, and let*

$$0 \rightarrow E \rightarrow F' \rightarrow G \rightarrow 0$$

be an exact sequence of holomorphic vector bundles on X . Assume that F' has a smooth positive definite Hermitian metric whose curvature form $\Theta_{F'}$ is semi-positive, and that the curvature form Θ_G of the induced metric on G vanishes. Then the orthogonal complement E^\perp of E with respect to the metric on F' is a holomorphic subbundle of F' which is isomorphic to G . In particular we have $F' \cong E \oplus^\perp G$ as holomorphic bundles with metric.

Proof We shall prove the dual statement; namely let now

$$0 \rightarrow S \xrightarrow{i} F \xrightarrow{p} Q \rightarrow 0$$

be an exact sequence of holomorphic vector bundles, where F has a smooth positive definite Hermitian metric whose curvature form Θ_F is semi-negative, and that the curvature form Θ_S of the induced metric on the subbundle S vanishes. Then we shall prove that the orthogonal complement S^\perp of S in F is a holomorphic subbundle which is isomorphic to the quotient bundle Q , hence $F \cong S \oplus^\perp Q$.

Let $D_F : A^0(F) \rightarrow A^1(F)$ be the connection of F which is compatible with the complex structure and the metric, and let $\sigma = pD_{Fi} : A^0(S) \rightarrow A^1(Q)$ be the second

fundamental form. Let $p_0: S^\perp \rightarrow Q$ be the restriction of p . p_0 is an isomorphism of C^∞ vector bundles with Hermitian metrics. Then $D_S = D_F - p_0^{-1}\sigma: A^0(S) \rightarrow A^1(S)$ and $D_Q = pD_F p_0^{-1}: A^0(Q) \rightarrow A^1(Q)$ are the connections which are compatible with the complex structures and the induced metrics. We have the formula

$$(\Theta_S u, v) = (\Theta_F u, v) - (\sigma(u), \sigma(v))$$

for $u, v \in A^0(S)$. Since $\Theta_S = 0$ and Θ_F is semi-negative while the metric is positive definite, we deduce that $\sigma = 0$. Thus $D_F(u) \in A^1(S)$ for $u \in A^0(S)$. It follows that $D_F(u) \in A^1(S^\perp)$ for $u \in A^0(S^\perp)$. If u is a holomorphic section of Q , then $D_Q(u) \in A^{1,0}(Q)$. It follows that $D_F(p_0^{-1}(u)) = p_0^{-1}D_Q(u) \in A^{1,0}(F)$. Hence $p_0^{-1}(u)$ is a holomorphic section of F . Therefore S^\perp is a holomorphic subbundle. \square

We can deduce from Theorem 2.1 the following well-known result:

Example 2.2 Let F be a locally free sheaf of rank 2 on an algebraic curve C which is a non-trivial extension of \mathcal{O}_C by a nef invertible sheaf. Then F is nef but F does not admit a smooth Hermitian metric with semi-positive curvature.

We apply the above theorem to the restriction $V_C^\circ, (U')_C^\circ$ and W_C° of V, U' and W , respectively, to the open part $C^\circ = C \cap Y^\circ$. Since the numerical semi-positivity of U' is a consequence of the semi-positivity of the curvature and the null boundary contributions, the condition $\deg(U') = 0$ implies that the curvature of $(U')_C^\circ$ vanishes. By Theorem 2.1, there is a holomorphic subbundle U_C° of V_C° such that $V_C^\circ = W_C^\circ \oplus^\perp U_C^\circ$, and such that U_C° is flat with respect to the connection associated to the positive definite Hermitian metric of V_C° . Thus U_C° is unitary flat with respect to the Gauss–Manin connection.

Since the local monodromies of the Gauss–Manin connection on $H_{\mathbb{Z}}^\circ$ are unipotent, the local monodromies of U_C° are trivial. It follows that U_C° extends to a unitary flat vector bundle U_C on C . Since the curve sections C are general, we can move C around so that the unitary flat bundles U_C can be extended to a unitary flat bundle U^{oo} on a big open subset Y^{oo} of Y , an open subset whose complement has codimension at least 2 by virtue of the following argument.

The union of general curves which are complete intersections of hypersurface sections of high degree is a constructible set Y^* , which contains an open subset of Y , else the curves are not general. Moreover the complement of Y^* does not contain an open subset in any irreducible divisor Δ since C is general. Thus the complement of Y^* has codimension at least 2. By the same token, at the general points of Y^* the tangents of the curves C fill out a big open subset. Hence we get a unitary flat bundle U^{oo} on a big open set Y^{oo} . Since $\pi_1(Y^{oo}) \cong \pi_1(Y)$, U^{oo} extends to a unitary flat bundle on the whole Y .

On the other hand, the canonical extension H of H° is characterized as a holomorphic vector bundle by the condition that the norms of its holomorphic sections have at most logarithmic growth. Therefore U is still a holomorphic subbundle of V , and we have a direct sum decomposition $V = W \oplus U$.

Remark 2.3 Keeping the same assumptions of Theorem 1.1, further use of Theorem 2.1 shows: if there is a generically surjective homomorphism $V \rightarrow Q$ to a reflexive sheaf with $\text{deg}(Q) = 0$, then there is a direct sum decomposition $V = W' \oplus Q'$ with $Q' \cong Q$ being a unitary flat bundle with respect to the natural Hermitian metric (see also the argument in the next section).

3 Proof of Theorem 1.2

In this section we fully extend Fujita’s decomposition theorem to the case of a higher dimensional base variety Y , proving Theorem 1.2.

In the proof we shall use a similar notation to the one of Theorem 1.1. We denote by D the degeneracy locus of f , the locus over which f is not smooth (also called the set of critical values of f), so that f is smooth over $Y^o = Y \setminus D$.

We take a log resolution $u: Y' \rightarrow Y$ for $D \subset Y$; Y' is smooth and projective, u is birational, and $D' = u^{-1}(D)$ is a simple normal crossing divisor. Since the direct image sheaf V does not depend on the choice of a birational model of X , we may assume that there is a morphism $f': X \rightarrow Y'$ such that $f = u \circ f'$.

Let $V' = f'_* \mathcal{O}_X(K_X - (f')^* K_{Y'})$. It is a locally free sheaf by [14, Theorem 4.1]. Moreover, the weak-positivity of V' and V was proved by Viehweg [14, Theorem III].

Since $K_{Y'} \geq u^* K_Y$, we have $u_* V' \subset V$. More precisely, if R is the ramification divisor of u , $K_{Y'} = u^*(K_Y) + R$, we have:

$$V = f_*(\mathcal{O}(K_X))(-K_Y) = u_*(f'_*(\mathcal{O}(K_X))(-K_{Y'} + R)) \supset u_*(V').$$

On the other hand, since these sheaves are equal outside of codimension 2 and V is torsion free, V embeds into the reflexive hull of $u_* V'$: $V \subset (u_* V')^{\vee\vee}$.

Our proof of (2) of Theorem 1.2 is similar to that of Theorem 1.1 in the following way. We take a Harder–Narasimhan filtration of V , let U' be the last quotient, and assume that $\text{deg}(U') = 0$. By Mehta–Ramanathan, we take a generic curve C and consider the restriction U'_C . We recall that $\text{deg}(U'_C)$ is the sum of the integration of the curvature on the open part $C^o = C \cap Y^o$ and the sum of the boundary contributions, as shown in [10, Lemma 21, p. 268]. Indeed the line bundle $L := \Lambda^{\text{top}}(U'|_C)$ on C has a smooth metric outside a finite set of points P , where the growth of the norm of a local generating section s_P is asymptotic to $|s_P| \sim |t_P|^{-\alpha_P} (\log |t_P|)^{\beta_P}$, where the α_P are non-negative rational numbers, in view of the choice of the branch for the canonical extension. We observe that each term is non-negative, due to the semipositivity of the curvature. Since $\text{deg}(U'_C) = 0$, both contributions vanish, i.e., all the numbers $\alpha_P = 0$, and the curvature vanishes.

Since $\alpha_P = 0$, the local monodromies are unipotent whence trivial (unipotent and unitary). Indeed, if the local monodromy is not unipotent, then the growth of the norm of the generating holomorphic sections, defined by using the multivalued flat sections, is not logarithmic, so for some of them it is greater than a constant times $|t_P|^{-a_P} (\log |t_P|)^{b_P}$ with $a_P > 0$, contradicting that the sum of the a_P ’s equals $\alpha_P = 0$.

By Theorem 2.1, there is a holomorphic subbundle $U_C^o \cong (U'_C)_{C^o}$ such that $V_C^o = W_C^o \oplus^\perp U_C^o$ on C^o . Since U_C^o is unitary flat with respect to the given Hermitian metric,

and with trivial local monodromies there is a locally free sheaf with a unitary flat connection U_C on C which is an extension of U_C^o . By the same argument as in the proof of Theorem 1.1, we obtain a locally free sheaf U on Y which is unitary flat.

We have to prove that U is a direct summand of V . By the same argument applied to $f': X \rightarrow Y'$, we obtain a locally free sheaf $U_{Y'}$ on Y' which is unitary flat with respect to the given metric. Since u^*U and $U_{Y'}$ are defined as extensions of the same unitary flat bundles on Y^o , we have $u^*U = U_{Y'}$. This time, since V' is a subbundle of the canonical extension defined by using the local monodromies, $U_{Y'}$ is a direct summand of V' ; $V' = U_{Y'} \oplus W_{Y'}$. By taking the direct image by u , we have

$$U \oplus^\perp u_* W_{Y'} \subset V \subset U \oplus^\perp (u_* W_{Y'})^{\vee\vee}.$$

Therefore we have $V = U \oplus^\perp W$. □

Proof of Proposition 1.3 In the general case, we construct the following bimeromorphically Cartesian diagram using [10, Theorem 17]:

$$\begin{array}{ccccc} X'' & \xrightarrow{v'} & X & \xrightarrow{\text{Id}} & X \\ \downarrow f'' & & \downarrow f' & & \downarrow f \\ Y'' & \xrightarrow{u'} & Y' & \xrightarrow{u} & Y, \end{array}$$

where X'', Y', Y'' are all smooth, such that

- (i) u is birational, and the degeneracy locus of f' is contained in a simple normal crossing divisor D' on Y' .
- (ii) u' is a finite surjective morphism with Abelian Galois group G , and $f'': X'' \rightarrow Y''$ satisfies the hypotheses of Theorem 1.1.

Denote by V'', V', V the respective images of the relative canonical sheaves. By [14] or by the definition of the canonical extension, we have a generically invertible homomorphism between the two vector bundles of the same rank $V'' \rightarrow u'^*(V')$. Thus we have an injective homomorphism $(u'_* V'')^G \rightarrow V'$ between locally free sheaves which is generically surjective.

By Theorem 1.1, there is an orthogonal splitting $V'' = U'' \oplus W''$, where U'' is unitary flat and W'' is generically ample. Let U'^{orb} be smallest subsheaf of V' containing $(u'_* U'')^G$ and such that V'/U'^{orb} is torsion free, i.e., the hull of $(u'_* U'')^G$ in V' . By the definition of the canonical extension, U'^{orb} coincides with the canonical extension of $(u'_* U'')^G|_{Y' \setminus D'}$, hence a locally free sheaf. We finish the proof by taking U^{orb} to be the hull of $u_*(u'_* U'')^G$ in V . □

References

1. Catanese, F., Dettweiler, M.: The direct image of the relative dualizing sheaf needs not be semiample. C. R. Math. Acad. Sci. Paris **352**(3), 241–244 (2014)

2. Catanese, F., Dettweiler, M.: Answer to a question by Fujita on variation of Hodge structures. In: Oguiso, K. et al. (eds.) *Higher Dimensional Algebraic Geometry*. *Advanced Studies in Pure Mathematics*, vol. 74, pp. 73–102. Mathematical Society of Japan, Tokyo (2017)
3. Catanese, F., Dettweiler, M.: Vector bundles on curves coming from variation of Hodge structures. *Int. J. Math.* **27**(7), # 1640001 (2016)
4. Deligne, P.: *Équations différentielles à points singuliers réguliers*. *Lecture Notes in Mathematics*, vol. 163. Springer, Berlin (1970)
5. Fujita, T.: Kähler fiber spaces over curves. *J. Math. Soc. Japan* **30**(4), 779–794 (1978)
6. Fujita, T.: The sheaf of relative canonical forms of a Kähler fiber space over a curve. *Proc. Japan Acad. Ser. A Math. Sci.* **54**(7), 183–184 (1978)
7. Griffiths, Ph.A.: Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. *Inst. Hautes Études Sci. Publ. Math.* **38**, 125–180 (1970)
8. Hartshorne, R.: Ample vector bundles on curves. *Nagoya Math. J.* **43**, 73–89 (1971)
9. Hosono, G.: Approximations and examples of singular Hermitian metrics on vector bundles. *Ark. Mat.* **55**, 131–153 (2017)
10. Kawamata, Y.: Characterization of abelian varieties. *Compositio Math.* **43**(2), 253–276 (1981)
11. Kawamata, Y.: Kodaira dimension of algebraic fiber spaces over curves. *Invent. Math.* **66**(1), 57–71 (1982)
12. Kawamata, Y.: Variation of mixed Hodge structures and the positivity for algebraic fiber spaces. In: Chen, J.A., et al. (eds.) *Algebraic Geometry in East Asia-Taipei 2011*. *Advanced Studies in Pure Mathematics*, vol. 65, pp. 27–58. Mathematical Society of Japan, Tokyo (2015)
13. Mehta, V.B., Ramanathan, A.: Semistable sheaves on projective varieties and their restriction to curves. *Math. Ann.* **258**(3), 213–224 (1981/1982)
14. Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In: *Algebraic Varieties and Analytic Varieties*. *Advanced Studies in Pure Mathematics*, vol. 1, pp. 329–353. North-Holland, Amsterdam (1983)