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Rigid group actions on complex tori are projective (after Ekedahl)

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In this paper, we give a detailed proof of a result due to Torsten Ekedahl, describing complex tori admitting a rigid group action and showing explicitly their projectivity and their structure in terms of CM-fields. In the appendix, joint with Claudon, we show, using a method of Green-Voisin, that all group actions on complex tori deform to projective ones.

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1. Introduction

The work of Kodaira [8, 9] lead to the question whether any compact Kähler manifold enjoys the property of admitting arbitrarily small deformations which are projective (Kodaira settled in [9] the case of surfaces).

Motivated by Kodaira's problem (see the final section and the appendix) the first author asked Torsten Ekedahl at an Oberwolfach conference around 1999 if there exists a rigid group action of a finite group $G \subset \mathrm{Bihol}(T)$ on a complex torus T (see Sec. 2 for definitions regarding deformations of group actions) which is not projective. Ekedahl answered this question and sketched a strategy of proof for the statement that the rigidity of the action (T, G) implies that T is projective (i.e. T is an abelian variety).

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Later Claire Voisin gave a counterexample to the general Kodaira problem showing in [15] the existence of a rigid compact Kähler manifold which is not projective (and later in [16] she even gave counterexamples which are not bimeromorphic to a projective manifold). Kodaira's property still remains a very interesting theme of research: understanding which compact Kähler manifolds or Kähler spaces with klt singularities satisfy Kodaira's property (see [5] for quite recent progress).

On the other hand Ekedahl's approach allows a rather explicit description of rigid actions on complex tori in terms of orders in CM-fields, hence providing explicitly given polarizations on them. Therefore his result turned out to be quite interesting and useful for other purposes (see [4] for applications to the classification theory of quotient manifolds of complex tori), and for this reason we find it important to publish here a complete proof.

Theorem 1 (Ekedahl). *Let* (T, G) *be a rigid group action of a finite group* $G \subset \mathrm{Bihol}(T)$ *on a complex torus* T. *Then* T (*or, equivalently,* T/G) *is projective. Moreover, if we write* $T = V^{1,0}/\Lambda$, *then*

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \oplus_j W_j^{n_j},$$

where W_j *is a Hodge structure on a CM field* F_j *and where* $\oplus_j F_j$ *is a subalgebra of the centre of the group algebra* $\mathbb{Q}[G]$.

The contents of the paper are as follows.

In Sec. 2, we briefly discuss deformations of group actions on complex manifolds.

Then, in the subsequent section, we develop the tools used in the proof of Theorem 1, mainly based on Hodge theory and representation theory.

The main ideas of the proof are the following: if $\mathcal{A}$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra, the rigidity of the action of $\mathcal{A}$ (cf. Definition 5) on a rational Hodge structure V of weight 1 can be determined by looking at the simple summands of $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C}$ appearing in $V^{1,0}$, respectively in $V^{0,1}$. A second ingredient is that, for $\mathcal{A} = \mathbb{Q}[G]$ with G finite (and also in a more general situation), we show that rigidity is equivalent to having a rigid action of the commutative subalgebra given by the centre $Z(\mathbb{Q}[G])$.

Then we apply Proposition 16, stating that, if $\mathcal{A} = Z(\mathbb{Q}[G])$ is the centre of the group algebra and the action of $\mathcal{A}$ on V is rigid, then the Hodge structure V is polarizable.

Finally, in the appendix, we show that every group action (T, G) on a complex torus admits arbitrarily small deformations which are projective.

2. Deformations of Group Actions

Let X be a compact complex manifold. Let $G \subset \mathrm{Bihol}(X)$ be a finite group, and denote by $\alpha : G \times X \to X$ the corresponding group action of G on X.

Definition 2. (1) A *deformation* (p, α') of the group action α of G on X consists of a deformation $p : (\mathfrak{X}, \mathfrak{X}_0) \to (B, t_0)$ of X (i.e. $\mathfrak{X}_0 := p^{-1}(t_0)$ and $X \cong \mathfrak{X}_0$) given together with $\alpha' : G \times \mathfrak{X} \to \mathfrak{X}$, a holomorphic group action commuting with p (here we let G act trivially on the base), such that the action on $\mathfrak{X}_0 \cong X$ induces the initially given action α.

(2) A deformation (p, α') is said to be *trivial* if its germ is isomorphic to the trivial deformation $X \times B \to B$, endowed with the action $\alpha \times \mathrm{id}_B$.

(3) The action α is said to be *rigid* if every deformation of α is trivial.

Kuranishi theory leads to an easy characterization of rigidity of an action α of a group G on X, see [1, p. 23; 2, Chap. 4; 10].

Denote by $p : \mathfrak{X} \to \mathrm{Def}(X)$ the Kuranishi family of X; then this characterization is related to the question: which condition on $t \in \mathrm{Def}(X)$ guarantees that G is a subgroup of $\mathrm{Aut}(\mathfrak{X}_t)$? It turns out (cf. [1, p. 23]) that $G \subset \mathrm{Bihol}(\mathfrak{X}_t)$ if and only if $g_* t = t$ for any $g \in G$, so that $t \in \mathrm{Def}(X) \cap H^1(X, \theta_X)^G$.

We then have (see [2, Proposition 4.5]):

Proposition 3. *Set* $\mathrm{Def}(X)^G := \mathrm{Def}(X) \cap H^1(X, \theta_X)^G$. *The group action* α *of* G *on* X *is rigid if and only if* $\mathrm{Def}(X)^G = 0$ (*as a set*). *A fortiori the action is rigid if* $H^1(X, \theta_X)^G = 0$ (*in this latter case we say that the action is infinitesimally rigid*).

In the upcoming chapter, we shall consider the case where $X = T$ is a complex torus: the rigidity of (T, G), amounting to the fact that the representation of G on $H^1(X, \Theta_X)$ contains no trivial summand, can then be read off explicitly from the action of G on the tangent bundle.

3. Rigid Actions on Rational Hodge Structures

Denote by $\mathcal{H}^1$ the category of rational Hodge structures of type $((1,0),(0,1))$ (a good reference for the following notions is the classic book [11]). An object of $\mathcal{H}^1$ is a finite-dimensional $\mathbb{Q}$-vector space V endowed with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = U \oplus \overline{U} =: V^{1,0} \oplus V^{0,1}.$$

The elements of $\mathcal{H}^1$ can be viewed as isogeny classes of complex tori

$$T := \frac{\Lambda \otimes_{\mathbb{Z}} \mathbb{C}}{\Lambda \oplus V^{0,1}},$$

where $\Lambda \subset V$ is an *order*, i.e. a free subgroup of maximal rank (by abuse of notation we shall also say that Λ is a lattice in V, observe that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$).

We have isogeny classes of Abelian varieties when a rational Hodge structure is polarizable, according to the following.

Definition 4. Let $V \in \mathcal{H}^1$ and write for short $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$.

A *polarization on* V is an alternating form $E : V \times V \to \mathbb{Q}$ satisfying the two Hodge–Riemann bilinear relations:

(i) The complexification $E_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ satisfies $E_{\mathbb{C}}(V^{1,0}, V^{1,0}) = 0$ (hence also $E_{\mathbb{C}}(V^{0,1}, V^{0,1}) = 0$)
(ii) For any nonzero vector $v \in V^{1,0}$, we have $-i \cdot E_{\mathbb{C}}(v, \overline{v}) > 0$ Equivalently, setting $E_{\mathbb{R}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$, we have:

(I) $E_{\mathbb{R}}(Jx, Jy) = E_{\mathbb{R}}(x, y)$
(II) the symmetric bilinear form $E_{\mathbb{R}}(x, Jy)$ is positive definite.

Here, if $x = u + \bar{u}$, $Jx := iu - i\bar{u}$ ($J^2 = -\mathrm{Id}$).

Let $\mathcal{A}$ be a semisimple and finite-dimensional $\mathbb{Q}$-algebra (for example the group algebra $\mathcal{A} = \mathbb{Q}[G]$ for a finite group G). We denote an action $r : \mathcal{A} \to \mathrm{End}_{\mathcal{H}^1}(V)$ for $V \in \mathcal{H}^1$ by a triple $(V, \mathcal{A}, r)$.

If $\Lambda \subset V$ is a lattice and $T = (V \otimes_{\mathbb{Q}} \mathbb{C})/(\Lambda \oplus V^{0,1})$ is the corresponding complex torus then $\mathcal{A}$ maps to $\mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 5. An action $(V, \mathcal{A}, r)$ is called *rigid*, if

$$\mathrm{Hom}_{\mathcal{A}}(V^{0,1}, V^{1,0}) = 0. \tag{1}$$

Rigidity 1 means, in view of what we saw in the previous section, and in view of

$$H^1(\Theta_T) = H^1(\mathcal{O}_T) \otimes_{\mathbb{C}} H^0(\Omega^1_T)^{\vee} = \overline{U}^{\vee} \otimes_{\mathbb{C}} U = \mathrm{Hom}_{\mathcal{A}}(V^{0,1}, V^{1,0}),$$

that there are no deformations of T preserving the $\mathcal{A}$-action.

We consider now some examples of the above notion.

Example 6. Let $\mathcal{A}$ be a totally imaginary number field F. This means that $[F : \mathbb{Q}] = 2k$ and F possesses $2k$ different embeddings $\sigma_j : F \to \mathbb{C}$, none of which is *real* (this means: $\sigma_j(F) \subset \mathbb{R}$).

Hence each σ_j is different from the complex conjugate, $\sigma_j \neq \overline{\sigma_j}$, and if we set $V := F$, with the obvious action of F, all the Hodge structures on V are rigid and correspond to the finite set of partitions of the set $\mathcal{E}$ of embeddings of F into two conjugate sets $\{\sigma_1, \dots, \sigma_k\}$ and $\{\overline{\sigma_1}, \dots, \overline{\sigma_k}\}$.

Since the F-module $F \otimes_{\mathbb{Q}} \mathbb{C}$ is the direct sum

$$F \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\sigma_j \in \mathcal{E}} \ \mathbb{C}_{\sigma_j},$$

where $\mathbb{C}_{\sigma_j}$ is the vector space $\mathbb{C}$ with left action of F given by:

$$x \cdot z := \sigma_j(x) \cdot z, \quad \forall\, x \in F, \ \ z \in \mathbb{C},$$

and choosing such a partition amounts to choosing $V^{1,0} := \oplus_{j=1,\dots,k} \mathbb{C}_{\sigma_j}$.

A particular case is given by the class of CM-fields.

Example 7. Recall that a *CM-field* is a totally imaginary quadratic extension F of a totally real number field K.

Equivalently, (cf. [13, Proposition 5.11]) F is a CM-field if it carries a non-trivial involution ρ such that $\sigma \circ \rho = \overline{\sigma}$ for all embeddings $\sigma : F \hookrightarrow \mathbb{C}$. In particular F is totally imaginary.

In this case any Hodge structure on $V := F$ is polarizable.

Let indeed $\sigma_1, \dots, \sigma_k : F \hookrightarrow \mathbb{C}$ be the embeddings of F occurring in $V^{1,0}$. Following [13, p. 128] choose $\zeta \in F$ satisfying the following conditions:

(a) ζ is imaginary, i.e., $\rho(\zeta) = -\zeta$,
(b) $\sigma_j(\zeta)$ is imaginary with positive imaginary part for each $j = 1, \dots, k$.

A polarization on V of F is then given, if we set $x_j := \sigma_j(x), y_j := \sigma_j(y)$, by the skew symmetric form (we set here $\sigma_{k+j} := \overline{\sigma_j}$)

$$E(x,y) := \mathrm{tr}_{F/\mathbb{Q}}(\zeta x \rho(y)) = \sum_{j=1}^{2k} \sigma_j(\zeta) x_j \overline{y_j} = \sum_{j=1}^{k} \sigma_j(\zeta)(x_j \overline{y_j} - \overline{x_j} y_j).$$

In fact, the first Riemann bilinear relation amounts to $E(Jx, Jy) = E(x, y)$, which is clearly satisfied, since $(Jx)_j = ix_j$, for $j = 1, \dots, k$, and the real part of the associated Hermitian form is the symmetric form

$$E(x, Jy) = \sum_{j=1}^{k} (-i)\sigma_j(\zeta)(x_j \overline{y_j} + \overline{x_j} y_j),$$

which is positive definite since

$$E(x, Jx) = \sum_{j=1}^{k} 2\,\mathrm{Im}(\sigma_j(\zeta))|x_j|^2 > 0$$

for $x \neq 0$.

Let us now proceed towards the proof of the main theorem.

An important step towards the main theorem is that in the case where

$$\mathcal{A} = \mathbb{Q}[G] \tag{2}$$

rigidity can be reduced to rigidity of the action restricted to the centre of the group algebra.

Proposition 8. *Let $\mathcal{A} = \mathbb{Q}[G]$ be the group algebra of a finite group G over the rationals.*

Then the triple $(V, \mathcal{A}, r)$ is rigid if and only if $(V, Z(\mathcal{A}), r')$ is rigid, where $Z(\mathcal{A})$ is the centre of $\mathcal{A}$ and r' is the restriction of r to $Z(\mathcal{A})$.

Proof. For each field K, $\mathbb{Q} \subset K \subset \mathbb{C}$, $\mathcal{A} \otimes_{\mathbb{Q}} K = K[G]$ has as centre $Z_K := Z(K[G])$, the vector space with basis $v_{\mathcal{C}}$, indexed by the conjugacy classes $\mathcal{C}$ of G, and where

$$v_{\mathcal{C}} := \sum_{g \in \mathcal{C}} g.$$

For $K = \mathbb{C}$, another more useful basis is indexed by the irreducible complex representations W_χ of G, and their characters χ (these form an orthonormal basis for the space of class functions, i.e. the space $Z_{\mathbb{C}}$ if we identify the element g to its characteristic function).

For each irreducible χ, the element

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) \cdot g \in \mathbb{C}[G]$$

is an idempotent in $Z(\mathbb{C}[G])$. Indeed, we even have that

$$Z(\mathbb{C}[G]) = \oplus_\chi \mathbb{C} \cdot e_\chi,$$

and the idempotents e_χ satisfy the orthogonality relations $e_{\chi'} \cdot e_\chi = 0$ for $\chi \neq \chi'$.

This leads directly to the decomposition

$$\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[G] = \oplus_{\chi \in \mathrm{Irr}(G)} A_\chi, \quad A_\chi := e_\chi \mathbb{C}[G] \cong \mathrm{End}(W_\chi),$$

where χ runs over all irreducible characters of G, and to the semisimplicity of the group algebra. Notice that e_χ acts as the identity on W_χ, and as 0 on $W_{\chi'}$ for $\chi' \neq \chi$.

In fact, we shall prove the stronger statement that for any two finitely generated $\mathbb{C}[G]$-modules M and N (note that $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[G]$)

$$\mathrm{Hom}_{\mathbb{C}[G]}(M, N) = 0 \Leftrightarrow \mathrm{Hom}_{Z(\mathbb{C}[G])}(M, N) = 0.$$

The right-hand side $\mathrm{Hom}_{Z(\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C})}(M, N)$ clearly contains the left-hand side.

By semisimplicity, each representation M splits as a direct sum of irreducible representations,

$$M = \sum_{\chi \in \mathrm{Irr}(G)} M_\chi, M_\chi = W_\chi \otimes_{\mathbb{C}[G]} (\mathbb{C}^r),$$

where $\mathbb{C}^r$ is a trivial representation of G.

By bilinearity we may assume that $M = W_\chi$ and $N = W_{\chi'}$ are simple modules associated to irreducible characters χ, χ' of G.

Then, by the lemma of Schur, the left-hand side $\mathrm{Hom}_{\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C}}(M, N)$ is $= 0$ for $\chi' \neq \chi$, and isomorphic to $\mathbb{C}$ for $\chi' = \chi$.

For the right-hand side, it suffices to prove that $\mathrm{Hom}_{Z(\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C})}(M, N) = 0$ for $\chi' \neq \chi$, when $M = W_\chi, N = W_{\chi'}$.

However, e_χ acts as the identity on M and as zero on N, hence $\psi \in \mathrm{Hom}_{Z(\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C})}(M, N)$ implies

$$\psi(v) = \psi(e_\chi v) = e_\chi(\psi(v)) = 0,$$

as we wanted to show.

This shows the statement. □

We have more generally:

Proposition 9. *Let $\mathcal{A}$ be a semisimple $\mathbb{Q}$-algebra of finite dimension, and let $(V, \mathcal{A}, r)$ be an action on a rational Hodge structure V, Then r is rigid if and only if $(V, Z(\mathcal{A}), r')$ is rigid; here $Z(\mathcal{A})$ is the centre of $\mathcal{A}$ and r' is the restriction of r.*

Proof. More generally, if M, N are $\mathcal{A} \otimes \mathbb{C}$-modules, then we claim that

$$\mathrm{Hom}_{\mathcal{A}\otimes\mathbb{C}}(M, N) = 0 \Leftrightarrow \mathrm{Hom}_{Z(\mathcal{A}\otimes\mathbb{C})}(M, N) = 0.$$

By bilinearity of both sides, and by semisimplicity (each module splits as a direct sum of irreducibles) we can assume that M, N are simple modules and that $\mathcal{A}$ is a simple algebra.

By Schur's lemma the left-hand side is nonzero exactly when M and N are isomorphic. The left-hand side is contained in the right-hand side, so it suffices to show that the right-hand side is nonzero exactly when M and N are isomorphic. But ([7, Lemma 1, p. 205]) any two irreducible modules over a simple Artininian ring are isomorphic. □

Remark 10. We have $\mathbb{C}[G] = \oplus_\chi \mathbb{C}[G] \cdot e_\chi$.

Working instead over a field K of characteristic 0, an algebraic extension of $\mathbb{Q}$ (so $\mathbb{Q} \subset K \subset \mathbb{C}$), the decomposition of $K[G]$ into simple summands is (see [17, Proposition 1.1]) again provided by central idempotents in $K[G]$,

$$K[G] = \oplus_{[\chi]} K[G] e_K(\chi), \quad e_K(\chi) := \sum_{\chi^\sigma \in [\chi]} e_{\chi^\sigma},$$

where the first sum runs over the set of Γ-orbits $[\chi]$ in the set all irreducible characters χ of G; here Γ is the Galois group $\mathrm{Gal}(K(\chi)/K)$ of the field extension $K(\chi)$ of K, generated by the values of all the characters χ, i.e. by $\{\chi(g) \,|\, g \in G, \chi \in \mathrm{Irr}(G)\}$.

And the centre of $K[G]$ is a direct sum of fields

$$Z(K[G]) = \oplus_{[\chi]} F_{[\chi]},$$

where the field $F_{[\chi]}$ is the centre (for the last isomorphism, see [17], Proposition 1.4)

$$F_{[\chi]} := Z(K[G]) e_K(\chi) \cong K(\{\chi(g) \,|\, g \in G\})$$

of the algebra $K[G]e_K(\chi)$, and enjoys the property that $F_{[\chi]} \otimes_K \mathbb{C} = \oplus_{\chi \in [\chi]} \mathbb{C} e_{\chi^\sigma}$.

The next lemma explains the relation occurring between finite groups and CM-fields.

Lemma 11. *The centre of the group algebra $Z(\mathbb{Q}[G])$ splits as a direct sum of number fields, $Z(\mathbb{Q}[G]) = F_1 \oplus \cdots \oplus F_l$ which are either totally real, or CM-fields.*

Proof. Write $m := |G|$, let ζ_m be a primitive mth root of unity and let d be the number of conjugacy classes in G, which equals the number of irreducible representations of G. Then

$$F_j \subset Z(\mathbb{Q}[G]) \subset Z(\mathbb{Q}(\zeta_m)[G]) \cong_{\mathbb{Q}-\text{alg.}} \mathbb{Q}(\zeta_m)^d,$$

where we used in the last isomorphism that every complex representation of G is defined over $\mathbb{Q}(\zeta_m)$. Hence F_j embeds into the cyclotomic field $\mathbb{Q}(\zeta_m)$. The extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ is Galois with group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$ (the isomorphism maps $\varphi_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, such that $\varphi_a(\zeta_m) = \zeta_m^a$, to $a \in (\mathbb{Z}/m\mathbb{Z})^*$), so by the Main Theorem of Galois Theory, there is a subgroup H of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, such that $F_j \cong \mathbb{Q}(\zeta_m)^H$ (the subfield of $\mathbb{Q}(\zeta_m)$ fixed by the action of H). If $-1 \in H$ (which corresponds to φ_{-1}, the complex conjugation), the field F_j is totally real, otherwise F_j is a CM-field. □

4. Proof of Theorem 1

Fix now an action $(V, \mathcal{A}, r)$ and assume that

$$\mathcal{A} \text{ is commutative.} \tag{3}$$

Since $\mathcal{A}$ is commutative, $\mathcal{A}$ is a direct sum of number fields,

$$\mathcal{A} = F_1 \oplus \cdots \oplus F_l.$$

Assume that we have a homomorphism of algebras $\sigma : \mathcal{A} \to \mathbb{C}$. For each idempotent e of $\mathcal{A}$, $\sigma(e)$ is an idempotent of $\mathbb{C}$, hence $\sigma(e) = 1$ or $\sigma(e) = 0$. In $\mathcal{A}$, the identity element 1 is a sum of idempotents

$$1 = 1_{F_1} + \cdots + 1_{F_l},$$

and if $\sigma \neq 0$, then $\sigma(1) = 1$. This implies that for such a homomorphism σ there is exactly one $j \in \{1, \ldots, l\}$, such that $\sigma(1_{F_j}) = 1$, and, for $i \neq j$, we have $\sigma(1_{F_i}) = 0$.

Let then $\mathcal{C} = \{\sigma_1, \ldots, \sigma_k\}$ be the set of all the distinct $\mathbb{Q}$-algebra homomorphisms $\mathcal{A} \to \mathbb{C}$: then these homomorphisms $\sigma_j : \mathcal{A} \to \mathbb{C}$ are obtained as the composition of one of the projections $\mathcal{A} \to F_h$ with an embedding $F_h \hookrightarrow \mathbb{C}$ (hence $k = \sum_h [F_h : \mathbb{Q}] = \dim_{\mathbb{Q}} \mathcal{A}$).

Define now (as in Example 6) the $\mathcal{A}$-module $\mathbb{C}_{\sigma_j}$ as the vector space $\mathbb{C}$ endowed with the action of $\mathcal{A}$ such that

$$x \cdot z := \sigma_j(x) \cdot z.$$

Hence we have a splitting of $\mathcal{A}$-modules

$$\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{j=1}^{l} (F_j \otimes_{\mathbb{Q}} \mathbb{C}) = \oplus_{j=1}^{k} \mathbb{C}_{\sigma_j}.$$

We now show that we have a splitting in the category of rational Hodge structures

$$V = V_1 \oplus \cdots \oplus V_l,$$

where V_i is an F_i-module, and an $\mathcal{A}$-module via the surjection $\mathcal{A} \to F_i$.

We simply define $V_j := 1_{F_j} \cdot V$. We have a splitting of modules

$$V = V_1 \oplus \cdots \oplus V_l,$$

since for $i \neq j$, $1_{F_i} 1_{F_j} = 0$, and

$$v = 1 \cdot v = (1_{F_1} + \cdots + 1_{F_l})v =: v_1 + \cdots + v_l.$$

It is a splitting in the category of rational Hodge structures because each element of $\mathcal{A}$ preserves the Hodge decomposition, hence V_j is a sub-Hodge structure of V.

Therefore, the action r is a direct sum of actions

$$r_j : F_j \to \mathrm{End}_{\mathcal{H}^1}(V_j).$$

Each r_j induces, by tensor product, a homomorphism of rings

$$F_j \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{End}(V_j \otimes_{\mathbb{Q}} \mathbb{C}) = \mathrm{End}(V_j^{1,0} \oplus V_j^{0,1}),$$

and a splitting of $\mathcal{A}$-modules

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} = \oplus_{j=1}^{k} (V_{\sigma_j}^{1,0} \oplus V_{\sigma_j}^{0,1}),$$

where V_{σ_j} is the character subspace on which $\mathcal{A}$ acts via $x \cdot v := \sigma_j(x) \cdot v$. This holds for the following reason: each V_j is an F_j module; and since F_j is a number field, then $F_j = \mathbb{Q}[x]/P(x)$, where P is irreducible, and $r_j(x)$ is an endomorphism a_j of V_j with minimal polynomial P (a polynomial with distinct roots). In particular, a_j is diagonalizable over $V_j \otimes_{\mathbb{Q}} \mathbb{C}$, and each diagonal entry yields some embedding σ_h of F_j into $\mathbb{C}$.

Remark 12. The rigidity of $(V, \mathcal{A}, r)$ is equivalent to the fact that for each $\sigma_j \in \mathcal{C}$ either $V_{\sigma_j}^{1,0}$ or $V_{\sigma_j}^{0,1}$ is zero, in particular, since $\overline{V_{\sigma_j}^{1,0}} = V_{\overline{\sigma_j}}^{0,1}$, no real σ_j appears either in $V^{1,0}$ or in $V^{0,1}$.

Following a terminology similar to the one introduced in [3], we define the notion of Hodge-type.

Definition 13. Define the *Hodge-type* of an action of $\mathcal{A}$ by the function $\tau_V : \mathcal{C} \to \mathbb{N}$, such that

$$\tau_V(\sigma) := \dim_{\mathbb{C}} V_{\sigma}^{1,0}.$$

Hodge symmetry translates into

$$(HS)\ \tau_V(\sigma) + \tau_V(\bar{\sigma}) = \dim_{\mathbb{C}} V_{\sigma},$$

which implies in particular that if we have a real embedding, i.e. $\sigma = \overline{\sigma}$, then $\tau_V(\sigma) = \frac{1}{2} \dim_{\mathbb{C}} V_{\sigma}$.

Moreover, if Hodge symmetry holds, the action is rigid if and only if

$$(R)\ \tau_V(\sigma)\cdot\tau_V(\bar\sigma)=0,\quad \forall\,\sigma.$$

Proposition 14. *If $(V,\mathcal{A},r)$ is rigid, then it is determined by the $\mathcal{A}$-module V and by the Hodge-type.*

Conversely, if V is an $\mathcal{A}$-module, and there is a Hodge structure such that

$$(HS)\ \tau_V(j)+\tau_V(\bar j)=\dim_{\mathbb{C}} V_{\sigma_j},$$

whenever $\sigma_{\bar j}=\overline{\sigma_j}$, and moreover

$$(R)\ \tau_V(j)\cdot\tau_V(\bar j)=0\quad\forall\, j,$$

then this Hodge structure determines a rigid action $(V,\mathcal{A},r)$.

Proof. In one direction, the Hodge-type determines $V^{0,1},V^{1,0}$, since, $\mathcal{A}$ being commutative, V splits into character spaces V_{σ_j}, and the function τ_V determines whether $V_{\sigma_j}\subset V^{0,1}$, or $V_{\sigma_j}\subset V^{1,0}$.

In the other direction, the given Hodge structure is preserved by the action of $\mathcal{A}$ hence we have an action in the category of rational Hodge structures. □

Lemma 15. *Assume that we have a rigid action $(V,\mathcal{A},r)$ of split type, where*

$$\mathcal{A}=F_1\oplus\cdots\oplus F_l$$

is commutative and each F_i is a field.

(i) *If $l=1$ (so $\mathcal{A}=:F$ is a field), $V\cong W^n$ in $\mathcal{H}^1$, where W is a Hodge structure on F.*

(ii) *The rational Hodge structure V splits as a direct sum*

$$V=W_1^{n_i}\oplus\cdots\oplus W_l^{n_l},$$

where W_j is a Hodge structure on F_j and $n_j\geq 0$.

Proof. Assertion (i): Here V is an F-vector space, and so $f: V\xrightarrow{\sim}F^n$ as vector spaces. As we observed the rigidity of (V,F,r) implies that all embeddings of F into $\mathbb{C}$ appear in either $V^{1,0}$ or $V^{0,1}$, hence F has no real ones. Let $\sigma_1,\ldots,\sigma_d$ be the embeddings of F appearing in $V^{1,0}$, so that $\overline{\sigma_1},\ldots,\overline{\sigma_d}$ are the ones appearing in $V^{0,1}$. Define a Hodge structure W on F according to the type of V, i.e. as follows:

$$W\otimes_{\mathbb{Q}}\mathbb{C}=W^{1,0}\oplus W^{0,1},\quad \text{where } W^{1,0}=\oplus_{j=1}^d\mathbb{C}_{\sigma_j},\quad W^{0,1}=\oplus_{j=1}^d\mathbb{C}_{\overline{\sigma_j}}.$$

Then $f_{\mathbb{C}}: V\otimes_{\mathbb{Q}}\mathbb{C}\to(W\otimes_{\mathbb{Q}}\mathbb{C})^n$ is an isomorphism of $\mathbb{C}$-vector spaces together with an F-action.

Assertion (ii) follows immediately from assertion (i), since we have the splittings $\mathcal{A}=F_1\oplus\cdots\oplus F_l$ and $V=V_1\oplus\cdots\oplus V_l$, and the $\mathcal{A}$-rigidity of V implies the F_j-rigidity of V_j for all $j=1,\ldots,l$, hence we can apply step (i) to each V_j. □

The crucial proposition from which the proof of Theorem 1 follows is now.

Proposition 16. *If* $(V, \mathbb{Q}[G], r)$ *is rigid, then it is polarizable.*

Proof. First of all, if $(V, \mathbb{Q}[G], r)$ is rigid, then $(V, Z(\mathbb{Q}[G]), r)$ is rigid by Proposition 8. The assumption that $(V, Z(\mathbb{Q}[G]), r)$ is rigid implies now that if some field F_j does not act as 0 on V, then F_j is necessarily a CM-field by Lemma 11 and the previous remarks. By Lemma 15, the rational Hodge structure V splits as a direct sum $W_1^{n_i} \oplus \cdots \oplus W_l^{n_l}$, where W_j is a Hodge structure on F_j and $n_j \geq 0$.

To give a polarization on V, it therefore suffices to show the existence of a polarization for a Hodge structure W_j on a CM-field F_j. But this was shown in Example 7. □

Ekedahl's Theorem is therefore proven.

5. Final Remarks

Assume that $X := T$ is a complex torus of dimension ≥ 3, and that $Y = T/G$ has only isolated singularities.

Schlessinger showed in [12, Theorem 3] that every deformation of the analytic germ of Y at each singular point of Y is trivial.

Hence for every deformation $\mathcal{Y} \to B$ of Y (we write informally $\mathcal{Y}$ as $\{Y_t\}_{t \in B}$) Y_t has the same singularities as Y, and in particular it follows easily that $Y_t \backslash \operatorname{Sing}(Y_t)$ and $Y \backslash \operatorname{Sing}(Y)$ are diffeomorphic and *a fortiori* one has an isomorphism $\pi_1(Y_t \backslash \operatorname{Sing}(Y_t)) \cong \pi_1(Y \backslash \operatorname{Sing}(Y)) \cong \pi_1(\mathcal{Y} \backslash \operatorname{Sing}(\mathcal{Y}))$. Therefore the surjection $\pi_1(Y \backslash \operatorname{Sing}(Y)) \to G$ induces a surjection $\pi_1(\mathcal{Y} \backslash \operatorname{Sing}(\mathcal{Y})) \to G$.

Whence, by Grauert's and Remmert's extension of Riemann's Existence Theorem, cf. [6, Satz 32], Y_t and $\mathcal{Y}$ have respective Galois covers X_t and $\mathcal{X}$ with group G. Hence, the action of G extends to the family $\mathcal{X}$, and each deformation of Y yields a deformation of the pair (T, G).

The conclusion is that Y is rigid if and only if the action of G on T is rigid. On the other hand, Ekedahl's theorem implies then that if Y is rigid, then Y is projective.

Therefore in this case one cannot get a counterexample to the Kodaira property via rigidity. We show more generally in the appendix that any such a quotient $Y = T/G$ with only isolated singularities satisfies the Kodaira property, since any action can be approximated by a projective one.

An interesting question is: in the case where Y is rigid, is it true that a minimal resolution of Y is also rigid?

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Appendix A

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Ekedahl's theorem has the advantage of elucidating the structure of (rigid and non-rigid) actions of a finite group G on a complex torus.

The method of period mappings, used by Green and Voisin (see proposition 17.20 and Lemma 17.21 of [14]) for showing the density of algebraic tori (non constructive, since it uses the implicit functions theorem), was used by Graf in [5] to obtain a general criterion, from which follows the following theorem.

Theorem A.1. *Let (T, G) be a group action on a complex torus. Then there are arbitrarily small deformations (T_t, G) of the action where T_t is projective.*

Proof. Given a complex torus

$$T := (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})/(\Lambda \oplus V^{1,0}),$$

set, as in Sec. 2,

$$V \otimes_{\mathbb{Q}} \mathbb{C} = U \oplus \overline{U} =: V^{1,0} \oplus V^{0,1}.$$

The Teichmüller space of T is an open set $\mathcal{T}$ in the Grassmann variety $Gr(n, V \otimes_{\mathbb{Q}} \mathbb{C})$,

$$\mathcal{T} = \{U_t \,|\, U_t \oplus \overline{U_t} = V \otimes_{\mathbb{Q}} \mathbb{C}\},$$

parametrizing Hodge structures. By abuse of notation we shall use the notation $t \in \mathcal{T}$ for the points of Teichmüller space.

The deformations of the pair (T, G) are parametrized by the submanifold $\mathcal{T}^G$ of the fixed points for the action of G, which correspond to the set of the subspaces U_t which are G-invariant.

The tangent space to $\mathcal{T}^G$ at the point (T, G) is, as seen in Sec. 2, the subspace

$$H^1(\Theta_T)^G \subset H^1(\Theta_T) = H^1(\mathcal{O}_T) \otimes_{\mathbb{C}} H^0(\Omega^1_T)^\vee = \overline{U}^\vee \otimes_{\mathbb{C}} U.$$

Over $\mathcal{T}^G$ we have the Hodge bundle

$$F^1 \subset \mathcal{T}^G \times \wedge^2(V \otimes_{\mathbb{Q}} \mathbb{C})^\vee \text{ s.t. } F^1_t = H^{1,1}(T_t) \oplus H^{2,0}(T_t).$$

Since the family of complex tori is differentiably trivial there is a canonical isomorphism

$$\wedge^2(V \otimes_{\mathbb{Q}} \mathbb{C})^\vee = H^2(T, \mathbb{C}) \cong H^2(T_t, \mathbb{C}).$$

This allows to define a holomorphic mapping $\psi : F^1 \to H^2(T, \mathbb{C})$ induced by the second projection.

We can indeed consider the subbundle (defined over $\mathcal{T}^G$)

$$(F^1)^G \subset \mathcal{T}^G \times H^2(T, \mathbb{C})^G \quad \text{s.t. } (F^1)^G_t = H^{1,1}(T_t)^G \oplus H^{2,0}(T_t)^G,$$

and the corresponding holomorphic mapping $\phi : (F^1)^G \to H^2(T, \mathbb{C})^G$ induced by the second projection.

Step 1: Let η be a Kähler metric on T. By averaging, we replace η by $\sum_g g^*(\eta)$ and we can assume that η is G-invariant.

Let $\omega \in H^{1,1}(T) \cap H^2(T_t, \mathbb{R})^G$ be the corresponding Kähler class.

Step 2: Setting $T =: T_0$, the map ϕ is a submersion at the point $(0, \omega)$.

Before proving step 2, let us see how the theorem follows.

Let $\mathcal{D}$ be a sufficiently small neighbourhood of ω inside

$$H^2(T, \mathbb{C})^G = H^2(T, \mathbb{Q})^G \otimes_{\mathbb{Q}} \mathbb{C}.$$

For each class $\xi \in H^2(T, \mathbb{Q})^G \cap \mathcal{D}$, there is therefore a (t, ξ) in a small neighbourhood $\mathcal{D}'$ of $(0, \omega)$ such that

$$\xi \in (F^1)^G_t = H^{1,1}(T_t)^G \oplus H^{2,0}(T_t)^G.$$

Since ξ is real, $\xi \in H^{1,1}(T_t)^G \cap H^2(T, \mathbb{Q})^G$. Taking $\mathcal{D}$ sufficiently small, the class ξ is also positive definite, hence ξ is the class of a polarization on T_t.

Shrinking $\mathcal{D}$ and $\mathcal{D}'$, we obtain that $t \in \mathcal{T}^G$ tends to 0 (the point corresponding to the torus T). Hence the assertion of the theorem is proven.

Proof of Step 2. The tangent space to $(F^1)^G$ at the point $(0, \omega)$ is the direct sum

$$H^1(\Theta_T)^G \oplus (F^1)^G_0 = H^1(\Theta_T)^G \oplus H^{1,1}(T)^G \oplus H^{2,0}(T)^G,$$

and the derivative of ϕ is the direct sum of $\cup\omega, \iota$, where ι is the inclusion $(F^1)^G_0 \subset H^2(T, \mathbb{C})^G$, while the cup product with $\omega \in$ yields a linear map

$$\beta : H^1(\Theta_T)^G \to H^2(T, \mathcal{O}_T)^G = H^{0,2}(T)^G \subset H^2(T, \mathbb{C})^G.$$

Whence ϕ is a submersion at $(0, \omega)$ if and only if β is surjective.

Now, β is surjective if the cup product with ω yields a surjection

$$\beta' : H^1(\Theta_T) \to H^2(T, \mathcal{O}_T)$$

(taking the subspace of G-invariants is an exact functor).

Observe that $H^2(T, \mathcal{O}_T) = \wedge^2(\overline{U}^\vee)$, while

$$H^{1,1}(T) = H^1(\Omega^1_T) = \overline{U}^\vee \otimes_{\mathbb{C}} U^\vee.$$

Cup product with ω is the composition of two linear maps

$$H^1(\Theta_T) \to H^2(\Theta_T \otimes_{\mathcal{O}_T} \Omega^1_T) \to H^2(T, \mathcal{O}_T),$$

where the second map is induced by contraction.

It can be also seen as the composition of three linear maps:

$$\begin{aligned} H^1(\Theta_T) &= \overline{U}^\vee \otimes_{\mathbb{C}} U \to (\overline{U}^\vee \otimes_{\mathbb{C}} U) \otimes_{\mathbb{C}} (\overline{U}^\vee \otimes_{\mathbb{C}} U^\vee) \\ &\to \overline{U}^\vee \otimes_{\mathbb{C}} \overline{U}^\vee \to \wedge^2(\overline{U}^\vee) = H^2(T, \mathcal{O}_T). \end{aligned}$$

Since the last linear map is a surjection, it suffices to show that the composition of the first two maps yields a surjection

$$b : \overline{U}^\vee \otimes_{\mathbb{C}} U \to \overline{U}^\vee \otimes_{\mathbb{C}} \overline{U}^\vee.$$

Since ω is a Kähler class, there exists a basis u_i of U such that

$$\omega = \sum_i \overline{u_i^\vee} \otimes_{\mathbb{C}} u_i^\vee.$$

Hence

$$\sum_{h,k} a_{h,k} \overline{u_h^\vee} \otimes_{\mathbb{C}} u_k \to \sum_{h,k} a_{h,k} \overline{u_h^\vee} \otimes_{\mathbb{C}} \overline{u_k^\vee}$$

and b is an isomorphism. □

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