

The double point formula with isolated singularities and canonical embeddings

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ABSTRACT

Motivated by the embedding problem of canonical models in small codimension, we extend Severi’s double point formula to the case of surfaces with rational double points, and we give more general double point formulae for varieties with isolated singularities. A concrete application is for surfaces with geometric genus $p_g = 5$: the canonical model is embedded in \mathbb{P}^4 if and only if we have a complete intersection of type (2,4) or (3,3).

Contents

1. Introduction	1
2. The classical double point formula	5
3. Extension of Severi’s statement	7
4. Comparing normal sheaf and quotient bundle	10
5. Surfaces with rational double points	12
6. Double point formulae via symplectic approximations	18
References	19

1. Introduction

Already for irreducible plane curves $C \subset \mathbb{P}_{\mathbb{C}}^2$ of degree d with δ ordinary double points (that is, with local analytical equation $xy = 0$) we have two types of double point formulae.

First there is the Plücker formula for the class $\omega_1(C)$, which is a projective invariant, the number of lines L passing through a general point $O \in \mathbb{P}^2$ and tangent to the curve C at some smooth point:

$$(I) \quad \omega_1(C) = d(d - 1) - 2\delta,$$

second, there is the genus formula:

$$(II) \quad 2g(C) - 2 = d(d - 3) - 2\delta,$$

where $2g(C)$ is the first Betti number of the normalization \tilde{C} of C , which is a topological invariant. The two formulae can be rewritten as expressing the number of double points in terms of the degree and the class, respectively, in terms of the degree and the genus:

$$2\delta = d(d - 1) - \omega_1(C) = d(d - 3) + 2 - 2g(C).$$

Both formulae can be generalized also for plane curves with more general singularities, but the number 2δ is replaced differently in each of the two formulae.

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In the genus formula, the number 2δ is replaced by the sum

$$\sum_P m_P(m_P - 1),$$

where the sum runs over all points P , also the infinitely near ones, and m_P is the multiplicity of (the strict transform of) the curve C in P .

In the Plücker formula appear not only the multiplicity of a singular point, but also the local classes of the branches of C at the singular points (cf. [17, pp. 277–282]), and, for instance, in the case of an ordinary *cuspidal* singularity, that is, when the local analytic equation of C is

$$y^2 - x^3 = 0,$$

the contribution for the Plücker formula is 3, the sum of the multiplicity (=2) and the local class (=1).

Another ‘philosophical’ aspect of the genus formula is that it explains that a smooth projective curve of genus g and degree d cannot be projected isomorphically to the plane \mathbb{P}^2 if

$$2g - 2 - d(d - 3) \neq 0.$$

It was classically known that every smooth subvariety $X' \subset \mathbb{P}^N$ of dimension n can be projected isomorphically to $X \subset \mathbb{P}^{2n+1}$, since the secant variety of X' has dimension $2n + 1$ and does not fill up the whole space if $N \geq 2n + 2$, so that we can project X' from a general point O isomorphically in \mathbb{P}^{2n+1} . But when you want to project to \mathbb{P}^{2n} there appear, for a general projection, δ *improper double points* (IDPs), that is isolated singularities consisting of two smooth branches intersecting transversally.

If we put ourselves in a situation similar to the one of plane curves with only IDPs as singularities (in this case IDPs are the same as before, ordinary double points), again we have two double point formulae.

The following Severi’s double point formula, established by Severi in 1902, is a generalization of the Plücker formula:

$$(I) \quad 2\delta = d(d - 1) - \sum_{i=1}^n \omega_i(X),$$

and is given in terms of projective invariants $\omega_i(X)$, called the *ceti* of X (see Section 2).

Later on, it was realized that these projective invariants are related to absolute invariants (similar to the genus g of a curve), and this led to the theory of characteristic classes, notably Chern classes. In terms of these, we have the Todd–Fulton–Laksov formula, which is the modern double point formula:

$$(II) \quad 2\delta = d^2 - c_n(N_f) = d^2 - [c(T_{X'})^{-1}(1 + H)^{2n+1}]_n,$$

where $f : X' \rightarrow X$ is the normalization map, N_f is the normal bundle to the immersion f , defined through the exact sequence

$$0 \rightarrow T_{X'} \rightarrow f^*(T_{\mathbb{P}^{2n}}) \rightarrow N_f \rightarrow 0,$$

where we denote by H the pull-back of the hyperplane class to X' , and where $[c_0 + c_1 + \cdots + c_n]_n$ denotes the part c_n of dimension $2n$ of the sum in the cohomology ring (or in the Chow ring) of X' .

This double point formula has, as we saw, two formulations (the second follows from multiplicativity of the Chern classes for exact sequences); the first has a topological flavour, and allows for generalizations: it says that d^2 , the self-intersection number of X in \mathbb{P}^{2n} , equals to 2δ plus the self-intersection number of the zero section in the normal bundle N_f (which maps via the exponential map onto a neighbourhood of X).

The second instead lends itself to concrete calculations, by which it is possible to exclude that a smooth variety X' of dimension n can be embedded in a projective space of dimension at most $2n$ by certain linear systems.

For instance, the first author and Luca Cesarano have proven elsewhere [10] the following result.

THEOREM 1.1. *Let D be an ample smooth divisor in an Abelian variety A of dimension $n + 1$. If D yields a polarization of type (m_1, \dots, m_{n+1}) , then the canonical map Φ_D of D is a morphism, and it can be an embedding only if $p_g(D) := h^0(K_D) \geq 2n + 2$, which means that the Pfaffian $m := m_1 \cdot m_2 \cdots m_{n+1}$ satisfies the inequality*

$$m \geq n + 2.$$

The main idea for proving the above theorem is, assuming that D is embedded by $H^0(K_D) = H^0(\mathcal{O}_D(D))$ and that $N := p_g(D) - 1 \leq 2n$, to apply the double point formula (in the special case $\delta = 0$) to D and to its iterated hyperplane sections. See also [11, 19] for other applications.

The main motivation for our work stems from surface theory, namely the desire to extend the following theorem of the first author [7].

THEOREM 1.2. *Let S be a complex projective surface with ample canonical divisor and with $p_g(S) = 5$. Then the canonical map is an embedding if and only if S is a smooth complete intersection in \mathbb{P}^4 of type $(2,4)$ or of type $(3,3)$.*

If we take instead a minimal surface S of general type with $p_g(S) = 5$, the canonical map factors through the canonical model S' of S , a normal surface with canonical singularities (rational double points), and we may ask the similar question: when do we get an embedding of the canonical model? The answer is analogous.

COROLLARY 1.1. *Let S be a minimal smooth surface of general type with $p_g(S) = 5$ such that the canonical map*

$$\Phi_{|K_S|} : S \rightarrow S' \subset \mathbb{P}^4$$

is a morphism with image isomorphic to the canonical model S' of S . Then, setting $d := (K_S^2)$,

$$12\chi(\mathcal{O}_S) = (17 - d)d.$$

In particular, S' is then a complete intersection of type $(2, 4)$ or $(3, 3)$, if either the base field is of characteristic 0 or the characteristic of the field k is $\neq 2$ and $h^1(\mathcal{O}_S) = 0$.

The idea is to extend the double point formula to the case of surfaces with singularities either IDPs or canonical singularities (and then setting $\delta = 0$ for the application): this is done in an elementary way in Theorem 5.1, Section 5 over any algebraically closed field of any characteristic. Then one establishes the above numerical formula, by which follows that $d = 8, 9$; then the proof proceeds exactly as in Theorem 1.2.

Since it would be interesting to extend this type of result also in higher dimension, it is desirable to extend the double point formula to varieties X with singularities. In this paper we restrict to varieties with isolated singularities.

Fulton and Laksov's generalization [16] allows in particular the variety X , outside the IDPs, to have local complete intersection singularities, a property which implies that X is locally smoothable at these points.

In this flavour we prove in the last section another generalization, of which the following theorem is a special case:

THEOREM 1.3. *Let $X \subset \mathbb{P}^{2n}$ be a variety of degree d with isolated singularities, of which δ are IDPs, and the other are normal singularities admitting a smoothing. Then X admits a global smoothing to a symplectic immersed manifold $M \subset \mathbb{P}^{2n}$, with exactly δ Improper Double Points, and we have, if $f : M' \rightarrow M$ is the immersion,*

$$d^2 = 2\delta + e(N_f),$$

where e is the Euler class of the oriented normal bundle to the map.

In spite of their elegance, the two above generalizations have the drawback that the top Chern class of the normal bundle of the partial resolution of X at the IDPs (respectively, the Euler class in Theorem 1.3) is not directly computable (at least this is our impression). This is due to the fact that Chern classes are multiplicative only for sheaves on smooth varieties, or, for general varieties, only for exact sequences of vector bundles, while Euler classes are not multiplicative.

It seems therefore worthwhile to go back to Severi's approach.

Severi established an inductive formula,

$$2\hat{\delta} = 2\delta + \omega_n(X),$$

where \hat{X} is obtained from X intersecting with a general hyperplane H , and then projecting $X \cap H$ from a general point $O \in H$ to \mathbb{P}^{2n-2} , to obtain \hat{X} with $\hat{\delta}$ IDPs.

First of all we explain in Section 2 how the two different double point formulae are related to each other, showing how to formulate Severi's inductive formula in terms of the Gauss map of the immersion $f : X' \rightarrow \mathbb{P}^{2n}$.

Indeed, if $Q_{X'}$ is the pull-back of the universal quotient bundle of the Grassmannian $Gr(n, 2n)$ (of projective subspaces of dimension n in \mathbb{P}^{2n}), then we have

$$\omega_n(X) = c_n(Q_{X'}),$$

and the relation between the two approaches is given by the simple formula

$$Q_{X'} = N_f(-1).$$

For worse singularities, we take a resolution Y of X on which the Gauss map becomes a morphism. Then $f : Y \rightarrow \mathbb{P}^{2n}$ is no longer an immersion and the two sheaves are different, yet there is a surjection $N_f(-1) \rightarrow Q_Y$, and the double point formula can be expressed in terms of the kernel sheaf \mathcal{F} , called *discrepancy sheaf*; see Theorem 4.1.

This is done in Section 4, using the results of Section 3 where we generalize Severi's double point formula by extending Severi's inductive formula almost verbatim, but using the notion of the *tangent star* of a singular point x : this is the closure of the union of the lines which are limits of secant lines joining a pair of points x_1, x_2 tending to x .

THEOREM 1.4. *Let $X \subset \mathbb{P} := \mathbb{P}^{2n}$ be an irreducible nondegenerate subvariety of dimension n having isolated singularities x_1, \dots, x_h .*

Assume that these singularities are either IDPs or have tangent star of dimension at most $2n - 1$.

Let δ be the number of the IDPs: then, in the notation of Theorem 2.1, we have

$$2\hat{\delta} = 2\delta + \omega_n(X).$$

Of course our above partial results raise several questions.

First of all, in Theorem 3.3 we observe that Improper Double Points have full tangent star (the tangent star is the whole space).

There is an obvious generalization of Theorem 3.3, for varieties with isolated singularities: but the main problem here is to calculate exactly the contribution of the points with full tangent

star to the double point formula. This problem is related to work of Flenner, O' Carroll and Vogel and others who defined a cycle which gives (as far as we understand) the contribution to this multiplicity; see [13].

This said, the topic of double point formulae can be treated in so many different ways, that it is a very challenging and currently active area of research to investigate the relations among the several approaches: our approach here was led by the problem we wanted to solve, and by the search of effectively computable formulae.

The expert reader may note that the resolution of indeterminacy of the Gauss map was later called Nash blowing up, and used by Mather to define Chern classes of singular varieties [22, 23]. However, Mather–Chern classes are defined by taking the push forward of the Chern classes of the pull-back (under the Gauss map) of the universal subbundle, whereas we need those of the quotient bundle to be pushed forward, and again we are missing multiplicativity. MacPherson [22] introduced functorial Chern classes of singular varieties; Piene [24] obtained expressions relating the polar classes to the Chern–Mather classes, and used these to calculate local Euler obstructions of hypersurfaces in terms of Milnor numbers. Recently Aluffi related the MacPherson Chern classes to the Fulton Chern classes first [1] in the case of hypersurfaces, and later, [3], in greater generality. There has been other related recent work, such as [2, 5].

It would be interesting to see whether some of these ideas shed some light on the problem of embeddings of canonical models in higher dimension, but we postpone this investigation to the future.

2. The classical double point formula

The classical double point formula established by Severi in 1902 concerns the following situation.

X is an irreducible subvariety of dimension n in \mathbb{P}^{2n} , whose singularities are only *IDPs*, that is isolated singularities consisting of two smooth branches intersecting transversally.

Letting X' be the normalization of X , X' is smooth projective, and we have a normalization map $\nu : X' \rightarrow \mathbb{P}^{2n}$ which is an immersion, hence we have the Gauss map

$$\gamma : X' \rightarrow Gr(n, 2n)$$

associating to $x \in X'$ the projective linear subspace image of the tangent space TX'_x under the derivative $D\nu_x$ of ν at x .

DEFINITION 2.1. The *n*th *ceto* $\omega_n(X)$ of $X \subset \mathbb{P}^{2n}$ is the number of subspaces $\gamma(x)$ passing through a general point $O \in \mathbb{P}^{2n}$.

More generally, for any projective variety X of dimension n in \mathbb{P}^N , $X \subset \mathbb{P}^N$, the *n*th *ceto* of X , $\omega_n(X)$, is the number of linear subspaces of dimension n tangent to X at smooth points, and intersecting a general linear subspace of codimension $2n$ in \mathbb{P}^N .

The *i*th *ceto* of X , $\omega_i(X)$, is the *i*th *ceto* of the intersection $X \cap L'$ of X with a general subspace L' of codimension $n - i$.

The following is Severi's main assertion ([25], see also [6] for a more detailed proof).

THEOREM 2.1 (Severi's inductive statement). *Let $X \subset \mathbb{P}^{2n}$ be a subvariety of dimension n having only δ IDPs as singularities, let $O \in \mathbb{P}^{2n}$ be a general point, let L be a general linear subspace of \mathbb{P}^{2n} of codimension 2, let H be the hyperplane spanned by O and L .*

Then, denoting by \hat{X}' the (smooth) intersection $X \cap H$, the projection with centre O to L of \hat{X}' is a variety \hat{X} with only $\hat{\delta}$ IDPs as singularities, and we have

$$2\hat{\delta} = 2\delta + \omega_n(X).$$

Using the inductive statement, and observing that, for a plane curve of degree d and with δ' double points, the 1-ceto is just its class, the number of tangents through a general point, hence equal to $d(d-1) - 2\delta'$, Severi obtains the following result.

THEOREM 2.2 (Severi's double point formula). *Let $X \subset \mathbb{P}^{2n}$ be a subvariety of dimension n and degree d having only δ IDPs as singularities: then we have*

$$2\delta = d(d-1) - \sum_{i=1}^n \omega_i(X).$$

The modern version of the double point formula uses holomorphic vector bundles and Chern classes, and can be geometrically explained as follows, over the complex numbers.

Define the normal bundle N_ν to the map ν through the exact sequence associated to the derivative of ν

$$0 \rightarrow T_{X'} \rightarrow \nu^*(T_{\mathbb{P}^{2n}}) \rightarrow N_\nu \rightarrow 0.$$

The exponential map yields an oriented submersion of a neighbourhood of the 0-section of N_ν (which we identify to X') onto a neighbourhood of X , and one can find a small perturbation of the 0-section yielding an oriented submanifold X'_t intersecting X' transversally in a finite number of points, so that the self-intersection $(X')^2$ of X' in N_ν equals the Euler class of the bundle N_ν , in turn equal to the top Chern class $c_n(N_\nu)$.

It is easy to see that the self-intersection of X equals $X^2 = 2\delta + (X')^2$: since X has degree d , we get (see [21] for an algebraic proof).

THEOREM 2.3 (Modern double point formula). *Let $X \subset \mathbb{P}^{2n}$ be a subvariety of dimension n and degree d having only δ IDPs as singularities: then we have*

$$(DP) \quad d^2 = 2\delta + c_n(N_\nu) = 2\delta + c_n(\nu^*(T_{\mathbb{P}^{2n}}) - T_{X'}).$$

Using then the multiplicativity of Chern classes for exact sequences, and that the total Chern class $c(T_{\mathbb{P}^{2n}}) = c(\mathcal{O}(1))^{2n+1}$, if we denote by H the pull-back of the hyperplane class to X' , we get

$$2\delta = d^2 - [c(T_{X'})^{-1}(1+H)^{2n+1}]_n,$$

where $[c_0 + c_1 + \dots + c_n]_n$ denotes the part c_n of dimension $2n$ of the sum in the cohomology ring (or in the Chow ring).

The most interesting (not only historical) question is then: how are the two formulae related?

One deals with projective invariants of X , the other with invariants of the abstract variety X' , but indeed the Gauss map γ produces some vector bundles on X' .

Denote as usual by V the complex vector space such that $\mathbb{P}(V) = \mathbb{P}^{2n}$ (here $\mathbb{P}(V)$ is the variety of one-dimensional subspaces of V): then on the Grassmann variety $G := Gr(n, 2n)$ we have an exact sequence of vector bundles

$$0 \rightarrow U \rightarrow V \otimes \mathcal{O}_G \rightarrow Q \rightarrow 0,$$

where U is the universal subbundle, whose fibre at a subspace $W \subset V$ is tautologically equal to W , while Q is the quotient bundle.

In our situation, a point $O \in \mathbb{P} := \mathbb{P}(V)$ is the class $[v]$ of a vector $v \in V \setminus \{0\}$, and v defines a regular section of $V \otimes \mathcal{O}_G$, and, by composition, $s_v : \mathcal{O}_G \rightarrow Q$. The section s_v vanishes exactly at the subspaces W which contain v .

Taking the pull-back under the Gauss map, we get

$$0 \rightarrow \gamma^*U \rightarrow V \otimes \mathcal{O}_{X'} \rightarrow \gamma^*Q \rightarrow 0,$$

and we see then that $\omega_n(X)$ equals to the numbers of zeroes of $\gamma^*(s_\nu)$, hence

$$\omega_n(X) = c_n(\gamma^*Q).$$

The Euler sequence in $\mathbb{P}(V) =: \mathbb{P}$ pulls back to X' yielding

$$0 \rightarrow \nu^*\mathcal{U} \rightarrow V \otimes \mathcal{O}_{X'} \rightarrow \nu^*T_{\mathbb{P}}(-1) \rightarrow 0.$$

Here $\mathcal{U} \cong \mathcal{O}_{\mathbb{P}}(-1)$ is the tautological subbundle on \mathbb{P} , and an easy but important observation is the inclusion

$$\nu^*\mathcal{U} \subset \gamma^*U.$$

Denoting by $\mathcal{O}_{X'}(H) := \nu^*\mathcal{O}_{\mathbb{P}}(1)$, the derivative $D\nu$ yields, after twisting, the exact sequence

$$(*) \quad 0 \rightarrow T_{X'}(-H) \rightarrow \nu^*(T_{\mathbb{P}}(-1)) \rightarrow N_\nu(-H) \rightarrow 0.$$

Moding out the other two previous exact sequences by $\nu^*\mathcal{U}$, we get

$$(**) \quad 0 \rightarrow (\gamma^*U)/(\nu^*\mathcal{U}) \rightarrow \nu^*(T_{\mathbb{P}}(-1)) \rightarrow \gamma^*Q \rightarrow 0.$$

Since ν is an immersion, $D\nu_x : TX'_x \rightarrow (\gamma^*U/\nu^*\mathcal{U})_x$ is an isomorphism, hence $(*)$ and $(**)$ are the same exact sequences. In particular, we have an isomorphism $\gamma^*Q \simeq N_\nu(-H)$.

Restricting N_ν to a hyperplane section $H = \hat{X}'$, we obtain

$$0 \rightarrow N_\nu(-H) \rightarrow N_\nu \rightarrow N_\nu|_{\hat{X}'} := N_\nu \otimes \mathcal{O}_{\hat{X}'} \rightarrow 0,$$

yielding $c(N_\nu) = c(\gamma^*Q) \cdot c(N_\nu|_{\hat{X}'})$ for the total Chern classes.

The upshot here is as follows: the restriction $N_\nu|_{\hat{X}'}$ is the normal bundle $N_{\nu'}$ of the map $\nu' : \hat{X}' \rightarrow \mathbb{P}^{2n-1}$.

On the other hand, projection from the point O yields a map

$$\hat{\nu} : \hat{X}' \rightarrow \mathbb{P}^{2n-2}$$

such that

$$N_{\nu'} = N_{\hat{\nu}} + \mathcal{O}_{\hat{X}'}(H)$$

in the K -group, hence

$$c(N_{\hat{\nu}}) = c(N_{\nu'})(1 + H)^{-1}$$

in the Chow ring. We can then calculate

$$\begin{aligned} c_n(\gamma^*Q) &= c_n(N_\nu(-H)) = c_n(N_\nu) - Hc_{n-1}(N_\nu) + \cdots + (-1)^n H^n \\ &= c_n(N_\nu) - H(c_{n-1}(N_\nu - \mathcal{O}_{X'}(H))) = c_n(N_\nu) - i_*(c_{n-1}((N_{\nu'} - \mathcal{O}_{\hat{X}'}(H)))) \\ &= c_n(N_\nu) - i_*(c_{n-1}(N_{\hat{\nu}})), \end{aligned}$$

where $i : \hat{X}' \rightarrow X'$ is the natural inclusion.

We have thus shown

$$\omega_n(X) = c_n(N_\nu) - i_*(c_{n-1}(N_{\hat{\nu}})),$$

which, using formula (DP) for the double points, gives Severi's inductive formula.

3. Extension of Severi's statement

In this section $X \subset \mathbb{P} := \mathbb{P}^{2n}$ is an irreducible nondegenerate subvariety of dimension n having isolated singularities. We set $X^* := X \setminus \text{Sing}(X)$, and let x_1, \dots, x_h be the singular points of X .

Severi's basic idea starts with the consideration of the secant map

$$\sigma : (X \times X) \setminus \Delta_X \rightarrow \mathcal{G} := Gr(1, 2n)$$

associating to $x \neq x'$ the line $\sigma(x, x') := x * x'$.

Taking the closure Σ of the graph of σ in $(X \times X) \times \mathcal{G}$, and on it the pull-back of the projectivized universal subbundle $\mathcal{L}_{\mathcal{G}} \subset \mathcal{G} \times \mathbb{P}$ of the Grassmannian, we obtain a line incidence relation

$$\mathcal{L}_X \subset \Sigma \times \mathbb{P} \subset (X \times X) \times \mathcal{G} \times \mathbb{P},$$

which is a \mathbb{P}^1 -bundle over Σ .

There is a natural surjective projection $p : \mathcal{L}_X \rightarrow (X \times X)$ and a natural projection $\pi : \mathcal{L}_X \rightarrow \mathbb{P}$.

DEFINITION 3.1. We define as customary (cf. [18]) for $x \in X$,

$$\mathcal{L}_x := p^{-1}(\{(x, x)\}),$$

and define the tangent star of X at x as

$$Star_x(X) := \pi(\mathcal{L}_x).$$

For any smooth point $x \in X$, the tangent star is the projective tangent space of X at x , and in general one has a series of inclusions:

$$\mathcal{T}_x(X) \subset Star_x(X) \subset T^{Zar} X_x,$$

in words: the tangent cone at x is contained in the tangent star at x , which is contained in the projective closure of the Zariski tangent space.

Since Σ has dimension $2n$, the fibre Σ_x over (x, x) has dimension at most $2n - 1$, hence in general the tangent star at x has dimension at most $2n$. Moreover, Σ_x consists of lines passing through x , hence

PROPOSITION 3.1. *The tangent star is a cone with vertex x . If $X \subset \mathbb{P} := \mathbb{P}^{2n}$ has dimension n , then the following assertions (1), (2), (3) are equivalent:*

- (1) *the tangent star at x equals \mathbb{P}^{2n} ;*
- (2) *Σ_x has dimension $2n - 1$;*
- (3) *Σ_x is the Schubert cell C_x of all lines through x .*

In particular the degree of the map $\mathcal{L}_x \rightarrow \mathbb{P}^{2n}$ is always either 0 or 1.

Proof. Assertion (1) amounts to $\dim(Star_x(X)) = 2n$, which implies $\dim(\Sigma_x) \geq 2n - 1$, which is equivalent to assertion (3) since $\Sigma_x \subset C_x = \mathbb{P}^{2n-1}$, where $C_x = \mathbb{P}^{2n-1}$ is the variety (Schubert cell) of lines through x . Assertion (3) obviously implies (1), and also the last assertion is clear, since the degree is 0 if and only if the map is not surjective. \square

REMARK 3.2. (i) The IDPs are isolated singularities with *full tangent star*, that is, whose tangent star is the whole space \mathbb{P}^{2n} .

(ii) Each point of embedding dimension less than $2n$ (that is, the Zariski tangent space has dimension at most $2n - 1$) has a nonfull tangent star.

(iii) Observe that \mathcal{L}_x is taken here with the reduced structure: in general it is part of the ramification locus of π .

(iv) See [26] for some related results concerning tangent stars of complete intersections.

We can now prove a generalization of Severi's theorem (Theorem 2.1, the inductive assertion).

THEOREM 3.3. *Let $X \subset \mathbb{P} := \mathbb{P}^{2n}$ be an irreducible nondegenerate subvariety of dimension n having isolated singularities x_1, \dots, x_h only.*

Assume that these singularities are either IDPs or have tangent star of dimension at most $2n - 1$.

Let δ be the number of the IDPs: then, in the notation of Theorem 2.1, we have

$$2\hat{\delta} = 2\delta + \omega_n(X).$$

Proof. Let Z be the image of Σ inside the Grassmannian \mathcal{G} . Observe that $q : \Sigma \rightarrow Z$ factors through the involution τ on Σ birationally induced by the involution exchanging the factors of $X \times X$.

Arguing as in [6, Lemma 1], we see that $\Sigma \rightarrow Z$ is generically finite of degree 2, hence Z is irreducible of dimension $2n$, and birational to Σ/τ .

Given a point $O \in \mathbb{P}$, we let C_O be the Schubert cell of the lines through O , and we consider the curve

$$C := q^{-1}(Z \cap C_O) \subset \Sigma \subset (X \times X) \times \mathcal{G}.$$

Denote by Γ_2 the projection of C inside $X \times X$, respectively, by Γ its projection to X (because of τ symmetry, the first and second projection give the same result).

Observe that, by our assumption, for general choice of O , Γ does not pass through the singular points of X which are not IDPs.

Let X' be the normalization of X precisely at the IDPs. Then we can construct an incidence variety $\Sigma' \subset (X' \times X') \times \mathcal{G}$ which is birational to Σ . Hence we can similarly define curves C', Γ'_2, Γ' .

The proof proceeds now exactly as in [6, pp. 766–773] (but with a slightly different notation).

(Step I) Proposition 2 shows that Γ'_2 is a smooth curve, and the graph of a birational involution τ on Γ' such that, if x'_i, x''_i are the two points lying over an IDP x_i , then $\tau(x'_i) = x''_i$.

(Step II) For general choice of O, L , setting $H = O * L$, $\hat{X}' = X \cap H$ is smooth and $\hat{X}' \subset L \cong \mathbb{P}^{2n-2}$ has only IDPs. Moreover $\Gamma \cap L = \emptyset$, H intersects Γ in $\deg(\Gamma)$ distinct smooth points of Γ , and different from the points $Q_1, \dots, Q_{\omega_n(X)}$ [†] where the tangent space to X does not pass through O , so that in particular τ has no fixed points on $H \cap \Gamma$. Denote these points by R_1, \dots, R_{2m} . We observe that $m = \hat{\delta}$: since the points in $\Gamma \cap H$ contribute in pairs to IDPs of \hat{X}' .

(Step III) Let ϕ_0, ϕ_1 be linear forms such that $H = \{\phi_0 = 0\}$, $L = \{\phi_0 = \phi_1 = 0\}$, and let $\phi : \mathbb{P} \setminus L \rightarrow \mathbb{P}^1$ the projection with centre L .

Then we have a morphism $\psi : \Gamma'_2 \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1)$, the composition of the projection into $X' \times X'$ with the map $\phi \times \phi$.

(Step IV) The points Q_i are the fixed points of the involution τ , hence if Δ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, then

$$\psi^{-1}(\Delta) = \{(R_i, \tau R_i), (P'_j, P''_j), (P''_j, P'_j), (Q_h, Q_h)\}.$$

As proven in [6, p. 772] the points $\psi(Q_h, Q_h)$ are distinct, equivalently, the points $\phi(Q_h)$ are distinct, in particular ψ yields a birational map from Γ'_2 to its image Γ^0 . The symmetry τ on Γ'_2 induces the coordinate exchange on Γ^0 , so that Γ^0 is symmetric, and of bidegree $(2\hat{\delta}, 2\hat{\delta})$, since $\Gamma \cap H$ consists of $2\hat{\delta}$ smooth points.

[†]O is chosen such that C_O intersects $\overline{\gamma(X')}$ transversally at the points $\gamma(Q_i)$.

Hence the intersection number of Γ^0 with Δ equals $4\hat{\delta}$; but the previous calculation of the inverse image of the diagonal on the smooth curve Γ'_2 shows

$$4\hat{\delta} = 2\hat{\delta} + 2\delta + \omega_n(X) \Leftrightarrow 2\hat{\delta} = 2\delta + \omega_n(X). \quad \square$$

REMARK 3.4. The above proof works for any algebraically closed field of characteristic not equal to 2.

REMARK 3.2. Assume that $X \subset \mathbb{P} := \mathbb{P}^{2n}$ is an irreducible nondegenerate subvariety of dimension n having isolated singularities x_1, \dots, x_h .

We have used here Severi's original approach rather than subsequent developments, as done in [13, 14], where the authors used the Stückrad–Vogel cycle.

It seems to us that from these works follows: one can determine, for each singular point $P = x_i$, a 'number of double points' δ_P (clearly such that $\delta_P = 0$ if $\text{Star}_P(X)$ has dimension at most $2n - 1$), with the property that, defining

$$\delta = \sum_{P \in X} \delta_P,$$

we have, in the notation of Theorem 2.1,

$$2\hat{\delta} = 2\delta + \omega_n(X).$$

The main question is how to determine these numbers explicitly.

EXAMPLE 3.3. Let $X \subset \mathbb{P}^4$ be the projective cone with vertex x_0 over the twisted cubic curve $D \subset \mathbb{P}^3$. Then x_0 is the only singular point of X , and clearly its tangent star is \mathbb{P}^4 .

Here D projects to a nodal plane cubic, hence $\hat{\delta} = 1$.

Here $\omega_2(X) = 0$, since the union of the tangent planes to X has dimension 3. Hence $\delta = 1$ and the normal point x_0 contributes 1 to the number of IDPs.

Replacing D by a nondegenerate smooth curve of genus g and degree d in \mathbb{P}^3 , we see that again $\omega_2(X) = 0$, hence $2\delta_{x_0} = (d - 1)(d - 2) - 2g$.

4. Comparing normal sheaf and quotient bundle

As in the preceding section (where we have computed the number δ of IDPs) we consider irreducible subvarieties $X \subset \mathbb{P} := \mathbb{P}^{2n}$ of dimension n and with isolated singularities, which are either IDPs or have degenerate tangent star (that is, of dimension at most $2n - 1$).

In this section we slightly change our notation and denote the normalization of X by X' .

In the case where X is a surface, we shall take $\tilde{X} = S$ to be a minimal resolution of singularities of X' , but in general \tilde{X} shall be a resolution of singularities of X' , and $Y \rightarrow \tilde{X}$ a further resolution such that the Gauss map becomes a morphism

$$\gamma : Y \rightarrow G = \text{Gr}(n, 2n).$$

We let $f : Y \rightarrow \mathbb{P}$ be the morphism composition of the two maps $Y \rightarrow X' \rightarrow X \subset \mathbb{P}$, and we shall use the following self-explanatory notation: for instance $Q_Y = \gamma^*Q$.

We set $\mathcal{O}_Y(H) := f^*\mathcal{O}_{\mathbb{P}}(1)$. Then, as in Section 2,

$$(*) \quad 0 \rightarrow T_Y(-H) \rightarrow f^*(T_{\mathbb{P}}(-1)) \rightarrow N_f(-H) \rightarrow 0$$

and

$$(**) \quad 0 \rightarrow U_Y/\mathcal{U}_Y \rightarrow f^*(T_{\mathbb{P}}(-1)) \rightarrow Q_Y \rightarrow 0.$$

Since f is generically an immersion, Df yields a sheaf inclusion $T_Y(-H) \subset U_Y/\mathcal{U}_Y$ and associated exact sequences

$$\begin{aligned} 0 &\rightarrow T_Y(-H) \rightarrow U_Y/\mathcal{U}_Y \rightarrow \mathcal{F} \rightarrow 0, \\ (***) \quad 0 &\rightarrow \mathcal{F} \rightarrow N_f(-H) \rightarrow Q_Y \rightarrow 0, \end{aligned}$$

where the sheaf \mathcal{F} is supported at the inverse image of the (finitely many) singular points of X which are different from the IDPs.

DEFINITION 4.1. We define the sheaf \mathcal{F} on Y to be the IDP discrepancy sheaf on Y .

Note that Y is not unique, since not always there is a minimal resolution of the closure of the Gauss map.

THEOREM 4.1. *Let $X \subset \mathbb{P}^{2n}$ be a subvariety of dimension n and degree d having only isolated singularities, of which δ are IDPs, and the rest are singular points with degenerate tangent star and whose associated discrepancy sheaf \mathcal{F} on Y is equal to zero $\mathcal{F} = 0$.*

Then we have

$$d^2 = 2\delta + c_n(N_f) = 2\delta + c_n(f^*(T_{\mathbb{P}^{2n}}) - T_Y).$$

If instead $\mathcal{F} \neq 0$ on Y , we have

$$d^2 = 2\delta + c_n(N_f) + c_n(N_f(-H) - \mathcal{F}) - c_n(N_f(-H))$$

or, equivalently,

$$d^2 = 2\delta + c_n(N_f) + [c(N_f(-H)) \cdot c(\mathcal{F})^{-1}]_n - c_n(N_f(-H)).$$

Proof. The proof of the first assertion follows from Theorem 3.3, and the argument given in Section 2 to show how to relate $c_n(Q_Y)$ with $c_n(N_f)$.

The proof of the second assertion follows with the same derivation as above plugging in the relation $c(Q_Y) = c(N_f(-H))c(\mathcal{F})^{-1}$. \square

Of course, for most applications, one has to determine the sheaf \mathcal{F} ; this can be done by explicit but sometimes lengthy calculations.

For instance, in the case of isolated hypersurface singularities

$$X = \{x | F(x) = 0\} \subset W := \mathbb{C}^{n+1},$$

the tangent plane at the smooth points $x \in X^* := X \setminus \{0\}$ is the dual (annulator) subspace to the gradient at x . Composing the gradient map with the projection onto $\mathbb{P}(W^\vee)$ (the projective space of lines in the vector space W^\vee),

$$\nabla F : X^* \rightarrow W^\vee \setminus \{0\}, \quad p : W^\vee \setminus \{0\} \rightarrow \mathbb{P}(W^\vee) =: \mathbb{P}, \quad f := p \circ \nabla F$$

and letting Z be the closure of the graph of f , we see that the pull-back Q_Z of the quotient bundle is locally the direct sum of a trivial summand with $f^*(\mathcal{O}_{\mathbb{P}}(1))$.

Indeed, if $L \cong \mathcal{O}_{\mathbb{P}}(-1)$ is the universal subbundle on \mathbb{P} , the exact sequence

$$0 \rightarrow L \rightarrow W^\vee \otimes \mathcal{O}_{\mathbb{P}} \rightarrow T_{\mathbb{P}}(-1) \rightarrow 0$$

dualizes to

$$0 \rightarrow L^{ann} = \Omega_{\mathbb{P}}^1(1) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}} \rightarrow L^\vee = \mathcal{O}_{\mathbb{P}}(1) \rightarrow 0.$$

For simplicity of notation we continue to denote by f the map of Y to $\mathbb{P}(W^\vee)$ obtained from the resolution $Y \rightarrow Z$.

Since N_f is the cokernel of $T_Y \rightarrow f^*(W \otimes \mathcal{O}_{\mathbb{P}})$, we have exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{F} \rightarrow N_f \rightarrow f^*(\mathcal{O}_{\mathbb{P}}(1)) \rightarrow 0, \\ 0 \rightarrow T_Y \rightarrow f^*(\Omega_{\mathbb{P}}^1(1)) \rightarrow \mathcal{F} \rightarrow 0. \end{aligned}$$

The second sequence shows that the discrepancy sheaf \mathcal{F} is a Cohen–Macaulay sheaf of codimension 1 on Y , supported on an exceptional divisor.

The first sequence instead identifies, in a neighbourhood of this divisor, the virtual sheaf $N_f(-H) - \mathcal{F}$ as a line bundle: $f^*(\mathcal{O}_{\mathbb{P}}(1))$.

EXAMPLE 4.2. Consider a hypersurface singularity in \mathbb{C}^3 , of multiplicity equal to d , and with tangent cone equal to the cone over a smooth plane curve C .

Let the local equation be

$$X := \{x = (x_1, x_2, x_3) \mid F(x) = 0, F(x) = P_d(x) + \dots\},$$

so that the blow-up Y of X at the origin is smooth,

$$Y \subset \mathbb{C}^3 \times \mathbb{P}^2, Y = \{(x, u) \mid x = \lambda u, \lambda^{-d}F(\lambda x) = 0\},$$

and the Gauss map is a morphism on Y , which we view as a hypersurface in the universal subbundle over \mathbb{P}^2 . Here $C = \{u \mid P(u) = 0\}$ is identified to the exceptional curve of the blow-up, and on Y we have $C^2 = -d$.

Here U/\mathcal{U} , at the points of C , equals the cone U' over the tangent line $T_u C$. If $\phi(t)$ is a local parametrization of C , $(t, \lambda) \mapsto \lambda\phi(t)$, hence the image of the tangent space to Y is the subspace generated by $\lambda\phi'(t)$ and $\phi(t)$, while U' is generated by $\phi'(t)$ and $\phi(t)$. Hence the cokernel \mathcal{F} is a line bundle on C , equal to $T_C(C) = \mathcal{O}_C((d-2)C)$.

Therefore $c(\mathcal{F})^{-1} = 1 - C + C(d-2)C$, and the contribution to 2δ equals

$$-c_1(N_f)(-C) - (d-2)C^2 = K_Y \cdot C + d(d-2) = -(d-2)C^2 + d(d-2) = 2d(d-2).$$

The previous calculation shows that we get a positive contribution to δ , equal to $d(d-2)$: this contribution is equal to 0 if and only if we have a double point.

REMARK 4.3. (a) Similar calculations could be done in order to show that the contribution we obtain equals 0 for all rational double points, which are hypersurface singularities with equation $z^2 = f(x, y)$, where f has multiplicity at most 3, and no infinitely near triple point. It is to observe that Y is not here a minimal resolution of the singularity.

We do not pursue this calculation here, since we shall give an elementary proof in a later section that rational double points do not contribute to δ .

(b) In the previous example, the Milnor number of the singularity equals $(d-1)^3$, while the topological Euler characteristic of the exceptional divisor C is $e(C) = -d(d-3)$, hence $e(Y) - e(X) = -[d(d-3) + 1]$.

Indeed, in this case the contribution equals $2[(d-1)^2 - 1]$ which is twice the Milnor number μ_1 of a general hyperplane section through the singular point, diminished by 1. This reminds of a formula by Piene in [24], end of page 25; however the number $1 - \mu_1$ is not zero for rational double points $z^2 - x^3 - y^3 = 0$, or $z^2 - x^3 - y^5 = 0$ (it is -1). This shows that our contribution is not the local Euler obstruction.

5. Surfaces with rational double points

In this section we work over any algebraically closed field k of characteristic $p \geq 0$ and we show the following theorem for surfaces.

THEOREM 5.1. *Let S be a smooth projective surface and let*

$$\mu : S \rightarrow X \subset \mathbb{P}^4$$

be the minimal resolution of a surface X whose singularities are only $\delta := \delta(X)$ IDPs, and some rational double points. Then

$$2\delta = d^2 - 10d + 12\chi(\mathcal{O}_S) - 5(H_S \cdot K_S)_S - 2(K_S^2)_S .$$

*Here $H_S = \mu^*H$ for the hyperplane class H of \mathbb{P}^4 and $d = (H_S^2)_S = \deg X$.*

Theorem 5.1 will be applied in the following natural situation.

Let S be a smooth minimal surface and let $f : S \rightarrow S'$ be either the morphism to the canonical model for S of general type, or a birational morphism onto a normal surface S' for S with numerically trivial canonical class. Then f is crepant and S' has only rational double points as its singularities. As S' is always embedded into some projective space \mathbb{P}^N ($N \geq 5$), we may assume that $S' \subset \mathbb{P}^N$. Then, as the embedded dimension of any rational double point is 3, it follows that any general linear projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^4$, as explained above, induces a morphism

$$\pi|_{S'} : S' \rightarrow X := \pi(S') \subset \mathbb{P}^4$$

such that $\pi|_{S'}$ is isomorphic at all rational double points of S' and X has only these rational double points and finitely many IDPs. So, the morphism

$$\pi|_{S'} \circ f : S \rightarrow X \subset \mathbb{P}^4$$

satisfies the assumption made in Theorem 5.1. In fact, this gives, among other things, a generalization of a result of the first author [7, Proposition 6.2, Corollary 6.3] see also [9, Theorem 0.2], and an affirmative answer to the conjecture made there.

COROLLARY 5.2. *Let S be a minimal smooth surface of general type with $p_g(S) = 5$ such that the canonical map*

$$\Phi_{|K_S|} : S \rightarrow S' \subset \mathbb{P}^4$$

is a morphism with image isomorphic to the canonical model S' of S . Then, setting $d := (K_S^2)_S$,

$$12\chi(\mathcal{O}_S) = (17 - d)d .$$

In particular, S' is then a complete intersection of type $(2, 4)$ or $(3, 3)$, if either the base field is of characteristic 0 or the characteristic of the field k is odd and $h^1(\mathcal{O}_S) = 0$.

Proof. We have $\delta = 0$, $(H_S^2)_S = (K_S^2)_S = d$ by our assumption. So, the first equality follows from Theorem 5.1. The last statement then follows from the proof of [9, Theorem 0.2]. Note that, in positive odd characteristic case, we have $h^1(mK_S) = 0$ for all $m \geq 1$ by our assumption (for $m = 1$) and [12, Theorem 1.7] (for $m \geq 2$); hence $\chi(\mathcal{O}_S) = 6$ by our assumption, and the rest of the proof follows as in [7, p. 41]. \square

PROBLEM 5.1. Are there exceptions to the above statement that S' is a complete intersection in odd positive characteristic for $H^1(\mathcal{O}_S) \neq 0$, or in characteristic $p = 2$?

The above corollary was actually an initial motivation for our study. The following fact will be used frequently in the proof of Theorem 5.1.

LEMMA 5.3. *Let $\pi : V \rightarrow W$ be the blow-up at $Q \in W$ of a smooth projective variety W of dimension 4, let $E = \pi^{-1}(Q) \simeq \mathbb{P}^3$ be the exceptional divisor and let $P \simeq \mathbb{P}^2$ be a plane in $E \simeq \mathbb{P}^3$. Then, in the Chow ring of V , we have:*

$$c_1(V) = \pi^*c_1(W) - 3E, \quad c_2(V) = \pi^*c_2(W) - 2P, \quad (P \cdot P)_V = -1.$$

Proof. As π is the blow-up at a smooth point of a fourfold, we have that the Chow ring $A(V)$ of V is generated by the pull-back of the Chow ring of W and by the exceptional divisor.

We have moreover $E^2 = E|_E = -P$. Thus

$$(P \cdot P)_V = (E^4)_V = ((E|_E)^3)_E = ((-P)^3)_E = -1.$$

This proves the last formula. As $c_1(V) = -K_V$ and $c_1(W) = -K_W$, the first formula is nothing but the canonical bundle formula under the blow-up.

Let us show the second formula. We have $A^2(V) = \pi^*A^2(W) \oplus \mathbb{Z}[P]$ and, evaluating on two-dimensional cycles avoiding the exceptional divisor, we infer that we can write $c_2(V) = \pi^*c_2(W) + aP$. In order to determine a , consider the exact sequence

$$0 \rightarrow T_P \rightarrow T_V|_P \rightarrow N_{V/P} \rightarrow 0.$$

By functoriality of the Chern class, we have

$$(A) \quad (c_2(V) \cdot P)_V = (c_1(P) \cdot c_1(N_{V/P}))_P + c_2(P) + c_2(N_{V/P}).$$

The left-hand side is then

$$(c_2(V) \cdot P)_V = (\pi^*c_2(W) + aP \cdot P) = a(P \cdot P) = -a.$$

We compute the right-hand side. Note that $c_1(P) = -K_P = 3l$ in the Chow ring, where l is a line in $P \simeq \mathbb{P}^2$. We have

$$(B) \quad c_1(V)|_P = c_1(P) + c_1(N_{V/P}) = 3l + c_1(N_{V/P})$$

from the exact sequence above. Note that, as $l, P \subset E$, we have

$$(E|_P \cdot l)_P = (E|_E \cdot l)_E = (-P \cdot l)_E = -1.$$

In particular, $E|_P = -l$ in the Chow ring of P . Combining this with the first equality, we have

$$c_1(V)|_P = (\pi^*c_1(W) - 3E)|_P = -3E|_P = 3l.$$

Hence $c_1(N_{V/P}) = 0$ by (B). Since $P \subset V$ is a smooth subvariety of codimension $2 = 4/2$, by the self-intersection formula and the last formula $(P^2)_V = -1$, we have $c_2(N_{V/P}) = (P^2)_V = -1$. As $P \simeq \mathbb{P}^2$, we have $c_2(P) = 3$. Substituting everything into formula (A), we obtain $-a = 0 + 3 - 1$, hence $a = -2$ as claimed. \square

The rest of this section is devoted to prove Theorem 5.1.

Proof. Let x_j , for $1 \leq j \leq \delta$, be the IDPs of X . Consider the blow-up

$$f : V_0 \rightarrow \mathbb{P}^4$$

of \mathbb{P}^4 at the points x_j , $1 \leq j \leq \delta$.

Let S_0 be the proper transform of X and $f|_{S_0} : S_0 \rightarrow X$ be the induced morphism. Then $f|_{S_0}$ is an isomorphism except over the points x_j , S_0 is smooth over the points x_j and $f|_{S_0}^{-1}(x_j)$ consists of two (-1) -curves, that is, two smooth rational curves of self-intersection number (-1) . We denote these two curves by l_{j1} and l_{j2} .

Next, we are going to take the minimal embedded resolution of $S_0 \subset V_0$ via a sequence of point blow-ups. Let

$$g_1 : V_1 \rightarrow V_0$$

be the blow-up of V_0 at a rational double point, say P , of S_0 and define S_1 to be the proper transform of S_0 under g_1 . Then we have an induced morphism $g_1|_{S_1} : S_1 \rightarrow S_0$ which is the same as the blow-up of S_0 at the maximal ideal of P . The special properties of rational double points that we need are that the canonical divisor K_{S_0} is Cartier, S_1 continues to have only rational double points as singularities, and the morphism $g_1|_{S_1}$ is crepant, that is, $(g_1|_{S_1})^*K_{S_0} = K_{S_1}$ (see [4]). Now we choose an embedded resolution of $S_0 \subset V_0$ inductively by

$$g_i : V_i \rightarrow V_{i-1}, \quad g_i|_{S_i} : S_i \rightarrow S_{i-1}$$

($i = 1, 2, \dots, n$), where g_i is the blow-up of S_{i-1} at some (singular) rational double point, S_i is the proper transform of S_{i-1} , and S_n is smooth.

We set

$$\tilde{S} := S_n, \quad g|_{\tilde{S}} : \tilde{S} \rightarrow S_0,$$

where $g|_{\tilde{S}}$ is the morphism induced by $g : V_n \rightarrow V_0$, the composition of the functions g_i . We also denote respective proper transforms of l_{j1}, l_{j2} on \tilde{S} by the same letter. This is harmless for us, as they are disjoint from the exceptional divisors of g .

Clearly \tilde{S} is birational to the original S in our Theorem 5.1. Recall that we started from the birational morphism $\mu : S \rightarrow X \subset \mathbb{P}^4$. Indeed, by our assumption, the birational map $\nu := \mu \circ f|_{S_0} \circ g|_{\tilde{S}} : \tilde{S} \rightarrow S$ is the contraction morphism of the 2δ (-1) -curves l_{j1}, l_{j2} . We denote the exceptional divisor of $g_i : V_i \rightarrow V_{i-1}$ by E_i and let $P_i \simeq \mathbb{P}^2$ be a plane of $E_i \simeq \mathbb{P}^3$.

LEMMA 5.4. *Under the above setting,*

$$S_i = g_i^*S_{i-1} - 2P_i, \quad (S_i \cdot P_i)_{V_i} = 2$$

in the Chow ring of V_i .

Proof. Since $A^2(V_i) = g_i^*A^2(V_{i-1}) \oplus \mathbb{Z}P_i$, one can write $S_i = g_i^*S_{i-1} + aP_i$ for some integer a . Then $(S_i \cdot P_i)_{V_i} = a(P_i \cdot P_i)_{V_i} = -a$ by the formula above and Lemma 5.3. On the other hand, as g_i is the blow-up at a rational double point of S_i , it follows that $S_i|_{E_i}$ is a plane conic curve in $E_i \simeq \mathbb{P}^3$, while P_i is a plane in $E_i \simeq \mathbb{P}^3$. Thus

$$(S_i \cdot P_i)_{V_i} = (S_i|_{E_i} \cdot P_i)_{E_i} = 2.$$

Therefore $a = -2$ as claimed. □

We set

$$\chi_i := (c_2(V_i) \cdot S_i)_{V_i} + (c_1(V_i)|_{S_i} \cdot K_{S_i})_{S_i} + 2(K_{S_i}^2)_{S_i} - (S_i \cdot S_i)_{V_i}$$

for all integers i with $0 \leq i \leq n-1$.

LEMMA 5.5. *In the above setting, $12\chi(\mathcal{O}_S) = \chi_{n-1}$.*

Proof. Since $S_n \subset V_n$ are both smooth, we have the exact sequence

$$0 \rightarrow T_{S_n} \rightarrow T_{V_n}|_{S_n} \rightarrow N_{V_n/S_n} \rightarrow 0,$$

and we obtain

$$(A) : c_1(V_n)|_{S_n} = c_1(S_n) + c_1(N_{V_n/S_n}) = -K_{S_n} + c_1(N_{V_n/S_n}).$$

By Lemma 5.3, we compute the left-hand side of (A) as

$$c_1(V_n)|_{S_n} = (g_n|_{S_n})^*(c_1(V_{n-1})|_{S_{n-1}}) - 3R_n,$$

where $R_n := E_n|_{S_n}$. Substituting this into (A), we obtain

$$(B) : c_1(N_{V_n/S_n}) = (g_n|_{S_n})_1^*(V_{n-1})|_{S_{n-1}} - 3R_n + K_{S_n}.$$

Again from the normal bundle sequence, we obtain

$$(C) : c_2(V_n)|_{S_n} = c_1(N_{V_n/S_n})c_1(S_n) + c_2(S_n) + c_2(N_{V_n/S_n}).$$

Using Lemma 5.3, we compute the left-hand side of (C) as

$$c_2(V_n)|_{S_n} = (g_n^*c_2(V_{n-1}) \cdot S_n)_{V_n} - 2(P_n \cdot S_n)_{V_n} = (c_2(V_{n-1}) \cdot S_{n-1})_{V_{n-1}} - 4.$$

We compute now each term of the right-hand side of (C). Since $g_n|_{S_n}$ is crepant, we have $(R_n \cdot K_{S_n})_{S_n} = 0$. Thus, by using (B), we compute

$$\begin{aligned} (c_1(N_{V_n/S_n}) \cdot c_1(S_n))_{S_n} &= ((g_n|_{S_n})^*(c_1(V_{n-1})|_{S_{n-1}}) \cdot (-K_{S_n}))_{S_n} - (K_{S_n}^2)_{S_n} \\ &= (c_1(V_{n-1})|_{S_{n-1}} \cdot -K_{S_{n-1}})_{S_{n-1}} - (K_{S_{n-1}}^2)_{S_{n-1}}. \end{aligned}$$

Using Noether's formula for the smooth projective surface S_n , the fact that $\chi(\mathcal{O}_S)$ is invariant under birational smooth modification, and the fact that $g_n|_{S_n}$ is crepant, we compute

$$c_2(S_n) = 12\chi(\mathcal{O}_{S_n}) - (K_{S_n}^2)_{S_n} = 12\chi(\mathcal{O}_S) - (K_{S_{n-1}}^2)_{S_{n-1}}.$$

By using again the self-intersection formula $c_2(N_{V_n/S_n}) = (S_n^2)_{V_n}$ and the formula $S_n = g_n^*S_{n-1} - 2P_n$ (Lemma 5.4), we have

$$c_2(N_{V_n/S_n}) = (S_n \cdot S_n)_{V_n} = (S_{n-1} \cdot S_{n-1})_{V_{n-1}} - 4.$$

Substituting all into equation (C), we obtain that

$$\begin{aligned} (c_2(V_{n-1}) \cdot S_{n-1})_{V_{n-1}} - 4 &= (c_1(V_{n-1})|_{S_{n-1}} \cdot -K_{S_{n-1}})_{S_{n-1}} - 2(K_{S_{n-1}}^2)_{S_{n-1}} \\ &\quad + (S_{n-1} \cdot S_{n-1})_{V_{n-1}} - 4 + 12\chi(\mathcal{O}_S). \end{aligned}$$

By the definition of χ_i , this equality is equivalent to $12\chi(\mathcal{O}_S) = \chi_{n-1}$. This completes the proof of Lemma 5.5. \square

LEMMA 5.6. *The number χ_i is independent of i , for all $0 \leq i \leq n-1$.*

Proof. Let $i \geq 1$. Then by Lemmas 5.3 and 5.4, we have

$$\begin{aligned} (c_2(V_i) \cdot S_i)_{V_i} &= ((g_i^*c_2(V_{i-1}) - 2P_i) \cdot (g_i^*S_{i-1} - 2P_i))_{V_i} = (c_2(V_{i-1}) \cdot S_{i-1})_{V_{i-1}} - 4, \\ (S_i \cdot S_i)_{V_i} &= ((g_i^*S_{i-1} - 2P_i) \cdot (g_i^*S_{i-1} - 2P_i))_{S_i} = (S_{i-1} \cdot S_{i-1})_{V_{i-1}} - 4. \end{aligned}$$

Using Lemma 5.3 and observing that $K_{S_i} = (g_i|_{S_i})^*K_{S_{i-1}}$, we have

$$(c_1(V_i)|_{S_i} \cdot K_{S_i})_{S_i} = ((g_i^*c_1(V_{i-1}) - 2P_i)|_{S_i} \cdot (g_i|_{S_i})^*K_{S_{i-1}}) = (c_1(V_{i-1})|_{S_{i-1}} \cdot K_{S_{i-1}})_{S_{i-1}},$$

and $(K_{S_i}^2)_{S_i} = (K_{S_{i-1}}^2)_{S_{i-1}}$. Substituting all these equalities into the formula in Lemma 5.6, we obtain that $\chi_i = \chi_{i-1}$. This implies the result. \square

LEMMA 5.7. *In the above setting,*

$$12\chi(\mathcal{O}_S) = (c_2(V_0) \cdot S_0)_{V_0} + (c_1(V_0)|_{S_0} \cdot K_{S_0})_{S_0} + 2(K_{S_0}^2)_{S_0} - (S_0 \cdot S_0)_{V_0}.$$

Proof. By Lemmas 5.5 and 5.4, we have $12\chi(\mathcal{O}_S) = \chi_{n-1} = \chi_0$. The right-hand side of the formula in Lemma 5.7 is by definition equal to χ_0 . This completes the proof. \square

LEMMA 5.8. *In the above situation, and with the usual notation where H_S is the pull-back of the hyperplane class H of \mathbb{P}^4 under $\mu : S \rightarrow X \subset \mathbb{P}^4$ and $d = (H_{\tilde{S}}^2)_S = \deg X$, we have*

$$12\chi(\mathcal{O}_S) = 10d - d^2 + 2(K_{\tilde{S}}^2)_S + 5(H_S \cdot K_S) + 2\delta.$$

Proof. We shall prove the formula by computing each term of the right-hand side of the equation in Lemma 5.7.

Let $F_j \simeq \mathbb{P}^3$ be the exceptional divisor of $f : V_0 \rightarrow \mathbb{P}^4$ over x_j and let $R_j \simeq \mathbb{P}^2$ be a plane in F_j .

Then $c_*(\mathbb{P}^4) = (1 + H)^5$ in the Chow ring of \mathbb{P}^4 , and

$$c_1(V_0) = f^*c_1(\mathbb{P}^4) - 3 \sum_{j=1}^{\delta} F_j,$$

$$c_2(V_0) = f^*c_2(\mathbb{P}^4) - 2 \sum_{j=1}^{\delta} R_j, \quad (R_j \cdot R_j)_{V_0} = -1$$

by Lemma 5.3, and

$$S_0 = f^*X - 2 \sum_{j=1}^{\delta} R_j = f^*(dH^2) - 2 \sum_{j=1}^{\delta} R_j$$

in the Chow ring of V_0 . Hence

$$(c_2(V_0) \cdot S_0)_{V_0} = ((10f^*H^2 - 2 \sum_{j=1}^{\delta} R_j) \cdot (df^*H^2 - 2 \sum_{j=1}^{\delta} R_j))_{V_0} = 10d - 4\delta,$$

$$(S_0 \cdot S_0) = ((df^*(H^2) - 2 \sum_{j=1}^{\delta} R_j)^2) = d^2 - 4\delta.$$

As $g : \tilde{S} \rightarrow S_0$ is crepant and $\tilde{S} \rightarrow S$ contracts exactly 2δ disjoint (-1) -curves, we obtain

$$(K_{S_0}^2)_{S_0} = (K_{\tilde{S}}^2)_{\tilde{S}} = (K_S^2)_S - 2\delta.$$

Finally we compute $(c_1(V_0)|_{S_0} \cdot K_{S_0})_{S_0}$. First of all, we have

$$\begin{aligned} (c_1(V_0)|_{S_0} \cdot K_{S_0})_{S_0} &= ((f \circ g)^*c_1(\mathbb{P}^4)|_{\tilde{S}} \cdot K_{\tilde{S}})_{\tilde{S}} \\ &= \left(5H_{\tilde{S}} - 3 \sum_{j=1}^{\delta} F_j|_{\tilde{S}} \right) \cdot K_{\tilde{S}} = (5H_{\tilde{S}} \cdot K_{\tilde{S}})_{\tilde{S}} - 3 \sum_{j=1}^{\delta} (l_{j1} + l_{j2}) \cdot K_{\tilde{S}}|_{\tilde{S}}. \end{aligned}$$

Here F_j is the proper transform of the exceptional divisor $f^{-1}(x_j)$ of f and we used the fact that $g|_{\tilde{S}} : \tilde{S} \rightarrow S_0$ is crepant for the second equality so there appears no exceptional divisor of g_i in the formula. As l_{j1} and l_{j2} are (-1) -curves on a smooth surface \tilde{S} , we have $(l_{j1} \cdot K_{\tilde{S}})_{\tilde{S}} = (l_{j2} \cdot K_{\tilde{S}})_{\tilde{S}} = -1$. Moreover, since the morphism $\nu : \tilde{S} \rightarrow S$ defined above is the contraction of exactly 2δ (-1) -curves l_{j1} and l_{j2} , it follows that

$$(H_{\tilde{S}} \cdot K_{\tilde{S}})_{\tilde{S}} = \left(\nu^*H_S \cdot \left(\nu^*K_S + \sum_{j=1}^{\delta} (l_{j1} + l_{j2}) \right) \right)_{\tilde{S}} = (H_S \cdot K_S)_S.$$

Substituting these two formulae into the last formula for $(c_1(V_0)|_{S_0} \cdot K_{S_0})_{S_0}$, we obtain that $(c_1(V_0)|_{S_0} \cdot K_{S_0})_{S_0} = 5(H_S \cdot K_S)$. Substituting the four formulae that we obtained for $(c_2(V_0) \cdot S_0)_{V_0}$, $(c_1(V_0)|_{S_0} \cdot K_{S_0})_{S_0}$, $(K_{S_0}^2)_{S_0}$ and $(S_0 \cdot S_0)_{V_0}$ inside the formula in Lemma 5.7, we obtain

$$12\chi(\mathcal{O}_S) = 10d - 4\delta + 5(H_S \cdot K_S) + 6\delta + 2(K_{\tilde{S}}^2) - 4\delta - d^2 + 4\delta.$$

Simplifying the right-hand side, we obtain

$$12\chi(\mathcal{O}_S) = 10d - d^2 + 2(K_S^2)_S + 5(H_S \cdot K_S)_S + 2\delta,$$

as claimed. \square

This completes the proof of Theorem 5.1. \square

6. Double point formulae via symplectic approximations

We introduce, inspired by a concept introduced by Kodaira in [20], a class of isolated singularities.

DEFINITION 6.1. An n -dimensional isolated singularity $O \in X \subset \mathbb{C}^{2n}$ is said to be *quasi-improper multiple point (QIMP)*, if X consists of r smooth branches X_1, \dots, X_r passing through the origin O .

To explain the notion, one can take a good projection yielding a splitting $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$, so that

$$X_i = \{(x, y) \in \mathbb{C}^n \oplus \mathbb{C}^n \mid y = F_i(x)\}.$$

Clearly the intersection points $X_i \cap X_j$ correspond to the set

$$\Gamma_{ij} := \{x \in \mathbb{C}^n \mid F_i(x) - F_j(x) = 0\}.$$

The hypothesis of isolated singularities amounts to the requirement that $0 \in \mathbb{C}^n$ is isolated in the locus Γ_{ij} , and, setting $\mathcal{O} := \mathcal{O}_{\mathbb{C}^n, 0}$, we consider the intersection multiplicity

$$m_{ij} := \dim_{\mathbb{C}}(\mathcal{O}/(F_i - F_j)\mathcal{O}^n),$$

where $F_i - F_j$ is considered as a $1 \times n$ matrix. It is clear that, for a generic perturbation of the branches X_i , the isolated singularity deforms to

$$\delta_O := \sum_{i < j} m_{ij}$$

IDPs. This is why we shall say that δ_O is the *local number of IDPs*.

THEOREM 6.1. *Let $X \subset \mathbb{P}^{2n}$ be a complex variety with isolated singularities, of which*

- (1) *h are quasi-improper-multiple points, such that the sum of the local numbers of IDPs equals δ ;*
- (2) *the other singular points are normal and locally smoothable.*

Then X admits a global smoothing to a symplectic immersed manifold $M \subset \mathbb{P}^{2n}$, with exactly δ IDPs, and we have, if $f : M' \rightarrow M$ is the immersion, that

$$d^2 = 2\delta + e(N_f),$$

where e is the Euler class of the oriented normal bundle to the map.

Proof. The result follows from [8, Theorem 1.2], and a local deformation at the nonnormal singularities, showing that we can deform X to a symplectic immersed manifold $M \subset \mathbb{P}^{2n}$, with exactly δ IDPs.

Letting $f : M' \rightarrow M$ be the immersion, we calculate the self-intersection d^2 of M in \mathbb{P}^{2n} (in the same way as explained in the introduction) as 2δ plus the self-intersection number

of the zero section in the normal bundle N_f (which maps via the exponential map onto a neighbourhood of X). \square

REMARK 6.2. A special case is the one where the normal singularities are isolated hypersurface singularities.

The following was the original idea which led us to realize that the Severi double point formula for smooth surfaces (that is, smooth outside of the IDPs) holds verbatim if we also allow rational double points, at least over \mathbb{C} .

In fact, for surface rational double points over \mathbb{C} , one has the fortunate coincidence that M' coincides with the minimal resolution $\pi : S \rightarrow X$, and then one has only to observe that π and f are a differentiable deformation of each other, which can be taken as the identity outside the inverse image S_B of a sufficiently small neighbourhood B of the normal singularities.

In this case, a possible way to show the equality between $e(N_f)$ and $c_2(N_\pi)$, following the arguments of [8], could be to argue that the Euler number of N_f is obtained by integrating the top Chern form of N_π outside S_B , plus adding an integral on the Milnor fibre M'_B , diffeomorphic to the neighbourhood S_B of the exceptional divisor: this integral should yield the top Chern class integral on the tangent bundle of S_B , and then the desired equality would follow.

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